Higher symmetries of relativistic wave equations in curved spacetime

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Declaration page

This is to certify that:

1. This thesis comprises of my original work.
2. Due acknowledgement has been made in the text to all other materials used.
3. This thesis consists of less than 60 pages inclusive of tables and figures but exclusive of references and appendices.

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Abstract

For a little over a century now, symmetry has played a foundational role in the development of theoretical physics. Loosely, a higher symmetry is a differential operator which takes a solution of the equations of motion for a system to a new solution of those equations of motion. While originally studied by experts in general relativity analysing the Kerr spacetime and subsequently by mathematicians in the context of separation of variables on manifolds, in recent years higher symmetries have garnered renewed interest in high energy physics due to the parallels between their algebra and higher spin algebras. In this thesis, I developed techniques - especially emphasizing spinor methods - for computing higher symmetries in curved spacetimes. As illustrative examples, the equations of motion I considered were the relativistic wave equations for spin-0 and spin-1/2 massless particles. I mainly studied the cases when the higher symmetry was a 1st or 2nd order differential operator. For both equations of motion I was able to uniquely determine physically admissible candidates for 1st and 2nd order higher symmetries in terms of conformal Killing vectors/tensors. However, only the 1st order candidates actually proved to be higher symmetries on arbitrary manifolds possessing a conformal Killing vector. Provided a conformal Killing tensor exists on the manifold, conformal flatness was a sufficient, but perhaps not necessary, condition for the 2nd order candidates to be higher symmetries. I finished by briefly exploring the potential for "conformal geometry" to improve the efficiency of the calculations presented. All calculations were performed on an arbitrary, four-dimensional, orientable, connected manifold of Lorentzian metric signature.

Acknowledgements

While I have written this thesis in 1st person singular for better flow and better grammatical sense (after all, it is a single student work), my thesis is hardly an individual effort. I could not have achieved anything without my supervisors. I would like to thank Sergei especially for plenty of guidance, teaching and many discussions on the material. Besides Sergei, in some of the long practical calculations I received assistance from Emmanouil Raptakis - to the point where he probably deserves a co-supervision credit himself. Most of all, I would like to thank my family for support and my friends for inspiration.

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Contents

Chapter 1

Introduction

Nothing in physics seems so hopeful to as the idea that it is possible for a theory to have a high degree of symmetry [and be] hidden from us in everyday life. The physicist's task is to find this deeper symmetry.
- Steven Weinberg

The last hundred years or so has seen something of a revolution in how theoretical physicists view the foundations of their subject. It is not just that we have more accurate or sophisticated theories now. While the days of Newton's laws are of course well and truly behind us, we have also moved on from the mindset of those days. The ultimate test of any theory is still its agreement with experiment, but there has been enormous upheaval in how theoreticians formulate their ideas in the first place. No longer do physicists simply postulate laws - equations of motion - to fit experiment like Kepler, Newton, Carnot or even Maxwell. Instead, one now starts from symmetry [1]. Simply put, a symmetry is a system's invariance under a particular transformation. The first benefactor of the new age thinking - indeed one its pioneers - was Einstein, in his formulation of special relativity. Rather than relativistic invariance being a consequence of Maxwell's equations, Maxwell's equations were now largely just a consequence of relativistic invariance. Ten years later, Einstein upturned the standard worldview again with general relativity. In some sense, Einstein's general theory of relativity was grounded in the belief that physics must be invariant under an arbitrary change to the reference system. Thus, the equations of motion had to transform covariantly under general coordinate transformations. Simple enough to state, but profoundly consequential in terms of physical implications to the description of gravity and the technical tools required to describe it.

However, several developments conspired to put symmetry on the pedestal it is placed today. Through the 19th century, mathematicians too came to court symmetry. The path laid by Galois, Cauchy and Cayley, pursued through to Klein's Erlangen program meant that group theory was here to stay. With group theory, physicists could not only appreciate thinking symmetrically, they could also describe it quantitatively. One of the first to wield the new power was Wigner. With many striking applications of group theory, Wigner brought symmetry to the fledgling quantum mechanics - in particular the theory of atomic spectra and isospin symmetry [2]. By the late 1920s, Wigner had already proven that symmetries in quantum mechanics are implemented by linear and unitary or - if time reversal was involved - antilinear and antiunitary operators. But in special relativity, the fundamental symmetries are Poincaré transformations. Therefore, to marry quantum mechanics and special relativity, one had to study the unitary representations of the Poincaré group. In his famous 1939 treatise [3], Wigner classified the irreducible representations by mass and spin - in the process explaining the origin of spin, elucidating the meaning of elementary particle, deriving a fundamental distinction between massive and massless particles and paving the path to modern formulations of quantum field theory - see e.g. [4].

Group theory and symmetry have since become ingrained in the psyche of the modern theoretical physicist. In 1956, while visiting Russia, even the famously austere Dirac once used his opportunity to present the honorary comment on a Moscow university blackboard by writing "a physical law must possess mathematical beauty" - an inscription no-one has since dared to erase. In this context, beauty is of course nothing but symmetry.

In summary, the story of modern theoretical physics is inextricably linked with the study of symmetry in its myriad manifestations. Rather than the spacetime symmetries of interest to Einstein and Wigner or the various internal and gauge symmetries present in quantum field theory, I will discuss symmetries of the equations of motion themselves with a concept called "higher symmetry." While precise definitions are deferred to chapter 2, loosely speaking, higher symmetries are differential operators, D, which map solutions of some other differential operator, $\mathfrak{D}$, to new solutions of the same operator - hence the term "symmetry." The operator, $\mathfrak{D}$, is typically an operator appearing in the equations of motion for some physically interesting system. In recent years there has been a renewed interest in the subject as various connections have emerged between higher spin algebras, the AdS/CFT correspondence and the algebra built from linear combinations and compositions of higher symmetries.

The main task of my thesis will be to develop and present techniques to compute higher symmetries in curved spacetimes. Rather than an algorithmic general theory, the formalism is best illustrated via specific examples. As such, I will be focused on two operators in particular. In chapter 3, I will consider the conformal d'Alembertian, $\mathfrak{D} = \Delta = \Box - \frac{1}{6}R$, acting on a scalar field, φ, and in chapter 4, I will consider the massless Dirac operator, $\gamma^a \nabla_a$, acting on a four-component Dirac spinor, $\boldsymbol{\Psi}$. These operators represent conformally invariant relativistic wave equations for massless spin-0 and spin-1/2 particles. Some of the longer proofs associated with results in these chapters are contained in appendices B and C. In finding higher symmetries, I will especially emphasize spinor methods. Not only are they natural when working with the Dirac operator, they also make many properties of some tensors and differential operator contractions more transparent. Given the heavy reliance on spinors - and especially the two-component formalism - I have provided a comprehensive account of spinors in appendix D. Further notational conventions and frequently used identities are listed in appendices E and F respectively. While I did most of my calculations using standard differential geometry, in chapter 5, I will present the case for "conformal geometry" as a superior alternative when dealing with conformally invariant operators and conformal field theories in general.

But before all that, I will begin in chapter 2 (while briefly referring to appendix A in the process) with a slightly more extended review of symmetry in general relativity. This will serve to give more quantitative motivation for the study of higher symmetries while also introducing many definitions and theorems which will be foundational to the later chapters.

Chapter 2

Symmetries and higher symmetries in general relativity

In this chapter I will briefly recount the remarkable story of symmetry within the context of general relativity and classical field theory more broadly. Concurrently, I will build towards the research topic I will be studying in subsequent chapters. Unlike the rest of the thesis, in this chapter I will occasionally devolve responsibility for proofs to various references.

With the benefit of hindsight, it is fair to say symmetry's most starring role in physics is its ability to constrain the form of action functionals and subsequently apply Noether's theorem to generate conserved quantities [5]. For a taster, first consider classical field theory in flat space - the domain governed by special relativity. Note that it will also almost certainly help to read appendices E and F at some point before the end of this chapter.

Theorem 2.1 (Noether)**.** *For every (infinitesimal) continuous transformation of matter fields, $\delta\varphi^I = X^I(\varphi, \partial_a\varphi)$, that changes the Lagrangian density, $\mathcal{L}$, by a total derivative*[1]*, $\delta\mathcal{L} = \partial_a F^a$, the vector,*

$$j^a = \frac{\partial\mathcal{L}}{\partial(\partial_a\varphi^I)} X^I - F^a \,, \tag{2.1}$$

is a conserved current, i.e. $\partial_a j^a = 0$.

Proof. Upon an infinitesimal variation to the matter field, $\delta\varphi^I = X^I(\varphi, \partial_a\varphi)$,

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\varphi^I} X^I + \frac{\partial\mathcal{L}}{\partial(\partial_a\varphi^I)} \partial_a(X^I) \,. \tag{2.2}$$

Therefore,

$$\partial_a F^a = \left(\frac{\partial\mathcal{L}}{\partial\varphi^I} - \partial_a\left(\frac{\partial\mathcal{L}}{\partial(\partial_a\varphi^I)}\right)\right) X^I + \partial_a\left(\frac{\partial\mathcal{L}}{\partial(\partial_a\varphi^I)} X^I\right) . \tag{2.3}$$

Hence, when the Euler-Lagrange equations hold,

$$0 = \partial_a\left(\frac{\partial\mathcal{L}}{\partial(\partial_a\varphi^I)} X^I - F^a\right) , \tag{2.4}$$

which is exactly $\partial_a j^a = 0$. □

[1] If $\mathcal{L}$ changes by a total derivative, the action is invariant, which is why $\delta\varphi^I = X^I(\varphi, \partial_a\varphi)$ can be called a "symmetry" of the system.

Corollary 2.1.1. $Q = \int j^0(\vec{x},t)\,\mathrm{d}^3\vec{x}$ *is conserved, i.e.* $\frac{\mathrm{d}Q}{\mathrm{d}t} = 0$.

Proof. Rewriting $\partial_a j^a = 0$ as $\partial_0 j^0 + \vec{\nabla}\cdot\vec{j} = 0$,

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \int \frac{\partial j^0(\vec{x},t)}{\partial t}\mathrm{d}^3\vec{x} = -\int \vec{\nabla}\cdot\vec{j}(\vec{x},t)\mathrm{d}^3\vec{x} = 0\,, \tag{2.5}$$

under the standard assumption that fields vanish sufficiently quickly at infinity. □

Integrating a continuous, infinitesimal symmetry, X^I, naturally leads to a finite symmetry; X^I effectively "generates" a finite symmetry.
Therefore, the study of symmetry in physics is intimately connected with the application of Lie group theory, where elements of the Lie algebra are the generators and the exponential map "integrates" the infinitesimal elements to produce a finite group element - see e.g. [6] complete mathematical details.

The fundamental postulates of special relativity can then be re-framed as saying the symmetry group of spacetime is the ten-dimensional, proper, orthochronous Poincaré group, $\mathrm{ISO}^{\uparrow}(3,1)$. A convenient choice of basis in its Lie algebra[2], $\mathfrak{io}(3,1)$, is one consisting of the generators of spacetime translations, P_a, and Lorentz transformations, $M_{ab} = -M_{ba}$, with defining Lie brackets,

$$\begin{aligned}
[M_{ab}, M_{cd}] &= 2\eta_{d[a}M_{b]c} - 2\eta_{c[a}M_{b]d}, \\
[P_a, M_{bc}] &= 2\eta_{a[b}P_{c]}, \\
[P_a, P_b] &= 0
\end{aligned} \tag{2.6}$$

Applying Noether's theorem to infinitesimal spacetime translations, rotations and boosts leads to the conservation of total four-momentum, total angular momentum and velocity of the centre of energy of the system respectively [5]. However, Noether's theorem is not just applicable for spacetime symmetries. One can consider gauge symmetries, internal symmetries etc. and there exists a generalised Noether's theorem to accommodate for them all - see e.g. [7]. While fascinating in its own right, it is somewhat tangential to my research topic.

From hereon, I will be working in curved space - the domain governed by general relativity. For the rest of the thesis, I will be working on an arbitrary, connected, four-dimensional, orientable manifold equipped with a Lorentzian metric. In curved space the story is more subtle. Not only are there matter fields, the metric, $g_{mn}(x)$, is itself dynamical[3]. Matter fields contribute to the energy momentum tensor, $T^{mn}(x)$, but the metric - and hence the gravitational field - does not directly. Rather than Noether's theorem, for spacetime symmetries in curved space, it will be more fruitful to take an alternative perspective.

$\mathrm{ISO}^{\uparrow}(3,1)$ is the symmetry group of flat space; the task is to find the equivalent for a curved spacetime. The main defining property of $\mathrm{ISO}^{\uparrow}(3,1)$ is that for any $(\Lambda, a) \in \mathrm{ISO}^{\uparrow}(3,1)$, the spacetime interval is preserved[4] under $x'^a = \Lambda^a{}_b x^b + a^a$, i.e. Poincaré transformations are isometries of the flat space metric, η_{ab}.
Thus, to find the symmetry group of a manifold, one must find its isometries. While objects

[2] As $\mathrm{IO}(3,1)$, $\mathrm{ISO}(3,1)$ and $\mathrm{ISO}^{\uparrow}(3,1)$ all share the same neighbourhood of the identity, all of $\mathfrak{io}(3,1)$, $\mathfrak{iso}(3,1)$ and $\mathfrak{iso}^{\uparrow}(3,1)$ are the same.

[3] For now, I will treat the metric as fundamental, rather than the vierbein/tetrad, $e_a{}^m$. This will not be the perspective I will take later, but it will be more convenient for now.

[4] The $\Lambda^0{}_0 \geq 1$ and $\det(\Lambda) = 1$ conditions pick out the connected component (based on the manifold structure of the Lie group) of the identity element, $(I, 0)$, and thereby prevent space or time inversions.

transform tensorially under general coordinate transformations in general relativity, there are actually no pre-conditions on the isometries of the metric. The same way Noether's theorem considers infinitesimal symmetries and then builds finite symmetries via the Lie group-Lie algebra correspondence, the standard approach is to consider infinitesimal spacetime transformations which leave the metric invariant.
Let $x'^m = x^m - \xi^m(x)$ be an infinitesimal spacetime transformation. By the tensor transformation law,

$$g'_{mn}(x') = \frac{\partial x^p}{\partial x'^m}\frac{\partial x^q}{\partial x'^n}g_{pq}(x) \tag{2.7}$$

Since I'm only working to 1st order in $\xi^m(x)$ for an infinitesimal transformation,

$$\frac{\partial x'^m}{\partial x^n} = \delta^m{}_n - \partial_n\xi^m(x) \implies \frac{\partial x^m}{\partial x'^n} = \delta^m{}_n + \partial_n\xi^m(x)\,. \tag{2.8}$$

Together, they imply

$$g'_{mn}(x') = g_{mn}(x) + \partial_m(\xi^p(x))g_{pn}(x) + \partial_n(\xi^p(x))g_{pm}(x)\,. \tag{2.9}$$

Then,

$$\begin{aligned}\delta g_{mn}(x) &= g'_{mn}(x) - g_{mn}(x)\\ &= g_{mn}(x+\xi) + \partial_m(\xi^p(x+\xi))g_{pn}(x+\xi) + \partial_n(\xi^p(x+\xi))g_{pm}(x+\xi) - g_{mn}(x)\\ &= \xi^p(x)\partial_p g_{mn}(x) + \partial_m(\xi^p(x))g_{pn}(x) + \partial_n(\xi^p(x))g_{pm}(x)\,.\end{aligned} \tag{2.10}$$

As with almost anything in differential geometry, tensorial equations are preferable. In this case, the required expression turns out to be $\nabla_m\xi_n + \nabla_n\xi_m$ since

$$\begin{aligned}\nabla_m\xi_n + \nabla_n\xi_m &= \partial_m\xi_n + \partial_n\xi_m - \Gamma^p{}_{mn}\xi_p - \Gamma^p{}_{nm}\xi_p\\ &= \partial_m(g_{pn}\xi^p) + \partial_n(g_{pm}\xi^p) - \xi^p(\partial_m g_{np} + \partial_n g_{pm} - \partial_p g_{mn})\\ &= \delta g_{mn}\,.\end{aligned} \tag{2.11}$$

Therefore, $\xi^m(x)$ induces an isometry of the metric if and only if $\nabla_m\xi_n + \nabla_n\xi_m = 0$.

Definition 2.2 (Killing vector)**.** *A four-vector, $\xi^m(x)$, is known as "Killing" if and only if*

$$\nabla_m\xi_n + \nabla_n\xi_m = 0 \iff \nabla_{(m}\xi_{n)} = 0\,. \tag{2.12}$$

Theorem 2.3. *The set of Killing vectors forms a Lie algebra.*

Proof. Since the defining condition, $\nabla_m\xi_n+\nabla_n\xi_m = 0$ is a linear PDE, the set of Killing vectors automatically forms a vector space. All that is left to show is that given two Killing vectors, ξ^m and ζ^m, their Lie bracket, $[\xi,\zeta]^m = \xi^n\partial_n\zeta^m - \zeta^n\partial_n\xi^m$, also satisfies the defining condition. See [8] for a proof of that property. □

Then, the Lie group-Lie algebra correspondence can be used to generate finite spacetime symmetries, $\mathrm{e}^{\xi^m(x)\partial_m}$, and such elements from a subgroup of the symmetry group of the manifold[5].

Just as Noether's theorem generates conserved quantities from symmetries, Killing vectors too generate conserved quantities - most famously along geodesics. Recall that if λ is an affine parameter for a geodesic, then a particle's position along the geodesic, $x^m(\lambda)$, satisfies

$$\frac{\mathrm{d}x^n(\lambda)}{\mathrm{d}\lambda}\nabla_n\frac{\mathrm{d}x^m(\lambda)}{\mathrm{d}\lambda} = 0 \iff \frac{\mathrm{d}^2x^m(\lambda)}{\mathrm{d}\lambda^2} + \Gamma^m{}_{np}(x(\lambda))\frac{\mathrm{d}x^n(\lambda)}{\mathrm{d}\lambda}\frac{\mathrm{d}x^p(\lambda)}{\mathrm{d}\lambda} = 0\,. \tag{2.13}$$

[5]The manifold can be "geodesically incomplete," so calculating $\mathrm{e}^{\xi^m(x)\partial_m}$ alone can leave the story of manifold symmetries incomplete too.

Theorem 2.4. *For any Killing vector, $\xi^m(x)$,*

$$\xi_m(x(\lambda))\frac{\mathrm{d}x^m(\lambda)}{\mathrm{d}\lambda} \tag{2.14}$$

is conserved along a geodesic.

Proof.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\xi_m\frac{\mathrm{d}x^m}{\mathrm{d}\lambda}\right) &= \frac{\mathrm{d}x^n}{\mathrm{d}\lambda}\partial_n\left(\xi_m\frac{\mathrm{d}x^m}{\mathrm{d}\lambda}\right) \\
&= \frac{\mathrm{d}x^n}{\mathrm{d}\lambda}\nabla_n\left(\xi_m\frac{\mathrm{d}x^m}{\mathrm{d}\lambda}\right) \quad \text{as } \xi_m\frac{\mathrm{d}x^m}{\mathrm{d}\lambda} \text{ is a scalar} \\
&= \frac{\mathrm{d}x^n}{\mathrm{d}\lambda}\frac{\mathrm{d}x^m}{\mathrm{d}\lambda}\nabla_n\xi_m + \xi^m\frac{\mathrm{d}x^n}{\mathrm{d}\lambda}\nabla_n\frac{\mathrm{d}x^m}{\mathrm{d}\lambda} \\
&= \frac{\mathrm{d}x^n}{\mathrm{d}\lambda}\frac{\mathrm{d}x^m}{\mathrm{d}\lambda}\nabla_{(n}\xi_{m)} + 0 \quad \text{by the geodesic equation} \\
&= 0 \quad \text{by the Killing condition}
\end{aligned} \tag{2.15}$$

□

In classical mechanics, one can circumvent differential equations by determining conserved quantities and analysing the subsequent algebraic equations. Likewise, in curved space, one can avoid solving the geodesic equation directly by finding the metric's Killing vectors and utilising the corresponding conserved quantities. However, sometimes a metric possesses fewer independent Killing vectors than required to completely determine a particle's motion.

The middle of the 20th century saw a spectacular new development in the analysis of free-fall trajectories. Unlike the Schwarzschild and Reissner-Nordstrøm metrics, which were discovered early in the development of general relativity, it took until 1963 to determine the Kerr metric[6],

$$\begin{aligned}
(\mathrm{d}s)^2 = -(\mathrm{d}t)^2 &+ \frac{r^2+a^2\cos^2(\theta)}{r^2-2GMr+a^2}(\mathrm{d}r)^2 + (r^2+a^2\cos^2(\theta))(\mathrm{d}\theta)^2 + (r^2+a^2)\sin^2(\theta)(\mathrm{d}\phi)^2 \\
&+ \frac{2GMr}{r^2+a^2\cos^2(\theta)}(a\sin^2(\theta)\mathrm{d}\phi - \mathrm{d}t)^2,
\end{aligned} \tag{2.16}$$

to describe a rotating black hole - see e.g. [9] for a brief review. Here, M is the black hole's mass and a is a constant measuring its rotation. The lack of spherical symmetry meant it was a much more technically challenging task to determine geodesics in the Kerr spacetime. There are still two Killing vectors, ∂_t and ∂_ϕ, but it turns out they are insufficient to completely specify trajectories[7]. The crucial piece of insight was that the Kerr metric possesses something higher order than a Killing vector.

Definition 2.5 (Killing tensor)**.** *A symmetric tensor, $\xi^{m_1\cdots m_a}(x)$, is called "Killing" if and only if*

$$\nabla^{(n}\xi^{m_1\cdots m_a)} = 0\,. \tag{2.17}$$

Theorem 2.6. *For any Killing tensor $\xi^{m_1\cdots m_a}(x)$,*

$$\xi_{m_1\cdots m_a}(x(\lambda))\frac{\mathrm{d}x^{m_1}(\lambda)}{\mathrm{d}\lambda}\cdots\frac{\mathrm{d}x^{m_a}(\lambda)}{\mathrm{d}\lambda} \tag{2.18}$$

is conserved along a geodesic.

[6] I have stated the Kerr metric here in Boyer-Lindquist coordinates.

[7] ∂_t and ∂_ϕ can be seen to be Killing vectors because none of the components of the metric are dependent on t or ϕ. Hence, translations along the t or ϕ direction leave the metric invariant.

Proof.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\xi_{m_1\cdots m_a}\frac{\mathrm{d}x^{m_1}}{\mathrm{d}\lambda}\cdots\frac{\mathrm{d}x^{m_a}}{\mathrm{d}\lambda}\right) &= \frac{\mathrm{d}x^n}{\mathrm{d}\lambda}\partial_n\left(\xi_{m_1\cdots m_a}\frac{\mathrm{d}x^{m_1}}{\mathrm{d}\lambda}\cdots\frac{\mathrm{d}x^{m_a}}{\mathrm{d}\lambda}\right)\\ &= \frac{\mathrm{d}x^n}{\mathrm{d}\lambda}\nabla_n\left(\xi_{m_1\cdots m_a}\frac{\mathrm{d}x^{m_1}}{\mathrm{d}\lambda}\cdots\frac{\mathrm{d}x^{m_a}}{\mathrm{d}\lambda}\right)\\ &= \frac{\mathrm{d}x^n}{\mathrm{d}\lambda}\frac{\mathrm{d}x^{m_1}}{\mathrm{d}\lambda}\cdots\frac{\mathrm{d}x^{m_a}}{\mathrm{d}\lambda}\nabla_n\xi_{m_1\cdots m_a}\\ &\quad+\xi_{m_1\cdots m_a}\sum_{i=1}^{a}\frac{\mathrm{d}x^{m_1}}{\mathrm{d}\lambda}\cdots\widehat{\frac{\mathrm{d}x^{m_i}}{\mathrm{d}\lambda}}\frac{\mathrm{d}x^{m_a}}{\mathrm{d}\lambda}\frac{\mathrm{d}x^n}{\mathrm{d}\lambda}\nabla_n\frac{\mathrm{d}x^{m_i}}{\mathrm{d}\lambda}\\ &= \frac{\mathrm{d}x^n}{\mathrm{d}\lambda}\frac{\mathrm{d}x^{m_1}}{\mathrm{d}\lambda}\cdots\frac{\mathrm{d}x^{m_a}}{\mathrm{d}\lambda}\nabla_{(n}\xi_{m_1\cdots m_a)}+0\\ &= 0 \end{aligned}\tag{2.19}$$

□

Not every Killing tensor of rank ≥ 2 can be written as a product of lower order Killing tensors or Killing vectors[8] [8]. Hence, going to higher orders has the potential to reveal previously undetected conserved quantities. Indeed, for the Kerr spacetime it was discovered

$$\begin{aligned}\xi_{mn} &= 2(r^2+a^2\cos^2(\theta))A_{(m}B_{n)}+r^2g_{mn} \quad \text{where}\\ A_m &\equiv \frac{1}{r^2-2GMr+a^2}\left(r^2+a^2, r^2-2GMr+a^2, 0, a\right) \quad \text{and}\\ B_m &\equiv \frac{1}{2(r^2+a^2\cos^2(\theta))}\left(r^2+a^2, -r^2+2GMr-a^2, 0, a\right),\end{aligned}\tag{2.20}$$

is a Killing tensor and greatly simplifies the analysis - see [9] for further detail. Likewise, in this thesis it will prove to be fruitful to work with higher rank tensors, rather than only vectors, to unearth higher symmetries.

However, my project will not deal with standard general relativity. Instead, I will be concerned with conformal field theory. To describe exactly what this means, I will begin by slightly changing the differential geometry perspective I employ. Rather than work with curved space indices, for the rest of my thesis it will be essential to work in the vierbein approach to differential geometry[9]. A vierbein is a new tangent space basis, $\{e_a{}^m(x)\partial_m\}_{a=0}^3$, such that $\eta_{ab} = e_a{}^m(x)e_b{}^n(x)g_{mn}(x)$ for all points, x, in the manifold. The vierbein is now the fundamental field and the metric, $g_{mn}(x) = e_m{}^a(x)e_n{}^b(x)\eta_{ab}$ where $e_m{}^a$ is the inverse matrix of $e_a{}^m$, is derived from the vierbein. Since a choice of vierbein is only unique up to local Lorentz transformations, $e'_a{}^m(x) = (\Lambda^{-1})^b{}_a(x)e_b{}^m(x)$ for $\Lambda^a{}_b(x) \in \mathrm{SO}^{\uparrow}(3,1)$ allows one to construct local representations of the Lorentz group.
Therefore, in the vierbein approach, field theories are covariant not just under general coordinate transformations, but also local Lorentz transformations.
Adopting the vierbein approach allows me to deploy the spinor formalism[10] and thereby describe the dynamics of half integer spin particles in curved space.

[8] In somewhat technical language, not all Killing tensors can be seen as elements of the universal enveloping algebras of lower order Killing tensors and Killing vectors.

[9] Although I didn't use it for that purpose, the vierbein approach works fine for general relativity too; it is nothing specific to conformal field theory.

[10] For my spinor conventions, see appendix E. For a comprehensive introduction to spinors, see appendix D.

The fundamental, new feature of conformal field theory lacking in general relativity is invariance under "Weyl transformations."

Definition 2.7 (Weyl transformation)**.** *A Weyl transformation is a change to the vierbein of the form,* ${e'}_a{}^m(x) = \mathrm{e}^{\sigma(x)} e_a{}^m(x)$*, for some scalar field,* $\sigma(x)$*.*

In a conformal field theory, the vierbein, inverse vierbein, metric etc. are only relevant up to scale. Note that the theory's invariance under Weyl transformations can be predicated on a corresponding transformation to the matter fields - this is exactly analogous to the quantum mechanics of particles in an electromagnetic field where a gauge transformation to the fields requires a point dependent phase transformation to the wavefunction for the equations of motion to remain unchanged.

In this thesis, rather than attempt to characterise symmetries of action functionals or spacetimes, I will be studying symmetries of the equations of motions themselves. In particular, I will analyse two differential operators which turn up in conformal field theory - namely the conformal d'Alembertian, $\Delta = \nabla^a \nabla_a - \frac{1}{6}R$, and the massless Dirac operator, $\gamma^a \nabla_a$. When quantised - although the actual quantisation is beyond the scope of my thesis - these equations turn up in the description of massless spin-0 and spin-1/2 particles respectively.

Analogous with the study of symmetry in general relativity, I will need the conformal versions of Killing vectors and tensors to describe the symmetries I will be studying.

Definition 2.8 (Conformal Killing vector)**.** *A vector,* $\xi^a(x)$*, is called "conformal Killing" if and only if*

$$\nabla^a \xi^b + \nabla^b \xi^a = \frac{1}{2} \eta^{ab} \nabla_c \xi^c \,. \tag{2.21}$$

Definition 2.9 (Conformal Killing tensor)**.** *A symmetric and traceless tensor,* $\xi^{a_1 \cdots a_n}(x)$*, is called "conformal Killing" if and only if the traceless part of* $\nabla^{(b} \xi^{a_1 \cdots a_n)}$ *is zero.*

Note that the former definition is just a special case of the latter definition.
Since I am interested in symmetries of the equations of motion, rather than study isometries of the vierbein or metric, it makes sense to study the transformations which leave the covariant derivative unchanged.

Theorem 2.10. *Under infinitesimal general coordinate, local Lorentz and Weyl transformations, i.e.* $x'^m = x^m - \xi^m(x)$*,* ${e'}_a{}^m(x) = e_a{}^m(x) + K_a{}^b(x) e_b{}^m(x)$ *with* $K_{ab} = -K_{ba}$ *and* ${e'}_a{}^m(x) = (1+\sigma(x)) e_a{}^m(x)$ *respectively (*ξ^m*,* K_{ab} *and* σ *all infinitesimal), the covariant derivative changes as*

$$\delta \nabla_a = \left[\xi^b \nabla_b + \frac{1}{2} K^{bc} M_{bc}, \nabla_a \right] + \sigma \nabla_a - \nabla^b(\sigma) M_{ab} \,. \tag{2.22}$$

Furthermore, $\delta \nabla_a = 0$ *if and only if* $\xi^a(x)$ *is a conformal Killing vector,* $K^{bc} = \frac{1}{2}(\nabla^b \xi^c - \nabla^c \xi^b)$ *and* $\sigma = \frac{1}{4} \nabla_a \xi^a$*.*

Proof. See [10] or appendix A. □

If one is dealing with a conformal field theory, the theorem means that under the conformal Killing vector based transformation just described, the physics is unchanged[11]. However, the

[11] General coordinate transformations, local Lorentz transformations and Weyl transformations are all part of the symmetry group of a conformal field theory.

matter fields must change to compensate. Since the equations of motion will be built on ∇_a, they will stay the same upon $\delta\nabla_a$.
Therefore, the new matter fields will satisfy the same equations of motion as the old matter fields, i.e. the conformal Killing vector based transformation induces a symmetry operation on the matter fields. It is the main task of my thesis to study the symmetries that may arise.

First consider $x'^m = x^m - \xi^m(x)$. Then, if a matter field is described by a tensor field[12], $T(x)$, since ξ^m is infinitesimal,

$$\begin{aligned}\delta T(x) &= T'(x) - T(x) \\ &= T'(x' + \xi) - T(x) \\ &= T'(x') + \xi^m(x')\partial_m T'(x') - T(x) \quad \text{to first order} \\ &= \xi^m(x)\partial_m T(x) \\ &= \xi^a(x)\nabla_a T(x) - \frac{1}{2}\xi^b(x)\omega_{bcd}(x)M^{cd}T \\ &= \xi^a(x)\nabla_a T(x) - \frac{1}{2}\tilde{K}^{bc}(x)M_{bc}T \end{aligned} \tag{2.23}$$

where $\tilde{K}^{bc} = \xi_d\omega^{dbc}$. Likewise, under a local Lorentz transformation,

$$\delta T = \frac{1}{2}K^{bc}(x)M_{bc}T, \tag{2.24}$$

and a more complicated transformation may arise based on the Weyl transformation. However, in all three cases, δT takes the form of a 1st order differential operator[13] acting on T. Furthermore, when K^{bc} and σ are determined in terms of ξ^a as per theorem 2.10, all "coefficients" in the differential operator are also determined in terms of ξ^a.
The net result is that in a conformal field theory, a conformal isometry induced by a conformal Killing vector, $\xi^a(x)$, induces a first order differential operator symmetry on the matter fields. Motivated by this result, I will be studying symmetries of the following type.

Definition 2.11 (Higher symmetry)**.** *Given a differential operator, $\mathfrak{D}$, acting on a tensor field, T, a higher symmetry is a scalar*[14]*, linear, differential operator, D, such that $\mathfrak{D}DT = 0$ whenever $\mathfrak{D}T = 0$.*

Corollary 2.11.1. *Provided D is a non-degenerate differential operator, D is a higher symmetry if and only if $\mathfrak{D}D = D'\mathfrak{D}$ for some other differential operator, D'.*

I will usually adopt the former definition in this thesis. These operators, D, are symmetries in that they take solutions to solutions. They are "higher" in the sense that D may not be a first order differential operator and there is no *a priori* link between D and conformal Killing vectors. A higher symmetry is a linear operator by definition. In principle, one could also look for non-linear transformations that take solutions to solutions, but I won't do that for simplicity and also because it can be shown that in many cases non-linear symmetries do not exist anyway [11].

