Higher symmetries of relativistic wave equations in curved spacetime

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Outline

This seminar will be loosely split up into three parts:

- Background and motivation
- Statement of results
- Further research

Symmetry in physics

The story of modern theoretical physics is inextricably linked with the study of symmetry in its myriad manifestations.

Some famous examples of symmetry and uses of symmetry in physics include

- Special and general relativity: Invariance under Poincaré and general coordinate transformations respectively dictate dynamics to a large extent.
- Noether's theorem: Every continuous symmetry of a system generates a conserved quantity.
- Wigner and group theory: Studying unitary irreducible representations of the Poincaré group explains the origin of spin and elementary particles in quantum field theory [1].

Reminder on differential geometry

A torsion-free, metric compatible, covariant derivative, ∇_m, is an extension to the partial derivative, ∂_m, for differential geometry. Then,

$$[\nabla_m, \nabla_n]V^p = R^p{}_{qmn}V^q,$$

where $R^p{}_{qmn}$ is the Riemann tensor. Some descendants include

$$\begin{aligned} &R_{mn} = R^p{}_{mpn}, \;\; R = R^m{}_m \;\text{ and} \\ &C_{mnpq} = R_{mnpq} + \frac{1}{2}(g_{mq}R_{np} + g_{np}R_{mq} - g_{mp}R_{nq} - g_{nq}R_{mp}) \\ &\qquad\qquad + \frac{1}{6}(g_{mp}g_{nq}R - g_{mq}g_{np}R)\,. \end{aligned}$$

e.g. The Einstein field equation is

$$R_{mn} - \frac{1}{2}g_{mn}R = 8\pi G T_{mn}\,.$$

Killing vectors

An isometry of the metric, $g_{mn}(x)$, is a change of coordinates, $x^m \to x'^m$ such that

$$g'_{mn}(x) = g_{mn}(x)\,.$$

When $x'^m = x^m - \xi^m(x)$ for infinitesimal $\xi^m(x)$,

$$\delta g_{mn}(x) = g'_{mn}(x) - g_{mn}(x) = \nabla_m \xi_n(x) + \nabla_n \xi_m(x)\,.$$

A Killing vector, $\xi^m(x)$, is one that satisfies

$$\nabla_m \xi_n(x) + \nabla_n \xi_m(x) = 0$$

and hence generates symmetries of a curved spacetime, e.g.

$$\xi_m(x(\lambda)) \frac{\mathrm{d}x^m(\lambda)}{\mathrm{d}\lambda}$$

is conserved along a geodesic.

Killing tensors

A symmetric tensor, $\xi^{m_1 \cdots m_a}(x)$, is called “Killing” if and only if

$$\nabla^{(n} \xi^{m_1 \cdots m_a)}(x) = 0 \,.$$

For any Killing tensor, $\xi^{m_1 \cdots m_a}(x)$,

$$\xi_{m_1 \cdots m_a}(x(\lambda)) \frac{\mathrm{d}x^{m_1}(\lambda)}{\mathrm{d}\lambda} \cdots \frac{\mathrm{d}x^{m_a}(\lambda)}{\mathrm{d}\lambda}$$

is conserved along a geodesic.

Conformal Killing vectors

A conformal Killing vector, $\xi^m(x)$, is one which preserves the metric up to scale. For that, when $x'^m = x^m - \xi^m(x)$,

$$\delta g_{mn}(x) = g'_{mn}(x) - g_{mn}(x) = \nabla_m \xi_n(x) + \nabla_n \xi_m(x) \propto g_{mn}(x)$$
$$\implies \nabla_m \xi_n + \nabla_n \xi_m = \frac{1}{2} g_{mn} \nabla_p \xi^p \,.$$

For a conformal Killing vector, $\xi^m(x)$,

$$\xi_m(x(\lambda)) \frac{\mathrm{d}x^m(\lambda)}{\mathrm{d}\lambda}$$

is conserved along a light-like geodesic, i.e. when

$$\frac{\mathrm{d}x^m(\lambda)}{\mathrm{d}\lambda} \frac{\mathrm{d}x_m(\lambda)}{\mathrm{d}\lambda} = 0 \,.$$

Conformal Killing tensors

A symmetric and traceless tensor, $\xi^{a_1 \cdots a_n}(x)$, is called "conformal Killing" if and only if the traceless part of $\nabla^{(b} \xi^{a_1 \cdots a_n)}$ is zero.

