Master's final presentation script

Virinchi Rallabhandi

Hello everyone, during my master's I explored a topic called "higher symmetry," which I'll give you an overview about today using the same illustrative examples I used in my study - namely relativistic wave equations for spin-0 and spin-1/2 massless particles.

Outline
I'll begin by outlining exactly what "higher symmetry" means, why we'd be interested in studying it and how it fits into the bigger picture of physics. Having established that, I'll briefly state the results of the calculations I did and finally speculate on future research directions.

Symmetry in physics
As a first step to higher symmetry, I'll present something of an advertorial for studying symmetry in physics. I know I don't have to motivate the study of symmetry to a room full of physicists, but it can't hurt. Simply put, a symmetry is a system's invariance under a particular transformation.

Special and general relativity were some of the first theories to really use symmetry as a foundational principle. Rather than trying to just fit equations of motion to experimental results, there was a realisation that spacetime's symmetries - whether Poincare symmetry or general coordinate symmetry - place highly non-trivial restrictions on action functionals and thus dynamics. For example, Maxwell's equations are in large part just a byproduct of Poincare invariance and the equivalence principle is built-in to the differential geometry of general relativity.

Another famous example in physics of leveraging the presence of symmetry is Noether's theorem, which states that every continuous symmetry of a system generates a conserved quantity. For example, in special relativity, symmetry under translations, rotations and boosts implies the conservation of 4-momentum, angular momentum and the velocity of the centre of energy.

Another big success of symmetry in the early 20th century was the work of Wigner in relativistic quantum mechanics. He first showed that the postulates of quantum mechanics imply that symmetries are encoded in linear and unitary operators - or in the special case of time inversion antilinear and antiunitary operators. Therefore, Poincare symmetry in quantum mechanics requires one to construct unitary irreducible representations of the Poincare group. The result was spectacular. He was able to explain the origin of spin, elementary particles and why massive and massless particles are so different. Since then, symmetry has become the bedrock of modern quantum field theory.

Reminder on differential geometry
Instead of the flat space of special relativity, my project was entirely in curved space. Working in curved space has become essential since the discovery of general relativity. So that I don't lose anyone in notation or something like that I have a brief reminder on differential geometry - the language of general relativity. In curved space, we can't use partial derivatives anymore.

But out replacement - the covariant derivative - doesn't commute. This is where the manifold's curvature - as quantified in the Riemann tensor - start's to show itself. In general relativity, the Riemann tensor is quite fundamental to everthing we do - e.g. its descendants are right there in the Einstein field equations. In understanding these definitions, I'll also remind you that this being theoretical physics, the Einstein summation convention and $c = \hbar = 1$ units are in place at all times. Another tensor I've got on this slide is the Weyl tensor, C_{mnpq}. It might look like a strange object to talk about now, but it'll be quite important later and that's why I've introduced it here. Very biefly, just as the Riemann tensor is so fundamental in general relativity, the Weyl tensor is fundamental in conformal field theory.

Killing tensors
In curved space, we don't immediately know what the metric's symmetries - or isometries - are in the same way that we know Poincare transformations are isometries of the Minkowski metric in flat space.

An isometry is a transformation that leaves the metric invariant [point to equation]. These are quite hard to find in general, but in the special case of an infinitesimal transformation [point to equation], we get a partial differential equation [point to equation], whose solutions are called Killing vectors. So Killing vectors, are generators of spacetime symmetries in curved space. Then, by the curved space version of Noether's theorem, Killing vectors generate conserved quantities. An example would be that this contraction [point to equation] is conserved along a geodesic.

Killing tensors
A Killing tensor [point to equation] is a higher order version of a Killing vector. Although they don't generate spacetime symmetries, they still generate conserved quantities along geodesics. The conserved quantity is this contraction [point to equation], which includes the conserved quantity on the previous slide as a special case, i.e. when the tensor is rank-1. This property was most famously used in the Kerr metric describing rotating black holes. Looking for Killing tensors alone was not enough to determine the geodesic motion. But then people found a rank-2 Killing tensor and it's conserved quantity then allowed the geodesic motion to be solved.

The moral from that story is that looking for a higher order quantity - like a Killing tensor - can lead to new physics not obvious for first order considerations. This is roughly the philosophy behind higher symmetry - which I'll define in a few slides time.