[12] I will suppress the indices on the tensor field, $T(x)$. The only assumption I make is that $T(x)$ does not have any curved space indices. This is fine because given any tensor, it can be converted into a tensor without curved space indices via vierbeins and inverse vierbeins. Thus, $T(x)$ is a general coordinate scalar.

[13] In the terminology I will employ in this thesis, Lorentz generators will count as 1st order differential operators because they appear in theorem 2.10 and they are related to the commutator of two covariant derivatives.

[14] D is a scalar in the sense that DT is the same tensor type as T.

Once one finds an nth order symmetry operator, D, it immediately implies the existence of symmetry operators of order kn for any $k \in \mathbb{N}$, namely D^k, D composed with itself k times. Therefore, an equation of motion possessing a 1st order symmetry operator - e.g. as equations of motions for matter fields in conformal field theories should by the reasoning above - possesses symmetry operators of all orders.
However, it may be the case that not all symmetry operators can be written as a composition of lower order symmetries. Thus, there is still the possibility of unearthing a truly "higher" symmetry, just as the Killing tensor in the Kerr metric provided a new conserved quantity not derivable from the Killing vectors alone. Hence, it still pays to study higher order symmetries and study symmetries of different orders separately.

The original motivation for studying higher symmetries was a purely mathematical task - the solution of partial differential equations on non-trivial manifolds. Unlike the textbook separation of variables typically taught in the undergraduate curriculum, the best that could be hoped for on arbitrary manifolds for most equations was the following [12, 13].

Definition 2.12 (R-separability)**.** *Let $\mathfrak{D}$ be a linear, partial differential operator acting on a tensor field*[15]*, $T(x)$. Then, $\mathfrak{D}$ is said to be R-separable if and only if both of the following conditions hold.*

- *$\exists$ four functions, $T^{(m)}(x^m)$, each depending on only one of x^0, x^1, x^2 or x^3 and $\exists$ a function, $R(x)$, such that*

$$T(x) = R(x) \prod_{m=0}^{3} T^{(m)}(x^m)\,. \tag{2.25}$$

- *For each $T^{(m)}(x^m)$, $\exists$ a linear, ordinary differential operator, $\mathfrak{D}^{(m)}$, such that $T(x)$ satisfies $\mathfrak{D}T(x) = 0$ whenever all four of the $T^{(m)}(x^m)$ satisfy $\mathfrak{D}^{(m)}T^{(m)}(x^m) = 0$.*

When $R(x) = 1$, this definition reduces to the textbook definition. On the surface, R-separability seems to have nothing to do with symmetry operators. However, by the 4th quarter of the 20th century many deep connections between the existence of R-separable coordinate systems, higher symmetries and Lie group theory were discovered, including attempts to classify separable coordinate systems based on higher symmetries' eigenfunctions and spectra [14, 7, 12]. In many ways this development was a throwback to Sophus Lie's original motivation for studying the concepts that today bear his name. Like my discussion on the existence of 1st order symmetries for conformally invariant equations of motion, he too was looking at the effect of local transformations on differential equations - trying to find similarities between equations previously thought to be disparate. In the early development of R-separability, much of the progress was made on the Kerr spacetime. It was convenient - it had a known metric of moderate symmetry and a known Killing tensor. But in subsequent years, the theory has grown to a wide variety of manifolds and encompasses many famous equations from mathematical physics [12]. Again, it is fascinating, but tangential to my thesis, so I will not dwell on it any longer.

Another reason to study higher symmetries is the associative algebra generated by composing and taking linear combinations of higher symmetries. In the last two decades or so, a number of deep connections have emerged between the algebra of higher symmetries and the algebra of various higher spin fields. In conjunction with the AdS/CFT correspondence, these developments have brought a renewed focus on computing the higher symmetries of the equations

[15] Again, I have suppressed indices on T.

of motion of free, massless particles [15, 16]. Before the higher symmetry algebra is useful for application though, it requires some refinement.

Definition 2.13 (Equivalence relation of symmetries)**.** *Given two higher symmetries, D_1 and D_2, of a differential operator, $\mathfrak{D}$, let $D_1 \sim D_2$ if and only if $D_1 - D_2 = d\,\mathfrak{D}$ for some other differential operator, d. Then, $\sim$ is an equivalence relation.*

Proof. $D_1 - D_1 = 0 = 0 \times \mathfrak{D} \implies \sim$ is reflexive.
If $D_1 \sim D_2$, then $\exists\, d$ such that $D_1 - D_2 = d\,\mathfrak{D} \implies D_2 - D_1 = -d\,\mathfrak{D} \implies \sim$ is symmetric.
Let $D_1 \sim D_2$ and $D_2 \sim D_3$. Hence $\exists\, d_1,\ d_2$ such that $D_1 - D_2 = d_1\,\mathfrak{D}$ and $D_2 - D_3 = d_2\,\mathfrak{D}$ $\implies D_1 - D_3 = (d_1 + d_2)\,\mathfrak{D} \implies \sim$ is transitive.
Therefore, $\sim$ is indeed an equivalence relation. □

Two symmetries linked by $\sim$ are essentially trivially related. It is not interesting to treat them as separate objects. Instead, the much richer algebra is the one consisting of equivalence classes of symmetries under $\sim$. Therefore, throughout the work, it will suffice to find a single representative for each equivalence class. Thinking in terms of equivalence classes is identical to the following.

Lemma 2.14. *The set of trivial higher symmetries, $D = d\mathfrak{D}$, forms a two-sided ideal in the algebra of higher symmetries.*

Proof. Let D be a trivial symmetry and let ρ be an arbitrary higher symmetry.
Therefore, $D = d\mathfrak{D}$ for some differential operator, d, and $\mathfrak{D}\rho = \rho'\mathfrak{D}$ for some differential operator, ρ'.
Hence, $\rho D = \rho d\mathfrak{D} = (\rho d)\mathfrak{D} \implies \rho D$ is a trivial symmetry.
Likewise, $D\rho = d\mathfrak{D}\rho = d\rho'\mathfrak{D} = (d\rho')\mathfrak{D} \implies D\rho$ is a trivial symmetry too. □

The set of equivalence classes discussed above is nothing but the algebra of higher symmetries quotient-ed by the two-sided ideal of trivial higher symmetries. As a matter of personal taste, I will largely talk in terms of equivalence classes as opposed to quotient algebras. However, the quotient algebra perspective has proven to be useful in higher spin field theory applications [16].

At a practical level, the equivalence relation will be extensively used in sections 3.2 and 4.2 to simplify the terms appearing in potential symmetry operators. For example, it is possible that $D_1 T \neq D_2 T$ unless $\mathfrak{D}T = 0$. That can only happen if $D_1 - D_2 = d\mathfrak{D}$ for some d, or equivalently $D_1 \sim D_2$. Even before the equivalence relation though, it might be that $D_1 \neq D_2$, but $D_1 T = D_2 T$ because of the specific form of the tensor[16], T. In such scenarios, I shall consider D_1 to be equivalent to D_2 provided it is clear only D_1 and D_2's actions on a particular tensor type are relevant.

But first, to make progress on separation of variables or higher spin fields in the ways I have outlined, one must actually know the higher symmetries of different differential operators; it will be my task to develop techniques to compute them. As aforementioned, I will be focused on two operators in particular. They are the conformal d'Alembertian, $\mathfrak{D} = \Delta = \Box - \frac{1}{6}R$, acting on a scalar field, φ, and the massless Dirac operator, $\gamma^a\nabla_a$, acting on a four-component Dirac spinor, $\boldsymbol{\Psi}$. Conformal Killing vectors and tensors will make numerous appearances in subsequent chapters, but actually finding a conformal Killing vector or tensor on a given manifold (if one exists) is beyond the scope of my thesis.

[16]For example, such a situation may arise if T is a scalar and D_1 and D_2 differ by a Lorentz generator. Lorentz generators annihilate scalars and so D_1 and D_2 still give the same result when acting on T, even though D_1 and D_2 are not strictly equal.

Several papers discussing similar topics have been published in recent years - see [17, 18, 19] for examples. The most complete account though, is [20], which contains some overlap with and extensions to the results I will derive about 2nd order symmetry operators. Despite some similarities in techniques, my work is completely independent[17].

[17] Also, they did most of their calculations on *Mathematica* where as I did them by hand.

Chapter 3

Higher symmetries of the conformal d'Alembertian

3.1 Action for the conformal d'Alembertian

The conformal d'Alembertian, Δ, is defined as

$$\Delta = \Box - \frac{1}{6}R = \nabla^a \nabla_a - \frac{1}{6}R\,. \tag{3.1}$$

In this thesis, I will only be concerned with the action of Δ on scalar fields, $\varphi(x)$. As a first step towards the equation I will be analysing, $\Delta\varphi = 0$, consider the theory of a free, massless, real, scalar field in flat space. It is described by the action,

$$S = -\frac{1}{2}\int \partial^a(\varphi)\partial_a(\varphi)\,\mathrm{d}^4x\,, \tag{3.2}$$

which has the massless Klein-Gordon equation, $\Box\varphi = 0$, as the equation of motion. When lifting an action to curved space, the standard procedure is to change partial derivatives to covariant derivatives and change the integration measure from d^4x to $e\,\mathrm{d}^4x$ where e is $\det({e_m}^a)$. However, this procedure is incomplete. Actions differing only by curvature factors coincide in flat space. For the free, massless, real, scalar field, one possible resolution is to impose that the action should be invariant under Weyl transformations in curved space. Then, it can be shown the curvature terms are fixed so that

$$S = -\frac{1}{2}\int \left(\nabla^a(\varphi)\nabla_a(\varphi) + \frac{1}{6}R\varphi^2\right) e\,\mathrm{d}^4x \quad \text{where } e = \det({e_m}^a) \tag{3.3}$$

and S is invariant under the Weyl transformation, ${e'_a}^m(x) = \mathrm{e}^{\sigma(x)}{e_a}^m(x)$, provided $\varphi'(x) = \mathrm{e}^{\sigma(x)}\varphi(x)$. Finally, the equation of motion for φ from this action is $\Delta\varphi = 0$.

This same action can be derived - along with its Weyl transformation properties - from an alternative, but equally interesting, perspective. Consider the following "Weyl invariant" formulation of (vacuum) general relativity described in [10]. The key idea is the observation that given an action, $S[{e_a}^m]$, constructed entirely out the vierbein, $S[{e_a}^m/\varphi]$ is invariant upon a Weyl transformation, ${e_a}^m \to \mathrm{e}^\sigma {e_a}^m$ and $\varphi \to \mathrm{e}^\sigma\varphi$ for arbitrary scalar fields, $\sigma(x)$. Hence, given any field theory, introducing a gauge field, $\varphi(x)$, makes the field theory Weyl invariant[1]. I will apply the formalism to the Einstein-Hilbert action,

$$S = \frac{1}{16\pi G}\int Re\,\mathrm{d}^4x\,. \tag{3.4}$$

[1]If $S[{e_a}^m/\varphi] = S[{e_a}^m]$, it means $S[{e_a}^m]$ was already a conformal field theory.

Hence, denoting all new variables with primes, $e' = \det(e_m{}^a\varphi) = e\varphi^4$. Since $e'_a{}^m = e_a{}^m/\varphi$ is formally identical to a Weyl transformation with $\mathrm{e}^\sigma = 1/\varphi$, one immediately gets

$$\begin{aligned} R' &= \frac{1}{\varphi^2}\left(R + 6\Box\left(\ln\left(\frac{1}{\varphi}\right)\right) - 6\nabla^a\left(\ln\left(\frac{1}{\varphi}\right)\right)\nabla_a\left(\ln\left(\frac{1}{\varphi}\right)\right)\right) \\ &= (R - 6\Box(\ln(\varphi)) - 6\nabla^a(\ln(\varphi))\nabla_a(\ln(\varphi)))/\varphi^2\,. \end{aligned} \tag{3.5}$$

Then, since $\nabla_a = e_a{}^m\nabla_m$ and $\nabla_m = \partial_m$ when acting on a scalar, $\nabla_a(\ln(\varphi)) = \nabla_a(\varphi)/\varphi$. Therefore,

$$\begin{aligned} \Box(\ln(\varphi)) &= \nabla^a(\nabla_a(\varphi)/\varphi) \\ &= -\frac{\nabla^a(\varphi)\nabla_a(\varphi)}{\varphi^2} + \frac{\Box(\varphi)}{\varphi}\,, \end{aligned} \tag{3.6}$$

which yields

$$\begin{aligned} R' &= \frac{R}{\varphi^2} + \frac{6\nabla^a(\varphi)\nabla_a(\varphi)}{\varphi^4} - \frac{6\Box(\varphi)}{\varphi^3} - \frac{6\nabla^a(\varphi)\nabla_a(\varphi)}{\varphi^4} \\ &= \frac{R}{\varphi^2} - \frac{6\Box(\varphi)}{\varphi^3}\,. \end{aligned} \tag{3.7}$$

Thus,

$$R'e' = Re\varphi^2 - 6e\varphi\Box(\varphi)\,. \tag{3.8}$$

It will be more convenient to re-write the 2nd term with $\varphi\Box(\varphi) = \nabla^a(\varphi\nabla_a\varphi) - \nabla^a(\varphi)\nabla_a(\varphi)$ since $\int \nabla^a(\varphi\nabla_a\varphi)e\,\mathrm{d}^4x = 0$ by the generalised Stokes' theorem. The action is then

$$S' = \frac{1}{16\pi G}\int R'e'\,\mathrm{d}^4x = \frac{1}{16\pi G}\int(R\varphi^2 + 6\nabla^a(\varphi)\nabla_a(\varphi))e\,\mathrm{d}^4x\,. \tag{3.9}$$

Finally varying S' with respect to φ,

$$\delta S' = \frac{1}{16\pi G}\int(2R\varphi\delta\varphi + 12\nabla^a(\varphi)\nabla_a(\delta\varphi))e\,\mathrm{d}^4x\,. \tag{3.10}$$

However, $\nabla^a(\varphi)\nabla_a(\delta\varphi) = \nabla_a(\nabla^a(\varphi)\delta\varphi) - \Box(\varphi)\delta\varphi$ and the first term on the RHS integrates to zero. Hence finally,

$$\delta S' = \frac{1}{8\pi G}\int(R\varphi - 6\Box(\varphi))\delta\varphi e\,\mathrm{d}^4x \tag{3.11}$$

and therefore the Euler-Lagrange equation for the gauge field, $\varphi(x)$, is $\Delta\varphi = 0$. It is this final equation whose symmetries I will consider in this chapter. Substantial work along this endeavour has already been attempted by Eastwood [17].

3.2 Structure of the symmetry operators

As per definition 2.11, candidate symmetries, D, must be scalar combinations of ∇_a, M_{ab} and tensor coefficients, $\xi^{a_1\cdots a_n}(x)$.

Lemma 3.1. *Lorentz generators do not appear in higher symmetries of φ.*

Proof. Since φ is a scalar, $M_{ab}\,\varphi = 0$.
Therefore, any M_{ab} that appear must appear on the left of derivatives acting on φ. However,

$$M_{ab}\nabla_{c_1}\cdots\nabla_{c_k}\varphi = \sum_{i=1}^{k}(\eta_{c_i a}\nabla_{c_1}\cdots\nabla_b\cdots\nabla_{c_k} - \eta_{c_i b}\nabla_{c_1}\cdots\nabla_a\cdots\nabla_{c_k})\varphi\,, \tag{3.12}$$

which is a sum of terms with k derivatives on φ, but no Lorentz generator.
Hence, all M_{ab}s can be absorbed into terms with no Lorentz generator on them. Thus by $\sim$, the equivalence 2.13, there is no need to consider symmetries with M_{ab} as there will be symmetries without Lorentz generators in the same equivalence class. □

Therefore, to ensure that the differential operator is scalar, and noting that derivatives acting on tensors other than φ can be absorbed into lower order terms, the most general nth order symmetry of Δ is

$$D^{(n)} = \sum_{k=0}^{n}\xi^{a_1\cdots a_k}\nabla_{a_1}\cdots\nabla_{a_k}\,. \tag{3.13}$$

Lemma 3.2. *Any coefficient of a term with multiple derivatives can be taken to be symmetric and traceless.*

Proof. Consider the action of a term with multiple derivatives, i.e.

$$\xi^{a_1\cdots a_k}\nabla_{a_1}\cdots\nabla_{a_k}, \tag{3.14}$$

on the scalar field, φ. Suppose there is an antisymmetric component between the ith and jth indices of $\xi^{a_1\cdots a_k}$. Then,

$$\begin{aligned}
&\xi^{a_1\cdots a_k}\nabla_{a_1}\cdots\nabla_{a_k}\\
&= \frac{1}{2}(\xi^{a_1\cdots a_i\cdots a_j\cdots a_k} + \xi^{a_1\cdots a_j\cdots a_i\cdots a_k})\nabla_{a_1}\cdots\nabla_{a_k} + \frac{1}{2}(\xi^{a_1\cdots a_i\cdots a_j\cdots a_k} - \xi^{a_1\cdots a_j\cdots a_i\cdots a_k})\nabla_{a_1}\cdots\nabla_{a_k}\\
&= \frac{1}{2}(\xi^{a_1\cdots a_i\cdots a_j\cdots a_k} + \xi^{a_1\cdots a_j\cdots a_i\cdots a_k})\nabla_{a_1}\cdots\nabla_{a_k}\\
&\quad + \frac{1}{2}\xi^{a_1\cdots a_k}(\nabla_{a_1}\cdots\nabla_{a_i}\cdots\nabla_{a_j}\cdots\nabla_{a_k} - \nabla_{a_1}\cdots\nabla_{a_j}\cdots\nabla_{a_i}\cdots\nabla_{a_k})\,.
\end{aligned} \tag{3.15}$$

But now, the 2nd term in the previous equation can be reduced to sum of commutators. Since commutators reduce the order of derivatives by 2, the antisymmetric contribution can be absorbed into lower order components. Therefore, the coefficients can be taken to be symmetric.
Next, suppose that $\xi^{a_1\cdots a_k} = \xi^{(a_1\cdots a_k)}$ has non-zero trace. Splitting into the trace and traceless components,

$$\begin{aligned}
&\xi^{a_1\cdots a_k}\nabla_{a_1}\cdots\nabla_{a_k}\varphi\\
&= \left(\xi^{a_1\cdots a_k}\nabla_{a_1} - \frac{1}{4}\eta^{a_1 a_2}\xi_b{}^{ba_3\cdots a_k}\right)\nabla_{a_1}\cdots\nabla_{a_k}\varphi + \frac{1}{4}\eta^{a_1 a_2}\xi_b{}^{ba_3\cdots a_k}\nabla_{a_1}\cdots\nabla_{a_k}\varphi\\
&= \left(\xi^{a_1\cdots a_k}\nabla_{a_1} - \frac{1}{4}\eta^{a_1 a_2}\xi_b{}^{ba_3\cdots a_k}\right)\nabla_{a_1}\cdots\nabla_{a_k}\varphi + \frac{1}{4}\xi_b{}^{ba_3\cdots a_k}\Box\nabla_{a_3}\cdots\nabla_{a_k}\varphi\\
&= \left(\xi^{a_1\cdots a_k}\nabla_{a_1} - \frac{1}{4}\eta^{a_1 a_2}\xi_b{}^{ba_3\cdots a_k}\right)\nabla_{a_1}\cdots\nabla_{a_k}\varphi + \frac{1}{4}\xi_b{}^{ba_3\cdots a_k}\nabla_{a_3}\cdots\nabla_{a_k}\Box\varphi\\
&\quad + \frac{1}{4}\xi_b{}^{ba_3\cdots a_k}[\Box, \nabla_{a_3}\cdots\nabla_{a_k}]\varphi\\
&= \left(\xi^{a_1\cdots a_k}\nabla_{a_1} - \frac{1}{4}\eta^{a_1 a_2}\xi_b{}^{ba_3\cdots a_k}\right)\nabla_{a_1}\cdots\nabla_{a_k}\varphi + \frac{1}{24}\xi_b{}^{ba_3\cdots a_k}\nabla_{a_3}\cdots\nabla_{a_k}(R\varphi)\\
&\quad + \frac{1}{4}\xi_b{}^{ba_3\cdots a_k}[\Box, \nabla_{a_3}\cdots\nabla_{a_k}]\varphi\,,
\end{aligned} \tag{3.16}$$

which shows given $\Delta\varphi = 0$, the trace component can be absorbed into terms with only $k-2$ derivatives on φ.
Therefore any trace in the components can be removed by $\sim$. □

Hence, by the lemma, the most general nth order symmetry of Δ is

$$D^{(n)} = \sum_{k=0}^{n} \xi^{a_1\cdots a_k} \nabla_{a_1} \cdots \nabla_{a_k} \tag{3.17}$$

with all $\xi^{a_1\cdots a_k}$ symmetric and traceless.

3.3 Top component for nth order symmetries

From here on, the two-component spinor formalism will be essential to the discussion. Again, if required, refer to appendix D for an overview of spinors and appendix E for my spinor conventions.

With the benefit of hindsight, I will begin with the following lemma.

Lemma 3.3. *A tensor, $\xi^{a_1\cdots a_n}$, is symmetric and traceless if and only if the corresponding spin tensor, $\xi^{\alpha_1\cdots\alpha_n\dot{\alpha}_1\cdots\dot{\alpha}_n}$, satisfies $\xi^{\alpha_1\cdots\alpha_n\dot{\alpha}_1\cdots\dot{\alpha}_n} = \xi^{(\alpha_1\cdots\alpha_n)(\dot{\alpha}_1\cdots\dot{\alpha}_n)}$.*

Proof. The lemma is vacuously true for $n = 1$.
I will begin by proving symmetric and traceless $\implies \xi^{\alpha_1\cdots\alpha_n\dot{\alpha}_1\cdots\dot{\alpha}_n} = \xi^{(\alpha_1\cdots\alpha_n)(\dot{\alpha}_1\cdots\dot{\alpha}_n)}$. For $n \geq 2$, I will prove it by induction.
By definition, $\xi_{\alpha_1\alpha_2\dot{\alpha}_1\dot{\alpha}_2} = (\sigma_{a_1})_{\alpha_1\dot{\alpha}_1}(\sigma_{a_2})_{\alpha_2\dot{\alpha}_2}\xi^{a_1a_2}$. Any type$-(2,2)$ spin tensor[2] can be "decomposed" as follows.

$$\begin{aligned}\xi_{\alpha_1\alpha_2\dot{\alpha}_1\dot{\alpha}_2} &= \xi_{[\alpha_1\alpha_2](\dot{\alpha}_1\dot{\alpha}_2)} + \xi_{(\alpha_1\alpha_2)[\dot{\alpha}_1\dot{\alpha}_2]} + \xi_{[\alpha_1\alpha_2][\dot{\alpha}_1\dot{\alpha}_2]} + \xi_{(\alpha_1\alpha_2)(\dot{\alpha}_1\dot{\alpha}_2)} \\ &= -\frac{1}{2}\varepsilon_{\alpha_1\alpha_2}\varepsilon^{\mu\nu}\xi_{\mu\nu(\dot{\alpha}_1\dot{\alpha}_2)} - \frac{1}{2}\varepsilon_{\dot{\alpha}_1\dot{\alpha}_2}\varepsilon^{\dot{\mu}\dot{\nu}}\xi_{(\alpha_1\alpha_2)\dot{\mu}\dot{\nu}} + \frac{1}{4}\varepsilon_{\alpha_1\alpha_2}\varepsilon_{\dot{\alpha}_1\dot{\alpha}_2}\varepsilon^{\mu\nu}\varepsilon^{\dot{\mu}\dot{\nu}}\xi_{\mu\nu\dot{\mu}\dot{\nu}} \\ &\quad + \xi_{(\alpha_1\alpha_2)(\dot{\alpha}_1\dot{\alpha}_2)}\end{aligned} \tag{3.18}$$

where the 2nd line follows from the fact that every antisymmetric rank-2 tensor is proportional to the Levi-Civita symbol with 2 indices; $\varepsilon^{\mu\nu}\xi_{\mu\nu(\dot{\alpha}_1\dot{\alpha}_2)}$, $\varepsilon^{\dot{\mu}\dot{\nu}}\xi_{(\alpha_1\alpha_2)\dot{\mu}\dot{\nu}}$ and $\varepsilon^{\mu\nu}\varepsilon^{\dot{\mu}\dot{\nu}}\xi_{\mu\nu\dot{\mu}\dot{\nu}}$ are the corresponding proportionality constants[3]. However,

$$\begin{aligned}\varepsilon^{\mu\nu}\xi_{\mu\nu(\dot{\alpha}_1\dot{\alpha}_2)} &= \frac{1}{2}\varepsilon^{\mu\nu}(\xi_{\mu\nu\dot{\alpha}_1\dot{\alpha}_2} + \xi_{\mu\nu\dot{\alpha}_2\dot{\alpha}_1}) \\ &= -\frac{1}{2}\varepsilon^{\nu\mu}(\xi_{\mu\nu\dot{\alpha}_1\dot{\alpha}_2} + \xi_{\mu\nu\dot{\alpha}_2\dot{\alpha}_1}) \\ &= -\frac{1}{2}\varepsilon^{\mu\nu}(\xi_{\nu\mu\dot{\alpha}_1\dot{\alpha}_2} + \xi_{\nu\mu\dot{\alpha}_2\dot{\alpha}_1}) \\ &= -\frac{1}{2}\varepsilon^{\mu\nu}(\xi_{\mu\nu\dot{\alpha}_2\dot{\alpha}_1} + \xi_{\mu\nu\dot{\alpha}_1\dot{\alpha}_2}) \text{ by } \xi^{a_1a_2}\text{'s symmetry} \\ &= -\varepsilon^{\mu\nu}\xi_{\mu\nu(\dot{\alpha}_1\dot{\alpha}_2)}\,,\end{aligned} \tag{3.19}$$

[2]When talking about spin tensors, I take type$-(m,n)$ to mean there are m undotted and n dotted indices, not that the tensor has m contravariant and n covariant indices.

[3]This can be explicitly checked using the identity, $\varepsilon_{\alpha\beta}\varepsilon^{\mu\nu} = \delta^{\nu}{}_{\alpha}\delta^{\mu}{}_{\beta} - \delta^{\mu}{}_{\alpha}\delta^{\nu}{}_{\beta}$, applied to equation 3.18.

which means $\varepsilon^{\mu\nu}\xi_{\mu\nu(\dot{\alpha_1}\dot{\alpha_2})} = 0$. Similarly, $\varepsilon^{\dot{\mu}\dot{\nu}}\xi_{(\alpha_1\alpha_2)\dot{\mu}\dot{\nu}} = 0$ as well. For the third term,

$$\begin{aligned}\varepsilon^{\mu\nu}\varepsilon^{\dot{\mu}\dot{\nu}}\xi_{\mu\nu\dot{\mu}\dot{\nu}} &= \xi_{\mu\dot{\mu}}{}^{\mu\dot{\mu}} \\ &= -2\xi_a{}^a \\ &= 0 \text{ as } \xi^{a_1a_2} \text{ is traceless}\,. \end{aligned} \tag{3.20}$$

And hence ultimately $\xi_{\alpha_1\alpha_2\dot{\alpha_1}\dot{\alpha_2}} = \xi_{(\alpha_1\alpha_2)(\dot{\alpha_1}\dot{\alpha_2})}$ as required.
Next, assume the statement holds for some $n = k$ to show that implies the statement holds for $n = k + 1$ as well.
By assumption, $\xi_{a_1\cdots a_{k+1}}$ is symmetric and traceless.
Therefore, $\xi_{a_1\cdots a_{k+1}}$ is symmetric and traceless in its first k indices alone.
Hence, by the inductive assumption, $\xi_{\alpha_1\cdots\alpha_{k+1}\dot{\alpha_1}\cdots\dot{\alpha}_{k+1}} = \xi_{(\alpha_1\cdots\alpha_k)(\dot{\alpha_1}\cdots\dot{\alpha_k})\alpha_{k+1}\dot{\alpha}_{k+1}}$. Like with the base case, I will re-write the last pair of indices in terms of symmetrisations and Levi-Civita symbols.
Decompose $\xi_{\alpha_1\cdots\alpha_{k+1}\dot{\alpha}_1\cdots\dot{\alpha}_{k+1}}$ as

$$\xi_{(\alpha_1\cdots\alpha_k)(\dot{\alpha_1}\cdots\dot{\alpha_k})\alpha_{k+1}\dot{\alpha}_{k+1}} = A_{\alpha_1\cdots\alpha_k}B_{\dot{\alpha}_1\cdots\dot{\alpha}_k}C_{\alpha_{k+1}}D_{\dot{\alpha}_{k+1}} \tag{3.21}$$

for some $A_{\alpha_1\cdots\alpha_k}, B_{\dot{\alpha}_1\cdots\dot{\alpha}_k}, C_{\alpha_{k+1}}$ and $D_{\dot{\alpha}_{k+1}}$ with $A_{\alpha_1\cdots\alpha_k}$ and $B_{\dot{\alpha}_1\cdots\dot{\alpha}_k}$ being symmetric in all their indices[4].

$$\begin{aligned} A_{(\alpha_1\cdots\alpha_k}C_{\alpha_{k+1})} &= \frac{1}{(k+1)!}\left(k!A_{\alpha_1\cdots\alpha_k}C_{\alpha_{k+1}} + k!\sum_{i=1}^{k} A_{\alpha_1\cdots\hat{\alpha}_i\cdots\alpha_{k+1}}C_{\alpha_i}\right) \\ &= \frac{1}{k+1}\left(A_{\alpha_1\cdots\alpha_k}C_{\alpha_{k+1}} + \sum_{i=1}^{k} A_{\alpha_1\cdots\hat{\alpha}_i\cdots\alpha_{k+1}}C_{\alpha_i}\right) \end{aligned} \tag{3.22}$$

To manipulate each of the terms in the sum,

$$\begin{aligned} \varepsilon_{\alpha_{k+1}\alpha_i}C^\beta A_{\beta\alpha_1\cdots\hat{\alpha}_i\cdots\alpha_k} &= \varepsilon_{\alpha_{k+1}\alpha_i}\varepsilon^{\beta\gamma}C_\gamma A_{\beta\alpha_1\cdots\hat{\alpha}_i\cdots\alpha_k} \\ &= (\delta^\gamma{}_{\alpha_{k+1}}\delta^\beta{}_{\alpha_i} - \delta^\beta{}_{\alpha_{k+1}}\delta^\gamma{}_{\alpha_i})C_\gamma A_{\beta\alpha_1\cdots\hat{\alpha}_i\cdots\alpha_k} \\ &= C_{\alpha_{k+1}}A_{\alpha_1\cdots\alpha_k} - C_{\alpha_i}A_{\alpha_1\cdots\hat{\alpha}_i\cdots\alpha_{k+1}}\,. \end{aligned} \tag{3.23}$$

Therefore,

$$C_{\alpha_i}A_{\alpha_1\cdots\hat{\alpha}_i\cdots\alpha_{k+1}} = C_{\alpha_{k+1}}A_{\alpha_1\cdots\alpha_k} - \varepsilon_{\alpha_{k+1}\alpha_i}C^\beta A_{\beta\alpha_1\cdots\hat{\alpha}_i\cdots\alpha_k} \tag{3.24}$$

Substituting this back into equation 3.22,

$$A_{\alpha_1\cdots\alpha_k}C_{\alpha_{k+1}} = A_{(\alpha_1\cdots\alpha_k}C_{\alpha_{k+1})} + \frac{1}{k+1}\sum_{i=1}^{k}\varepsilon_{\alpha_{k+1}\alpha_i}C^\beta A_{\beta\alpha_1\cdots\hat{\alpha}_i\cdots\alpha_k}\,. \tag{3.25}$$

Therefore,

$$B_{\dot{\alpha}_1\cdots\dot{\alpha}_k}D_{\dot{\alpha}_{k+1}} = B_{(\dot{\alpha}_1\cdots\dot{\alpha}_k}D_{\dot{\alpha}_{k+1})} + \frac{1}{k+1}\sum_{i=1}^{k}\varepsilon_{\dot{\alpha}_{k+1}\dot{\alpha}_i}D^{\dot{\beta}}B_{\dot{\beta}\dot{\alpha}_1\cdots\hat{\dot{\alpha}}_i\cdots\dot{\alpha}_k} \tag{3.26}$$

[4]Technically $A_{\alpha_1\cdots\alpha_k}$ is a function of the index, α_{k+1}, on $C_{\alpha_{k+1}}$ to get a true tensor product. This is because in a tensor product, bases elements are products, e.g. $e_1 \otimes \cdots \otimes e_k \otimes e_{k+1}$; the elements are not just products of lower order tensors. However, $A_{\alpha_1\cdots\alpha_k}C_{\alpha_{k+1}}$ is a convenient notation to explicitly show the separation/independence between the two sets of indices. The alternative would be something like $\xi_{\alpha_1\cdots\alpha_k,\alpha_{k+1}}$.