For a conformal Killing tensor, $\xi^{a_1 \cdots a_n}(x)$,

$$\xi_{m_1 \cdots m_a}(x(\lambda)) \frac{\mathrm{d}x^{m_1}(\lambda)}{\mathrm{d}\lambda} \cdots \frac{\mathrm{d}x^{m_a}(\lambda)}{\mathrm{d}\lambda}$$

is conserved along light-like geodesics.

Towards higher symmetry

A vierbein is a new tangent space basis, $\{e_a{}^m(x)\partial_m\}_{a=0}^3$, such that $g_{mn}(x)e_a{}^m(x)e_b{}^n(x) = \eta_{ab}$. Indices are converted by

$$V_a(x) = e_a{}^m(x)V_m(x) \text{ and } V_m(x) = e_m{}^a(x)V_a(x)$$

where $e_m{}^a(x) = (e_a{}^m(x))^{-1}$ as matrices.

Under infinitesimal general coordinate, local Lorentz and Weyl transformations, i.e. $x'^m = x^m - \xi^m(x)$,
$e'_a{}^m(x) = e_a{}^m(x) + K_a{}^b(x)e_b{}^m(x)$ with $K_{ab} = -K_{ba}$ and
$e'_a{}^m(x) = (1 + \sigma(x))e_a{}^m(x)$ respectively (ξ^m, K_{ab} and σ all infinitesimal), the covariant derivative changes as

$$\delta\nabla_a = \left[\xi^b\nabla_b + \frac{1}{2}K^{bc}M_{bc}, \nabla_a\right] + \sigma\nabla_a - \nabla^b(\sigma)M_{ab}\,.$$

Furthermore, $\delta\nabla_a = 0$ if and only if $\xi^a(x)$ is a conformal Killing vector, $K^{bc} = \frac{1}{2}(\nabla^b\xi^c - \nabla^c\xi^b)$ and $\sigma = \frac{1}{4}\nabla_a\xi^a$ [2].

Higher symmetries

Given a differential operator, $\mathcal{F}$, acting on a tensor field, $T(x)$, a higher symmetry is a scalar, linear, differential operator, D, such that $\mathcal{F}DT = 0$ whenever $\mathcal{F}T = 0$.

Besides their motivation as symmetries in their own right, higher symmetries also have applications in

- Finding solutions of partial differential equations on arbitrary manifolds via separation of variables [3].
- Parallel's between the algebra of higher symmetries and higher spin algebras [4].

Goal: Develop techniques to compute higher symmetries in curved spacetime.

Conformal d'Alembertian

The action for a free, massless, real, scalar field, $\varphi(x)$, is

$$S = -\frac{1}{2} \int \partial^a(\varphi)\partial_a(\varphi)\mathrm{d}^4 x$$

in flat space. Lifting to curved space, a conformally invariant action is

$$S = -\frac{1}{2} \int \left(\nabla^a(\varphi)\nabla_a(\varphi) + \frac{1}{6} R\varphi^2 \right) e\, \mathrm{d}^4 x \quad \text{where } e = \det({e_m}^a)\,,$$

provided $\varphi'(x) = \mathrm{e}^{\sigma(x)}\varphi(x)$ upon ${e'_a}^m(x) = \mathrm{e}^{\sigma(x)}{e_a}^m(x)$.

The equation of motion for the matter field, φ, is then

$$\Delta\varphi = \left(\Box - \frac{1}{6} R \right) \varphi = 0\,.$$

Conformal d'Alembertian - 1st order

The conformal d'Alembertian, $\Delta = \Box - \frac{1}{6}R$, has a unique 1st order higher symmetry,

$$D^{(1)} = \xi^a \nabla_a + \frac{1}{4}\nabla_a(\xi^a) + \xi,$$

where $\xi^a(x)$ is an arbitrary conformal Killing vector of the manifold and ξ is any constant.

Conformal d'Alembertian - 2nd order

At 2nd order, Δ has a unique (up to the addition of 1st order symmetries) physically admissible higher symmetry candidate,

$$D^{(2)} = \xi^{ab}\nabla_a\nabla_b + \frac{2}{3}\nabla_b(\xi^{ab})\nabla_a + \frac{1}{15}\nabla_a\nabla_b(\xi^{ab}) - \frac{3}{10}R_{ab}\xi^{ab},$$

where $\xi^{ab}(x)$ is an arbitrary conformal Killing tensor of the manifold. However, $D^{(2)}$ may not be a symmetry in general. Instead,

$$\begin{aligned}\Delta D^{(2)}\varphi = &\left(\frac{4}{15}C^a{}_{bcd}\nabla^c(\xi^{bd}) + \frac{4}{5}\nabla^d(C^a{}_{bcd})\xi^{bc}\right)\nabla_a(\varphi)\\ &+\left(\frac{2}{15}C_{abcd}\nabla^a\nabla^c(\xi^{bd}) + \frac{2}{5}\nabla^c\nabla_d(C^d{}_{abc})\xi^{ab}\right.\\ &\left.+\frac{4}{15}\nabla_d(C^d{}_{abc})\nabla^c(\xi^{ab})\right)\varphi\\ \neq &\ 0 \ \text{ in general.}\end{aligned}$$