Conformal Killing vectors
But before that, I have to talk about a generalisation called a "conformal Killing tensor." The operators that I studied in my project - the conformal d'Alembertian and massless Dirac operator - come from conformal field theory, rather than normal general relativity. In general relativity we have invariance under general coordinate transformations and local Lorentz transformations. But in conformal field theory, the metric only matters up to scale. That means there is an additional symmetry - called Weyl symmetry - which scales the metric. A well known theory that's invariant under Weyl transformations is Maxwell's electrodynamics in curved space.

Anyhow, this symmetry means that in the infinitesimal transformation I considered earlier [point to equation], as long as the variation of the metric [point to equation] is proportional to the metric itself [point to equation], the transformation is a symmetry. Taking traces of [point to equation] and [point to equation] gives the proportionality constant subsequently this equation [point to equation]. This is the defining equation of a conformal Killing vector. As you can probably see, this equation is a generalisation of the equation defining a Killing vector. This contraction from before [point to equation] is conserved, but this time only along light-like geodesics.

Another way of expressing the conformal Killing equation [point to equation] is to say that the symmetric and traceless part of $\nabla_m \xi_n$ [point to equation] is zero.

Conformal Killing tensors
Finally, extending that definition to arbitrary rank tensors defines a conformal Killing tensor [point to equation]. Again, the quantity from before [point to equation] is conserved, but only along light-like geodesics.

Towards higher symmetry
Rather than work with conformal isometries of the metric, another way to think about conformal Killing vectors is to see how they transform the covariant derivative. For what follows, it will help to change my differential geometry perspective slightly. Rather than "world indices" - like m, n etc. [point to equation] - will be better to work with vierbeins. They are a new tangent space basis, that means the metric looks locally Minkowski at every point [point to equation]. It's straightforward to change the notation of indices between the two as so [point to equation].

This theorem I've stated [point to equation] looks quite technical, but all it says is that under a conformal Killing vector based transformation, the covariant derivative which usually changes by this expression [point to equation], doesn't actually change at all.

Since the equations of motion are built from covariant derivatives, the physics - the equations of motion - will be unchanged upon this conformal Killing vector transformation. However, the matter fields will change. This means the new matter fields will solve the same equations of motion as the old matter fields. Since this $\delta\nabla_a$ expression is just a differential operator when expanded out, it means that in a conformal field theory there should be a conformal Killing vector based transformation which maps solutions of the equations of motion to new solutions of the equations of motion.

Higher symmetries
This is the idea behind higher symmetry - which I can now finally define rigorously. Given a differential operator, $\mathcal{F}$, where $\mathcal{F}T = 0$ will be the equation of motion for some system, a higher symmetry is another differential operator, D, such that DT solves the equations of motion whenever T does - so D maps solutions to solutions. I have already outlined the considerable value of studying symmetry for its own sake, but besides that there are actually a couple of specific known applications of higher symmetry. In the late 20th century, there was considerable interest in higher symmetries among mathematicians trying to solve partial differential equations by separation of variables. When working on manifolds there is a lot of freedom in the coordinate systems and charts we choose to use but the PDE may only be separable in some of those coordinate systems and charts. Over time a lot of links emerged between the existence of higher symmetries, the eigenvalues, eigenfunctions etc. of the higher symmetries and the coordinate systems in which separation of variables is possible. In more recent years, high energy physicists have also become interested in higher symmetries, because there were several parallels discovered between higher spin algebras and the algebra generated by composing and taking linear combinations of higher symmetries.

The main aim of my project was to develop techniques to compute higher symmetries in curved space. I used the relativistic wave equation for spin-0 and spin-1/2 massless particles as illustrative examples in this process. My method relied heavily on spinors and Weyl transformation properties. I can't go into any more detail because my thesis has approximately 1400 lines of equations and it will take me a half a month, rather than half an hour to really explain everything properly.

Conformal d'Alembertian
For spin-0, I had to analyse the conformal d'Alembertian [point to equation] acting on a scalar field [point to equation]. This is a curved space, conformal analogue of the massless Klein-Gordon equation. To see that, take the flat space limit of this action [point to equation]. Then, the result is just the action for a free, massless, real, scalar field used in quantum field theory [point to equation], which is why this equation [point to equation] is a relativistic wave equation for a massless spin-0 particle in the first place.

Conformal d'Alembertian - 1st order
I started with the case where the higher symmetry [point to equation] was a first order differential operator. I found that there was a unique higher symmetry, it was written in terms of a conformal Killing vector as so [point to equation] and this operator works on all manifolds which possess a conformal Killing vector. More precisely, by uniqueness, I mean the form is unique. There may still be multiple conformal Killing vectors on the manifolds and you're free to choose any one of them. Likewise, you can choose any constant, ξ. This operator is in fact just that $\delta\nabla_a$ expression I showed earlier unwrapped.