as well. Putting both parts together,

$$\xi_{(\alpha_1\cdots\alpha_k)(\dot{\alpha}_1\cdots\dot{\alpha}_k)\alpha_{k+1}\dot{\alpha}_{k+1}} = \left(A_{(\alpha_1\cdots\alpha_k}C_{\alpha_{k+1})} + \frac{1}{k+1}\sum_{i=1}^{k}\varepsilon_{\alpha_{k+1}\alpha_i}C^{\beta}A_{\beta\alpha_1\cdots\hat{\alpha}_i\cdots\alpha_k}\right)$$
$$\times\left(B_{(\dot{\alpha}_1\cdots\dot{\alpha}_k}D_{\dot{\alpha}_{k+1})} + \frac{1}{k+1}\sum_{j=1}^{k}\varepsilon_{\dot{\alpha}_{k+1}\dot{\alpha}_j}D^{\dot{\beta}}B_{\dot{\beta}\dot{\alpha}_1\cdots\hat{\dot{\alpha}}_j\cdots\dot{\alpha}_k}\right). \tag{3.27}$$

Re-arranging,

$$\begin{aligned}\xi_{\alpha_1\cdots\alpha_{k+1}\dot{\alpha}_1\cdots\dot{\alpha}_{k+1}} &= \xi_{(\alpha_1\cdots\alpha_{k+1})(\dot{\alpha}_1\cdots\dot{\alpha}_{k+1})} + \frac{1}{k+1}\sum_{i=1}^{k}\varepsilon_{\dot{\alpha}_{k+1}\dot{\alpha}_i}\xi_{(\alpha_1\cdots\alpha_{k+1})\dot{\alpha}_1\cdots\hat{\dot{\alpha}}_i\cdots\dot{\alpha}_k\dot{\beta}}{}^{\dot{\beta}} \\ &+ \frac{1}{k+1}\sum_{i=1}^{k}\varepsilon_{\alpha_{k+1}\alpha_i}\xi_{\alpha_1\cdots\hat{\alpha}_i\cdots\alpha_k\beta}{}^{\beta}{}_{(\dot{\alpha}_1\cdots\dot{\alpha}_{k+1})} \\ &+ \frac{1}{(k+1)^2}\sum_{i=1}^{k}\sum_{j=1}^{k}\varepsilon_{\alpha_{k+1}\alpha_i}\varepsilon_{\dot{\alpha}_{k+1}\dot{\alpha}_j}\xi_{\alpha_1\cdots\hat{\alpha}_i\cdots\alpha_k\beta}{}^{\beta}{}_{\dot{\alpha}_1\cdots\hat{\dot{\alpha}}_i\cdots\dot{\alpha}_k\dot{\beta}}{}^{\dot{\beta}}\,. \end{aligned}\tag{3.28}$$

It is now time to use the fact that $\xi^{a_1\cdots a_{k+1}}$ is symmetric and traceless. From the tracelessness,

$$\begin{aligned}\xi_{\alpha_1\cdots\hat{\alpha}_i\cdots\alpha_k\beta}{}^{\beta}{}_{\dot{\alpha}_1\cdots\hat{\dot{\alpha}}_i\cdots\dot{\alpha}_k\dot{\beta}}{}^{\dot{\beta}} &= (\sigma^a)_{\beta\dot{\beta}}(\tilde{\sigma}^b)^{\dot{\beta}\beta}\xi_{\alpha_1\cdots\hat{\alpha}_i\cdots\alpha_k\dot{\alpha}_1\cdots\hat{\dot{\alpha}}_i\cdots\dot{\alpha}_k ab} \\ &= -2\eta^{ab}\xi_{\alpha_1\cdots\hat{\alpha}_i\cdots\alpha_k\dot{\alpha}_1\cdots\hat{\dot{\alpha}}_i\cdots\dot{\alpha}_k ab} \\ &= -2\xi_{\alpha_1\cdots\hat{\alpha}_i\cdots\alpha_k\dot{\alpha}_1\cdots\hat{\dot{\alpha}}_i\cdots\dot{\alpha}_k a}{}^{a} \\ &= 0 \text{ as } \xi^{a_1\cdots a_{k+1}} \text{ is traceless.}\end{aligned}\tag{3.29}$$

Next, from the symmetry, one can swap any pair of indices, $(\alpha_i, \dot{\alpha}_i)$, with any other pair, $(\alpha_j, \dot{\alpha}_j)$. One can also freely swap indices within a symmetrisation. Thus,

$$\begin{aligned}\varepsilon_{\alpha_{k+1}\alpha_i}\xi_{\alpha_1\cdots\hat{\alpha}_i\cdots\alpha_k\beta}{}^{\beta}{}_{(\dot{\alpha}_1\cdots\dot{\alpha}_{k+1})} &= \varepsilon_{\alpha_i\alpha_{k+1}}\xi_{\alpha_1\cdots\hat{\alpha}_i\cdots\alpha_k\beta}{}^{\beta}{}_{(\dot{\alpha}_1\cdots\dot{\alpha}_{k+1}\cdots\dot{\alpha}_i)} \\ &= -\varepsilon_{\alpha_{k+1}\alpha_i}\xi_{\alpha_1\cdots\hat{\alpha}_i\cdots\alpha_k\beta}{}^{\beta}{}_{(\dot{\alpha}_1\cdots\dot{\alpha}_{k+1})}\,,\end{aligned}\tag{3.30}$$

meaning $\varepsilon_{\alpha_{k+1}\alpha_i}\xi_{\alpha_1\cdots\hat{\alpha}_i\cdots\alpha_k\beta}{}^{\beta}{}_{(\dot{\alpha}_1\cdots\dot{\alpha}_{k+1})} = 0$.
Similarly, $\varepsilon_{\dot{\alpha}_{k+1}\dot{\alpha}_i}\xi_{(\alpha_1\cdots\alpha_{k+1})\dot{\alpha}_1\cdots\hat{\dot{\alpha}}_i\cdots\dot{\alpha}_k\dot{\beta}}{}^{\dot{\beta}} = 0$ as well.
Therefore, $\xi_{\alpha_1\cdots\alpha_{k+1}\dot{\alpha}_1\cdots\dot{\alpha}_{k+1}} = \xi_{(\alpha_1\cdots\alpha_{k+1})(\dot{\alpha}_1\cdots\dot{\alpha}_{k+1})}$.
Hence, the induction is complete and symmetric and traceless $\implies$ $\xi_{\alpha_1\cdots\alpha_n\dot{\alpha}_1\cdots\dot{\alpha}_n} = \xi_{(\alpha_1\cdots\alpha_n)(\dot{\alpha}_1\cdots\dot{\alpha}_n)}$, or $\xi^{\alpha_1\cdots\alpha_n\dot{\alpha}_1\cdots\dot{\alpha}_n} = \xi^{(\alpha_1\cdots\alpha_n)(\dot{\alpha}_1\cdots\dot{\alpha}_n)}$ with indices raised, $\forall n \in \mathbb{N}$.

It remains to show $\xi^{\alpha_1\cdots\alpha_n\dot{\alpha}_1\cdots\dot{\alpha}_n} = \xi^{(\alpha_1\cdots\alpha_n)(\dot{\alpha}_1\cdots\dot{\alpha}_n)} \implies$ symmetric and traceless for $n \geq 2$.
Symmetry is automatic since that only requires that swapping pairs, $(\alpha_i, \dot{\alpha}_i)$ and $(\alpha_j, \dot{\alpha}_j)$, leaves the tensor unchanged. However, here there is already symmetry in the dotted and and

undotted indices independently, not just in pairs. For the trace,

$$\begin{aligned}
\xi^{a}{}_{aa_2\cdots a_n} &= -\frac{1}{2}\xi^{\alpha\dot{\alpha}}{}_{\alpha\dot{\alpha}a_2\cdots a_n} \\
&= -\frac{1}{2}\varepsilon^{\alpha\beta}\varepsilon^{\dot{\alpha}\dot{\beta}}\xi_{\alpha\beta\dot{\alpha}\dot{\beta}a_2\cdots a_n} \\
&= -\frac{1}{2}\varepsilon^{\alpha\beta}\varepsilon^{\dot{\alpha}\dot{\beta}}\xi_{\beta\alpha\dot{\alpha}\dot{\beta}a_2\cdots a_n} \\
&= \frac{1}{2}\varepsilon^{\beta\alpha}\varepsilon^{\dot{\alpha}\dot{\beta}}\xi_{\beta\alpha\dot{\alpha}\dot{\beta}a_2\cdots a_n} \\
&= \frac{1}{2}\varepsilon^{\beta\alpha}\varepsilon^{\dot{\beta}\dot{\alpha}}\xi_{\beta\alpha\dot{\beta}\dot{\alpha}a_2\cdots a_n} \\
&= \frac{1}{2}\xi^{\alpha\dot{\alpha}}{}_{\alpha\dot{\alpha}a_2\cdots a_n} \\
&= -\xi^{a}{}_{aa_2\cdots a_n}\,, \qquad (3.31)
\end{aligned}$$

Therefore, $\xi^{a}{}_{aa_2\cdots a_n} = 0$, thereby completing all parts of the proof. □

Corollary 3.3.1. *Hence, by definition, a symmetric and traceless tensor, $\xi^{a_1\cdots a_n}$, is conformal Killing if and only if*

$$\nabla^{(\beta(\dot{\beta}}\xi^{\alpha_1\cdots\alpha_n)\dot{\alpha}_1\cdots\dot{\alpha}_n)} = 0\,. \qquad (3.32)$$

Having established these preliminaries, I am ready to prove the main theorem of this subchapter.

Theorem 3.4. *For any higher symmetry, $D^{(n)}$, of Δ, the top component, $\xi^{a_1\cdots a_n}$, is conformal Killing.*

Proof. In vector notation,

$$D^{(n)} = \sum_{k=0}^{n}\xi^{a_1\cdots a_k}\nabla_{a_1}\cdots\nabla_{a_k}\,. \qquad (3.33)$$

However, the same operator can be re-written in spinor notation as

$$D^{(n)} = \sum_{k=0}^{n}\xi^{\alpha_1\cdots\alpha_k\dot{\alpha}_1\cdots\dot{\alpha}_k}\nabla_{\alpha_1\dot{\alpha}_1}\cdots\nabla_{\alpha_k\dot{\alpha}_k}\,, \qquad (3.34)$$

where I have scaled each of the $\xi^{\alpha_1\cdots\alpha_k\dot{\alpha}_1\cdots\dot{\alpha}_k}$ by $(-2)^k$ without loss of generality so that the previous equation does not have any unnecessary numerical factors in it. Likewise, in spinor notation, $\Delta = \Box - \frac{1}{6}R = -\frac{1}{2}(\nabla^{\alpha\dot{\alpha}}\nabla_{\alpha\dot{\alpha}} + \frac{1}{3}R)$ and the leading factor of $-1/2$ may be ignored as both $\Delta\varphi$ and $\Delta D^{(n)}\varphi$ are being equated to zero. Then, $\Delta\varphi = 0 \iff \nabla^{\alpha\dot{\alpha}}\nabla_{\alpha\dot{\alpha}}\varphi = -\frac{1}{3}R\varphi$

and since $D^{(n)}$ is a higher symmetry,

$$
\begin{aligned}
0 &= \Delta D^{(n)}\varphi \\
&= \left(\nabla^{\alpha\dot{\alpha}}\nabla_{\alpha\dot{\alpha}} + \frac{1}{3}R\right)\left(\sum_{k=0}^{n}\xi^{\alpha_1\cdots\alpha_k\dot{\alpha}_1\cdots\dot{\alpha}_k}\nabla_{\alpha_1\dot{\alpha}_1}\cdots\nabla_{\alpha_k\dot{\alpha}_k}\right)\varphi \\
&= \sum_{k=0}^{n}\xi^{\alpha_1\cdots\alpha_k\dot{\alpha}_1\cdots\dot{\alpha}_k}\nabla^{\alpha\dot{\alpha}}\nabla_{\alpha\dot{\alpha}}\nabla_{\alpha_1\dot{\alpha}_1}\cdots\nabla_{\alpha_k\dot{\alpha}_k}\varphi + 2\sum_{k=0}^{n}\nabla^{\alpha\dot{\alpha}}(\xi^{\alpha_1\cdots\alpha_k\dot{\alpha}_1\cdots\dot{\alpha}_k})\nabla_{\alpha\dot{\alpha}}\nabla_{\alpha_1\dot{\alpha}_1}\cdots\nabla_{\alpha_k\dot{\alpha}_k}\varphi \\
&\quad + \sum_{k=0}^{n}\nabla^{\alpha\dot{\alpha}}\nabla_{\alpha\dot{\alpha}}(\xi^{\alpha_1\cdots\alpha_k\dot{\alpha}_1\cdots\dot{\alpha}_k})\nabla_{\alpha_1\dot{\alpha}_1}\cdots\nabla_{\alpha_k\dot{\alpha}_k}\varphi + \sum_{k=0}^{n}\frac{1}{3}R\xi^{\alpha_1\cdots\alpha_k\dot{\alpha}_1\cdots\dot{\alpha}_k}\nabla_{\alpha_1\dot{\alpha}_1}\cdots\nabla_{\alpha_k\dot{\alpha}_k}\varphi \\
&= \sum_{k=0}^{n}\xi^{\alpha_1\cdots\alpha_k\dot{\alpha}_1\cdots\dot{\alpha}_k}[\nabla^{\alpha\dot{\alpha}}\nabla_{\alpha\dot{\alpha}}, \nabla_{\alpha_1\dot{\alpha}_1}\cdots\nabla_{\alpha_k\dot{\alpha}_k}]\varphi + 2\sum_{k=0}^{n}\nabla^{\alpha\dot{\alpha}}(\xi^{\alpha_1\cdots\alpha_k\dot{\alpha}_1\cdots\dot{\alpha}_k})\nabla_{\alpha\dot{\alpha}}\nabla_{\alpha_1\dot{\alpha}_1}\cdots\nabla_{\alpha_k\dot{\alpha}_k}\varphi \\
&\quad + \sum_{k=0}^{n}\nabla^{\alpha\dot{\alpha}}\nabla_{\alpha\dot{\alpha}}(\xi^{\alpha_1\cdots\alpha_k\dot{\alpha}_1\cdots\dot{\alpha}_k})\nabla_{\alpha_1\dot{\alpha}_1}\cdots\nabla_{\alpha_k\dot{\alpha}_k}\varphi\,. \qquad (3.35)
\end{aligned}
$$

Now, since a commutator reduces the number of derivatives by two (at the expense curvature terms), the term in the previous equation with the highest number of derivatives on φ is $2\nabla^{\alpha\dot{\alpha}}(\xi^{\alpha_1\cdots\alpha_n\dot{\alpha}_1\cdots\dot{\alpha}_n})\nabla_{\alpha\dot{\alpha}}\nabla_{\alpha_1\dot{\alpha}_1}\cdots\nabla_{\alpha_n\dot{\alpha}_n}\varphi$. This term has $n+1$ derivatives on φ and all other terms in the sums have fewer than $n+1$ derivatives on φ.
Therefore, to get $\Delta D^{(n)}\varphi = 0$, either this term must vanish on its own or further manipulation needs to be done to reduce the number of derivatives so that this term can cancel with some of the lower order terms in the sums.
By lemmas 3.2 and 3.3, $\xi^{\alpha_1\cdots\alpha_k\dot{\alpha}_1\cdots\dot{\alpha}_n} = \xi^{(\alpha_1\cdots\alpha_k)(\dot{\alpha}_1\cdots\dot{\alpha}_n)}$.
Hence, in exactly the same way that equation 3.28 was derived,

$$
\begin{aligned}
&\nabla^{\alpha\dot{\alpha}}(\xi^{\alpha_1\cdots\alpha_n\dot{\alpha}_1\cdots\dot{\alpha}_n})\nabla_{\alpha\dot{\alpha}}\nabla_{\alpha_1\dot{\alpha}_1}\cdots\nabla_{\alpha_n\dot{\alpha}_n}(\varphi) \\
&= \Bigg(\nabla^{(\alpha(\dot{\alpha}}\xi^{\alpha_1\cdots\alpha_n)\dot{\alpha}_1\cdots\dot{\alpha}_n)} + \frac{1}{n+1}\sum_{i=1}^{n}\varepsilon^{\dot{\alpha}\dot{\alpha}_i}\nabla^{(\alpha}{}_{\dot{\beta}}\xi^{\alpha_1\cdots\alpha_n)\dot{\alpha}_1\cdots\hat{\dot{\alpha}}_i\cdots\dot{\alpha}_n\dot{\beta}} \\
&\quad + \frac{1}{n+1}\sum_{i=1}^{n}\varepsilon^{\alpha\alpha_i}\nabla_{\beta}{}^{(\dot{\alpha}}\xi^{\alpha_1\cdots\hat{\alpha}_i\cdots\alpha_n\beta\dot{\alpha}_1\cdots\dot{\alpha}_n)} \\
&\quad + \frac{1}{(n+1)^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\varepsilon^{\alpha\alpha_i}\varepsilon^{\dot{\alpha}\dot{\alpha}_j}\nabla_{\beta\dot{\beta}}\xi^{\alpha_1\cdots\hat{\alpha}_i\cdots\alpha_n\beta\dot{\alpha}_1\cdots\hat{\dot{\alpha}}_j\cdots\dot{\alpha}_n\dot{\beta}}\Bigg)\nabla_{\alpha\dot{\alpha}}\nabla_{\alpha_1\dot{\alpha}_1}\cdots\nabla_{\alpha_n\dot{\alpha}_n}(\varphi)\,. \qquad (3.36)
\end{aligned}
$$

I will try to reduce the number of derivatives on φ for each of the terms in the sum.

$$
\begin{aligned}
&\sum_{i=1}^{n}\varepsilon^{\dot{\alpha}\dot{\alpha}_i}\nabla^{(\alpha}{}_{\dot{\beta}}\xi^{\alpha_1\cdots\alpha_n)\dot{\alpha}_1\cdots\hat{\dot{\alpha}}_i\cdots\dot{\alpha}_n\dot{\beta}}(\nabla_{\alpha\dot{\alpha}}\nabla_{\alpha_1\dot{\alpha}_1}\cdots\nabla_{\alpha_n\dot{\alpha}_n}\varphi) \\
&= \sum_{i=1}^{n}\varepsilon^{\dot{\alpha}\dot{\alpha}_i}\nabla^{(\alpha}{}_{\dot{\beta}}\xi^{\alpha_1\cdots\alpha_n)\dot{\alpha}_1\cdots\hat{\dot{\alpha}}_i\cdots\dot{\alpha}_n\dot{\beta}}(\nabla_{\alpha\dot{\alpha}}\nabla_{\alpha_i\dot{\alpha}_i}\nabla_{\alpha_1\dot{\alpha}_1}\cdots\nabla_{\alpha_n\dot{\alpha}_n}\varphi) \\
&\quad + \sum_{i=1}^{n}\varepsilon^{\dot{\alpha}\dot{\alpha}_i}\nabla^{(\alpha}{}_{\dot{\beta}}\xi^{\alpha_1\cdots\alpha_n)\dot{\alpha}_1\cdots\hat{\dot{\alpha}}_i\cdots\dot{\alpha}_n\dot{\beta}}(\nabla_{\alpha\dot{\alpha}}[\nabla_{\alpha_1\dot{\alpha}_1}\cdots\nabla_{\alpha_{i-1}\dot{\alpha}_{i-1}}, \nabla_{\alpha_i\dot{\alpha}_i}]\nabla_{\alpha_n\dot{\alpha}_n}\varphi) \qquad (3.37)
\end{aligned}
$$

The 2nd term/sum in the previous equation now only has $n-1$ derivatives on φ (at the cost of some curvature terms) and hence need not be analysed further. As for the other terms, using the antisymmetry of ε and the freedom to re-order indices within a symmetrisation,

$$\begin{aligned}
&\sum_{i=1}^{n} \varepsilon^{\dot\alpha\dot\alpha_i} \nabla^{(\alpha}{}_{\dot\beta} \xi^{\alpha_1\cdots\alpha_n)\dot\alpha_1\cdots\hat{\dot\alpha}_i\cdots\dot\alpha_n\dot\beta} (\nabla_{\alpha\dot\alpha}\nabla_{\alpha_i\dot\alpha_i}\nabla_{\alpha_1\dot\alpha_1}\cdots\nabla_{\alpha_n\dot\alpha_n}\varphi) \\
&= -\sum_{i=1}^{n} \varepsilon^{\dot\alpha_i\dot\alpha} \nabla^{(\alpha}{}_{\dot\beta} \xi^{\alpha_1\cdots\alpha_n)\dot\alpha_1\cdots\hat{\dot\alpha}_i\cdots\dot\alpha_n\dot\beta} (\nabla_{\alpha\dot\alpha}\nabla_{\alpha_i\dot\alpha_i}\nabla_{\alpha_1\dot\alpha_1}\cdots\nabla_{\alpha_n\dot\alpha_n}\varphi) \\
&= -\sum_{i=1}^{n} \varepsilon^{\dot\alpha\dot\alpha_i} \nabla^{(\alpha}{}_{\dot\beta} \xi^{\alpha_1\cdots\alpha_n)\dot\alpha_1\cdots\hat{\dot\alpha}_i\cdots\dot\alpha_n\dot\beta} (\nabla_{\alpha_i\dot\alpha_i}\nabla_{\alpha\dot\alpha}\nabla_{\alpha_1\dot\alpha_1}\cdots\nabla_{\alpha_n\dot\alpha_n}\varphi) \\
&= -\sum_{i=1}^{n} \varepsilon^{\dot\alpha\dot\alpha_i} \nabla^{(\alpha}{}_{\dot\beta} \xi^{\alpha_1\cdots\alpha_n)\dot\alpha_1\cdots\hat{\dot\alpha}_i\cdots\dot\alpha_n\dot\beta} (\nabla_{\alpha\dot\alpha}\nabla_{\alpha_i\dot\alpha_i}\nabla_{\alpha_1\dot\alpha_1}\cdots\nabla_{\alpha_n\dot\alpha_n}\varphi \\
&\qquad + [\nabla_{\alpha_i\dot\alpha_i}, \nabla_{\alpha\dot\alpha}]\nabla_{\alpha_1\dot\alpha_1}\cdots\nabla_{\alpha_n\dot\alpha_n}\varphi)\,.
\end{aligned} \tag{3.38}$$

Therefore,

$$\begin{aligned}
&\sum_{i=1}^{n} \varepsilon^{\dot\alpha\dot\alpha_i} \nabla^{(\alpha}{}_{\dot\beta} \xi^{\alpha_1\cdots\alpha_n)\dot\alpha_1\cdots\hat{\dot\alpha}_i\cdots\dot\alpha_n\dot\beta} (\nabla_{\alpha\dot\alpha}\nabla_{\alpha_i\dot\alpha_i}\nabla_{\alpha_1\dot\alpha_1}\cdots\nabla_{\alpha_n\dot\alpha_n}\varphi) \\
&= -\frac{1}{2}\sum_{i=1}^{n} \varepsilon^{\dot\alpha\dot\alpha_i} \nabla^{(\alpha}{}_{\dot\beta} \xi^{\alpha_1\cdots\alpha_n)\dot\alpha_1\cdots\hat{\dot\alpha}_i\cdots\dot\alpha_n\dot\beta} ([\nabla_{\alpha_i\dot\alpha_i}, \nabla_{\alpha\dot\alpha}]\nabla_{\alpha_1\dot\alpha_1}\cdots\nabla_{\alpha_n\dot\alpha_n}\varphi)\,,
\end{aligned} \tag{3.39}$$

which also has only $n-1$ derivatives on φ. Similarly,

$$\begin{aligned}
&\sum_{i=1}^{n} \varepsilon^{\alpha\alpha_i} \nabla_{\beta}{}^{(\dot\alpha} \xi^{\alpha_1\cdots\hat\alpha_i\cdots\alpha_n\beta\dot\alpha_1\cdots\dot\alpha_n)} (\nabla_{\alpha\dot\alpha}\nabla_{\alpha_1\dot\alpha_1}\cdots\nabla_{\alpha_n\dot\alpha_n}\varphi) \\
&= \sum_{i=1}^{n} \varepsilon^{\alpha\alpha_i} \nabla_{\beta}{}^{(\dot\alpha} \xi^{\alpha_1\cdots\hat\alpha_i\cdots\alpha_n\beta\dot\alpha_1\cdots\dot\alpha_n)} (\nabla_{\alpha\dot\alpha}\nabla_{\alpha_i\dot\alpha_i}\nabla_{\alpha_1\dot\alpha_1}\cdots\nabla_{\alpha_n\dot\alpha_n}\varphi) \\
&\quad + \sum_{i=1}^{n} \varepsilon^{\alpha\alpha_i} \nabla_{\beta}{}^{(\dot\alpha} \xi^{\alpha_1\cdots\hat\alpha_i\cdots\alpha_n\beta\dot\alpha_1\cdots\dot\alpha_n)} (\nabla_{\alpha\dot\alpha}[\nabla_{\alpha_1\dot\alpha_1}\cdots\nabla_{\alpha_{i-1}\dot\alpha_{i-1}}, \nabla_{\alpha_i\dot\alpha_i}]\nabla_{\alpha_n\dot\alpha_n}\varphi)\,,
\end{aligned} \tag{3.40}$$

where again the 2nd term/sum has has only $n-1$ derivatives on φ and analogously with the calculation above, the 1st term can be re-written as

$$\begin{aligned}
&\sum_{i=1}^{n} \varepsilon^{\alpha\alpha_i} \nabla_{\beta}{}^{(\dot\alpha} \xi^{\alpha_1\cdots\hat\alpha_i\cdots\alpha_n\beta\dot\alpha_1\cdots\dot\alpha_n)} (\nabla_{\alpha\dot\alpha}\nabla_{\alpha_i\dot\alpha_i}\nabla_{\alpha_1\dot\alpha_1}\cdots\nabla_{\alpha_n\dot\alpha_n}\varphi) \\
&= -\frac{1}{2}\sum_{i=1}^{n} \varepsilon^{\alpha\alpha_i} \nabla_{\beta}{}^{(\dot\alpha} \xi^{\alpha_1\cdots\hat\alpha_i\cdots\alpha_n\beta\dot\alpha_1\cdots\dot\alpha_n)} ([\nabla_{\alpha_i\dot\alpha_i}, \nabla_{\alpha\dot\alpha}]\nabla_{\alpha_1\dot\alpha_1}\cdots\nabla_{\alpha_n\dot\alpha_n}\varphi)\,,
\end{aligned} \tag{3.41}$$

which likewise has $n-1$ derivatives on φ.
The only remaining sum in equation 3.36 is

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \varepsilon^{\alpha\alpha_i}\varepsilon^{\dot\alpha\dot\alpha_j} \nabla_{\beta\dot\beta}(\xi^{\alpha_1\cdots\hat\alpha_i\cdots\alpha_n\beta\dot\alpha_1\cdots\hat{\dot\alpha}_j\cdots\dot\alpha_n\dot\beta})\nabla_{\alpha\dot\alpha}\nabla_{\alpha_1\dot\alpha_1}\cdots\nabla_{\alpha_n\dot\alpha_n}(\varphi)\,. \tag{3.42}$$

When $i=j$ in the previous sum,

$$\begin{aligned}
&\varepsilon^{\alpha\alpha_i}\varepsilon^{\dot\alpha\dot\alpha_i} \nabla_{\beta\dot\beta}(\xi^{\alpha_1\cdots\hat\alpha_i\cdots\alpha_n\beta\dot\alpha_1\cdots\hat{\dot\alpha}_j\cdots\dot\alpha_n\dot\beta})\nabla_{\alpha\dot\alpha}\nabla_{\alpha_1\dot\alpha_1}\cdots\nabla_{\alpha_n\dot\alpha_n}(\varphi) \\
&= \nabla_{\beta\dot\beta}(\xi^{\alpha_1\cdots\hat\alpha_i\cdots\alpha_n\beta\dot\alpha_1\cdots\hat{\dot\alpha}_j\cdots\dot\alpha_n\dot\beta})\nabla_{\alpha\dot\alpha}\nabla_{\alpha_1\dot\alpha_1}\cdots\nabla^{\alpha\dot\alpha}\cdots\nabla_{\alpha_n\dot\alpha_n}(\varphi) \\
&= \nabla_{\beta\dot\beta}(\xi^{\alpha_1\cdots\hat\alpha_i\cdots\alpha_n\beta\dot\alpha_1\cdots\hat{\dot\alpha}_j\cdots\dot\alpha_n\dot\beta})\nabla_{\alpha_1\dot\alpha_1}\cdots\nabla_{\alpha_n\dot\alpha_n}\nabla^{\alpha\dot\alpha}\nabla_{\alpha\dot\alpha}(\varphi) + \text{commutators} \\
&= -\frac{1}{3}\nabla_{\beta\dot\beta}(\xi^{\alpha_1\cdots\hat\alpha_i\cdots\alpha_n\beta\dot\alpha_1\cdots\hat{\dot\alpha}_j\cdots\dot\alpha_n\dot\beta})\nabla_{\alpha_1\dot\alpha_1}\cdots\nabla_{\alpha_n\dot\alpha_n}(R\varphi) + \text{commutators}\,,
\end{aligned} \tag{3.43}$$

and all the terms in the last line have $n-1$ derivatives on φ. When $i \neq j$,

$$\begin{aligned}
&\varepsilon^{\alpha\alpha_i}\varepsilon^{\dot\alpha\dot\alpha_j}\nabla_{\beta\dot\beta}(\xi^{\alpha_1\cdots\hat\alpha_i\cdots\alpha_n\beta\dot\alpha_1\cdots\hat{\dot\alpha}_j\cdots\dot\alpha_n\dot\beta})\nabla_{\alpha\dot\alpha}\nabla_{\alpha_1\dot\alpha_1}\cdots\nabla_{\alpha_n\dot\alpha_n}(\varphi)\\
&=\varepsilon^{\alpha\alpha_i}\varepsilon^{\dot\alpha\dot\alpha_j}\nabla_{\beta\dot\beta}(\xi^{\alpha_1\cdots\hat\alpha_i\cdots\alpha_n\beta\dot\alpha_1\cdots\hat{\dot\alpha}_j\cdots\dot\alpha_n\dot\beta})\nabla_{\alpha\dot\alpha}\nabla_{\alpha_i\dot\alpha_i}\nabla_{\alpha_1\dot\alpha_1}\cdots\nabla_{\alpha_n\dot\alpha_n}(\varphi)+\text{commutators}\,. \qquad (3.44)
\end{aligned}$$

Again, any term with commutators reduces the number of derivatives on φ, so can be ignored as far as analysing terms with $n+1$ derivatives on φ. For the remaining term in the previous line, a special case of equation 3.28 gives

$$\begin{aligned}
&\varepsilon^{\alpha\alpha_i}\varepsilon^{\dot\alpha\dot\alpha_j}\nabla_{\alpha\dot\alpha}\nabla_{\alpha_i\dot\alpha_i}\\
&=\varepsilon^{\alpha\alpha_i}\varepsilon^{\dot\alpha\dot\alpha_j}\left(\nabla_{(\alpha(\dot\alpha}\nabla_{\alpha_i)\dot\alpha_i)}+\frac{1}{2}\varepsilon_{\alpha\alpha_i}\nabla^{\beta}{}_{(\dot\alpha}\nabla_{\beta\dot\alpha_i)}+\frac{1}{2}\varepsilon_{\dot\alpha\dot\alpha_i}\nabla_{(\alpha}{}^{\dot\beta}\nabla_{\alpha_i)\dot\beta}+\frac{1}{4}\varepsilon_{\alpha\alpha_i}\varepsilon_{\dot\alpha\dot\alpha_i}\nabla^{\beta\dot\beta}\nabla_{\beta\dot\beta}\right). \qquad (3.45)
\end{aligned}$$

In the previous line $\varepsilon^{\alpha\alpha_i}\nabla_{(\alpha(\dot\alpha}\nabla_{\alpha_i)\dot\alpha_i)}=0$ and $\varepsilon^{\alpha\alpha_i}\nabla_{(\alpha}{}^{\dot\beta}\nabla_{\alpha_i)\dot\beta}=0$ because of the contraction between α and α_i (in ε there is antisymmetry in those indices where as in the ∇s those indices are symmetrised). Also in the last term, I have created a $\nabla^{\beta\dot\beta}\nabla_{\beta\dot\beta}$. This term can be pushed to the front of the queue of derivatives via commutators. The commutators reduce the number of derivatives by 2 and the remaining term also does the same by $\nabla^{\beta\dot\beta}\nabla_{\beta\dot\beta}\varphi=-R\varphi/3$. Hence, the only term left in the previous equation is

$$\begin{aligned}
\frac{1}{2}\varepsilon^{\alpha\alpha_i}\varepsilon^{\dot\alpha\dot\alpha_j}\varepsilon_{\alpha\alpha_i}\nabla^{\beta}{}_{(\dot\alpha}\nabla_{\beta\dot\alpha_i)}&=-\varepsilon^{\dot\alpha\dot\alpha_j}\nabla^{\beta}{}_{(\dot\alpha}\nabla_{\beta\dot\alpha_i)}\\
&=-\frac{1}{2}\varepsilon^{\dot\alpha\dot\alpha_j}(\nabla^{\beta}{}_{\dot\alpha}\nabla_{\beta\dot\alpha_i}+\nabla^{\beta}{}_{\dot\alpha_i}\nabla_{\beta\dot\alpha})\\
&=\frac{1}{2}(\nabla^{\beta\dot\alpha_j}\nabla_{\beta\dot\alpha_i}+\nabla^{\beta}{}_{\dot\alpha_i}\nabla_{\beta}{}^{\dot\alpha_j})\\
&=\frac{1}{2}(\nabla^{\beta\dot\alpha_j}\nabla_{\beta\dot\alpha_i}-\nabla_{\beta\dot\alpha_i}\nabla^{\beta\dot\alpha_j})\\
&=\frac{1}{2}[\nabla^{\beta\dot\alpha_j},\nabla_{\beta\dot\alpha_i}]\,, \qquad (3.46)
\end{aligned}$$

which again reduces the number of derivatives on φ by 2.
Going back to equation 3.36, the only term remaining is

$$\nabla^{(\alpha(\dot\alpha}\xi^{\alpha_1\cdots\alpha_n)\dot\alpha_1\cdots\dot\alpha_n)}(\nabla_{\alpha\dot\alpha}\nabla_{\alpha_1\dot\alpha_1}\cdots\nabla_{\alpha_n\dot\alpha_n}\varphi)\,. \qquad (3.47)$$

The only way to reduce the number of derivatives on φ - like I already have for the other terms - is to exploit the antisymmetry of ε to create a commutator or create a trace (with a pair of εs) and generate a $\nabla^{\alpha\dot\alpha}\nabla_{\alpha\dot\alpha}$ and use $\Delta\varphi=0$. However, neither of these techniques is applicable to $\nabla^{(\alpha(\dot\alpha}\xi^{\alpha_1\cdots\alpha_n)\dot\alpha_1\cdots\dot\alpha_n)}(\nabla_{\alpha\dot\alpha}\nabla_{\alpha_1\dot\alpha_1}\cdots\nabla_{\alpha_n\dot\alpha_n}\varphi)$ since $\nabla^{(\alpha(\dot\alpha}\xi^{\alpha_1\cdots\alpha_n)\dot\alpha_1\cdots\dot\alpha_n)}$ is symmetric and traceless by lemma 3.3.
Therefore, $\nabla^{(\alpha(\dot\alpha}\xi^{\alpha_1\cdots\alpha_n)\dot\alpha_1\cdots\dot\alpha_n)}(\nabla_{\alpha\dot\alpha}\nabla_{\alpha_1\dot\alpha_1}\cdots\nabla_{\alpha_n\dot\alpha_n}\varphi)$ is the only term left in $\Delta D^{(n)}\varphi$ with $n+1$ derivatives (the maximum) on φ.
Thus, $\Delta D^{(n)}\varphi=0$ is only possible if $\nabla^{(\alpha(\dot\alpha}\xi^{\alpha_1\cdots\alpha_n)\dot\alpha_1\cdots\dot\alpha_n)}=0$.
Hence, $\xi^{a_1\cdots a_n}$ is conformal Killing by corollary 3.3.1. □

This theorem was also given in [17], but by very different means. By construction, equation 3.9 describes a conformal field theory and its conformal invariance is predicated on $\varphi\to e^{\sigma}\varphi$ upon a Weyl transformation, $e_a{}^m\to e^{\sigma}e_a{}^m$. Hence, to have any physical significance, the symmetry operator, $D^{(n)}$, must be such that $D^{(n)}\varphi\to e^{\sigma}D^{(n)}\varphi$ as well upon a Weyl transformation. As I will show in the subsequent subchapters, this condition is sufficient to fix the possible forms of lower order terms in terms of the top component, $\xi^{a_1\cdots a_n}$. But to do that, I have to know how $\xi^{a_1\cdots a_n}$ transforms under a Weyl transformation. The required transformation is found by noting that as I have a conformal field theory, theorem 3.4 must be preserved upon a Weyl transformation. To that end, there is the following lemma.