Massless Dirac operator

The action for a free spinor field is

$$S[e_a{}^m, \mathbf{\Psi}] = -\frac{\mathrm{i}}{2}\int \overline{\mathbf{\Psi}}\gamma^a\nabla_a(\mathbf{\Psi})e\,\mathrm{d}^4x \quad \text{where } e = \det(e_m{}^a)\,.$$

The equation of motion for the matter field, $\mathbf{\Psi}$, is then

$$\gamma^a\nabla_a\mathbf{\Psi} = 0\,.$$

The action is Weyl invariant provided $\mathbf{\Psi}' = \mathrm{e}^{3\sigma/2}\mathbf{\Psi}$ upon $e'_a{}^m = \mathrm{e}^{\sigma}e_a{}^m$.

Massless Dirac operator - 1st order

To convert between spinor notation and vector notation,

$$V_{\alpha\dot{\alpha}} = (\sigma_a)_{\alpha\dot{\alpha}} V^a, \ V_a = -\frac{1}{2}(\tilde{\sigma}_a)^{\dot{\alpha}\alpha} V_{\alpha\dot{\alpha}} \, ,$$

where $(\sigma_a)_{\alpha\dot{\alpha}} \equiv (I, \vec{\sigma})$ and $(\tilde{\sigma}_a)^{\dot{\alpha}\alpha} \equiv (I, -\vec{\sigma})$ [5].

The massless Dirac operator, $\gamma^a \nabla_a$, has a unique 1st order symmetry,

$$\begin{aligned} D^{(1)} = {} & \xi^{\alpha\dot{\alpha}} \nabla_{\alpha\dot{\alpha}} + \frac{1}{2} \nabla^{(\alpha}{}_{\dot{\alpha}} \xi^{\beta)\dot{\alpha}} M_{\alpha\beta} + \frac{1}{2} \nabla_{\alpha}{}^{(\dot{\alpha}} \xi^{\alpha\dot{\beta})} \overline{M}_{\dot{\alpha}\dot{\beta}} \\ & + \frac{3}{8} \nabla_{\alpha\dot{\alpha}} (\xi^{\alpha\dot{\alpha}}) + \xi, \end{aligned}$$

where $\xi^{\alpha\dot{\alpha}}(x)$ is an arbitrary conformal Killing vector of the manifold and ξ is an arbitrary constant.

Massless Dirac operator - 2nd order

At 2nd order, $\gamma^a \nabla_a$ has a unique (up to the addition of 1st order symmetries) physically admissible higher symmetry candidate,

$$
\begin{aligned}
&D^{(2)} \\
&= \xi^{\alpha\beta\dot{\alpha}\dot{\beta}} \nabla_{\alpha\dot{\alpha}} \nabla_{\beta\dot{\beta}} + \frac{2}{3} \nabla^{(\alpha}{}_{\dot{\beta}} \xi^{\beta\gamma)\dot{\alpha}\dot{\beta}} \nabla_{\alpha\dot{\alpha}} M_{\beta\gamma} + \frac{2}{3} \nabla_{\beta}{}^{(\dot{\alpha}} \xi^{\alpha\beta\dot{\beta}\dot{\gamma})} \nabla_{\alpha\dot{\alpha}} \overline{M}_{\dot{\beta}\dot{\gamma}} \\
&\quad + \frac{8}{9} \nabla_{\beta\dot{\beta}} (\xi^{\alpha\beta\dot{\alpha}\dot{\beta}}) \nabla_{\alpha\dot{\alpha}} + \left(\frac{2}{9} \nabla^{(\alpha}{}_{\dot{\alpha}} \nabla_{\gamma\dot{\beta}} \xi^{\beta)\gamma\dot{\alpha}\dot{\beta}} + \frac{1}{3} E^{(\alpha}{}_{\gamma\dot{\alpha}\dot{\beta}} \xi^{\beta)\gamma\dot{\alpha}\dot{\beta}} \right) M_{\alpha\beta} \\
&\quad + \left(\frac{2}{9} \nabla_{\alpha}{}^{(\dot{\alpha}} \nabla_{\beta\dot{\gamma}} \xi^{\alpha\beta\dot{\beta})\dot{\gamma}} + \frac{1}{3} E_{\alpha\beta\dot{\gamma}}{}^{(\dot{\alpha}} \xi^{\alpha\beta\dot{\beta})\dot{\gamma}} \right) \overline{M}_{\dot{\alpha}\dot{\beta}} \\
&\quad + \frac{2}{15} \nabla_{\alpha\dot{\alpha}} \nabla_{\beta\dot{\beta}} (\xi^{\alpha\beta\dot{\alpha}\dot{\beta}}) - \frac{7}{10} E_{\alpha\beta\dot{\alpha}\dot{\beta}} \xi^{\alpha\beta\dot{\alpha}\dot{\beta}},
\end{aligned}
$$