Conformal d'Alembertian - 2nd order
Next, I considered the case where the higher symmetry was a 2nd order differential operator [point to equation]. Here, I showed that if a higher symmetry exists, then it can only have this form [point to equation] - which is written in terms of a conformal Killing tensor. However, when I substituted this into the equations of motion, I didn't get zero, I got this expression [point to equation] written in terms of the Weyl tensor, C_{abcd} [point to equation]. If the Weyl tensor is zero, that is the manifold is conformally flat, then this expression collapses to zero and $D^{(2)}$ really is a higher symmetry. However, this doesn't happen on all manifolds. Therefore, although 1st order symmetries exist on any manifold with a conformal Killing vector, the higher order symmetries can only exist on some special manifolds. On these particular manifolds, the conformal d'Alembertian - or the massless Klein-Gordon equation - possesses additional symmetry that isn't present in general.

Massless Dirac operator
Next, I repeated the whole story for the massless Dirac equation. Again, I started with a Weyl invariant action [point to equation], and derived the massless Dirac equation as its equation of motion [point to equation]. This action is the action for a free, massless Dirac spinor field in quantum field theory - that's why it describes spin-1/2 particles.

Massless Dirac operator - 1st order
Again, I started with 1st order higher symmetries. I was able to show that on any manifold with a conformal Killing vector, there exists a unique higher symmetry - unique in the same sense as before - and it is given by this formula [point to equation]. When working with a spinor field, it makes sense to just stay in spinor notation the whole time. I've provided a very short and somewhat incomplete "dictionary" to translate between spinor notation and vector

notation. Very briefly, the two-component spinor formalism is built on using the elements of the extended Pauli matrices in a clever way. The Lorentz generators [point to them] have a slightly more complicated conversion, but there's not that much use of going into the technical details here. In general, there are lots of identities associated with two-component spinors; they made my life a lot easier in the calculations, but don't really matter too much for the purposes of this presentation. Again, this $D^{(1)}$ [point to equation] is essentially just an unwrapping of the $\delta\nabla_a$ from before.

Massless Dirac operator - 2nd order

The last calculation I did was to find 2nd order higher symmetries of the massless Dirac equation. First, I showed that if a higher symmetry exists, then it can only take the form of this differential operator [point to equation], written in terms of a conformal Killing tensor. As you can see, the operator is quite complicated, so if it does work as a higher symmetry, then it's really quite a non-trivial and hidden symmetry that's unearthed. Unfortunately, like with the conformal d'Alembertian, when I substitute my candidate symmetry operator [point to equation] into the equation of motion, I don't get zero.

Massless Dirac operator - 2nd order continued

Instead, after about 30 pages of algebra, I get this mess. Although this is undoubtedly a complicated expression, every term has a factor of C or $\overline{C}$ on it. These are the spinor versions of the Weyl tensor. Therefore, on conformally flat manifolds, this whole expression collapses to zero. That means, just like with the conformal d'Alembertian, on conformally flat spaces, the equations of motion have an extra symmetry not present on arbitrary manifolds.

Further research

The next logical step is actually fairly obvious ... it's to read Emmanouil's work; he's doing calculations far more advanced than these. But more broadly, there are still a number of unanswered questions at the end of my project. The first one is regarding the necessary conditions of the manifold for the 2nd order symmetries to exist. As I've shown, if the Weyl tensor is zero, then there are higher symmetries at 2nd order. However, maybe that's more restriction than necessary. Maybe, there is a weaker condition that means the operators I constructed are true higher symmetries. There has been some research in this direction, but I didn't have time to verify the claims made in the literature.

The biggest question though is the generalisation to higher orders. As I've indicated the derivation of the results is quite long, so going to third order could be quite taxing. There is one immediate improvement that can be made. In conformal field theory there are two extra symmetries - dilatations and special conformal transformations - not present in general relativity. The method of "conformal geometry" better leverages these symmetries and I know it will definitely help with the problems I considered in my project because Emmanouil has used to it to derive symmetries of conformal d'Alembertian to arbitrary order in conformally flat spaces. But even then, working on manifolds that aren't conformally flat can be quite challenging. With the techniques I developed in my project it should be possible to analyse 3rd order symmetries in general, but going beyond that will take a more systematic and intelligent approach than what has been presented so far in my work or in the literature.

References

Thank you for listening and I'll now take questions if there are any.