Lemma 3.5. *For $\xi^{a_1\cdots a_n}$ to remain conformal Killing upon a Weyl transformation, ${e'}_a{}^m = \mathrm{e}^\sigma e_a{}^m$, $\xi^{a_1\cdots a_n}$ must transform by $\xi'^{a_1\cdots a_n} = \mathrm{e}^{-n\sigma}\xi^{a_1\cdots a_n}$.*

Proof. I will prove the theorem in the infinitesimal case. The exponential map can be used to lift the result to the finite case[5] given in the lemma statement. Since $\xi^{a_1\cdots a_n}$ is symmetric and traceless, $\delta\xi^{a_1\cdots a_n}$ must also be symmetric and traceless to maintain the conformal Killing condition.

$$\begin{aligned}
\nabla'^{a_{n+1}}\xi'^{a_1\cdots a_n} &= (\nabla^{a_{n+1}} + \sigma\nabla^{a_{n+1}} - \nabla_b(\sigma)M^{a_{n+1}b})(\xi^{a_1\cdots a_n} + \delta\xi^{a_1\cdots a_n}) \\
&= (1+\sigma)\nabla^{a_{n+1}}\xi^{a_1\cdots a_n} + \nabla^{a_{n+1}}\delta\xi^{a_1\cdots a_n} \\
&\quad - \sum_{i=1}^n \nabla_b(\sigma)(\eta^{a_i a_{n+1}}\xi^{a_1\cdots\hat{\alpha}_i b\cdots a_n} - \eta^{a_i b}\xi^{a_1\cdots\hat{\alpha}_i a_{n+1}\cdots a_n}) \\
&= (1+\sigma)\nabla^{a_{n+1}}\xi^{a_1\cdots a_n} + \nabla^{a_{n+1}}\delta\xi^{a_1\cdots a_n} \\
&\quad - \sum_{i=1}^n (\nabla_b(\sigma)\eta^{a_i a_{n+1}}\xi^{a_1\cdots\hat{\alpha}_i\cdots a_n b} - \nabla^{a_i}(\sigma)\xi^{a_1\cdots\hat{a}_i\cdots a_{n+1}}
\end{aligned} \tag{3.48}$$

It will be easiest to impose the conformal Killing condition in spinor notation as per corollary 3.3.1. When going to spinors,

$$\begin{aligned}
(\sigma_{a_i})_{\alpha_i\dot{\alpha}_i}(\sigma_{a_{n+1}})_{\alpha_{n+1}\dot{\alpha_{n+1}}}\eta^{a_i a_{n+1}} &= (\sigma_{a_i})_{\alpha_i\dot{\alpha}_i}(\sigma_{a_{n+1}})_{\alpha^{a_i}\dot{\alpha_{n+1}}}\eta^{a_i a_{n+1}} \\
&= -2\varepsilon_{\alpha_i\alpha_{n+1}}\varepsilon_{\dot{\alpha}_i\dot{\alpha}_{n+1}}\,.
\end{aligned} \tag{3.49}$$

Plugging this into the expression for $\nabla'^{a_{n+1}}\xi'^{a_1\cdots a_n}$ above,

$$\begin{aligned}
{\nabla'}_{\alpha_{n+1}\dot{\alpha}_{n+1}}{\xi'}_{\alpha_1\cdots\alpha_n\dot{\alpha}_1\cdots\dot{\alpha}_n} &= (1+\sigma)\nabla_{\alpha_{n+1}\dot{\alpha}_{n+1}}\xi_{\alpha_1\cdots\alpha_n\dot{\alpha}_1\cdots\dot{\alpha}_n} + \nabla_{\alpha_{n+1}\dot{\alpha}_{n+1}}\delta\xi_{\alpha_1\cdots\alpha_n\dot{\alpha}_1\cdots\dot{\alpha}_n} \\
&\quad - \sum_{i=1}^n (\nabla_{\beta\dot{\beta}}(\sigma)\varepsilon_{\alpha_i\alpha_{n+1}}\varepsilon_{\dot{\alpha}_i\dot{\alpha}_{n+1}}\xi_{\alpha_1\cdots\hat{\alpha}_i\cdots\alpha_n}{}^{\beta}{}_{\dot{\alpha}_1\cdots\hat{\dot{\alpha}}_i\cdots\dot{\alpha}_n}{}^{\dot{\beta}} \\
&\qquad - \nabla_{\alpha_i\dot{\alpha}_i}(\sigma)\xi_{\alpha_1\cdots\hat{\alpha}_i\cdots\alpha_{n+1}\dot{\alpha}_1\cdots\hat{\dot{\alpha}}_i\cdots\dot{\alpha}_{n+1}})\,.
\end{aligned} \tag{3.50}$$

Corollary 3.3.1 gives ${\nabla'}_{(\alpha_{n+1}(\dot{\alpha}_{n+1}}{\xi'}_{\alpha_1\cdots\alpha_n)\dot{\alpha}_1\cdots\dot{\alpha}_n)} = 0$. Since $\xi^{a_1\cdots a_n}$ is itself conformal Killing, $\nabla_{(\alpha_{n+1}(\dot{\alpha}_{n+1}}\xi_{\alpha_1\cdots\alpha_n)\dot{\alpha}_1\cdots\dot{\alpha}_n)} = 0$ already. Any terms with a Levi-Civita symbol go to zero upon symmetrisation. Finally, that leaves

$$\begin{aligned}
0 &= \nabla_{(\alpha_{n+1}(\dot{\alpha}_{n+1}}\delta\xi_{\alpha_1\cdots\alpha_n)\dot{\alpha}_1\cdots\dot{\alpha}_n)} + \sum_{i=1}^n \nabla_{(\alpha_i(\dot{\alpha}_i}(\sigma)\xi_{\alpha_1\cdots\hat{\alpha}_i\cdots\alpha_{n+1})\dot{\alpha}_1\cdots\hat{\dot{\alpha}}_i\cdots\dot{\alpha}_{n+1})} \\
&= \nabla_{(\alpha_{n+1}(\dot{\alpha}_{n+1}}\delta\xi_{\alpha_1\cdots\alpha_n)\dot{\alpha}_1\cdots\dot{\alpha}_n)} + n\nabla_{(\alpha_{n+1}(\dot{\alpha}_{n+1}}(\sigma)\xi_{\alpha_1\cdots\alpha_n)\dot{\alpha}_1\cdots\dot{\alpha}_n)} \\
&= \nabla_{(\alpha_{n+1}(\dot{\alpha}_{n+1}}\zeta_{\alpha_1\cdots\alpha_n)\dot{\alpha}_1\cdots\dot{\alpha}_n)}\,,
\end{aligned} \tag{3.51}$$

where $\zeta_{\alpha_1\cdots\alpha_n\dot{\alpha}_1\cdots\dot{\alpha}_n} = \delta\xi_{\alpha_1\cdots\alpha_n\dot{\alpha}_1\cdots\dot{\alpha}_n} + n\sigma\xi_{\alpha_1\cdots\alpha_n\dot{\alpha}_1\cdots\dot{\alpha}_n}$. This last step is possible since

$$\begin{aligned}
&\nabla_{(\alpha_{n+1}(\dot{\alpha}_{n+1}}\xi_{\alpha_1\cdots\alpha_n)\dot{\alpha}_1\cdots\dot{\alpha}_n)} = 0 \\
&\quad\implies \nabla_{(\alpha_{n+1}(\dot{\alpha}_{n+1}}(\sigma)\xi_{\alpha_1\cdots\alpha_n)\dot{\alpha}_1\cdots\dot{\alpha}_n)} = \nabla_{(\alpha_{n+1}(\dot{\alpha}_{n+1}}(\sigma\xi_{\alpha_1\cdots\alpha_n)\dot{\alpha}_1\cdots\dot{\alpha}_n)})\,.
\end{aligned} \tag{3.52}$$

[5]This is essentially an application of the the Lie group - Lie algebra correspondence.

Since $\delta\xi_{\alpha_1\cdots\alpha_n\dot{\alpha}_1\cdots\dot{\alpha}_n}$ shares all the symmetries of $\xi_{\alpha_1\cdots\alpha_n\dot{\alpha}_1\cdots\dot{\alpha}_n}$, the only way to get $\nabla_{(\alpha_{n+1}(\dot{\alpha}_{n+1}}\zeta_{\alpha_1\cdots\alpha_n)\dot{\alpha}_1\cdots\dot{\alpha}_n)} = 0$ is to have $\zeta_{\alpha_1\cdots\alpha_n\dot{\alpha}_1\cdots\dot{\alpha}_n} = 0$, or equivalently

$$\delta\xi_{\alpha_1\cdots\alpha_n\dot{\alpha}_1\cdots\dot{\alpha}_n} = -n\sigma\xi_{\alpha_1\cdots\alpha_n\dot{\alpha}_1\cdots\dot{\alpha}_n}\,, \tag{3.53}$$

which completes the proof. □

I have already found the most general form of the top component (conformal Killing) in theorem 3.4. Actually finding all (or indeed any) conformal Killing tensors, $\xi^{a_1\cdots a_n}$, for a given manifold is beyond the scope of this thesis. However, given $\xi^{a_1\cdots a_n}$, finding the lower order components can be achieved via a matter of informed guesswork.

Lemma 3.6. *Given the top component, $\xi^{a_1\cdots a_n}$, of $D^{(n)}$, the lower order components are unique up to the addition of lower order symmetries.*

Proof. Let

$$D_1^{(n)} = \sum_{k=0}^{n} \xi_1^{a_1\cdots a_k}\nabla_{a_1}\cdots\nabla_{a_k} \text{ and } D_2^{(n)} = \sum_{k=0}^{n} \xi_2^{a_1\cdots a_k}\nabla_{a_1}\cdots\nabla_{a_k} \tag{3.54}$$

both be symmetries of Δ such that $\xi_1^{a_1\cdots a_n} = \xi_2^{a_1\cdots a_n}$. Since $D_1^{(n)}$ and $D_2^{(n)}$ are both symmetries, whenever $\Delta\varphi = 0$, $\Delta D_1^{(n)}\varphi = 0$ and $\Delta D_2^{(n)}\varphi = 0$. Thus,

$$\begin{aligned} 0 &= \Delta D_1^{(n)}\varphi - \Delta D_2^{(n)}\varphi \\ &= \Delta(D_1^{(n)} - D_2^{(n)})\varphi \\ &= \Delta\sum_{k=0}^{n-1}(\xi_1^{a_1\cdots a_k}\nabla_{a_1}\cdots\nabla_{a_k} - \xi_2^{a_1\cdots a_k}\nabla_{a_1}\cdots\nabla_{a_k})\varphi \end{aligned} \tag{3.55}$$

Therefore, $D_1^{(n)} - D_2^{(n)}$ is a symmetry of order $n-1$. Hence, $D_1^{(n)}$ and $D_2^{(n)}$ differ by only a lower order symmetry, hence proving the lemma. □

This lemma is perhaps not very insightful, but it is useful nonetheless because there is actually a guide to guessing the lower order components in terms of the top component. Namely, in the action - equation 3.9 - φ must transform as $\varphi \to \mathrm{e}^{\sigma}\varphi$ under a Weyl transformation, ${e_a}^m \to \mathrm{e}^{\sigma}{e_a}^m$. As I will concretely demonstrate in the next two subchapters, this is sufficient to find the possible forms of the lower order components.

3.4 1st order symmetries

From theorem 3.4,

$$D^{(1)} = \xi^a\nabla_a + \xi \tag{3.56}$$

with ξ^a a conformal Killing vector. Next, to apply lemma 3.6, I have to first find all 0th order symmetries, i.e. scalar fields, ξ, such that $\Delta\xi\varphi = 0$ given $\Delta\varphi$.

Lemma 3.7. *The only 0th order symmetries of Δ are constants.*

Proof. $\Delta\xi\varphi = (\Box - R/6)(\xi\varphi) = \xi\Box\varphi + 2\nabla_a(\xi)\nabla^a(\varphi) + \varphi\Box(\xi) - R\xi\varphi/6 = 2\nabla_a(\xi)\nabla^a(\varphi) + \varphi\Box(\xi)$. Then, since φ and $\nabla^a(\varphi)$ are linearly independent, $\nabla_a(\xi) = 0$ and hence $\Box(\xi) = 0$.
Finally, as ξ is a scalar, $0 = \nabla_a\xi = {e_a}^m\partial_m\xi \implies \xi$ is a constant. □

Hence, the ξ in equation 3.56 is unique up to a constant. I will find its possible form via the requirement that $D^{(1)}\varphi \to \mathrm{e}^{\sigma} D^{(1)}\varphi$ upon a Weyl transformation.

Lemma 3.8. *The only physically admissible*[6] *1st order symmetry operator (up to the addition of a constant) is*

$$D^{(1)} = \xi^a \nabla_a + \frac{1}{4}\nabla_a(\xi^a)\,, \tag{3.57}$$

under the requirement that $D'^{(1)}\varphi' = \mathrm{e}^{\sigma} D^{(1)}\varphi$ *upon a Weyl transformation.*

Proof. As before, one can equivalently work with the infinitesimal case, ${e'}_a{}^m = (1+\sigma)e_a{}^m$. Then, also recalling lemma 3.5,

$$\begin{aligned}
D'^{(1)}\varphi' &= (\xi'^a \nabla'_a + \xi')(\varphi') \\
&= ((1-\sigma)\xi^a((1+\sigma)\nabla_a - \nabla^b(\sigma)M_{ab}) + \xi + \delta\xi)(1+\sigma)\varphi \\
&= \xi^a\nabla_a\varphi - \sigma\xi^a\nabla_a\varphi + \xi^a\sigma\nabla_a\varphi - 0 + \xi\varphi + \delta\xi\varphi + \xi^a\nabla_a(\sigma\varphi) + \xi\sigma\varphi \\
&= (1+\sigma)D^{(1)}\varphi + (\delta\xi + \xi^a\nabla_a(\sigma))\varphi
\end{aligned} \tag{3.58}$$

Therefore, to get the required transformation property, ξ should be constructed from ξ^a such that $\delta\xi = -\xi^a\nabla_a(\sigma)$. Physically, ξ^m functions as an infinitesimal generator of conformal symmetries of $g_{mn} \implies$ my ansatz for ξ should be constructed from ξ^a without products of ξ^a. This is what I mean by physically admissible. Hence, I need an object constructed from the metric/vierbein alone with one local Lorentz index only, to contract with the index of ξ^a. The only possible ansatz is thus $\xi = A\nabla_a(\xi^a)$ for some constant, A. Under a Weyl transformation,

$$\begin{aligned}
A{\nabla'}_a(\xi'^a) &= A((1+\sigma)\nabla_a - \nabla^b(\sigma)M_{ab})((1-\sigma)\xi^a) \\
&= A\nabla_a\xi^a + A\sigma\nabla_a\xi^a - A\nabla^b(\sigma)M_{ab}(\xi^a) - A\nabla_a(\sigma\xi^a)\,.
\end{aligned} \tag{3.59}$$

Hence,

$$\begin{aligned}
\delta(A\nabla_a\xi^a) &= A\sigma\nabla_a\xi^a - A\nabla^b(\sigma)M_{ab}(\xi^a) - A\nabla_a(\sigma\xi^a) \\
&= A\sigma\nabla_a\xi^a - A\nabla^b(\sigma)(\delta^a{}_a\xi_b - \delta^a{}_b\xi_a) - A\sigma\nabla_a\xi^a - A\nabla_a(\sigma)\xi^a \\
&= -4A\xi^a\nabla_a(\sigma)\,,
\end{aligned} \tag{3.60}$$

which gives the desired result, $\delta\xi = -\xi^a\nabla_a(\sigma) \implies A = 1/4$. □

Thus, it must now be checked whether

$$D^{(1)} = \xi^a\nabla_a + \frac{1}{4}\nabla_a(\xi^a) + \xi, \tag{3.61}$$

where ξ^a is an arbitrary conformal Killing vector of the manifold and ξ is a constant, really is a symmetry of Δ.

Theorem 3.9. $D^{(1)} = \xi^a\nabla_a + \frac{1}{4}\nabla_a(\xi^a) + \xi$ *is always a symmetry of* Δ.

Proof. In vector notation, ξ^a being conformal Killing is equivalent to $\nabla_a\xi_b + \nabla_b\xi_a = \frac{1}{2}\eta_{ab}\nabla_c\xi^c$ by setting the symmetric and traceless part to zero. The constant, ξ, is already a symmetry, so

[6] The meaning of physically admissible is given in the proof.

only $D^{(1)} = \xi^a\nabla_a + \frac{1}{4}\nabla_a\xi^a$ remains. Let $\Delta\varphi = 0 \iff \Box(\varphi) = \frac{1}{6}R\varphi$. Then, also remembering that $\nabla_a\nabla_b\varphi = \nabla_b\nabla_a\varphi$,

$$\begin{aligned}
\Delta D^{(1)}\varphi &= (\Box - \frac{1}{6}R)(\xi^a\nabla_a + \frac{1}{4}\nabla_a(\xi^a))\varphi \\
&= \Box(\xi^a\nabla_a\varphi) + \frac{1}{4}\Box(\nabla_a(\xi^a)\varphi) - \frac{1}{6}R\xi^a\nabla_a(\varphi) - \frac{1}{24}R\nabla_a(\xi^a)\varphi \\
&= \xi^a\Box\nabla_a(\varphi) + 2\nabla^b(\xi^a)\nabla_b\nabla_a(\varphi) + \Box(\xi^a)\nabla_a(\varphi) + \frac{1}{4}\Box\nabla_a(\xi^a)\varphi \\
&\quad + \frac{1}{2}\nabla^b\nabla_a(\xi^a)\nabla_b(\varphi) + \frac{1}{4}\nabla_a(\xi^a)\Box(\varphi) - \frac{1}{6}R\xi^a\nabla_a(\varphi) - \frac{1}{24}R\nabla_a(\xi^a)\varphi \\
&= \xi^a\Box\nabla_a(\varphi) + 2\nabla^b(\xi^a)\nabla_b\nabla_a(\varphi) + \Box(\xi^a)\nabla_a(\varphi) + \frac{1}{4}\Box\nabla_a(\xi^a)\varphi \\
&\quad + \frac{1}{2}\nabla^b\nabla_a(\xi^a)\nabla_b(\varphi) - \frac{1}{6}R\xi^a\nabla_a(\varphi)\,.
\end{aligned} \tag{3.62}$$

$$\begin{aligned}
\Box\nabla_a\varphi &= \nabla^b\nabla_b\nabla_a\varphi \\
&= \nabla^b\nabla_a\nabla_b\varphi \\
&= \nabla_a\Box\varphi + [\nabla_b, \nabla_a]\nabla^b\varphi \\
&= \frac{1}{6}\nabla_a(R\varphi) + R^b{}_{cba}\nabla^c\varphi \\
&= \frac{1}{6}R\nabla_a\varphi + \frac{1}{6}\varphi\nabla_a R + R_{ab}\nabla^b\varphi
\end{aligned} \tag{3.63}$$

Next, using the conformal Killing condition,

$$\begin{aligned}
\nabla^b(\xi^a)\nabla_b\nabla_a(\varphi) &= -\nabla^a(\xi^b)\nabla_b\nabla_a(\varphi) + \frac{1}{2}\eta^{ab}\nabla_c(\xi^c)\nabla_b\nabla_a(\varphi) \\
&= -\nabla^b(\xi^a)\nabla_b\nabla_a(\varphi) + \frac{1}{12}\nabla_a(\xi^a)R\varphi\,.
\end{aligned} \tag{3.64}$$

Therefore,

$$\nabla^b(\xi^a)\nabla_b\nabla_a(\varphi) = \frac{1}{24}\nabla_a(\xi^a)R\varphi \tag{3.65}$$

Then, putting all these pieces together,

$$\Delta D^{(1)}\varphi = \{\Box(\xi^a) + \frac{1}{2}\nabla^a\nabla_b(\xi^b) + R^{ab}\xi_b\}\nabla_a(\varphi) + \frac{1}{12}\{2\xi^a\nabla_a(R) + \nabla_a(\xi^a)R + 3\Box\nabla_a(\xi^a)\}\varphi \tag{3.66}$$

Let $\{1\}$ and $\{0\}$ denote the coefficients of $\nabla_a(\varphi)$ and φ respectively. In $\{1\}$,

$$\begin{aligned}
\nabla^a\nabla_b\xi^b &= \nabla_b\nabla^a\xi^b + [\nabla^a, \nabla_b]\xi^b \\
&= -\nabla_b\nabla^b\xi^a + \nabla_b\left(\frac{1}{2}\eta^{ab}\nabla_c\xi^c\right) + R^b{}_c{}^a{}_b\xi^c \\
&= -\Box\xi^a + \frac{1}{2}\nabla^a\nabla_b\xi^b - R^{ab}\xi_b\,.
\end{aligned} \tag{3.67}$$

Hence,

$$\frac{1}{2}\nabla^a\nabla_b\xi^b = -\Box\xi^a - R^{ab}\xi_b \iff \{1\} = 0\,. \tag{3.68}$$

Next, to simplify $\{0\}$, the previous equation yields

$$\begin{aligned}\Box\nabla_a\xi^a &= \nabla_a\nabla^a\nabla_b\xi^b \\ &= -2\nabla_a\Box\xi^a - 2\nabla_a(R^{ab}\xi_b) \\ &= -2\Box\nabla_a\xi^a - 2[\nabla_a,\Box]\xi^a - 2\nabla_a(R^{ab}\xi_b)\,, \end{aligned} \tag{3.69}$$

and thereby,

$$3\Box\nabla_a\xi^a = -2[\nabla_a,\Box]\xi^a - 2\nabla_a(R^{ab}\xi_b)\,. \tag{3.70}$$

$$\begin{aligned}[\nabla_a,\Box]\xi^a &= \nabla^b[\nabla_a,\nabla_b]\xi^a + [\nabla_a,\nabla_b]\nabla^b\xi^a \\ &= \nabla^b(R^a{}_{cab}\xi^c) + R^b{}_{cab}\nabla^c\xi^a + R^a{}_{cab}\nabla^b\xi^c \\ &= \nabla_a(R^{ab}\xi_b) \end{aligned} \tag{3.71}$$

Substituting that back,

$$\begin{aligned}3\Box\nabla_a\xi^a &= -4\nabla_a(R^{ab}\xi_b) \\ &= -4\nabla_a(R^{ab})\xi_b - 4R^{ab}\nabla_a(\xi_b) \\ &= -2\xi^a\nabla_a(R) - 2R^{ab}(\nabla_a(\xi_b) + \nabla_b(\xi_a)) \\ &= -2\xi^a\nabla_a(R) - R^{ab}\eta_{ab}\nabla_c\xi^c \\ &= -2\xi^a\nabla_a(R) - R\nabla_a\xi^a\,, \end{aligned} \tag{3.72}$$

which rearranges to $\{0\} = 0$.
Therefore, $\Delta D^{(1)}\varphi = 0 \iff D^{(1)}$ is a symmetry of Δ. □

Corollary 3.9.1. *$D^{(1)} = \xi^a\nabla_a + \frac{1}{4}\nabla_a(\xi^a) + \xi$ is the only 1st order symmetry of Δ.*

Proof. Theorem 3.4, lemma 3.6 and lemma 3.7 together immediately lead to the corollary. □

3.5 2nd order symmetries

This time, from theorem 3.4,

$$D^{(2)} = \xi^{ab}\nabla_a\nabla_b + \xi^a\nabla_a + \xi \tag{3.73}$$

for a conformal Killing tensor, ξ^{ab}.

Lemma 3.10. *To get $D'^{(2)}\varphi' = \mathrm{e}^{\sigma}D^{(2)}\varphi$ under a Weyl transformation, the only physically admissible 2nd order symmetry (up to the addition of 1st order symmetries) is*

$$D^{(2)} = \xi^{ab}\nabla_a\nabla_b + \frac{2}{3}\nabla_b(\xi^{ab})\nabla_a + \frac{1}{15}\nabla_a\nabla_b(\xi^{ab}) - \frac{3}{10}R_{ab}\xi^{ab}\,. \tag{3.74}$$

Proof. Again, I will form an ansatz for ξ^a and ξ by enforcing $D'^{(2)}\varphi' = (1+\sigma)D^{(2)}\varphi$ upon an infinitesimal Weyl transformation, ${e'}_a{}^m = (1+\sigma)e_a{}^m$. Hence, by lemma 3.5,

$$\begin{aligned}
D'^{(2)}\varphi' &= (\xi'^{ab}\nabla'_a\nabla'_b + \xi'^a\nabla'_a + \xi')\varphi' \\
&= ((1-2\sigma)\xi^{ab}((1+\sigma)\nabla_a - \nabla^c(\sigma)M_{ac})((1+\sigma)\nabla_b - \nabla^d(\sigma)M_{bd}) \\
&\quad + (\xi^a + \delta\xi^a)((1+\sigma)\nabla_a - \nabla^b(\sigma)M_{ab}) + \xi + \delta\xi)(1+\sigma)\varphi \\
&= \xi^{ab}\nabla_a\nabla_b\varphi - 2\sigma\xi^{ab}\nabla_a\nabla_b\varphi + \xi^{ab}\sigma\nabla_a\nabla_b\varphi - \xi^{ab}\nabla^c(\sigma)M_{ac}(\nabla_b\varphi) + \xi^{ab}\nabla_a(\sigma\nabla_b\varphi) \\
&\quad - \xi^{ab}\nabla_a(\nabla^d(\sigma)M_{bd}\varphi) + +\xi^{ab}\nabla_a\nabla_b(\sigma\varphi) + \xi^a\nabla_a\varphi + \delta\xi^a\nabla_a\varphi + \xi^a\sigma\nabla_a\varphi \\
&\quad - \xi^a\nabla_a(\nabla^b(\sigma)M_{ab}\varphi) + \xi^a\nabla_a(\sigma\varphi) + \xi\varphi + \delta\xi\varphi + \xi\sigma\varphi \\
&= (1+\sigma)D^{(2)}\varphi - \xi^{ab}\nabla^c(\sigma)(\eta_{ba}\nabla_c\varphi - \eta_{bc}\nabla_a\varphi) + \xi^{ab}\nabla_a(\sigma)\nabla_b(\varphi) \\
&\quad + 2\xi^{ab}\nabla_a(\sigma)\nabla_b(\varphi) + \xi^{ab}\nabla_a\nabla_b(\sigma)\varphi + \delta\xi^a\nabla_a\varphi + \sigma\xi^a\nabla_a\varphi + \xi^a\nabla_a(\sigma)\varphi + \delta\xi\varphi \\
&= (1+\sigma)D^{(2)}\varphi + (4\xi^{ab}\nabla_b(\sigma) + \sigma\xi^a + \delta\xi^a)\nabla_a(\varphi) \\
&\quad + (\xi^{ab}\nabla_a\nabla_b + \xi^a\nabla_a(\sigma) + \delta\xi)\varphi\,.
\end{aligned} \tag{3.75}$$

Therefore,

$$\delta\xi^a = -4\xi^{ab}\nabla_b(\sigma) - \sigma\xi^a \text{ and } \delta\xi = -\xi^{ab}\nabla_a\nabla_b - \xi^a\nabla_a(\sigma) \tag{3.76}$$

to get the required transformation property. As with the 1st order symmetries, ξ^a and ξ should be constructed from ξ^{ab} without products of ξ^{ab}. Hence, I need objects constructed from the metric/vierbein alone with one and two local Lorentz indices respectively to contract with the two indices of ξ^{ab} to give ξ^a and ξ.
Hence, the most general ansatz is $\xi^a = A\nabla_b(\xi^{ab})$ and $\xi = B\nabla_a(\xi^a) + CR_{ab}\xi^{ab}$.

$$\begin{aligned}
\xi'^a &= A\nabla'_b\xi'^{ab} \\
&= A(\nabla_b + \sigma\nabla_b - \nabla^c(\sigma)M_{bc})(\xi^{ab} - 2\sigma\xi^{ab}) \\
&= (1-\sigma)A\nabla_b\xi^{ab} - 2A\xi^{ab}\nabla_b(\sigma) - A\nabla^c(\sigma)M_{bc}(\xi^{ab})
\end{aligned} \tag{3.77}$$

Thus,

$$\begin{aligned}
\delta\xi^a &= -\sigma\xi^a - 2A\xi^{ab}\nabla_b(\sigma) - A\nabla^c(\sigma)(\delta^a{}_b\xi_c{}^b - \delta^a{}_c\xi_b{}^b + \delta^b{}_b\xi^a{}_c - \delta^b{}_c\xi^a{}_b) \\
&= -\sigma\xi^a - 2A\xi^{ab}\nabla_b(\sigma) - A\nabla_b(\sigma)\xi^{ab} + 0 - 4A\nabla_b(\sigma)\xi^{ab} + A\nabla_b(\sigma)\xi^{ab} \\
&= -\sigma\xi^a - 6A\xi^{ab}\nabla_b(\sigma)\,,
\end{aligned} \tag{3.78}$$

which implies $A = 2/3$ to get the required $\delta\xi^a = -4\xi^{ab}\nabla_b(\sigma) - \sigma\xi^a$.
Next, $\xi' = B\nabla'_a\xi'^a + CR'_{ab}\xi'^{ab}$.

$$\begin{aligned}
\nabla'_a\xi'^a &= (\nabla_a + \sigma\nabla_a - \nabla^c(\sigma)M_{ac})(\xi^a - \sigma\xi^a - 4\xi^{ab}\nabla_b(\sigma)) \\
&= \nabla_a\xi^a - \xi^a\nabla_a(\sigma) - 4\xi^{ab}\nabla_a\nabla_b(\sigma) - 4\nabla_a(\xi^{ab})\nabla_b(\sigma) - \nabla^c(\sigma)(\delta^a{}_a\xi_c - \delta^a{}_c\xi_a) \\
&= \nabla_a\xi^a - 10\xi^a\nabla_a(\sigma) - 4\xi^{ab}\nabla_a\nabla_b(\sigma)
\end{aligned} \tag{3.79}$$

$$\begin{aligned}
R'_{ab}\xi'^{ab} &= (R_{ab} + 2\sigma R_{ab} + \eta_{ab}\Box(\sigma) + 2\nabla_a\nabla_b(\sigma))(\xi^{ab} - 2\sigma\xi^{ab}) \\
&= R_{ab}\xi^{ab} + \xi^a{}_a\Box(\sigma) + 2\xi^{ab}\nabla_a\nabla_b(\sigma) \\
&= R_{ab}\xi^{ab} + 2\xi^{ab}\nabla_a\nabla_b(\sigma)
\end{aligned} \tag{3.80}$$

With these expressions,

$$\xi' = B\nabla_a\xi^a - 10B\xi^a\nabla_a(\sigma) - 4B\xi^{ab}\nabla_a\nabla_b(\sigma) + CR_{ab}\xi^{ab} + 2C\xi^{ab}\nabla_a\nabla_b(\sigma) \tag{3.81}$$

and hence

$$\delta\xi = -10B\xi^a\nabla_a(\sigma) + (2C - 4B)\xi^{ab}\nabla_a\nabla_b(\sigma)\,. \tag{3.82}$$

Therefore, $B = 1/10$ and $C = -3/10$ to get the required $\delta\xi = -\xi^{ab}\nabla_a\nabla_b - \xi^a\nabla_a(\sigma)$. Combined with the expression for ξ^a in terms of ξ^{ab}, one finally gets

$$D^{(2)} = \xi^{ab}\nabla_a\nabla_b + \frac{2}{3}\nabla_b(\xi^{ab})\nabla_a + \frac{1}{15}\nabla_a\nabla_b(\xi^{ab}) - \frac{3}{10}R_{ab}\xi^{ab} \tag{3.83}$$

where ξ^{ab} is an arbitrary conformal Killing tensor of the manifold, as the only possible physically admissible symmetry of Δ (modulo 1st order symmetries). □

Theorem 3.11. *$D^{(2)}$ may not be a symmetry of Δ on an arbitrary manifold. Instead, given $\Delta\varphi = 0$,*

$$\begin{aligned}\Delta D^{(2)}\varphi = &\left(\frac{4}{15}C^a{}_{bcd}\nabla^c(\xi^{bd}) + \frac{4}{5}\nabla^d(C^a{}_{bcd})\xi^{bc}\right)\nabla_a(\varphi) \\ &+ \left(\frac{2}{15}C_{abcd}\nabla^a\nabla^c(\xi^{bd}) + \frac{2}{5}\nabla^c\nabla_d(C^d{}_{abc})\xi^{ab} + \frac{4}{15}\nabla_d(C^d{}_{abc})\nabla^c(\xi^{ab})\right)\varphi\,.\end{aligned} \tag{3.84}$$

Proof. The proof is long and largely follows the same techniques as the 1st order case, so I've presented the calculation in appendix B rather than present it here. □

Corollary 3.11.1. *$D^{(2)}$ is a symmetry of Δ on a conformally flat manifold.*

Proof. A manifold is conformally flat if and only the Weyl tensor is zero. □

3.6 Remarks on nth order symmetries

Based on the discussion following theorem 2.10, I knew 1st order symmetries of Δ would exist on any manifold possessing a conformal Killing vector and indeed that was the result I found in theorem 3.9. A truly "higher" symmetry would be one which could not written as a product of 1st order symmetries. Since not every 2nd rank conformal Killing tensor can be written as a product of two conformal Killing vectors, I have shown via theorem 3.11 that truly "higher" symmetries of Δ do exist on conformally flat spaces. However, I have not shown that is the most general case; it is sufficient, but perhaps not necessary. See [20] for further discussion on necessary conditions. Given that higher symmetries of Δ do not exist even at 2nd order on all manifolds possessing a conformal Killing tensor, I find it unlikely that higher symmetries would exist on arbitrary manifolds for $n > 2$.