where $\xi^{\alpha\beta\dot{\alpha}\dot{\beta}}$ is an arbitrary conformal Killing tensor of the manifold.

Massless Dirac operator - 2nd order continued

However, $D^{(2)}$ may not be a symmetry in general. Instead,

$$
\begin{aligned}
&\gamma^a \nabla_a D^{(2)} \begin{bmatrix} \psi_\alpha \\ \overline{\chi}^{\dot\alpha} \end{bmatrix} \\
&= \Bigg[\frac{1}{3}(\overline{C}_{\dot\alpha\dot\beta}{}^{\dot\gamma\dot\mu}\xi_{\alpha\beta\dot\gamma\dot\mu} - C_{\alpha\beta}{}^{\gamma\mu}\xi_{\gamma\mu\dot\alpha\dot\beta})\nabla^{\beta\dot\beta}\overline{\chi}^{\dot\alpha} + \Bigg(\frac{4}{15} C^{\mu\gamma\beta}{}_{\alpha}\nabla_{(\beta}{}^{\dot\beta}\xi_{\gamma\mu)\dot\alpha\dot\beta} \\
&\quad - \frac{1}{15}\overline{C}_{\dot\alpha}{}^{\dot\beta\dot\gamma\dot\mu}\nabla^{\beta}{}_{(\dot\mu}\xi_{\alpha\beta\dot\gamma\dot\beta)} - \frac{2}{15}\xi^{\gamma\beta\dot\gamma}{}_{\dot\alpha}\nabla^{\mu}{}_{\dot\gamma}(C_{\alpha\beta\gamma\mu}) \\
&\quad - \frac{7}{15}\xi_{\alpha}{}^{\gamma\dot\beta\dot\gamma}\nabla_{\gamma}{}^{\dot\mu}(\overline{C}_{\dot\alpha\dot\beta\dot\gamma\dot\mu})\Bigg)\overline{\chi}^{\dot\alpha}, \\
&\quad \frac{1}{3}(C^{\alpha\beta}{}_{\gamma\mu}\xi^{\gamma\mu\dot\alpha\dot\beta} - \overline{C}^{\dot\alpha\dot\beta}{}_{\dot\gamma\dot\mu}\xi^{\alpha\beta\dot\gamma\dot\mu})\nabla_{\beta\dot\beta}\psi_\alpha + \Bigg(\frac{4}{15}\overline{C}_{\dot\mu\dot\gamma\dot\beta}{}^{\dot\alpha}\nabla_{\beta}{}^{(\dot\beta}\xi^{\alpha\beta\dot\gamma\dot\mu)} \\
&\quad - \frac{1}{15}C^{\alpha}{}_{\beta\gamma\mu}\nabla^{(\mu}{}_{\dot\beta}\xi^{\gamma\beta)\dot\alpha\dot\beta} - \frac{2}{15}\xi^{\alpha}{}_{\gamma\dot\beta\dot\gamma}\nabla^{\gamma}{}_{\dot\mu}(\overline{C}^{\dot\alpha\dot\beta\dot\gamma\dot\mu}) \\
&\quad - \frac{7}{15}\xi_{\gamma\beta\dot\gamma}{}^{\dot\alpha}\nabla_{\mu}{}^{\dot\gamma}(C^{\alpha\beta\gamma\mu})\Bigg)\psi_\alpha \Bigg]^T .
\end{aligned}
$$

Further research

There are two major unanswered questions at the end of the project

- Necessary conditions of the manifold so that the 2nd order operators I derived do result in symmetries
- Generalisation to higher orders

Techniques such as “conformal geometry” which are better tailored to the operators considered will make calculations easier.

References

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