For any n, theorem 3.4 shows that $\xi^{a_1\cdots a_n}$ is conformal Killing. From there, the next task is to determine the lower order components. Key to my construction of those components was enforcing that $D'^{(n)}\varphi' = \mathrm{e}^{\sigma}D^{(n)}\varphi$ under a Weyl transformation; I was able to show $D^{(1)}$ and $D^{(2)}$ were uniquely determined this way. A starting point on that path was lemma 3.5, but form there my approach was somewhat ad hoc. Going to nth order - where I have not shown existence or uniqueness of $D^{(n)}$ - will require a more systematic approach. Luckily, options exist. Using a variation of a formalism I will discuss in section 5, Eastwood[7] [17] has generalised the construction of the lower components in terms of Weyl transformation properties to all n and

[7] Eastwood's approach is usually dubbed "tractor calculus."

claims “it is easily verified that [his] formulae” are correct and uniquely determined. Unfortunately, it may be easy for him, but it is not for me and I do not understand his construction. Using methods quite different to mine, Eastwood has shown his $D^{(n)}$ are higher symmetries of Δ $\forall n$ in flat space, but concludes “it is difficult to say whether they are symmetry operators” in curved space - a question I have fully answered for $n = 1$ and partially answered for $n = 2$.

Another challenging extension - motivated by applications in higher spin field theory - to the calculation I have considered is the computation of higher symmetries of supersymmetric extensions of the d’Alembertian. There has been some success in this endeavour [21], but it is well beyond the scope of my thesis.

Chapter 4

Higher symmetries of the massless Dirac operator

4.1 Action for the massless Dirac operator

The general Dirac equation is

$$(\mathrm{i}\gamma^a \nabla_a - q\gamma^a A_a - m)\boldsymbol{\Psi} = 0 \tag{4.1}$$

where m is the particle's mass and q is its charge[1]. In this thesis I will only be considering the case of a massless particle in the absence of an external electromagnetic field.
Therefore, the Dirac equation reduces to $\mathrm{i}\gamma^a \nabla_a \boldsymbol{\Psi} = 0 \iff \gamma^a \nabla_a \boldsymbol{\Psi} = 0$.
As with the conformal d'Alembertian, the task in finding higher symmetries is to determine scalar, linear, differential operators, $D^{(n)}$, such that $\gamma^a \nabla_a D^{(n)} \boldsymbol{\Psi} = 0$ for any four-component spinor, $\boldsymbol{\Psi}$, satisfying $\gamma^a \nabla_a \boldsymbol{\Psi} = 0$.

Analogous to the last chapter, before hunting for higher symmetries, I will first derive $\gamma^a \nabla_a \boldsymbol{\Psi} = 0$ as the equation of motion for a matter field and find the properties of the matter field required to make the corresponding action Weyl invariant. Consider the action for a free, massless spinor field used in curved space quantum field theory,

$$S[e_a{}^m, \boldsymbol{\Psi}] = -\frac{\mathrm{i}}{2} \int \overline{\boldsymbol{\Psi}} \gamma^a \nabla_a(\boldsymbol{\Psi}) e \, \mathrm{d}^4 x \quad \text{where } e = \det(e_m{}^a)\,. \tag{4.2}$$

Under a variation to $\boldsymbol{\Psi}$,

$$\begin{aligned} S' &= -\frac{\mathrm{i}}{2} \int (\overline{\boldsymbol{\Psi} + \delta\boldsymbol{\Psi}}) \gamma^a \nabla_a (\boldsymbol{\Psi} + \delta\boldsymbol{\Psi}) e \, \mathrm{d}^4 x \\ &= S - \frac{\mathrm{i}}{2} \int (\delta\overline{\boldsymbol{\Psi}} \gamma^a \nabla_a(\boldsymbol{\Psi}) + \overline{\boldsymbol{\Psi}} \gamma^a \nabla_a (\delta\boldsymbol{\Psi})) e \, \mathrm{d}^4 x \,. \end{aligned} \tag{4.3}$$

Hence,

$$\delta S = -\frac{\mathrm{i}}{2} \int (\overline{\delta\boldsymbol{\Psi}} \gamma^a \nabla_a(\boldsymbol{\Psi}) + \overline{\boldsymbol{\Psi}} \gamma^a \nabla_a(\delta\boldsymbol{\Psi})) e \, \mathrm{d}^4 x \,. \tag{4.4}$$

[1] Strictly speaking, the zero on the RHS of the Dirac equation is a four-component spinor as well and thus should be denoted $\mathbf{0}$. However, I will ignore such pedantry and just call it 0.

To proceed, it is easiest to expand the matrix algebra contained within the action.

$$\begin{aligned}\overline{\delta\boldsymbol{\Psi}}\gamma^a\nabla_a(\boldsymbol{\Psi}) &= \begin{bmatrix}\delta\chi^\alpha & \delta\bar{\psi}_{\dot\alpha}\end{bmatrix}\begin{bmatrix}0 & (\sigma^a)_{\alpha\dot\alpha} \\ (\tilde\sigma^a)^{\dot\alpha\alpha} & 0\end{bmatrix}\begin{bmatrix}\nabla_a\psi_\alpha \\ \nabla_a\bar\chi^{\dot\alpha}\end{bmatrix} \\ &= \begin{bmatrix}\delta\chi^\alpha & \delta\bar{\psi}_{\dot\alpha}\end{bmatrix}\begin{bmatrix}\nabla_{\alpha\dot\alpha}\bar\chi^{\dot\alpha} \\ \nabla^{\alpha\dot\alpha}\psi_\alpha\end{bmatrix} \\ &= \delta\chi^\alpha\nabla_{\alpha\dot\alpha}\bar\chi^{\dot\alpha} + \delta\bar\psi_{\dot\alpha}\nabla^{\alpha\dot\alpha}\psi_\alpha \end{aligned} \tag{4.5}$$

$$\begin{aligned}\bar{\boldsymbol{\Psi}}\gamma^a\nabla_a(\delta\boldsymbol{\Psi}) &= \begin{bmatrix}\chi^\alpha & \bar{\psi}_{\dot\alpha}\end{bmatrix}\begin{bmatrix}0 & (\sigma^a)_{\alpha\dot\alpha} \\ (\tilde\sigma^a)^{\dot\alpha\alpha} & 0\end{bmatrix}\begin{bmatrix}\nabla_a\delta\psi_\alpha \\ \nabla_a\delta\bar\chi^{\dot\alpha}\end{bmatrix} \\ &= \begin{bmatrix}\chi^\alpha & \bar{\psi}_{\dot\alpha}\end{bmatrix}\begin{bmatrix}\nabla_{\alpha\dot\alpha}\delta\bar\chi^{\dot\alpha} \\ \nabla^{\alpha\dot\alpha}\delta\psi_\alpha\end{bmatrix} \\ &= \chi^\alpha\nabla_{\alpha\dot\alpha}\delta\bar\chi^{\dot\alpha} + \bar\psi_{\dot\alpha}\nabla^{\alpha\dot\alpha}\delta\psi_\alpha \\ &= \nabla_{\alpha\dot\alpha}(\chi^\alpha\delta\bar\chi^{\dot\alpha}) + \nabla^{\alpha\dot\alpha}(\bar\psi_{\dot\alpha}\delta\psi_\alpha) - \nabla_{\alpha\dot\alpha}(\chi^\alpha)\delta\bar\chi^{\dot\alpha} - \nabla^{\alpha\dot\alpha}(\bar\psi_{\dot\alpha})\delta\psi_\alpha \end{aligned} \tag{4.6}$$

Thus, the variation is

$$\begin{aligned}\delta S = -\frac{\mathrm{i}}{2}\int &(\nabla_{\alpha\dot\alpha}(\chi^\alpha\delta\bar\chi^{\dot\alpha}) + \nabla^{\alpha\dot\alpha}(\bar\psi_{\dot\alpha}\delta\psi_\alpha) - \nabla_{\alpha\dot\alpha}(\chi^\alpha)\delta\bar\chi^{\dot\alpha} - \nabla^{\alpha\dot\alpha}(\bar\psi_{\dot\alpha})\delta\psi_\alpha \\ &+ \delta\chi^\alpha\nabla_{\alpha\dot\alpha}\bar\chi^{\dot\alpha} + \delta\bar\psi_{\dot\alpha}\nabla^{\alpha\dot\alpha}\psi_\alpha)e\,\mathrm{d}^4x\,. \end{aligned} \tag{4.7}$$

The 1st two terms in the last equation integrate to zero by the generalised Stokes theorem since the variations vanish at the boundary. Therefore,

$$\delta S = -\frac{\mathrm{i}}{2}\int(\delta\chi^\alpha\nabla_{\alpha\dot\alpha}\bar\chi^{\dot\alpha} + \delta\bar\psi_{\dot\alpha}\nabla^{\alpha\dot\alpha}\psi_\alpha - \nabla_{\alpha\dot\alpha}(\chi^\alpha)\delta\bar\chi^{\dot\alpha} - \nabla^{\alpha\dot\alpha}(\bar\psi_{\dot\alpha})\delta\psi_\alpha)e\,\mathrm{d}^4x\,. \tag{4.8}$$

Since z and z^* form a "basis" for $\mathbb{C}$ and $\bar\psi_{\dot\alpha} = (\psi_\alpha)^*$ & $\chi^\alpha = (\bar\chi^{\dot\alpha})^*$ by definition, $\delta S = 0 \iff \nabla_{\alpha\dot\alpha}\bar\chi^{\dot\alpha} = 0$, $\nabla^{\alpha\dot\alpha}\psi_\alpha = 0$, $\nabla_{\alpha\dot\alpha}\chi^\alpha = 0$ and $\nabla^{\alpha\dot\alpha}\bar\psi_{\dot\alpha} = 0$. The latter two equations are complex conjugates of the former two, so really $\delta S = 0 \iff \nabla_{\alpha\dot\alpha}\bar\chi^{\dot\alpha} = 0$ and $\nabla^{\alpha\dot\alpha}\psi_\alpha = 0$. These two conditions can be summarised as $\gamma^a\nabla_a\boldsymbol{\Psi} = 0$ since

$$\gamma^a\nabla_a\boldsymbol{\Psi} = \begin{bmatrix}0 & (\sigma^a)_{\alpha\dot\alpha} \\ (\tilde\sigma^a)^{\dot\alpha\alpha} & 0\end{bmatrix}\begin{bmatrix}\nabla_a\psi_\alpha \\ \nabla_a\bar\chi^{\dot\alpha}\end{bmatrix} = \begin{bmatrix}\nabla_{\alpha\dot\alpha}\bar\chi^{\dot\alpha} \\ \nabla^{\alpha\dot\alpha}\psi_\alpha\end{bmatrix}\,. \tag{4.9}$$

Hence, the equation of motion for $\boldsymbol{\Psi}$ from S is the massless Dirac equation[2].

Next, I have to find the Weyl transformation properties of S. Consider a Weyl transformation, ${e'}_a{}^m = (1+\sigma)e_a{}^m$ for infinitesimal σ. Then, since $e = 1/\det(e_a{}^m)$,

$$e' = \frac{1}{\det((1+\sigma)e_a{}^m)} = \frac{1}{\det(e_a{}^m)\det(I+\sigma I)} = \frac{e}{1+\mathrm{tr}(\sigma I)} = \frac{e}{1+4\sigma} = (1-4\sigma)e\,. \tag{4.10}$$

The Weyl transformed action is then

$$\begin{aligned}S' &= -\frac{\mathrm{i}}{2}\int(1-4\sigma)(\overline{\boldsymbol{\Psi}+\delta\boldsymbol{\Psi}})\gamma^a((1+\sigma)\nabla_a - \nabla^b(\sigma)M_{ab})(\boldsymbol{\Psi}+\delta\boldsymbol{\Psi})e\,\mathrm{d}^4x \\ &= S + 2\mathrm{i}\int\sigma\bar{\boldsymbol{\Psi}}\gamma^a\nabla_a(\boldsymbol{\Psi})e\,\mathrm{d}^4x - \frac{\mathrm{i}}{2}\int\overline{\delta\boldsymbol{\Psi}}\gamma^a\nabla_a(\boldsymbol{\Psi})e\,\mathrm{d}^4x - \frac{\mathrm{i}}{2}\int\bar{\boldsymbol{\Psi}}\gamma^a\sigma\nabla_a(\boldsymbol{\Psi})e\,\mathrm{d}^4x \\ &\quad + \frac{\mathrm{i}}{2}\int\bar{\boldsymbol{\Psi}}\gamma^a\nabla^b(\sigma)M_{ab}(\boldsymbol{\Psi})e\,\mathrm{d}^4x - \frac{\mathrm{i}}{2}\int\bar{\boldsymbol{\Psi}}\gamma^a\nabla_a(\delta\boldsymbol{\Psi})e\,\mathrm{d}^4x\,. \end{aligned} \tag{4.11}$$

[2]Of course, this result is already used in quantum field theory.

Therefore,

$$\delta S = \frac{3\mathrm{i}}{2}\int \sigma\overline{\mathbf{\Psi}}\gamma^a\nabla_a(\mathbf{\Psi})e\,\mathrm{d}^4x - \frac{\mathrm{i}}{2}\int \overline{\delta\mathbf{\Psi}}\gamma^a\nabla_a(\mathbf{\Psi})e\,\mathrm{d}^4x + \frac{\mathrm{i}}{2}\int \overline{\mathbf{\Psi}}\gamma^a\nabla^b(\sigma)M_{ab}(\mathbf{\Psi})e\,\mathrm{d}^4x$$
$$- \frac{\mathrm{i}}{2}\int \overline{\mathbf{\Psi}}\gamma^a\nabla_a(\delta\mathbf{\Psi})e\,\mathrm{d}^4x\,. \tag{4.12}$$

In the 3rd term on the RHS,

$$\begin{aligned}
\gamma^a M_{ab}(\mathbf{\Psi}) &= \begin{bmatrix} 0 & (\sigma^a)_{\alpha\dot\alpha} \\ (\tilde\sigma^a)^{\dot\alpha\alpha} & 0 \end{bmatrix}\begin{bmatrix} M_{ab}\psi_\alpha \\ M_{ab}\overline{\chi}^{\dot\alpha} \end{bmatrix} \\
&= \begin{bmatrix} 0 & (\sigma^a)_{\alpha\dot\alpha} \\ (\tilde\sigma^a)^{\dot\alpha\alpha} & 0 \end{bmatrix}\begin{bmatrix} (\sigma_{ab})_\alpha{}^\beta\psi_\beta \\ (\tilde\sigma_{ab})^{\dot\alpha}{}_{\dot\beta}\overline{\chi}^{\dot\beta} \end{bmatrix} \\
&= \begin{bmatrix} (\sigma^a)_{\alpha\dot\alpha}(\tilde\sigma_{ab})^{\dot\alpha}{}_{\dot\beta}\overline{\chi}^{\dot\beta} \\ (\tilde\sigma^a)^{\dot\alpha\alpha}(\sigma_{ab})_\alpha{}^\beta\psi_\beta \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} (\delta^a{}_a(\sigma_b)_{\alpha\dot\beta} - \delta^a{}_b(\sigma_a)_{\alpha\dot\beta} - \mathrm{i}\varepsilon^a{}_{abc}(\sigma^c)_{\alpha\dot\beta})\overline{\chi}^{\dot\beta} \\ (\delta^a{}_a(\tilde\sigma_b)^{\dot\alpha\beta} - \delta^a{}_b(\tilde\sigma_a)^{\dot\alpha\beta} + \mathrm{i}\varepsilon^a{}_{abc}(\tilde\sigma^c)^{\dot\alpha\beta})\psi_\beta \end{bmatrix} \\
&= \frac{3}{2}\begin{bmatrix} (\sigma_b)_{\alpha\dot\beta}\overline{\chi}^{\dot\beta} \\ (\tilde\sigma_b)^{\dot\alpha\beta}\psi_\beta \end{bmatrix} \\
&= \frac{3}{2}\gamma_b\mathbf{\Psi}\,.
\end{aligned} \tag{4.13}$$

Substituting this back into the last expression for δS,

$$\begin{aligned}
\delta S &= \frac{3\mathrm{i}}{2}\int \sigma\overline{\mathbf{\Psi}}\gamma^a\nabla_a(\mathbf{\Psi})e\,\mathrm{d}^4x - \frac{\mathrm{i}}{2}\int \overline{\delta\mathbf{\Psi}}\gamma^a\nabla_a(\mathbf{\Psi})e\,\mathrm{d}^4x + \frac{3\mathrm{i}}{4}\int \nabla_a(\sigma)\overline{\mathbf{\Psi}}\gamma^a\mathbf{\Psi}e\,\mathrm{d}^4x \\
&\quad - \frac{\mathrm{i}}{2}\int \overline{\mathbf{\Psi}}\gamma^a\nabla_a(\delta\mathbf{\Psi})e\,\mathrm{d}^4x \\
&= -\frac{\mathrm{i}}{2}\int \left[(\delta\overline{\mathbf{\Psi}} - 3\sigma\overline{\mathbf{\Psi}})\gamma^a\nabla_a(\mathbf{\Psi}) + \overline{\mathbf{\Psi}}\gamma^a\left(\nabla_a(\delta\mathbf{\Psi}) - \frac{3}{2}\mathbf{\Psi}\nabla_a(\sigma)\right)\right]e\,\mathrm{d}^4x\,.
\end{aligned} \tag{4.14}$$

Let $\mathbf{\Phi} = \delta\mathbf{\Psi} - \frac{3}{2}\sigma\mathbf{\Psi} \iff \delta\mathbf{\Psi} = \frac{3}{2}\sigma\mathbf{\Psi} + \mathbf{\Phi}$.
Therefore, $\nabla_a(\delta\mathbf{\Psi}) = \frac{3}{2}\sigma\nabla_a\mathbf{\Psi} + \frac{3}{2}\nabla_a(\sigma)\mathbf{\Psi} + \nabla_a(\mathbf{\Phi})$ and

$$\begin{aligned}
\delta S &= -\frac{\mathrm{i}}{2}\int \Bigg[\left(\frac{3}{2}\sigma\overline{\mathbf{\Psi}} + \overline{\mathbf{\Phi}} - 3\sigma\overline{\mathbf{\Psi}}\right)\gamma^a\nabla_a(\mathbf{\Psi}) \\
&\qquad + \overline{\mathbf{\Psi}}\gamma^a\left(\frac{3}{2}\sigma\nabla_a\mathbf{\Psi} + \frac{3}{2}\nabla_a(\sigma)\mathbf{\Psi} + \nabla_a(\mathbf{\Phi}) - \frac{3}{2}\mathbf{\Psi}\nabla_a(\sigma)\right)\Bigg]e\,\mathrm{d}^4x \\
&= -\frac{\mathrm{i}}{2}\int (\overline{\mathbf{\Phi}}\gamma^a\nabla_a\mathbf{\Psi} + \overline{\mathbf{\Psi}}\gamma^a\nabla_a\mathbf{\Phi})e\,\mathrm{d}^4x\,.
\end{aligned} \tag{4.15}$$

Denote the components of $\mathbf{\Phi}$ by $\mathbf{\Phi} = (\phi_\alpha, \overline{\varphi}^{\dot\alpha})^{\mathrm{T}}$. Then,

$$\begin{aligned}
\overline{\mathbf{\Psi}}\gamma^a\nabla_a\mathbf{\Phi} &= \chi^\alpha\nabla_{\alpha\dot\alpha}\overline{\varphi}^{\dot\alpha} + \overline{\psi}_{\dot\alpha}\nabla^{\alpha\dot\alpha}\phi_\alpha \\
&= \nabla_{\alpha\dot\alpha}(\chi^\alpha\overline{\varphi}^{\dot\alpha}) + \nabla^{\alpha\dot\alpha}(\overline{\psi}_{\dot\alpha}\phi_\alpha) - \nabla_{\alpha\dot\alpha}(\chi^\alpha)\overline{\varphi}^{\dot\alpha} - \nabla^{\alpha\dot\alpha}(\overline{\psi}_{\dot\alpha})\phi_\alpha\,.
\end{aligned} \tag{4.16}$$

The 1st two terms on the RHS integrate to zero in the expression for δS by the generalised Stokes' theorem, provided spacetime does not have a boundary or the fields decay sufficiently rapidly at infinity. In either case,

$$\begin{aligned}\delta S &= -\frac{\mathrm{i}}{2}\int (\varphi^{\alpha}\nabla_{\alpha\dot\alpha}(\bar\chi^{\dot\alpha}) + \bar\phi_{\dot\alpha}\nabla^{\alpha\dot\alpha}(\psi_\alpha) - \nabla_{\alpha\dot\alpha}(\chi^\alpha)\bar\varphi^{\dot\alpha} - \nabla^{\alpha\dot\alpha}(\bar\psi_{\dot\alpha})\phi_\alpha) e\,\mathrm{d}^4x \\ &= -\frac{\mathrm{i}}{2}\int ((\nabla_{\alpha\dot\alpha}(\chi^\alpha)\bar\varphi^{\dot\alpha})^* + (\nabla^{\alpha\dot\alpha}(\bar\psi_{\dot\alpha})\phi_\alpha)^* - \nabla_{\alpha\dot\alpha}(\chi^\alpha)\bar\varphi^{\dot\alpha} - \nabla^{\alpha\dot\alpha}(\bar\psi_{\dot\alpha})\phi_\alpha) e\,\mathrm{d}^4x\,. \end{aligned} \tag{4.17}$$

Again, since z and z^* form a "basis" for $\mathbb{C}$ and $\nabla_{\alpha\dot\alpha}(\chi^\alpha)$ & $\nabla^{\alpha\dot\alpha}(\bar\psi_{\dot\alpha})$ are arbitrary, $\delta S = 0 \iff \phi_\alpha = 0$ & $\bar\varphi^{\dot\alpha} = 0 \iff \mathbf{\Phi} = 0 \iff \delta\mathbf{\Psi} = \frac{3}{2}\sigma\mathbf{\Psi}$.
Therefore, for S to describe a conformal field theory, the matter field, $\mathbf{\Psi}$, must transform as $\mathbf{\Psi}' = \mathrm{e}^{3\sigma/2}\mathbf{\Psi}$ upon a Weyl transformation, ${e'}_a{}^m = \mathrm{e}^\sigma e_a{}^m$.

4.2 Structure of the symmetry operators

Given $\mathbf{\Psi}$ is a spinor field, it will be more convenient to simply stick to spinor notation for the remainder of this chapter. Because $\mathbf{\Psi}$ is not a scalar, but a four-component object, the most general linear differential operator is actually a matrix of differential operators, e.g.

$$D\mathbf{\Psi} = \begin{bmatrix} D_{(1)\alpha}{}^{\beta} & D_{(2)\alpha\dot\beta} \\ D_{(3)}{}^{\dot\alpha\beta} & D_{(4)}{}^{\dot\alpha}{}_{\dot\beta}\end{bmatrix}\begin{bmatrix}\psi_\beta \\ \bar\chi^{\dot\beta}\end{bmatrix}\,. \tag{4.18}$$

Then, although D is a scalar in the sense of definition 2.11, none of $D_{(1)\alpha}{}^{\beta}$, $D_{(2)\alpha\dot\beta}$, $D_{(3)}{}^{\dot\alpha\beta}$ or $D_{(4)}{}^{\dot\alpha}{}_{\dot\beta}$ is a scalar in the same sense. Variants of this "matrix of differential operators" approach are undertaken in [18, 22, 19, 20] and require "conformal Killing-Yano tensors" - an extension to the concept I have already introduced. However, I will focus on the restricted case where the higher symmetry acts the same way on all of $\mathbf{\Psi}$'s components, i.e.

$$D\mathbf{\Psi} = \begin{bmatrix} D\psi_1 \\ D\psi_2 \\ D\bar\chi^{\dot 1} \\ D\bar\chi^{\dot 2}\end{bmatrix}\,. \tag{4.19}$$

Like in section 3.2, the next task is to find the most general form of D from products and contractions of $\nabla_{\alpha\dot\alpha}$, $M_{\alpha\beta}$, $\overline{M}_{\alpha\beta}$ and $\xi^{\alpha_1\cdots\alpha_k\dot\alpha_1\cdots\dot\alpha_k}$ while taking into account the equivalence relation, 2.13. By the same logic as applied in equation 3.12, any Lorentz generator can be "pushed to the front," e.g. any $M_{\beta\gamma}\nabla_{\alpha\dot\alpha}$ like terms can be replaced with terms like $\nabla_{\alpha\dot\alpha}M_{\beta\gamma}$.

Lemma 4.1. *Any terms in the symmetry operator with more than two Lorentz generators can be removed by the equivalence relation, $\sim$.*

Proof. First notice that whenever there are both dotted and undotted Lorentz generators, they annihilate $\mathbf{\Psi}$ since $M_{\alpha\beta}\chi^{\dot\alpha} = 0$, $\overline{M}_{\dot\alpha\dot\beta}\psi_\alpha = 0$ and $[M_{\alpha\beta}, \overline{M}_{\dot\alpha\dot\beta}] = 0$. Therefore, only terms of the form $M_{\alpha\beta}M_{\mu\nu}$ or $\overline{M}_{\dot\alpha\dot\beta}\overline{M}_{\dot\mu\dot\nu}$ need to be considered[3]. With D being a "scalar" of the form in equation 4.19, the Lorentz generators must appear with appropriate coefficients, i.e. as $\xi^{\alpha\beta\mu\nu}M_{\alpha\beta}M_{\mu\nu}$ with $\xi^{\alpha\beta\mu\nu} = \xi^{(\alpha\beta)(\mu\nu)}$ and $\xi^{\dot\alpha\dot\beta\dot\mu\dot\nu}\overline{M}_{\dot\alpha\dot\beta}\overline{M}_{\dot\mu\dot\nu}$ with $\xi^{\dot\alpha\dot\beta\dot\mu\dot\nu} = \xi^{(\dot\alpha\dot\beta)(\dot\mu\dot\nu)}$ respectively. Consider the decomposition of these coefficients. Applying equation 3.25 repeatedly, if

[3] If two Lorentz generators can be reduced to one or fewer, then by induction a term with n Lorentz generators can also be reduced to terms with one Lorentz generator or fewer.

$A_{\alpha\beta} = A_{(\alpha\beta)}$, then

$$\begin{aligned}
A_{\beta\gamma}C_\mu D_\nu &= A_{(\beta\gamma}C_{\mu)}D_\nu + \frac{1}{3}(\varepsilon_{\mu\beta}A_{\rho\gamma}C^\rho + \varepsilon_{\mu\gamma}A_{\rho\beta}C^\rho)D_\nu \\
&= A_{(\beta\gamma}C_\mu D_{\nu)} + \frac{1}{4}(\varepsilon_{\nu\beta}A_{(\rho\gamma}C_{\mu)}D^\rho + \varepsilon_{\nu\gamma}A_{(\beta\rho}C_{\mu)}D^\rho + \varepsilon_{\nu\mu}A_{(\beta\gamma}C_{\rho)}D^\rho) \\
&\quad + \frac{1}{3}(\varepsilon_{\mu\beta}A_{(\rho\gamma}D_{\nu)}C^\rho + \varepsilon_{\mu\gamma}A_{(\rho\beta}D_{\nu)}C^\rho) + \frac{1}{9}(\varepsilon_{\mu\beta}\varepsilon_{\nu\rho}A_{\lambda\gamma}C^\rho D^\lambda + \varepsilon_{\mu\beta}\varepsilon_{\nu\gamma}A_{\rho\lambda}C^\rho D^\lambda \\
&\quad + \varepsilon_{\mu\gamma}\varepsilon_{\nu\rho}A_{\lambda\beta}C^\rho D^\lambda + \varepsilon_{\mu\gamma}\varepsilon_{\nu\beta}A_{\rho\lambda}C^\rho D^\lambda) \\
&= A_{(\beta\gamma}C_\mu D_{\nu)} + \frac{1}{4}(\varepsilon_{\nu\beta}A_{(\rho\gamma}C_{\mu)}D^\rho + \varepsilon_{\nu\gamma}A_{(\beta\rho}C_{\mu)}D^\rho + \varepsilon_{\nu\mu}A_{(\beta\gamma}C_{\rho)}D^\rho) \\
&\quad + \frac{1}{3}(\varepsilon_{\mu\beta}A_{(\rho\gamma}D_{\nu)}C^\rho + \varepsilon_{\mu\gamma}A_{(\rho\beta}D_{\nu)}C^\rho) + \frac{1}{9}(\varepsilon_{\mu\beta}A_{\rho\gamma}C_\nu D^\rho + \varepsilon_{\mu\beta}\varepsilon_{\nu\gamma}A_{\rho\lambda}C^\rho D^\lambda \\
&\quad + \varepsilon_{\mu\gamma}A_{\rho\beta}C_\nu D^\rho + \varepsilon_{\mu\gamma}\varepsilon_{\nu\beta}A_{\rho\lambda}C^\rho D^\lambda)\,. \qquad (4.20)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\xi_{\beta\gamma\mu\nu} &= \xi_{(\beta\gamma\mu\nu)} + \frac{1}{4}(\varepsilon_{\nu\beta}\xi_{(\rho\gamma\mu)}{}^\rho + \varepsilon_{\nu\gamma}\xi_{(\beta\rho\mu)}{}^\rho + \varepsilon_{\nu\mu}\xi_{(\beta\gamma\rho)}{}^\rho) + \frac{1}{3}(\varepsilon_{\mu\beta}\xi_{(\rho\gamma}{}^\rho{}_{\nu)} + \varepsilon_{\mu\gamma}\xi_{(\rho\beta}{}^\rho{}_{\nu)}) \\
&\quad + \frac{1}{9}(\varepsilon_{\mu\beta}\xi_{\rho\gamma\nu}{}^\rho + \varepsilon_{\mu\beta}\varepsilon_{\nu\gamma}\xi_{\rho\lambda}{}^{\rho\lambda} + \varepsilon_{\mu\gamma}\xi_{\rho\beta\nu}{}^\rho + \varepsilon_{\mu\gamma}\varepsilon_{\nu\beta}\xi_{\rho\lambda}{}^{\rho\lambda})\,. \qquad (4.21)
\end{aligned}$$

Since $\xi_{\alpha\beta\mu\nu} = \xi_{\alpha\beta\nu\mu}$, the previous expression can be simplified to

$$\begin{aligned}
\xi_{\beta\gamma\mu\nu} &= \xi_{(\beta\gamma\mu\nu)} + \frac{7}{24}(\varepsilon_{\mu\beta}\xi_{(\rho\gamma\nu)}{}^\rho + \varepsilon_{\mu\gamma}\xi_{(\rho\beta\nu)}{}^\rho + \varepsilon_{\nu\beta}\xi_{(\rho\gamma\mu)}{}^\rho + \varepsilon_{\nu\gamma}\xi_{(\rho\beta\mu)}{}^\rho) + \frac{1}{9}(\varepsilon_{\mu\beta}\varepsilon_{\nu\gamma} + \varepsilon_{\nu\beta}\varepsilon_{\mu\gamma})\xi_{\rho\lambda}{}^{\rho\lambda} \\
&\quad + \frac{1}{18}(\varepsilon_{\mu\beta}\xi_{\rho\gamma\nu}{}^\rho + \varepsilon_{\nu\beta}\xi_{\rho\gamma\mu}{}^\rho + \varepsilon_{\mu\gamma}\xi_{\rho\beta\nu}{}^\rho + \varepsilon_{\nu\gamma}\xi_{\rho\beta\mu}{}^\rho)\,. \qquad (4.22)
\end{aligned}$$

Meanwhile,

$$\begin{aligned}
M^{\beta\gamma}M^{\mu\nu}\psi_\alpha &= \frac{1}{2}M^{\beta\gamma}(-\delta^\mu{}_\alpha\psi^\nu - \delta^\nu{}_\alpha\psi^\mu) \\
&= -\frac{1}{2}\delta^\mu{}_\alpha M^{\beta\gamma}\psi^\nu - \frac{1}{2}\delta^\nu{}_\alpha M^{\beta\gamma}\psi^\mu \\
&= \frac{1}{4}(\delta^\mu{}_\alpha\varepsilon^{\nu\beta}\psi^\gamma + \delta^\mu{}_\alpha\varepsilon^{\nu\gamma}\psi^\beta + \delta^\nu{}_\alpha\varepsilon^{\mu\beta}\psi^\gamma + \delta^\nu{}_\alpha\varepsilon^{\mu\gamma}\psi^\beta,\,. \qquad (4.23)
\end{aligned}$$

Putting the different pieces together,

$$\begin{aligned}
&\xi^{\beta\gamma\mu\nu}M_{\beta\gamma}M_{\mu\nu}\psi_\alpha \\
&= \xi_{\beta\gamma\mu\nu}M^{\beta\gamma}M^{\mu\nu}\psi_\alpha \\
&= \frac{1}{4}\Bigg(\xi_{(\beta\gamma\mu\nu)} + \frac{7}{24}(\varepsilon_{\mu\beta}\xi_{(\rho\gamma\nu)}{}^\rho + \varepsilon_{\mu\gamma}\xi_{(\rho\beta\nu)}{}^\rho + \varepsilon_{\nu\beta}\xi_{(\rho\gamma\mu)}{}^\rho + \varepsilon_{\nu\gamma}\xi_{(\rho\beta\mu)}{}^\rho) + \frac{1}{9}(\varepsilon_{\mu\beta}\varepsilon_{\nu\gamma} + \varepsilon_{\nu\beta}\varepsilon_{\mu\gamma})\xi_{\rho\lambda}{}^{\rho\lambda} \\
&\quad + \frac{1}{18}(\varepsilon_{\mu\beta}\xi_{\rho\gamma\nu}{}^\rho + \varepsilon_{\nu\beta}\xi_{\rho\gamma\mu}{}^\rho + \varepsilon_{\mu\gamma}\xi_{\rho\beta\nu}{}^\rho + \varepsilon_{\nu\gamma}\xi_{\rho\beta\mu}{}^\rho)\Bigg) \\
&\quad \times (\delta^\mu{}_\alpha\varepsilon^{\nu\beta}\psi^\gamma + \delta^\mu{}_\alpha\varepsilon^{\nu\gamma}\psi^\beta + \delta^\nu{}_\alpha\varepsilon^{\mu\beta}\psi^\gamma + \delta^\nu{}_\alpha\varepsilon^{\mu\gamma}\psi^\beta)\,. \qquad (4.24)
\end{aligned}$$

This last expression actually simplifies a lot because

$$\begin{aligned}
&\xi_{(\beta\gamma\mu\nu)}(\delta^\mu{}_\alpha\varepsilon^{\nu\beta}\psi^\gamma + \delta^\mu{}_\alpha\varepsilon^{\nu\gamma}\psi^\beta + \delta^\nu{}_\alpha\varepsilon^{\mu\beta}\psi^\gamma + \delta^\nu{}_\alpha\varepsilon^{\mu\gamma}\psi^\beta) = 0 \\
&\varepsilon_{\mu\beta}\xi_{(\rho\gamma\nu)}{}^\rho(\delta^\mu{}_\alpha\varepsilon^{\nu\beta}\psi^\gamma + \delta^\mu{}_\alpha\varepsilon^{\nu\gamma}\psi^\beta + \delta^\nu{}_\alpha\varepsilon^{\mu\beta}\psi^\gamma + \delta^\nu{}_\alpha\varepsilon^{\mu\gamma}\psi^\beta) \\
&\quad = -\xi_{(\rho\gamma\alpha)}{}^\rho\psi^\gamma + 0 - 2\xi_{(\rho\gamma\alpha)}{}^\rho\psi^\gamma - \xi_{(\rho\gamma\alpha)}{}^\rho\psi^\gamma \\
&\quad = -4\xi_{(\gamma\beta\alpha)}{}^\gamma\psi^\beta\,. \qquad (4.25)
\end{aligned}$$

Likewise, $\varepsilon_{\mu\gamma}\xi_{(\rho\beta\nu)}{}^{\rho}$, $\varepsilon_{\nu\beta}\xi_{(\rho\gamma\mu)}{}^{\rho}$ and $\varepsilon_{\nu\gamma}\xi_{(\rho\beta\mu)}{}^{\rho}$ multiplied with $(\delta^{\mu}{}_{\alpha}\varepsilon^{\nu\beta}\psi^{\gamma}+\delta^{\mu}{}_{\alpha}\varepsilon^{\nu\gamma}\psi^{\beta}+\delta^{\nu}{}_{\alpha}\varepsilon^{\mu\beta}\psi^{\gamma}+\delta^{\nu}{}_{\alpha}\varepsilon^{\mu\gamma}\psi^{\beta})$ all equal $-4\xi_{(\gamma\beta\alpha)}{}^{\gamma}\psi^{\beta}$ as well because the former three are the same as $\varepsilon_{\mu\beta}\xi_{(\rho\gamma\nu)}{}^{\rho}$ but with a μ and ν swapped or β and γ swapped while the last factor is symmetric under μ & ν and β & γ swaps.

$$
\begin{aligned}
&\varepsilon_{\mu\beta}\varepsilon_{\nu\gamma}\xi_{\rho\lambda}{}^{\rho\lambda}(\delta^{\mu}{}_{\alpha}\varepsilon^{\nu\beta}\psi^{\gamma}+\delta^{\mu}{}_{\alpha}\varepsilon^{\nu\gamma}\psi^{\beta}+\delta^{\nu}{}_{\alpha}\varepsilon^{\mu\beta}\psi^{\gamma}+\delta^{\nu}{}_{\alpha}\varepsilon^{\mu\gamma}\psi^{\beta})\\
&=\xi_{\rho\lambda}{}^{\rho\lambda}(-\psi_{\alpha}-2\psi_{\alpha}-2\psi_{\alpha}-\psi_{\alpha})\\
&=-6\xi^{\beta\gamma}{}_{\beta\gamma}\psi_{\alpha}
\end{aligned}
\tag{4.26}
$$

Therefore $\varepsilon_{\nu\beta}\varepsilon_{\mu\gamma}\xi_{\rho\lambda}{}^{\rho\lambda}(\delta^{\mu}{}_{\alpha}\varepsilon^{\nu\beta}\psi^{\gamma}+\delta^{\mu}{}_{\alpha}\varepsilon^{\nu\gamma}\psi^{\beta}+\delta^{\nu}{}_{\alpha}\varepsilon^{\mu\beta}\psi^{\gamma}+\delta^{\nu}{}_{\alpha}\varepsilon^{\mu\gamma}\psi^{\beta})=-6\xi^{\beta\gamma}{}_{\beta\gamma}\psi_{\alpha}$ too.

$$
\begin{aligned}
&\varepsilon_{\mu\beta}\xi_{\rho\gamma\nu}{}^{\rho}(\delta^{\mu}{}_{\alpha}\varepsilon^{\nu\beta}\psi^{\gamma}+\delta^{\mu}{}_{\alpha}\varepsilon^{\nu\gamma}\psi^{\beta}+\delta^{\nu}{}_{\alpha}\varepsilon^{\mu\beta}\psi^{\gamma}+\delta^{\nu}{}_{\alpha}\varepsilon^{\mu\gamma}\psi^{\beta})\\
&=-\xi_{\rho\gamma\alpha}{}^{\rho}\psi^{\gamma}-\xi^{\beta\gamma}{}_{\beta\gamma}\psi_{\alpha}-2\xi_{\rho\gamma\alpha}{}^{\rho}\psi^{\gamma}-\xi_{\rho\gamma\alpha}{}^{\rho}\psi^{\gamma}\\
&=-4\xi_{\gamma\beta\alpha}{}^{\gamma}\psi^{\beta}-\xi^{\beta\gamma}{}_{\beta\gamma}\psi_{\alpha}
\end{aligned}
\tag{4.27}
$$

Similarly, $\varepsilon_{\nu\beta}\xi_{\rho\gamma\mu}{}^{\rho}$, $\varepsilon_{\mu\gamma}\xi_{\rho\beta\nu}{}^{\rho}$ and $\varepsilon_{\nu\gamma}\xi_{\rho\beta\mu}{}^{\rho}$ contracted with $(\delta^{\mu}{}_{\alpha}\varepsilon^{\nu\beta}\psi^{\gamma}+\delta^{\mu}{}_{\alpha}\varepsilon^{\nu\gamma}\psi^{\beta}+\delta^{\nu}{}_{\alpha}\varepsilon^{\mu\beta}\psi^{\gamma}+\delta^{\nu}{}_{\alpha}\varepsilon^{\mu\gamma}\psi^{\beta})$ also give the same result as the last line for the reason outlined above. Anyhow, putting all the parts together,

$$
\begin{aligned}
\xi^{\beta\gamma\mu\nu}M_{\beta\gamma}M_{\mu\nu}\psi_{\alpha}&=-\frac{7}{6}\xi_{(\gamma\beta\alpha)}{}^{\gamma}\psi^{\beta}-\frac{1}{3}\xi^{\beta\gamma}{}_{\beta\gamma}\psi_{\alpha}-\frac{2}{9}\xi_{\gamma\beta\alpha}{}^{\gamma}\psi^{\beta}-\frac{1}{18}\xi^{\beta\gamma}{}_{\beta\gamma}\psi_{\alpha}\\
&=-\frac{7}{6}\xi_{(\gamma\beta\alpha)}{}^{\gamma}\psi^{\beta}-\frac{2}{9}\xi_{\gamma\beta\alpha}{}^{\gamma}\psi^{\beta}-\frac{7}{18}\xi^{\beta\gamma}{}_{\beta\gamma}\psi_{\alpha}\,.
\end{aligned}
\tag{4.28}
$$

However, $\xi_{\gamma\beta\alpha}{}^{\gamma}\psi^{\beta}=\xi_{\gamma(\beta\alpha)}{}^{\gamma}\psi^{\beta}+\xi_{\gamma[\beta\alpha]}{}^{\gamma}\psi^{\beta}=\xi_{\gamma(\beta\alpha)}{}^{\gamma}\psi^{\beta}+\frac{1}{2}\xi\varepsilon_{\alpha\beta}\xi_{\gamma\rho}{}^{\rho\gamma}\psi^{\beta}=\xi_{\gamma(\beta\alpha)}{}^{\gamma}\psi^{\beta}+\frac{1}{2}\xi^{\beta\gamma}{}_{\beta\gamma}\psi_{\alpha}$. Thus,

$$
\begin{aligned}
\xi^{\beta\gamma\mu\nu}M_{\beta\gamma}M_{\mu\nu}\psi_{\alpha}&=-\frac{7}{6}\xi_{(\gamma\beta\alpha)}{}^{\gamma}\psi^{\beta}-\frac{2}{9}\xi_{\gamma(\beta\alpha)}{}^{\gamma}\psi^{\beta}-\frac{1}{2}\xi^{\beta\gamma}{}_{\beta\gamma}\psi_{\alpha}\\
&=\zeta_{\alpha\beta}\psi^{\beta}+\zeta\psi_{\alpha}
\end{aligned}
\tag{4.29}
$$

for some some symmetric tensor, ζ_{ab}, and a scalar, ζ. The $\zeta\psi_{\alpha}$ term has no Lorentz generators; it is simply a 0th order differential operator (i.e. a scalar) acting on ψ_{α}. The $\zeta_{\alpha\beta}\psi^{\beta}$ term can be absorbed into a term[4] with one Lorentz generator since

$$
\xi^{\beta\gamma}M_{\beta\gamma}=\frac{1}{2}\xi^{\beta\gamma}(\varepsilon_{\alpha\beta}\psi_{\gamma}+\varepsilon_{\alpha\gamma}\psi_{\beta})=\xi_{\alpha}{}^{\beta}\psi_{\beta}=-\xi_{\alpha\beta}\psi^{\beta}\,.
\tag{4.30}
$$

Hence, $\xi^{\beta\gamma\mu\nu}M_{\beta\gamma}M_{\mu\nu}$ is related by $\sim$ to an operator with one or fewer Lorentz generators in each term. Likewise,

$$
\xi^{\dot{\beta}\dot{\gamma}\dot{\mu}\dot{\nu}}\overline{M}_{\dot{\beta}\dot{\gamma}}\overline{M}_{\dot{\mu}\dot{\nu}}\chi^{\dot{\alpha}}=\zeta^{\dot{\alpha}\dot{\beta}}\chi_{\dot{\beta}}+\zeta\chi^{\dot{\alpha}}
\tag{4.31}
$$

for some $\zeta^{\dot{\alpha}\dot{\beta}}$ and ζ as the algebra is the same, just with dots and the free index in an upstairs position. □

Lemma 4.2. *Any terms in the symmetry operator with a contraction between a covariant derivative and a Lorentz generator can be removed by* $\sim$.

[4]Absorbing into lower order terms implies the two differential operators are related by $\sim$.

Proof. To prove this lemma it will be necessary to use the Dirac equation, equation 4.9, which says $\gamma^a \nabla_a \mathbf{\Psi} = 0 \iff \nabla_{\alpha\dot{\alpha}} \chi^{\dot{\alpha}} = 0$ and $\nabla^{\alpha\dot{\alpha}} \psi_\alpha = 0$. This is fine since applying the Dirac equation to simplify a symmetry operator keeps one in the same equivalence class.
Given the previous lemma, there are two possible contractions, $\xi^{\beta\dot{\alpha}} \nabla^\gamma{}_{\dot{\alpha}} M_{\beta\gamma}$ and $\xi^{\alpha\dot{\beta}} \nabla_\alpha{}^{\dot{\gamma}} \overline{M}_{\dot{\beta}\dot{\gamma}}$.

$$\begin{aligned}
\xi^{\beta\dot{\alpha}} \nabla^\gamma{}_{\dot{\alpha}} M_{\beta\gamma} \psi_\alpha &= \frac{1}{2} \xi^{\beta\dot{\alpha}} \nabla^\gamma{}_{\dot{\alpha}} (\varepsilon_{\alpha\beta} \psi_\gamma + \varepsilon_{\alpha\gamma} \psi_\beta) \\
&= \frac{1}{2} \xi_\alpha{}^{\dot{\alpha}} \nabla^\gamma{}_{\dot{\alpha}} \psi_\gamma + \frac{1}{2} \xi^{\beta\dot{\alpha}} \nabla_{\alpha\dot{\alpha}} \psi_\beta \\
&= 0 + \frac{1}{2} \xi^{\beta\dot{\alpha}} \nabla_{\alpha\dot{\alpha}} \psi_\beta \qquad (4.32) \\
\text{However, } \nabla_{\alpha\dot{\alpha}} \psi_\beta &= \nabla_{(\alpha\dot{\alpha}} \psi_{\beta)} + \nabla_{[\alpha\dot{\alpha}} \psi_{\beta]} \\
&= \nabla_{(\alpha\dot{\alpha}} \psi_{\beta)} + \frac{1}{2} \varepsilon_{\alpha\beta} \nabla^\gamma{}_{\dot{\alpha}} \psi_\gamma \\
&= \nabla_{(\alpha\dot{\alpha}} \psi_{\beta)} + 0 \qquad (4.33) \\
\implies \nabla_{\alpha\dot{\alpha}} \psi_\beta &= \nabla_{\beta\dot{\alpha}} \psi_\alpha \qquad (4.34)
\end{aligned}$$

Therefore,

$$\xi^{\beta\dot{\alpha}} \nabla^\gamma{}_{\dot{\alpha}} M_{\beta\gamma} \psi_\alpha = \frac{1}{2} \xi^{\beta\dot{\beta}} \nabla_{\beta\dot{\beta}} \psi_\alpha \qquad (4.35)$$

and thus a $\nabla^\gamma{}_{\dot{\alpha}} M_{\beta\gamma}$ term is equivalent to a single derivative, $\nabla_{\beta\dot{\beta}}$. Similarly,

$$\begin{aligned}
\xi^{\alpha\dot{\beta}} \nabla_\alpha{}^{\dot{\gamma}} \overline{M}_{\dot{\beta}\dot{\gamma}} \chi^{\dot{\alpha}} &= \frac{1}{2} \xi^{\alpha\dot{\beta}} \nabla_\alpha{}^{\dot{\gamma}} (\delta^{\dot{\alpha}}{}_{\dot{\beta}} \chi_{\dot{\gamma}} + \delta^{\dot{\alpha}}{}_{\dot{\gamma}} \chi_{\dot{\beta}}) \\
&= \frac{1}{2} \xi^{\alpha\dot{\alpha}} \nabla_\alpha{}^{\dot{\gamma}} \chi_{\dot{\gamma}} + \frac{1}{2} \xi^{\alpha\dot{\beta}} \nabla_\alpha{}^{\dot{\alpha}} \chi_{\dot{\beta}} \\
&= 0 - \frac{1}{2} \xi^\alpha{}_{\dot{\beta}} \nabla_\alpha{}^{\dot{\alpha}} \chi^{\dot{\beta}} \qquad (4.36)
\end{aligned}$$

Again, $\nabla_\alpha{}^{\dot{\alpha}} \chi^{\dot{\beta}} = \nabla_\alpha{}^{(\dot{\alpha}} \chi^{\dot{\beta})} + \nabla_\alpha{}^{[\dot{\alpha}} \chi^{\dot{\beta}]} = \nabla_\alpha{}^{(\dot{\alpha}} \chi^{\dot{\beta})} + \frac{1}{2} \varepsilon^{\dot{\alpha}\dot{\beta}} \nabla_{\alpha\dot{\gamma}} \chi^{\dot{\gamma}} = \nabla_\alpha{}^{(\dot{\alpha}} \chi^{\dot{\beta})}$. Hence,

$$\xi^{\alpha\dot{\beta}} \nabla_\alpha{}^{\dot{\gamma}} \overline{M}_{\dot{\beta}\dot{\gamma}} \chi^{\dot{\alpha}} = -\frac{1}{2} \xi^\alpha{}_{\dot{\beta}} \nabla_\alpha{}^{\dot{\beta}} \chi^{\dot{\alpha}} = \frac{1}{2} \xi^{\beta\dot{\beta}} \nabla_{\beta\dot{\beta}} \chi^{\dot{\alpha}} \qquad (4.37)$$

and thus $\nabla_\alpha{}^{\dot{\gamma}} \overline{M}_{\dot{\beta}\dot{\gamma}}$ is also equivalent to $\nabla_{\beta\dot{\beta}}$. □

Corollary 4.2.1. *The coefficient of any term with both a derivative and a Lorentz generator can be fully symmetrised in the indices common to the derivative and Lorentz generator.*

Proof. A term with both a derivative and a Lorentz generator can be represented as $\xi^{\alpha\beta\gamma\dot{\alpha}} \nabla_{\alpha\dot{\alpha}} M_{\beta\gamma}$ or $\xi^{\alpha\dot{\alpha}\dot{\beta}\dot{\gamma}} \nabla_{\alpha\dot{\alpha}} \overline{M}_{\dot{\beta}\dot{\gamma}}$ respectively[5]. $\xi^{\alpha\beta\gamma\dot{\alpha}} = \xi^{\alpha(\beta\gamma)\dot{\alpha}}$ and $\xi^{\alpha\dot{\alpha}\dot{\beta}\dot{\gamma}} = \xi^{\alpha\dot{\alpha}(\dot{\beta}\dot{\gamma})}$ already since $M_{\beta\gamma} = M_{\gamma\beta}$ and $\overline{M}_{\dot{\beta}\dot{\gamma}} = \overline{M}_{\dot{\gamma}\dot{\beta}}$. Then, by equation 3.25,

$$\begin{aligned}
\xi^{\alpha\beta\gamma\dot{\alpha}} \nabla_{\alpha\dot{\alpha}} M_{\beta\gamma} &= \xi^{(\alpha\beta\gamma)\dot{\alpha}} \nabla_{\alpha\dot{\alpha}} M_{\beta\gamma} + \frac{1}{3} \varepsilon^{\alpha\beta} \xi_\mu{}^{\mu\gamma\dot{\alpha}} \nabla_{\alpha\dot{\alpha}} M_{\beta\gamma} + \frac{1}{3} \varepsilon^{\alpha\gamma} \xi_\mu{}^{\beta\mu\dot{\alpha}} \nabla_{\alpha\dot{\alpha}} M_{\beta\gamma} \\
&= \xi^{(\alpha\beta\gamma)\dot{\alpha}} \nabla_{\alpha\dot{\alpha}} M_{\beta\gamma} - \frac{2}{3} \xi_\mu{}^{\mu\gamma\dot{\alpha}} \nabla^\beta{}_{\dot{\alpha}} M_{\beta\gamma}\,, \qquad (4.38)
\end{aligned}$$

[5] The overall operator/term may be of a higher order, but I am only interested here in the interaction of a Lorentz generator with any one derivative and hence the coefficient can be restricted to only the one Lorentz generator and one derivative case.

where the 2nd term in the previous line is equivalent to a term with a single derivative by the lemma. Likewise,

$$\xi^{\alpha\dot\alpha\dot\beta\dot\gamma}\nabla_{\alpha\dot\alpha}\overline{M}_{\dot\beta\dot\gamma} = \xi^{\alpha(\dot\alpha\dot\beta\dot\gamma)}\nabla_{\alpha\dot\alpha}\overline{M}_{\dot\beta\dot\gamma} + \frac{1}{3}\varepsilon^{\dot\alpha\dot\beta}\xi^{\alpha}{}_{\dot\mu}{}^{\dot\mu\dot\gamma}\nabla_{\alpha\dot\alpha}\overline{M}_{\dot\beta\dot\gamma} + \frac{1}{3}\varepsilon^{\dot\alpha\dot\gamma}\xi^{\alpha}{}_{\dot\mu}{}^{\dot\beta\dot\mu}\nabla_{\alpha\dot\alpha}\overline{M}_{\dot\beta\dot\gamma}$$

$$= \xi^{\alpha(\dot\alpha\dot\beta\dot\gamma)}\nabla_{\alpha\dot\alpha}\overline{M}_{\dot\beta\dot\gamma} - \frac{2}{3}\xi^{\alpha}{}_{\dot\mu}{}^{\dot\beta\dot\mu}\nabla_{\alpha}{}^{\dot\gamma}\overline{M}_{\dot\beta\dot\gamma}\,, \tag{4.39}$$

where again the 2nd term can be simplified by the lemma.
In either case, only the fully symmetrised form of the coefficient, $\xi^{(\alpha\beta\gamma)\dot\alpha}$ and $\xi^{\alpha(\dot\alpha\dot\beta\dot\gamma)}$ respectively, is left acting on $\nabla_{\alpha\dot\alpha}M_{\beta\gamma}$ or $\nabla_{\alpha\dot\alpha}\overline{M}_{\dot\beta\dot\gamma}$ respectively. ☐

Lemma 4.3. *For terms with only derivatives, the coefficient can be taken to be symmetric and traceless.*

Proof. Since $\gamma^a\nabla_a\mathbf{\Psi} = 0$,

$$\begin{aligned}0 &= \gamma^b\nabla_b\gamma^a\nabla_a\mathbf{\Psi}\\ &= \frac{1}{2}(\gamma^b\gamma^a + \gamma^a\gamma^b)\nabla_b\nabla_a\mathbf{\Psi} + \frac{1}{2}(\gamma^b\gamma^a - \gamma^a\gamma^b)\nabla_b\nabla_a\mathbf{\Psi}\\ &= -\eta^{ba}\nabla_b\nabla_a\mathbf{\Psi} + \frac{1}{2}\gamma^a\gamma^b[\nabla_a,\nabla_b]\mathbf{\Psi}\\ &= -\Box\mathbf{\Psi} + \frac{1}{4}\gamma^a\gamma^b R_{ab}{}^{cd}M_{cd}\mathbf{\Psi}\,.\end{aligned} \tag{4.40}$$

Therefore,

$$\begin{aligned}\Box\mathbf{\Psi} &= \frac{1}{4}\begin{bmatrix}0 & (\sigma^a)_{\alpha\dot\alpha}\\ (\tilde\sigma^a)^{\dot\alpha\alpha} & 0\end{bmatrix}\begin{bmatrix}0 & (\sigma^b)_{\alpha\dot\beta}\\ (\tilde\sigma^b)^{\dot\alpha\beta} & 0\end{bmatrix}R_{ab}{}^{cd}M_{cd}\mathbf{\Psi}\\ &= \frac{1}{4}\begin{bmatrix}(\sigma^a)_{\alpha\dot\alpha}(\tilde\sigma^b)^{\dot\alpha\beta}R_{ab}{}^{cd}M_{cd} & 0\\ 0 & (\tilde\sigma^a)^{\dot\alpha\alpha}(\sigma^b)_{\alpha\dot\beta}R_{ab}{}^{cd}M_{cd}\end{bmatrix}\mathbf{\Psi}\\ &= \frac{1}{4}\begin{bmatrix}R_{\alpha\dot\alpha}{}^{\beta\dot\alpha cd}M_{cd}\psi_\beta\\ R^{\alpha\dot\alpha}{}_{\alpha\dot\beta}{}^{cd}M_{cd}\overline{\chi}^{\dot\beta}\end{bmatrix}\end{aligned} \tag{4.41}$$

$$\begin{aligned}\frac{1}{4}R_{\alpha\dot\alpha}{}^{\beta\dot\alpha cd}M_{cd}\psi_\beta &= \frac{1}{2}(R_{\alpha\dot\alpha}{}^{\beta\dot\alpha\mu\nu}M_{\mu\nu}\psi_\beta + \overline{R}_{\alpha\dot\alpha}{}^{\beta\dot\alpha\dot\mu\dot\nu}\overline{M}_{\dot\mu\dot\nu}\psi_\beta)\\ &= \frac{1}{2}R_{\alpha\dot\alpha}{}^{\beta\dot\alpha\mu}{}_{\beta}\psi_\mu\\ &= \frac{1}{2}(-\delta^{\dot\alpha}{}_{\dot\alpha}C_\alpha{}^{\beta\mu\nu} - \delta^{\beta}{}_{\alpha}E^{\mu}{}_{\beta\dot\alpha}{}^{\dot\alpha} - \delta^{\dot\alpha}{}_{\dot\alpha}(-\delta^{\mu}{}_{\alpha}\delta^{\beta}{}_{\beta} + \varepsilon^{\mu\beta}\varepsilon_{\alpha\beta})F)\psi_\mu\\ &= 3F\psi_\alpha\\ &= \frac{1}{4}R\psi_\alpha\end{aligned} \tag{4.42}$$

$$\begin{aligned}\frac{1}{4}R^{\alpha\dot\alpha}{}_{\alpha\dot\beta}{}^{cd}M_{cd}\overline{\chi}^{\dot\beta} &= \frac{1}{2}(R^{\alpha\dot\alpha}{}_{\alpha\dot\beta}{}^{\mu\nu}M_{\mu\nu}\overline{\chi}^{\dot\beta} + \overline{R}^{\alpha\dot\alpha}{}_{\alpha\dot\beta}{}^{\dot\mu\dot\nu}\overline{M}_{\dot\mu\dot\nu}\overline{\chi}^{\dot\beta})\\ &= \frac{1}{2}\overline{R}^{\alpha\dot\alpha}{}_{\alpha\dot\beta}{}^{\dot\beta\dot\mu}\overline{\chi}_{\dot\mu}\\ &= (\delta^{\alpha}{}_{\alpha}\overline{C}^{\dot\alpha}{}_{\dot\beta}{}^{\dot\beta\dot\mu} + \delta^{\dot\alpha}{}_{\dot\beta}E^{\alpha}{}_{\alpha}{}^{\dot\beta\dot\mu} + \delta^{\alpha}{}_{\alpha}(-\varepsilon^{\dot\beta\dot\alpha}\delta^{\dot\mu}{}_{\dot\beta} - \delta^{\dot\beta}{}_{\dot\beta}\varepsilon^{\dot\mu\dot\alpha})F)\overline{\chi}_{\dot\mu}\\ &= 3F\overline{\chi}^{\dot\alpha}\\ &= \frac{1}{4}R\overline{\chi}^{\dot\alpha}\,,\end{aligned} \tag{4.43}$$

which finally gives

$$\Box\mathbf{\Psi} = \frac{1}{4}R\mathbf{\Psi}\,. \tag{4.44}$$

Hence, every component of $\mathbf{\Psi}$ satisfies an equation identical to the conformal d'Alembertian except for a change of $\frac{1}{6}\to\frac{1}{4}$. This change occurs because $(\Box-\frac{1}{6}R)\varphi=0$ transforms covariantly under Weyl transformations given $\varphi'=\mathrm{e}^{\sigma}\varphi$; $\mathbf{\Psi}$ instead transforms as $\mathbf{\Psi}'=\mathrm{e}^{3\sigma/2}\mathbf{\Psi}$ and for that $(\Box-\frac{1}{4}R)\mathbf{\Psi}=0$ is the covariant equation. In any case, the present lemma then follows directly from lemma 3.2 because the $\frac{1}{6}\to\frac{1}{4}$ change is inconsequential in the proof of lemma 3.2. $\Box$

Hence, given lemma 3.3 and the results & comments in this section, it follows that the most general nth order symmetry operator I will need to consider is

$$\begin{aligned}D^{(n)} &= \sum_{k=0}^{n}\xi^{\alpha_1\cdots\alpha_k\dot\alpha_1\cdots\dot\alpha_k}\nabla_{\alpha_1\dot\alpha_1}\cdots\nabla_{\alpha_k\dot\alpha_k}+\sum_{k=0}^{n-1}\xi^{\alpha_1\cdots\alpha_k\beta\gamma\dot\alpha_1\cdots\dot\alpha_k}\nabla_{\alpha_1\dot\alpha_1}\cdots\nabla_{\alpha_k\dot\alpha_k}M_{\beta\gamma}\\ &\quad+\sum_{k=0}^{n-1}\xi^{\alpha_1\cdots\alpha_k\dot\alpha_1\cdots\dot\alpha_k\dot\beta\dot\gamma}\nabla_{\alpha_1\dot\alpha_1}\cdots\nabla_{\alpha_k\dot\alpha_k}\overline{M}_{\dot\beta\dot\gamma}\end{aligned} \tag{4.45}$$

where $\xi^{\alpha_1\cdots\alpha_k\dot\alpha_1\cdots\dot\alpha_k}=\xi^{(\alpha_1\cdots\alpha_k)(\dot\alpha_1\cdots\dot\alpha_k)}$, $\xi^{\alpha_1\cdots\alpha_k\beta\gamma\dot\alpha_1\cdots\dot\alpha_k}=\xi^{(\alpha_1\cdots\alpha_k\beta\gamma)(\dot\alpha_1\cdots\dot\alpha_k)}$ and $\xi^{\alpha_1\cdots\alpha_k\dot\alpha_1\cdots\dot\alpha_k\dot\beta\dot\gamma}=\xi^{(\alpha_1\cdots\alpha_k)(\dot\alpha_1\cdots\dot\alpha_k\dot\beta\dot\gamma)}$.
For example, in the $n=1,2$ cases, the symmetry operators are

$$D^{(1)}=\xi^{\alpha\dot\alpha}\nabla_{\alpha\dot\alpha}+\xi^{\alpha\beta}M_{\alpha\beta}+\xi^{\dot\alpha\dot\beta}\overline{M}_{\dot\alpha\dot\beta}+\xi \tag{4.46}$$

where $\xi^{\alpha\beta}=\xi^{(\alpha\beta)}$ & $\xi^{\dot\alpha\dot\beta}=\xi^{(\dot\alpha\dot\beta)}$ and

$$D^{(2)}=\xi^{\alpha\beta\dot\alpha\dot\beta}\nabla_{\alpha\dot\alpha}\nabla_{\beta\dot\beta}+\xi^{\alpha\beta\gamma\dot\alpha}\nabla_{\alpha\dot\alpha}M_{\beta\gamma}+\xi^{\alpha\dot\alpha\dot\beta\dot\gamma}\nabla_{\alpha\dot\alpha}\overline{M}_{\dot\beta\dot\gamma}+\xi^{\alpha\dot\alpha}\nabla_{\alpha\dot\alpha}+\xi^{\alpha\beta}M_{\alpha\beta}+\xi^{\dot\alpha\dot\beta}\overline{M}_{\dot\alpha\dot\beta}+\xi \tag{4.47}$$

where $\xi^{\alpha\beta\dot\alpha\dot\beta}=\xi^{(\alpha\beta)(\dot\alpha\dot\beta)}$, $\xi^{\alpha\beta\gamma\dot\alpha}=\xi^{(\alpha\beta\gamma)\dot\alpha}$, $\xi^{\alpha\dot\alpha\dot\beta\dot\gamma}=\xi^{\alpha(\dot\alpha\dot\beta\dot\gamma)}$, $\xi^{\alpha\beta}=\xi^{(\alpha\beta)}$ and $\xi^{\dot\alpha\dot\beta}=\xi^{(\dot\alpha\dot\beta)}$.

4.3 1st order symmetries

Lemma 4.4. *In $D^{(1)}$, $\xi^{\alpha\dot\alpha}$ must be conformal Killing, $\xi^{\alpha\beta}=\frac{1}{2}\nabla^{(\alpha}{}_{\dot\alpha}\xi^{\beta)\dot\alpha}$ and $\xi^{\dot\alpha\dot\beta}=\frac{1}{2}\nabla_{\alpha}{}^{(\dot\alpha}\xi^{\alpha\dot\beta)}$.*

Proof. First, let $\mathbf{\Psi}$ satisfy $\gamma^a\nabla_a\mathbf{\Psi}=0$. That is,

$$0=\begin{bmatrix}0 & (\sigma^a)_{\alpha\dot\alpha}\\ (\tilde\sigma^a)^{\dot\alpha\alpha} & 0\end{bmatrix}\begin{bmatrix}\nabla_a\psi_\alpha\\ \nabla_a\bar\chi^{\dot\alpha}\end{bmatrix}=\begin{bmatrix}\nabla_{\alpha\dot\alpha}\bar\chi^{\dot\alpha}\\ \nabla^{\dot\alpha\alpha}\psi_\alpha\end{bmatrix}. \tag{4.48}$$

In components, $\nabla_{\alpha\dot\alpha}\bar\chi^{\dot\alpha} = 0$ and $\nabla^{\alpha\dot\alpha}\psi_\alpha = 0$. Then, applying the candidate symmetry operator,

$$
\begin{aligned}
&\gamma^a \nabla_a D^{(1)}\boldsymbol{\Psi} \\
&= \begin{bmatrix} 0 & (\sigma^a)_{\alpha\dot\alpha} \\ (\tilde\sigma^a)^{\dot\alpha\alpha} & 0 \end{bmatrix} \nabla_a \left((\xi^{\beta\dot\beta}\nabla_{\beta\dot\beta} + \xi^{\beta\gamma}M_{\beta\gamma} + \xi^{\dot\beta\dot\gamma}\overline{M}_{\dot\beta\dot\gamma} + \xi) \begin{bmatrix} \psi_\alpha \\ \bar\chi^{\dot\alpha} \end{bmatrix} \right) \\
&= \begin{bmatrix} \nabla_{\alpha\dot\alpha}((\xi^{\beta\dot\beta}\nabla_{\beta\dot\beta} + \xi^{\beta\gamma}M_{\beta\gamma} + \xi^{\dot\beta\dot\gamma}\overline{M}_{\dot\beta\dot\gamma} + \xi)\bar\chi^{\dot\alpha}) \\ \nabla^{\alpha\dot\alpha}((\xi^{\beta\dot\beta}\nabla_{\beta\dot\beta} + \xi^{\beta\gamma}M_{\beta\gamma} + \xi^{\dot\beta\dot\gamma}\overline{M}_{\dot\beta\dot\gamma} + \xi)\psi_\alpha) \end{bmatrix} \\
&= \begin{bmatrix} \nabla_{\alpha\dot\alpha}((\xi^{\beta\dot\beta}\nabla_{\beta\dot\beta} + \xi^{\dot\beta\dot\gamma}\overline{M}_{\dot\beta\dot\gamma} + \xi)\bar\chi^{\dot\alpha}) \\ \nabla^{\alpha\dot\alpha}((\xi^{\beta\dot\beta}\nabla_{\beta\dot\beta} + \xi^{\beta\gamma}M_{\beta\gamma} + \xi)\psi_\alpha) \end{bmatrix} \\
&= \begin{bmatrix} \nabla_{\alpha\dot\alpha}(\xi^{\beta\dot\beta}\nabla_{\beta\dot\beta}\bar\chi^{\dot\alpha} + \tfrac{1}{2}\xi^{\dot\beta\dot\gamma}(\delta^{\dot\alpha}{}_{\dot\beta}\bar\chi_{\dot\gamma} + \delta^{\dot\alpha}{}_{\dot\gamma}\bar\chi_{\dot\beta}) + \xi\bar\chi^{\dot\alpha}) \\ \nabla^{\alpha\dot\alpha}(\xi^{\beta\dot\beta}\nabla_{\beta\dot\beta}\psi_\alpha + \tfrac{1}{2}\xi^{\beta\gamma}(\varepsilon_{\alpha\beta}\psi_\gamma + \varepsilon_{\alpha\gamma}\psi_\beta) + \xi\psi_\alpha) \end{bmatrix} \\
&= \begin{bmatrix} \nabla_{\alpha\dot\alpha}(\xi^{\beta\dot\beta}\nabla_{\beta\dot\beta}\bar\chi^{\dot\alpha} + \xi^{\dot\alpha\dot\beta}\bar\chi_{\dot\beta} + \xi\bar\chi^{\dot\alpha}) \\ \nabla^{\alpha\dot\alpha}(\xi^{\beta\dot\beta}\nabla_{\beta\dot\beta}\psi_\alpha + \xi_\alpha{}^\beta\psi_\beta + \xi\psi_\alpha) \end{bmatrix} \\
&= \begin{bmatrix} \nabla_{\alpha\dot\alpha}(\xi^{\beta\dot\beta})\nabla_{\beta\dot\beta}\bar\chi^{\dot\alpha} + \xi^{\beta\dot\beta}\nabla_{\alpha\dot\alpha}\nabla_{\beta\dot\beta}\bar\chi^{\dot\alpha} + \nabla_{\alpha\dot\alpha}(\xi^{\dot\alpha\dot\beta})\bar\chi_{\dot\beta} + \xi^{\dot\alpha\dot\beta}\nabla_{\alpha\dot\alpha}(\bar\chi_{\dot\beta}) + \nabla_{\alpha\dot\alpha}(\xi)\bar\chi^{\dot\alpha} \\ \nabla^{\alpha\dot\alpha}(\xi^{\beta\dot\beta})\nabla_{\beta\dot\beta}\psi_\alpha + \xi^{\beta\dot\beta}\nabla^{\alpha\dot\alpha}\nabla_{\beta\dot\beta}\psi_\alpha + \nabla^{\alpha\dot\alpha}(\xi_\alpha{}^\beta)\psi_\beta + \xi_\alpha{}^\beta\nabla^{\alpha\dot\alpha}\psi_\beta + \nabla^{\alpha\dot\alpha}(\xi)\psi_\alpha \end{bmatrix} ,
\end{aligned} \tag{4.49}
$$

where I have used $\nabla_{\alpha\dot\alpha}\bar\chi^{\dot\alpha} = 0$, $\nabla^{\alpha\dot\alpha}\psi_\alpha = 0$, $M_{\beta\gamma}\bar\chi^{\dot\alpha} = 0$ and $\overline{M}_{\dot\beta\dot\gamma}\psi_\alpha = 0$. In the expression above,

$$
\begin{aligned}
\xi^{\beta\dot\beta}\nabla_{\alpha\dot\alpha}\nabla_{\beta\dot\beta}\bar\chi^{\dot\alpha} &= \xi^{\beta\dot\beta}\nabla_{\beta\dot\beta}\nabla_{\alpha\dot\alpha}\bar\chi^{\dot\alpha} + \xi^{\beta\dot\beta}[\nabla_{\alpha\dot\alpha}, \nabla_{\beta\dot\beta}]\bar\chi^{\dot\alpha} \\
&= 0 + \xi^{\beta\dot\beta}(R_{\alpha\dot\alpha\beta\dot\beta}{}^{\mu\nu}M_{\mu\nu}\bar\chi^{\dot\alpha} + \bar R_{\alpha\dot\alpha\beta\dot\beta}{}^{\dot\mu\dot\nu}\overline{M}_{\dot\mu\dot\nu}\bar\chi^{\dot\alpha}) \\
&= \xi^{\beta\dot\beta}\bar R_{\alpha\dot\alpha\beta\dot\beta}{}^{\dot\alpha\dot\gamma}\bar\chi_{\dot\gamma} \\
&= \xi^{\beta\dot\beta}(\varepsilon_{\alpha\beta}\bar C_{\dot\alpha\dot\beta}{}^{\dot\alpha\dot\gamma} + \varepsilon_{\dot\alpha\dot\beta}E_{\alpha\beta}{}^{\dot\alpha\dot\gamma} + \varepsilon_{\alpha\beta}(\delta^{\dot\alpha}{}_{\dot\alpha}\delta^{\dot\gamma}{}_{\dot\beta} + \delta^{\dot\gamma}{}_{\dot\alpha}\delta^{\dot\alpha}{}_{\dot\beta})F)\bar\chi_{\dot\gamma} \\
&= \xi^{\beta\dot\beta}E_{\alpha\beta\dot\alpha\dot\beta}\bar\chi^{\dot\alpha} - 3F\xi_{\alpha\dot\alpha}\bar\chi^{\dot\alpha} \,.
\end{aligned} \tag{4.50}
$$

$$
\begin{aligned}
\xi^{\beta\dot\beta}\nabla^{\alpha\dot\alpha}\nabla_{\beta\dot\beta}\psi_\alpha &= \xi^{\beta\dot\beta}\nabla_{\beta\dot\beta}\nabla^{\alpha\dot\alpha}\psi_\alpha + \xi^{\beta\dot\beta}[\nabla^{\alpha\dot\alpha}, \nabla_{\beta\dot\beta}]\psi_\alpha \\
&= 0 + \xi^{\beta\dot\beta}(R^{\alpha\dot\alpha}{}_{\beta\dot\beta}{}^{\mu\nu}M_{\mu\nu}\psi_\alpha + \bar R^{\alpha\dot\alpha}{}_{\beta\dot\beta}{}^{\dot\mu\dot\nu}\overline{M}_{\dot\mu\dot\nu}\psi_\alpha) \\
&= \xi^{\beta\dot\beta}R^{\alpha\dot\alpha}{}_{\beta\dot\beta\alpha}{}^{\gamma}\psi_\gamma \\
&= \xi^{\beta\dot\beta}(\delta^{\dot\alpha}{}_{\dot\beta}C^{\alpha}{}_{\beta\alpha}{}^{\gamma} + \delta^\alpha{}_\beta E_\alpha{}^{\gamma\dot\alpha}{}_{\dot\beta} + \delta^{\dot\alpha}{}_{\dot\beta}(-\delta^\alpha{}_\alpha\delta^\gamma{}_\beta + \varepsilon^{\gamma\alpha}\varepsilon_{\beta\alpha})F)\psi_\gamma \\
&= \xi_{\beta\dot\beta}E^{\alpha\beta\dot\alpha\dot\beta}\psi_\alpha - 3F\xi^{\alpha\dot\alpha}\psi_\alpha
\end{aligned} \tag{4.51}
$$

Further simplifications can be made in equation 4.49 using equation 3.28.

$$
\begin{aligned}
&\nabla^{\alpha\dot\alpha}(\xi^{\beta\dot\beta})\nabla_{\beta\dot\beta}\psi_\alpha \\
&= \left(\nabla^{(\alpha(\dot\alpha}(\xi^{\beta)\dot\beta)}) + \frac{1}{2}\varepsilon^{\alpha\beta}\nabla_\gamma{}^{(\dot\alpha}\xi^{\gamma\dot\beta)} + \frac{1}{2}\varepsilon^{\dot\alpha\dot\beta}\nabla^{(\alpha}{}_{\dot\gamma}\xi^{\beta)\dot\gamma} + \frac{1}{4}\varepsilon^{\alpha\beta}\varepsilon^{\dot\alpha\dot\beta}\nabla_{\gamma\dot\gamma}\xi^{\gamma\dot\gamma} \right) \nabla_{\beta\dot\beta}\psi_\alpha \\
&= \nabla^{(\alpha(\dot\alpha}(\xi^{\beta)\dot\beta)})\nabla_{\beta\dot\beta}\psi_\alpha + \frac{1}{2}\nabla^{(\alpha}{}_{\dot\gamma}\xi^{\beta)\dot\gamma}\nabla_\beta{}^{\dot\alpha}\psi_\alpha
\end{aligned} \tag{4.52}
$$

as $\varepsilon^{\alpha\beta}\nabla_{\beta\dot\beta}\psi_\alpha = \nabla^\alpha{}_{\dot\beta}\psi_\alpha = 0$. Similarly,

$$\begin{aligned}
&\nabla_{\alpha\dot\alpha}(\xi^{\beta\dot\beta})\nabla_{\beta\dot\beta}\bar\chi^{\dot\alpha} \\
&= \nabla_{\alpha\dot\alpha}(\xi_{\beta\dot\beta})\nabla^{\beta\dot\beta}\bar\chi^{\dot\alpha} \\
&= \left(\nabla_{(\alpha(\dot\alpha}(\xi_{\beta)\dot\beta)}) + \frac{1}{2}\varepsilon_{\alpha\beta}\nabla^\gamma{}_{(\dot\alpha}\xi_{\gamma\dot\beta)} + \frac{1}{2}\varepsilon_{\dot\alpha\dot\beta}\nabla_{(\alpha}{}^{\dot\gamma}\xi_{\beta)\dot\gamma} + \frac{1}{4}\varepsilon_{\alpha\beta}\varepsilon_{\dot\alpha\dot\beta}\nabla_{\gamma\dot\gamma}\xi^{\gamma\dot\gamma}\right)\nabla^{\beta\dot\beta}\bar\chi^{\dot\alpha} \\
&= \nabla_{(\alpha(\dot\alpha}(\xi_{\beta)\dot\beta)})\nabla^{\beta\dot\beta}\bar\chi^{\dot\alpha} + \frac{1}{2}\nabla^\gamma{}_{(\dot\alpha}\xi_{\gamma\dot\beta)}\nabla_\alpha{}^{\dot\beta}\bar\chi^{\dot\alpha}\,.
\end{aligned} \tag{4.53}$$

Plugging all this back into equation 4.49,

$$\begin{aligned}
\gamma^a\nabla_a D^{(1)}\mathbf{\Psi} &= \Big[\nabla_{(\alpha(\dot\alpha}(\xi_{\beta)\dot\beta)})\nabla^{\beta\dot\beta}\bar\chi^{\dot\alpha} + \frac{1}{2}\nabla^\gamma{}_{(\dot\alpha}\xi_{\gamma\dot\beta)}\nabla_\alpha{}^{\dot\beta}\bar\chi^{\dot\alpha} + \xi^{\beta\dot\beta}E_{\alpha\beta\dot\alpha\dot\beta}\bar\chi^{\dot\alpha} - 3F\xi_{\alpha\dot\alpha}\bar\chi^{\dot\alpha} \\
&\quad + \nabla_{\alpha\dot\alpha}(\xi^{\dot\alpha\dot\beta})\bar\chi_{\dot\beta} + \xi^{\dot\alpha\dot\beta}\nabla_{\alpha\dot\alpha}(\bar\chi_{\dot\beta}) + \nabla_{\alpha\dot\alpha}(\xi)\bar\chi^{\dot\alpha}, \\
&\quad \nabla^{(\alpha(\dot\alpha}(\xi^{\beta)\dot\beta)})\nabla_{\beta\dot\beta}\psi_\alpha + \frac{1}{2}\nabla^{(\alpha}{}_{\dot\gamma}\xi^{\beta)\dot\gamma}\nabla_\beta{}^{\dot\alpha}\psi_\alpha + \xi_{\beta\dot\beta}E^{\alpha\beta\dot\alpha\dot\beta}\psi_\alpha - 3F\xi^{\alpha\dot\alpha}\psi_\alpha \\
&\quad + \nabla^{\alpha\dot\alpha}(\xi_\alpha{}^\beta)\psi_\beta + \xi_\alpha{}^\beta\nabla^{\alpha\dot\alpha}\psi_\beta + \nabla^{\alpha\dot\alpha}(\xi)\psi_\alpha\Big]^T \\
&= \Big[\left(\nabla_{(\alpha(\dot\alpha}(\xi_{\beta)\dot\beta)}) + \varepsilon_{\alpha\beta}\left(\frac{1}{2}\nabla^\gamma{}_{(\dot\alpha}\xi_{\gamma\dot\beta)} + \xi_{\dot\alpha\dot\beta}\right)\right)\nabla^{\beta\dot\beta}\bar\chi^{\dot\alpha} \\
&\quad + (\nabla_{\alpha\dot\alpha}(\xi) + \nabla_\alpha{}^{\dot\beta}(\xi_{\dot\alpha\dot\beta}) + \xi^{\beta\dot\beta}E_{\alpha\beta\dot\alpha\dot\beta} - 3F\xi_{\alpha\dot\alpha})\bar\chi^{\dot\alpha}, \\
&\quad \left(\nabla^{(\alpha(\dot\alpha}(\xi^{\beta)\dot\beta)}) + \varepsilon^{\dot\alpha\dot\beta}\left(\frac{1}{2}\nabla^{(\alpha}{}_{\dot\gamma}\xi^{\beta)\dot\gamma} - \xi^{\alpha\beta}\right)\right)\nabla_{\beta\dot\beta}\psi_\alpha \\
&\quad + (\nabla^{\alpha\dot\alpha}(\xi) - \nabla_\beta{}^{\dot\alpha}(\xi^{\alpha\beta}) + \xi_{\beta\dot\beta}E^{\alpha\beta\dot\alpha\dot\beta} - 3F\xi^{\alpha\dot\alpha})\psi_\alpha\Big]^T.
\end{aligned} \tag{4.54}$$

The $\gamma^a\nabla_a\mathbf{\Psi} = 0$ property can no longer be used to simplify the expression because the coefficients of $\nabla^{\beta\dot\beta}\bar\chi^{\dot\alpha}$ and $\nabla_{\beta\dot\beta}\psi_\alpha$ are symmetric in $\dot\alpha$ & $\dot\beta$ and α & β respectively. However, for $D^{(1)}$ to be a symmetry, $\gamma^a\nabla_a D^{(1)}\mathbf{\Psi}$ must equal zero. Hence, given $\mathbf{\Psi}$ is an arbitrary solution, the only way to get $\gamma^a\nabla_a D^{(1)}\mathbf{\Psi} = 0$ is to get

$$\begin{aligned}
0 &= \nabla_{(\alpha(\dot\alpha}\xi_{\beta)\dot\beta)} + \varepsilon_{\alpha\beta}\left(\frac{1}{2}\nabla^\gamma{}_{(\dot\alpha}\xi_{\gamma\dot\beta)} + \xi_{\dot\alpha\dot\beta}\right), \\
0 &= \nabla_{\alpha\dot\alpha}(\xi) + \nabla_\alpha{}^{\dot\beta}(\xi_{\dot\alpha\dot\beta}) + \xi^{\beta\dot\beta}E_{\alpha\beta\dot\alpha\dot\beta} - 3F\xi_{\alpha\dot\alpha}, \\
0 &= \nabla^{(\alpha(\dot\alpha}\xi^{\beta)\dot\beta)} + \varepsilon^{\dot\alpha\dot\beta}\left(\frac{1}{2}\nabla^{(\alpha}{}_{\dot\gamma}\xi^{\beta)\dot\gamma} - \xi^{\alpha\beta}\right) \quad \text{and} \\
0 &= \nabla^{\alpha\dot\alpha}(\xi) - \nabla_\beta{}^{\dot\alpha}(\xi^{\alpha\beta}) + \xi_{\beta\dot\beta}E^{\alpha\beta\dot\alpha\dot\beta} - 3F\xi^{\alpha\dot\alpha}\,.
\end{aligned} \tag{4.55}$$

Notice that in the 1st and 3rd of these equations, the term with the ε coefficient cannot cancel out the other term because the other term is symmetric in the indices of the ε. Therefore,

$$\begin{aligned}
0 = \nabla_{(\alpha(\dot\alpha}\xi_{\beta)\dot\beta)} &\iff 0 = \nabla^{(\alpha(\dot\alpha}\xi^{\beta)\dot\beta)}\,, \\
0 = \frac{1}{2}\nabla^\gamma{}_{(\dot\alpha}\xi_{\gamma\dot\beta)} + \xi_{\dot\alpha\dot\beta} &\iff \xi^{\dot\alpha\dot\beta} = \frac{1}{2}\nabla_\gamma{}^{(\dot\alpha}\xi^{\gamma\dot\beta)} \quad \text{and} \\
0 = \frac{1}{2}\nabla^{(\alpha}{}_{\dot\gamma}\xi^{\beta)\dot\gamma} - \xi^{\alpha\beta} &\iff \xi^{\alpha\beta} = \frac{1}{2}\nabla^{(\alpha}{}_{\dot\gamma}\xi^{\beta)\dot\gamma}\,.
\end{aligned} \tag{4.56}$$

The first equation implies $\xi^{\alpha\dot\alpha}$ is conformal Killing by corollary 3.3.1 and the latter two equations are two of the relations to be proven. □

Corollary 4.4.1. *Given $\xi^{\alpha\dot{\alpha}}$, $D^{(1)}$ is unique up to the addition of a constant.*

Proof. In $D^{(1)}$, given $\xi^{\alpha\dot{\alpha}}$, $\xi^{\alpha\beta}$ and $\xi^{\dot{\alpha}\dot{\beta}}$ are both determined by the lemma. Hence, only ξ remains to be found. Let $\xi = A$ and $\xi = B$ be two solutions. By equation 4.54, to be a symmetry,

$$0 = \begin{bmatrix} (\nabla_{\alpha\dot{\alpha}}(\xi) + \nabla_{\alpha}{}^{\dot{\beta}}(\xi_{\dot{\alpha}\dot{\beta}}) + \xi^{\beta\dot{\beta}}E_{\alpha\beta\dot{\alpha}\dot{\beta}} - 3F\xi_{\alpha\dot{\alpha}})\bar{\chi}^{\dot{\alpha}} \\ (\nabla^{\alpha\dot{\alpha}}(\xi) - \nabla_{\beta}{}^{\dot{\alpha}}(\xi^{\alpha\beta}) + \xi_{\beta\dot{\beta}}E^{\alpha\beta\dot{\alpha}\dot{\beta}} - 3F\xi^{\alpha\dot{\alpha}})\psi_{\alpha} \end{bmatrix} . \tag{4.57}$$

Then, since $\mathbf{\Psi}$ is an arbitrary solution,

$$\begin{aligned} 0 &= \nabla_{\alpha\dot{\alpha}}(\xi) + \nabla_{\alpha}{}^{\dot{\beta}}(\xi_{\dot{\alpha}\dot{\beta}}) + \xi^{\beta\dot{\beta}}E_{\alpha\beta\dot{\alpha}\dot{\beta}} - 3F\xi_{\alpha\dot{\alpha}} \quad \text{and} \\ 0 &= \nabla^{\alpha\dot{\alpha}}(\xi) - \nabla_{\beta}{}^{\dot{\alpha}}(\xi^{\alpha\beta}) + \xi_{\beta\dot{\beta}}E^{\alpha\beta\dot{\alpha}\dot{\beta}} - 3F\xi^{\alpha\dot{\alpha}} . \end{aligned} \tag{4.58}$$

It will suffice to consider either one of these equations alone; I will choose the first. Then,

$$\begin{aligned} &\nabla_{\alpha\dot{\alpha}}(A) = \nabla_{\alpha\dot{\alpha}}(B) = -\nabla_{\alpha}{}^{\dot{\beta}}(\xi_{\dot{\alpha}\dot{\beta}}) - \xi^{\beta\dot{\beta}}E_{\alpha\beta\dot{\alpha}\dot{\beta}} + 3F\xi_{\alpha\dot{\alpha}} \\ \implies &\nabla_{\alpha\dot{\alpha}}(A - B) = 0 \\ \implies &\partial_{\alpha\dot{\alpha}}(A - B) = 0 \quad \text{as } A \text{ and } B \text{ are scalars.} \end{aligned} \tag{4.59}$$

Therefore, $A - B$ is a constant.
Hence, any two solutions for ξ differ at most by a constant $\implies$ given $\xi^{\alpha\dot{\alpha}}$, $D^{(1)}$ is determined up to a constant. ☐

The lemma means the only term left to constrain in $D^{(1)}$ is ξ. As in the previous chapter, ξ can be determined by ensuring $D^{(1)}\mathbf{\Psi}$ transforms the same way as $\mathbf{\Psi}$ under Weyl transformations. By the exact same reasoning as in the paragraph following equation 3.58, the only physically admissible ansatz for ξ is $A\nabla_{\alpha\dot{\alpha}}\xi^{\alpha\dot{\alpha}}$ for a constant, $A \in \mathbb{R}$.

Lemma 4.5. *In $D^{(1)}$, if $\xi = A\nabla_{\alpha\dot{\alpha}}\xi^{\alpha\dot{\alpha}}$, then $A = 3/8$ to get $D'^{(1)}\mathbf{\Psi}' = \mathrm{e}^{3\sigma/2}D^{(1)}\mathbf{\Psi}$ under a Weyl transformation.*

Proof. Under a Weyl transformation,

$$\nabla'_a = (1+\sigma)\nabla_a - \nabla^b(\sigma)M_{ab} . \tag{4.60}$$

Translating to spinors,

$$\begin{aligned} \nabla'_{\alpha\dot{\alpha}} &= (1+\sigma)\nabla_{\alpha\dot{\alpha}} + \frac{1}{2}(\sigma^a)_{\alpha\dot{\alpha}}(\tilde{\sigma}^b)^{\dot{\beta}\beta}\nabla_{\beta\dot{\beta}}(\sigma)((\sigma_{ab})^{\mu\nu}M_{\mu\nu} - (\tilde{\sigma}_{ab})^{\dot{\mu}\dot{\nu}}\overline{M}_{\dot{\mu}\dot{\nu}}) \\ &= (1+\sigma)\nabla_{\alpha\dot{\alpha}} - \frac{1}{8}(\sigma^a)_{\alpha\dot{\alpha}}(\tilde{\sigma}^b)^{\dot{\beta}\beta}\nabla_{\beta\dot{\beta}}(\sigma)(\varepsilon^{\mu\gamma}((\sigma_a)_{\gamma\dot{\gamma}}(\tilde{\sigma}_b)^{\dot{\gamma}\nu} - (\sigma_b)_{\gamma\dot{\gamma}}(\tilde{\sigma}_a)^{\dot{\gamma}\nu})M_{\mu\nu} \\ &\quad - \varepsilon^{\dot{\nu}\dot{\gamma}}((\tilde{\sigma}_a)^{\dot{\mu}\gamma}(\sigma_b)_{\gamma\dot{\gamma}} - (\tilde{\sigma}_b)^{\dot{\mu}\gamma}(\sigma_a)_{\gamma\dot{\gamma}})\overline{M}_{\dot{\mu}\dot{\nu}}) \\ &= (1+\sigma)\nabla_{\alpha\dot{\alpha}} - \frac{1}{2}\nabla_{\beta\dot{\beta}}(\sigma)(\varepsilon^{\mu\gamma}(\varepsilon_{\alpha\gamma}\varepsilon_{\dot{\alpha}\dot{\gamma}}\varepsilon^{\nu\beta}\varepsilon^{\dot{\gamma}\dot{\beta}} - \delta^{\dot{\gamma}}{}_{\dot{\alpha}}\delta^{\nu}{}_{\alpha}\delta^{\dot{\beta}}{}_{\dot{\gamma}}\delta^{\beta}{}_{\gamma})M_{\mu\nu} \\ &\quad - \varepsilon^{\dot{\nu}\dot{\gamma}}(\delta^{\gamma}{}_{\alpha}\delta^{\dot{\mu}}{}_{\dot{\alpha}}\delta^{\beta}{}_{\gamma}\delta^{\dot{\beta}}{}_{\dot{\gamma}} - \varepsilon_{\alpha\gamma}\varepsilon_{\dot{\alpha}\dot{\gamma}}\varepsilon^{\beta\gamma}\varepsilon^{\dot{\beta}\dot{\mu}})\overline{M}_{\dot{\mu}\dot{\nu}}) \\ &= (1+\sigma)\nabla_{\alpha\dot{\alpha}} + \nabla^{\beta}{}_{\dot{\alpha}}(\sigma)M_{\alpha\beta} + \nabla_{\alpha}{}^{\dot{\beta}}(\sigma)\overline{M}_{\dot{\alpha}\dot{\beta}} . \end{aligned} \tag{4.61}$$

Next, since $\xi^{\alpha\dot{\alpha}}$ is conformal Killing, by lemma 3.5, $\xi'^{\alpha\dot{\alpha}} = (1-\sigma)\xi^{\alpha\dot{\alpha}}$. Thus,

$$\begin{aligned} \xi'^{\alpha\dot{\alpha}}\nabla'_{\alpha\dot{\alpha}} &= (1-\sigma)\xi^{\alpha\dot{\alpha}}((1+\sigma)\nabla_{\alpha\dot{\alpha}} + \nabla^{\beta}{}_{\dot{\alpha}}(\sigma)M_{\alpha\beta} + \nabla_{\alpha}{}^{\dot{\beta}}(\sigma)\overline{M}_{\dot{\alpha}\dot{\beta}}) \\ &= \xi^{\alpha\dot{\alpha}}\nabla_{\alpha\dot{\alpha}} + \xi^{\alpha\dot{\alpha}}\nabla^{\beta}{}_{\dot{\alpha}}(\sigma)M_{\alpha\beta} + \xi^{\alpha\dot{\alpha}}\nabla_{\alpha}{}^{\dot{\beta}}(\sigma)\overline{M}_{\dot{\alpha}\dot{\beta}} . \end{aligned} \tag{4.62}$$

$$\begin{aligned}\nabla'_{\alpha\dot\alpha}\xi'^{\alpha\dot\alpha} &= ((1+\sigma)\nabla_{\alpha\dot\alpha} + \nabla^{\beta}{}_{\dot\alpha}(\sigma)M_{\alpha\beta} + \nabla_{\alpha}{}^{\dot\beta}(\sigma)\overline{M}_{\dot\alpha\dot\beta})((1-\sigma)\xi^{\alpha\dot\alpha}) \\ &= \nabla_{\alpha\dot\alpha}\xi^{\alpha\dot\alpha} - \xi^{\alpha\dot\alpha}\nabla_{\alpha\dot\alpha}(\sigma) + \frac{1}{2}\nabla^{\beta}{}_{\dot\alpha}(\sigma)(\delta^{\alpha}{}_{\alpha}\xi_{\beta}{}^{\dot\alpha} + \delta^{\alpha}{}_{\beta}\xi_{\alpha}{}^{\dot\alpha}) \\ &\quad + \frac{1}{2}\nabla_{\alpha}{}^{\dot\beta}(\sigma)(\delta^{\dot\alpha}{}_{\dot\alpha}\xi^{\alpha}{}_{\dot\beta} + \delta^{\dot\alpha}{}_{\dot\beta}\xi^{\alpha}{}_{\dot\alpha}) \\ &= \nabla_{\alpha\dot\alpha}\xi^{\alpha\dot\alpha} - 4\xi^{\alpha\dot\alpha}\nabla_{\alpha\dot\alpha}(\sigma) \end{aligned} \tag{4.63}$$

Then, by lemma 4.4,

$$\begin{aligned}\xi'^{\alpha\beta} &= \frac{1}{2}\nabla'^{(\alpha}{}_{\dot\alpha}\xi'^{\beta)\dot\alpha} \\ &= \frac{1}{2}((1+\sigma)\nabla^{(\alpha}{}_{\dot\alpha} + \nabla^{\gamma}{}_{\dot\alpha}(\sigma)M^{(\alpha}{}_{\gamma} + \nabla^{(\alpha\dot\beta}(\sigma)\overline{M}_{\dot\alpha\dot\beta})((1-\sigma)\xi^{\beta)\dot\alpha}) \\ &= \xi^{\alpha\beta} - \frac{1}{2}\nabla^{(\alpha}{}_{\dot\alpha}(\sigma)\xi^{\beta)\dot\alpha} + \frac{1}{4}\nabla^{\gamma}{}_{\dot\alpha}(\sigma)(\varepsilon^{(\alpha\beta)}\xi_{\gamma}{}^{\dot\alpha} + \delta^{(\beta}{}_{\gamma}\xi^{\alpha)\dot\alpha}) \\ &\quad + \frac{1}{4}\nabla^{(\alpha\dot\beta}(\sigma)(\delta^{\dot\alpha}{}_{\dot\alpha}\xi^{\beta)}{}_{\dot\beta} + \delta^{\dot\alpha}{}_{\dot\beta}\xi^{\beta)}{}_{\dot\alpha}) \\ &= \xi^{\alpha\beta} - \nabla^{(\alpha}{}_{\dot\alpha}(\sigma)\xi^{\beta)\dot\alpha}\,. \end{aligned} \tag{4.64}$$

$$\begin{aligned}\xi'^{\dot\alpha\dot\beta} &= \frac{1}{2}\nabla'_{\alpha}{}^{(\dot\alpha}\xi'^{\alpha\dot\beta)} \\ &= \frac{1}{2}((1+\sigma)\nabla_{\alpha}{}^{(\dot\alpha} + \nabla^{\beta(\dot\alpha}(\sigma)M_{\alpha\beta} + \nabla_{\alpha}{}^{\dot\gamma}(\sigma)\overline{M}^{(\dot\alpha}{}_{\dot\gamma})((1-\sigma)\xi^{\alpha\dot\beta)}) \\ &= \xi^{\dot\alpha\dot\beta} - \frac{1}{2}\nabla_{\alpha}{}^{(\dot\alpha}(\sigma)\xi'^{\alpha\dot\beta)} + \frac{1}{4}\nabla^{\beta(\dot\alpha}(\sigma)(\delta^{\alpha}{}_{\alpha}\xi_{\beta}{}^{\dot\beta)} + \delta^{\alpha}{}_{\beta}\xi_{\alpha}{}^{\dot\beta)}) \\ &\quad + \frac{1}{4}\nabla_{\alpha}{}^{\dot\gamma}(\sigma)(\varepsilon^{(\dot\alpha\dot\beta)}\xi^{\alpha}{}_{\dot\gamma} + \delta^{(\dot\beta}{}_{\dot\gamma}\xi^{\alpha\dot\alpha)}) \\ &= \xi^{\dot\alpha\dot\beta} - \nabla_{\alpha}{}^{(\dot\alpha}(\sigma)\xi'^{\alpha\dot\beta)} \end{aligned} \tag{4.65}$$

Finally, putting all of these pieces together,

$$\begin{aligned} &D'^{(1)}\mathbf{\Psi}' \\ &= (\xi^{\alpha\dot\alpha}\nabla_{\alpha\dot\alpha} + \xi^{\alpha\dot\alpha}\nabla^{\beta}{}_{\dot\alpha}(\sigma)M_{\alpha\beta} + \xi^{\alpha\dot\alpha}\nabla_{\alpha}{}^{\dot\beta}(\sigma)\overline{M}_{\dot\alpha\dot\beta} + (\xi^{\alpha\beta} - \nabla^{(\alpha}{}_{\dot\alpha}(\sigma)\xi^{\beta)\dot\alpha})M_{\alpha\beta} \\ &\quad + (\xi^{\dot\alpha\dot\beta} - \nabla_{\alpha}{}^{(\dot\alpha}(\sigma)\xi'^{\alpha\dot\beta)})\overline{M}_{\dot\alpha\dot\beta} + A\nabla_{\alpha\dot\alpha}(\xi^{\alpha\dot\alpha}) - 4A\xi^{\alpha\dot\alpha}\nabla_{\alpha\dot\alpha}(\sigma))\left(\left(1+\frac{3}{2}\sigma\right)\mathbf{\Psi}\right) \\ &= \left(1+\frac{3}{2}\sigma\right)D^{(1)}\mathbf{\Psi} + \frac{3}{2}\xi^{\alpha\dot\alpha}\nabla_{\alpha\dot\alpha}(\sigma)\mathbf{\Psi} + \xi^{\alpha\dot\alpha}\nabla^{\beta}{}_{\dot\alpha}(\sigma)M_{\alpha\beta}\mathbf{\Psi} + \xi^{\alpha\dot\alpha}\nabla_{\alpha}{}^{\dot\beta}(\sigma)\overline{M}_{\dot\alpha\dot\beta}\mathbf{\Psi} \\ &\quad - \nabla^{(\alpha}{}_{\dot\alpha}(\sigma)\xi^{\beta)\dot\alpha}M_{\alpha\beta}\mathbf{\Psi} - \nabla_{\alpha}{}^{(\dot\alpha}(\sigma)\xi'^{\alpha\dot\beta)}\overline{M}_{\dot\alpha\dot\beta}\mathbf{\Psi} - 4A\xi^{\alpha\dot\alpha}\nabla_{\alpha\dot\alpha}(\sigma)\mathbf{\Psi} \\ &= \left(1+\frac{3}{2}\sigma\right)D^{(1)}\mathbf{\Psi} + \left(\frac{3}{2}-4A\right)\xi^{\alpha\dot\alpha}\nabla_{\alpha\dot\alpha}(\sigma)\mathbf{\Psi} + \xi^{\alpha\dot\alpha}\nabla^{\beta}{}_{\dot\alpha}(\sigma)M_{\alpha\beta}\mathbf{\Psi} + \xi^{\alpha\dot\alpha}\nabla_{\alpha}{}^{\dot\beta}(\sigma)\overline{M}_{\dot\alpha\dot\beta}\mathbf{\Psi} \\ &\quad - \nabla^{\beta}{}_{\dot\alpha}(\sigma)\xi^{\alpha\dot\alpha}M_{\alpha\beta}\mathbf{\Psi} - \nabla_{\alpha}{}^{\dot\beta}(\sigma)\xi'^{\alpha\dot\alpha}\overline{M}_{\dot\alpha\dot\beta}\mathbf{\Psi} \\ &= \left(1+\frac{3}{2}\sigma\right)D^{(1)}\mathbf{\Psi} + \left(\frac{3}{2}-4A\right)\xi^{\alpha\dot\alpha}\nabla_{\alpha\dot\alpha}(\sigma)\mathbf{\Psi}\,. \end{aligned} \tag{4.66}$$

Therefore, to get the required transformation, it must be that $3/2-4A=0 \iff A=3/8$. □

Hence, the most general 1st order symmetry of the massless Dirac operator is,

$$D^{(1)} = \xi^{\alpha\dot\alpha}\nabla_{\alpha\dot\alpha} + \frac{1}{2}\nabla^{(\alpha}{}_{\dot\alpha}\xi^{\beta)\dot\alpha}M_{\alpha\beta} + \frac{1}{2}\nabla_{\alpha}{}^{(\dot\alpha}\xi^{\alpha\dot\beta)}\overline{M}_{\dot\alpha\dot\beta} + \frac{3}{8}\nabla_{\alpha\dot\alpha}(\xi^{\alpha\dot\alpha}) \tag{4.67}$$

up to the addition of a constant.

Theorem 4.6. $D^{(1)}$ *is always a symmetry of the massless Dirac operator.*

Proof. Continuing from the proof of lemma 4.4, and in particular equation 4.54 with $\xi^{\alpha\beta} = \frac{1}{2}\nabla^{(\alpha}{}_{\dot\alpha}\xi^{\beta)\dot\alpha}$, $\xi^{\dot\alpha\dot\beta} = \frac{1}{2}\nabla_{\alpha}{}^{(\dot\alpha}\xi^{\alpha\dot\beta)}$, $\xi = \nabla_{\alpha\dot\alpha}\xi^{\alpha\dot\alpha}$ and $\xi^{\alpha\dot\alpha}$ conformal Killing,

$$\begin{aligned}\gamma^a\nabla_a D^{(1)}\mathbf{\Psi} &= \begin{bmatrix}(\nabla_{\alpha\dot\alpha}(\xi) + \nabla_{\alpha}{}^{\dot\beta}(\xi_{\dot\alpha\dot\beta}) + \xi^{\beta\dot\beta}E_{\alpha\beta\dot\alpha\dot\beta} - 3F\xi_{\alpha\dot\alpha})\bar\chi^{\dot\alpha} \\ (\nabla^{\alpha\dot\alpha}(\xi) - \nabla_{\beta}{}^{\dot\alpha}(\xi^{\alpha\beta}) + \xi_{\beta\dot\beta}E^{\alpha\beta\dot\alpha\dot\beta} - 3F\xi^{\alpha\dot\alpha})\psi_\alpha\end{bmatrix} \\ &= \begin{bmatrix}(\frac{3}{8}\nabla_{\alpha\dot\alpha}\nabla_{\beta\dot\beta}(\xi^{\beta\dot\beta}) - \frac{1}{2}\nabla_{\alpha}{}^{\dot\beta}\nabla^{\beta}{}_{(\dot\alpha}\xi_{\beta\dot\beta)} + \xi^{\beta\dot\beta}E_{\alpha\beta\dot\alpha\dot\beta} - 3F\xi_{\alpha\dot\alpha})\bar\chi^{\dot\alpha} \\ (\frac{3}{8}\nabla^{\alpha\dot\alpha}\nabla_{\beta\dot\beta}(\xi^{\beta\dot\beta}) - \frac{1}{2}\nabla_{\beta}{}^{\dot\alpha}\nabla^{(\alpha}{}_{\dot\beta}\xi^{\beta)\dot\beta} + \xi_{\beta\dot\beta}E^{\alpha\beta\dot\alpha\dot\beta} - 3F\xi^{\alpha\dot\alpha})\psi_\alpha\end{bmatrix}.\end{aligned} \tag{4.68}$$

Then, apply the conformal Killing equation and equation 3.28 along the way,

$$\begin{aligned}\nabla_{\alpha\dot\alpha}\nabla_{\beta\dot\beta}\xi^{\beta\dot\beta} &= \nabla^{\beta\dot\beta}\nabla_{\alpha\dot\alpha}\xi_{\beta\dot\beta} + [\nabla_{\alpha\dot\alpha}, \nabla_{\beta\dot\beta}]\xi^{\beta\dot\beta} \\ &= \nabla^{\beta\dot\beta}\left(\nabla_{(\alpha(\dot\alpha}\xi_{\beta)\dot\beta)} + \frac{1}{2}\varepsilon_{\dot\alpha\dot\beta}\nabla_{(\alpha}{}^{\dot\gamma}\xi_{\beta)\dot\gamma} + \frac{1}{2}\varepsilon_{\alpha\beta}\nabla^{\gamma}{}_{(\dot\alpha}\xi_{\gamma\dot\beta)} + \frac{1}{4}\varepsilon_{\alpha\beta}\varepsilon_{\dot\alpha\dot\beta}\nabla^{\gamma\dot\gamma}\xi_{\gamma\dot\gamma}\right) \\ &\quad + (R_{\alpha\dot\alpha\beta\dot\beta}{}^{\mu\nu}M_{\mu\nu} + \overline{R}_{\alpha\dot\alpha\beta\dot\beta}{}^{\dot\mu\dot\nu}M_{\dot\mu\dot\nu})\xi^{\beta\dot\beta} \\ &= \frac{1}{2}\nabla^{\beta}{}_{\dot\alpha}\nabla_{(\alpha}{}^{\dot\beta}\xi_{\beta)\dot\beta} + \frac{1}{2}\nabla_{\alpha}{}^{\dot\beta}\nabla^{\beta}{}_{(\dot\alpha}\xi_{\beta\dot\beta)} + \frac{1}{4}\nabla_{\alpha\dot\alpha}\nabla_{\beta\dot\beta}\xi^{\beta\dot\beta} \\ &\quad - R_{\alpha\dot\alpha\beta\dot\beta\mu}{}^{\beta}\xi^{\mu\dot\beta} - \overline{R}_{\alpha\dot\alpha\beta\dot\beta\dot\mu}{}^{\dot\beta}\xi^{\beta\dot\mu}.\end{aligned} \tag{4.69}$$

Rearranging,

$$\begin{aligned}\nabla_{\alpha\dot\alpha}\nabla_{\beta\dot\beta}\xi^{\beta\dot\beta} &= \frac{2}{3}\nabla^{\beta}{}_{\dot\alpha}\nabla_{(\alpha}{}^{\dot\beta}\xi_{\beta)\dot\beta} + \frac{2}{3}\nabla_{\alpha}{}^{\dot\beta}\nabla^{\beta}{}_{(\dot\alpha}\xi_{\beta\dot\beta)} \\ &\quad - \frac{4}{3}(\varepsilon_{\dot\alpha\dot\beta}C_{\alpha\beta\mu}{}^{\beta} + \varepsilon_{\alpha\beta}E_{\mu}{}^{\beta}{}_{\dot\alpha\dot\beta} + \varepsilon_{\dot\alpha\dot\beta}(-\varepsilon_{\alpha\mu}\delta^{\beta}{}_{\beta} - \delta^{\beta}{}_{\alpha}\varepsilon_{\beta\mu})F)\xi^{\mu\dot\beta} \\ &\quad - \frac{4}{3}(\varepsilon_{\alpha\beta}\overline{C}_{\dot\alpha\dot\beta\dot\mu}{}^{\dot\beta} + \varepsilon_{\dot\alpha\dot\beta}E_{\alpha\beta\dot\mu}{}^{\dot\beta} + \varepsilon_{\alpha\beta}(-\varepsilon_{\dot\alpha\dot\mu}\delta^{\dot\beta}{}_{\dot\beta} - \delta^{\dot\beta}{}_{\dot\alpha}\varepsilon_{\dot\beta\dot\mu})F)\xi^{\beta\dot\mu} \\ &= \frac{2}{3}\nabla^{\beta}{}_{\dot\alpha}\nabla_{(\alpha}{}^{\dot\beta}\xi_{\beta)\dot\beta} + \frac{2}{3}\nabla_{\alpha}{}^{\dot\beta}\nabla^{\beta}{}_{(\dot\alpha}\xi_{\beta\dot\beta)} - \frac{8}{3}\xi^{\beta\dot\beta}E_{\alpha\beta\dot\alpha\dot\beta} + 8F\xi_{\alpha\dot\alpha}.\end{aligned} \tag{4.70}$$

By raising the indices (to use in the bottom two components of equation 4.68),

$$\nabla^{\alpha\dot\alpha}\nabla_{\beta\dot\beta}\xi^{\beta\dot\beta} = \frac{2}{3}\nabla_{\beta}{}^{\dot\alpha}\nabla^{(\alpha}{}_{\dot\beta}\xi^{\beta)\dot\beta} + \frac{2}{3}\nabla^{\alpha}{}_{\dot\beta}\nabla_{\beta}{}^{(\dot\alpha}\xi^{\beta\dot\beta)} - \frac{8}{3}\xi_{\beta\dot\beta}E^{\alpha\beta\dot\alpha\dot\beta} + 8F\xi^{\alpha\dot\alpha}. \tag{4.71}$$

The preceding two equations can be simplified as follows.

$$\begin{aligned}\nabla^{\beta}{}_{\dot\alpha}\nabla_{(\alpha}{}^{\dot\beta}\xi_{\beta)\dot\beta} &= \nabla^{\beta}{}_{\dot\alpha}\nabla_{\alpha}{}^{\dot\beta}\xi_{\beta\dot\beta} - \frac{1}{2}\varepsilon_{\alpha\beta}\nabla^{\beta}{}_{\dot\alpha}\nabla^{\gamma\dot\beta}\xi_{\gamma\dot\beta} \\ &= \nabla_{\alpha}{}^{\dot\beta}\nabla^{\beta}{}_{\dot\alpha}\xi_{\beta\dot\beta} + [\nabla^{\beta}{}_{\dot\alpha}, \nabla_{\alpha}{}^{\dot\beta}]\xi_{\beta\dot\beta} - \frac{1}{2}\nabla_{\alpha\dot\alpha}\nabla_{\beta\dot\beta}\xi^{\beta\dot\beta} \\ &= \nabla_{\alpha}{}^{\dot\beta}\nabla^{\beta}{}_{(\dot\alpha}\xi_{\beta\dot\beta)} + \frac{1}{2}\varepsilon_{\dot\alpha\dot\beta}\nabla_{\alpha}{}^{\dot\beta}\nabla^{\beta\dot\gamma}\xi_{\beta\dot\gamma} - \frac{1}{2}\nabla_{\alpha\dot\alpha}\nabla_{\beta\dot\beta}\xi^{\beta\dot\beta} \\ &\quad + (R^{\beta}{}_{\dot\alpha\alpha}{}^{\dot\beta\mu\nu}M_{\mu\nu} + \overline{R}^{\beta}{}_{\dot\alpha\alpha}{}^{\dot\beta\dot\mu\dot\nu}\overline{M}_{\dot\mu\dot\nu})\xi_{\beta\dot\beta} \\ &= \nabla_{\alpha}{}^{\dot\beta}\nabla^{\beta}{}_{(\dot\alpha}\xi_{\beta\dot\beta)} + (R^{\beta}{}_{\dot\alpha\alpha}{}^{\dot\beta}{}_{\beta}{}^{\mu}\xi_{\mu\dot\beta} + \overline{R}^{\beta}{}_{\dot\alpha\alpha}{}^{\dot\beta}{}_{\dot\beta}{}^{\dot\mu}\xi_{\beta\dot\mu})\end{aligned} \tag{4.72}$$

Thus,

$$\begin{aligned}\nabla^{\beta}{}_{\dot\alpha}\nabla_{(\alpha}{}^{\dot\beta}\xi_{\beta)\dot\beta} &= \nabla_{\alpha}{}^{\dot\beta}\nabla^{\beta}{}_{(\dot\alpha}\xi_{\beta\dot\beta)} + (-\delta^{\dot\beta}{}_{\dot\alpha}C^{\beta}{}_{\alpha\beta}{}^{\mu} + \delta^{\beta}{}_{\alpha}E_{\beta}{}^{\mu}{}_{\dot\alpha}{}^{\dot\beta} - \delta^{\dot\beta}{}_{\dot\alpha}(-\delta^{\beta}{}_{\beta}\delta^{\mu}{}_{\alpha} + \varepsilon_{\alpha\beta}\varepsilon^{\mu\beta})F)\xi_{\mu\dot\beta} \\ &\quad + (\delta^{\beta}{}_{\alpha}\overline{C}_{\dot\alpha}{}^{\dot\beta}{}_{\dot\beta}{}^{\dot\mu} - \delta^{\dot\beta}{}_{\dot\alpha}E^{\beta}{}_{\alpha\dot\beta}{}^{\dot\mu} + \delta^{\beta}{}_{\alpha}(\varepsilon_{\dot\alpha\dot\beta}\varepsilon^{\dot\mu\dot\beta} - \delta^{\dot\mu}{}_{\dot\alpha}\delta^{\dot\beta}{}_{\dot\beta})F)\xi_{\beta\dot\mu} \\ &= \nabla_{\alpha}{}^{\dot\beta}\nabla^{\beta}{}_{(\dot\alpha}\xi_{\beta\dot\beta)}\end{aligned} \tag{4.73}$$

Hence,

$$\begin{aligned}\nabla_{\alpha\dot{\alpha}}\nabla_{\beta\dot{\beta}}\xi^{\beta\dot{\beta}} &= \frac{4}{3}\nabla_{\alpha}{}^{\dot{\beta}}\nabla^{\beta}{}_{(\dot{\alpha}}\xi_{\beta\dot{\beta})} - \frac{8}{3}\xi^{\beta\dot{\beta}}E_{\alpha\beta\dot{\alpha}\dot{\beta}} + 8F\xi_{\alpha\dot{\alpha}} \\ \iff \frac{3}{8}\nabla_{\alpha\dot{\alpha}}\nabla_{\beta\dot{\beta}}\xi^{\beta\dot{\beta}} &= \frac{1}{2}\nabla_{\alpha}{}^{\dot{\beta}}\nabla^{\beta}{}_{(\dot{\alpha}}\xi_{\beta\dot{\beta})} - \xi^{\beta\dot{\beta}}E_{\alpha\beta\dot{\alpha}\dot{\beta}} + 3F\xi_{\alpha\dot{\alpha}} \,. \end{aligned} \tag{4.74}$$

The previous equation combined with equation 4.73 immediately leads to

$$\frac{3}{8}\nabla^{\alpha\dot{\alpha}}\nabla_{\beta\dot{\beta}}\xi^{\beta\dot{\beta}} = \frac{1}{2}\nabla_{\beta}{}^{\dot{\alpha}}\nabla^{(\alpha}{}_{\dot{\beta}}\xi^{\beta)\dot{\beta}} - \xi_{\beta\dot{\beta}}E^{\alpha\beta\dot{\alpha}\dot{\beta}} + 3F\xi^{\alpha\dot{\alpha}} \,. \tag{4.75}$$

Then, substituting equations 4.74 and 4.75 into equation 4.68 immediately results in $\gamma^a\nabla_a D^{(1)}\mathbf{\Psi} = 0$, i.e. $D^{(1)}$ is a symmetry of $\gamma^a\nabla_a$. □

4.4 2nd order symmetries

Theorem 4.7. *The only candidate for a physically admissible, 2nd order higher symmetry of the massless Dirac operator such that $D'^{(2)}\mathbf{\Psi}' = \mathrm{e}^{3\sigma/2}D^{(2)}\mathbf{\Psi}$ under a Weyl transformation is*

$$\begin{aligned} D^{(2)} = {} & \xi^{\alpha\beta\dot{\alpha}\dot{\beta}}\nabla_{\alpha\dot{\alpha}}\nabla_{\beta\dot{\beta}} + \frac{2}{3}\nabla^{(\alpha}{}_{\dot{\beta}}\xi^{\beta\gamma)\dot{\alpha}\dot{\beta}}\nabla_{\alpha\dot{\alpha}}M_{\beta\gamma} + \frac{2}{3}\nabla_{\beta}{}^{(\dot{\alpha}}\xi^{\alpha\beta\dot{\beta}\dot{\gamma})}\nabla_{\alpha\dot{\alpha}}\overline{M}_{\dot{\beta}\dot{\gamma}} + \frac{8}{9}\nabla_{\beta\dot{\beta}}(\xi^{\alpha\beta\dot{\alpha}\dot{\beta}})\nabla_{\alpha\dot{\alpha}} \\ & + \left(\frac{2}{9}\nabla^{(\alpha}{}_{\dot{\alpha}}\nabla_{\gamma\dot{\beta}}\xi^{\beta)\gamma\dot{\alpha}\dot{\beta}} + \frac{1}{3}E^{(\alpha}{}_{\gamma\dot{\alpha}\dot{\beta}}\xi^{\beta)\gamma\dot{\alpha}\dot{\beta}}\right)M_{\alpha\beta} \\ & + \left(\frac{2}{9}\nabla_{\alpha}{}^{(\dot{\alpha}}\nabla_{\beta\dot{\gamma}}\xi^{\alpha\beta\dot{\beta})\dot{\gamma}} + \frac{1}{3}E_{\alpha\beta\dot{\gamma}}{}^{(\dot{\alpha}}\xi^{\alpha\beta\dot{\beta})\dot{\gamma}}\right)\overline{M}_{\dot{\alpha}\dot{\beta}} + \frac{2}{15}\nabla_{\alpha\dot{\alpha}}\nabla_{\beta\dot{\beta}}(\xi^{\alpha\beta\dot{\alpha}\dot{\beta}}) - \frac{7}{10}E_{\alpha\beta\dot{\alpha}\dot{\beta}}\xi^{\alpha\beta\dot{\alpha}\dot{\beta}} \,. \end{aligned} \tag{4.76}$$

However, $D^{(2)}$ may not be a symmetry of $\gamma^a\nabla_a$ in general. Instead, given $\gamma^a\nabla_a\mathbf{\Psi} = 0$,

$$\begin{aligned} \gamma^a\nabla_a D^{(2)}\mathbf{\Psi} = \Bigg[& \frac{1}{3}(\overline{C}_{\dot{\alpha}\dot{\beta}}{}^{\dot{\gamma}\dot{\mu}}\xi_{\alpha\beta\dot{\gamma}\dot{\mu}} - C_{\alpha\beta}{}^{\gamma\mu}\xi_{\gamma\mu\dot{\alpha}\dot{\beta}})\nabla^{\beta\dot{\beta}}\bar{\chi}^{\dot{\alpha}} + \bigg(\frac{4}{15}C^{\mu\gamma\beta}{}_{\alpha}\nabla_{(\beta}{}^{\dot{\beta}}\xi_{\gamma\mu)\dot{\alpha}\dot{\beta}} \\ & - \frac{1}{15}\overline{C}_{\dot{\alpha}}{}^{\dot{\beta}\dot{\gamma}\dot{\mu}}\nabla^{\beta}{}_{(\dot{\mu}}\xi_{\alpha\beta\dot{\gamma}\dot{\beta})} - \frac{2}{15}\xi^{\gamma\beta\dot{\gamma}}{}_{\dot{\alpha}}\nabla^{\mu}{}_{\dot{\gamma}}(C_{\alpha\beta\gamma\mu}) - \frac{7}{15}\xi_{\alpha}{}^{\gamma\beta\dot{\gamma}}\nabla_{\gamma}{}^{\dot{\mu}}(\overline{C}_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\mu}})\bigg)\bar{\chi}^{\dot{\alpha}}, \\ & \frac{1}{3}(C^{\alpha\beta}{}_{\gamma\mu}\xi^{\gamma\mu\dot{\alpha}\dot{\beta}} - \overline{C}^{\dot{\alpha}\dot{\beta}}{}_{\dot{\gamma}\dot{\mu}}\xi^{\alpha\beta\dot{\gamma}\dot{\mu}})\nabla_{\beta\dot{\beta}}\psi_{\alpha} + \bigg(\frac{4}{15}\overline{C}_{\dot{\mu}\dot{\gamma}\dot{\beta}}{}^{\dot{\alpha}}\nabla_{\beta}{}^{(\dot{\beta}}\xi^{\alpha\beta\dot{\gamma}\dot{\mu})} \\ & - \frac{1}{15}C^{\alpha}{}_{\beta\gamma\mu}\nabla^{(\mu}{}_{\dot{\beta}}\xi^{\gamma\beta)\dot{\alpha}\dot{\beta}} - \frac{2}{15}\xi^{\alpha}{}_{\gamma\dot{\beta}\dot{\gamma}}\nabla^{\gamma}{}_{\dot{\mu}}(\overline{C}^{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\mu}}) - \frac{7}{15}\xi_{\gamma\beta\dot{\gamma}}{}^{\dot{\alpha}}\nabla_{\mu}{}^{\dot{\gamma}}(C^{\alpha\beta\gamma\mu})\bigg)\psi_{\alpha}\Bigg]^T \,. \end{aligned} \tag{4.77}$$

Proof. This is easily the longest and most technical proof in my thesis. I have presented it in full in appendix C. □

Corollary 4.7.1. *$D^{(2)}$ is a symmetry of $\gamma^a\nabla_a$ on conformally flat spaces.*

Proof. The Weyl tensor is zero on conformally flat spaces. □

Again, like with Δ, a truly "higher" symmetry - i.e. one which is not a composition of 1st order symmetries - can be guaranteed on conformally flat spaces, but not on arbitrary manifolds.

Chapter 5

Conformal geometry and the path forward

I have successfully constructed the most general symmetry operator candidates for Δ and $\gamma^a\nabla_a$ at the 1st and 2nd order and found they are indeed higher symmetries on arbitrary manifolds and conformally flat manifolds respectively. However, I think it is clear from reading appendices B and C that extending to 3rd order and higher will be impossible within the limits of human endurance and perseverance.
Therefore, a more efficient and elegant approach will be required to reduce the computational complexity of the task. As it happens, such an approach exists - although I met it too late in the progress of my master's to use it. The vierbein approach to differential geometry makes manifest an invariance under both general coordinate transformations and local Lorentz transformations. But, $\Delta\varphi = 0$ and $\gamma^a\nabla_a\mathbf{\Psi} = 0$ are both conformally invariant equations of motion. The additional Weyl symmetry is not naturally accounted for in $\nabla_a = e_a{}^m\partial_m + \frac{1}{2}\omega_{abc}M^{bc}$. The formalism of "conformal geometry" seeks to rectify this shortfall[1]. Here I will briefly recount its features as discussed in [23] and even more briefly sketch its application to the problems I have considered[2].

First, consider the Poincare algebra, $\mathfrak{io}(3,1)$. With an appropriate choice of basis, it can be defined as an abstract Lie algebra with

$$[M_{ab}, M_{cd}] = 2\eta_{d[a}M_{b]c} - 2\eta_{c[a}M_{b]d},\ [P_a, M_{bc}] = 2\eta_{a[b}P_{c]} \text{ and } [P_a, P_b] = 0 \tag{5.1}$$

as the fundamental Lie brackets. Physically, P_a generate translations and M_{ab} generate Lorentz transformations. The form of the covariant derivative,

$$\nabla_a = \partial_a + \frac{1}{2}\omega_{abc}M^{bc}, \tag{5.2}$$

is directly determined by the generators of $\mathfrak{io}(3,1)$ as follows. One cannot use partial derivatives in differential geometry because given a (non-scalar) tensor, T, $\partial_a T$ no longer transforms covariantly under local Lorentz transformations[3]. The resolution is of course well known - add a compensating field. Different tensors transform differently under Lorentz transformations though.
Hence, the compensating field must itself depend on the tensor ∇_a acts on, i.e. the compensating field must itself be an operator.

[1] Of course, if one is not dealing with a conformal field theory, then there is no shortfall. Conformal geometry is only useful for conformally invariant theories.

[2] My notational conventions will be somewhat different to [23] though.

[3] It can be assumed T is a scalar with respect to general coordinate transformations because any curved space indices can be converted to local Lorentz indices by vierbeins and inverse vierbeins.

Since it is local Lorentz transformations that are the issue, the compensating field must be proportional to the Lorentz generators, thereby resulting in the form, $\nabla_a = \partial_a + \frac{1}{2}\omega_{abc}M^{bc}$. A compensating field proportional to P_a need not be added because under a Poincare transformation, (Λ, a), a tensor transforms as

$$T'(x) = (R[\Lambda]T)((\Lambda, a)^{-1}x) \tag{5.3}$$

for some representation, R, of $\mathrm{SO}^{\uparrow}(3,1)$. Note that R is only a representation of $\mathrm{SO}^{\uparrow}(3,1)$, not $\mathrm{ISO}^{\uparrow}(3,1)$. Indeed the effect of translation is already accounted for in ∂_a because under an infinitesimal translation, $x'^a = x^a - \xi^a$,

$$\begin{aligned} T'(x) &= T(x+\xi) \\ &= T(x) + \xi^a \partial_a T(x) \\ \iff \delta T(x) &= \xi^a \partial_a T(x)\,. \end{aligned} \tag{5.4}$$

Thus, the partial derivative is already the generator of translations, hence justifying why only a Lorentz generator compensating field is necessary. Finally, the exact form of ω_{abc} is fixed by enforcing that ∇_a is torsion free and $\nabla_a T$ transforms covariantly for any tensor, T.

However, there is no reason why this logic in constructing ∇_a cannot be applied to a larger gauge group. Consider the conformal algebra, $\mathfrak{io}(4,2)$, which can be defined as an abstract Lie algebra with

$$\begin{aligned} &[M_{ab}, M_{cd}] = 2\eta_{d[a}M_{b]c} - 2\eta_{c[a}M_{b]d},\ [P_a, M_{bc}] = 2\eta_{a[b}P_{c]},\ [K_a, M_{bc}] = 2\eta_{a[b}K_{c]}, \\ &[\mathbb{D}, P_a] = P_a,\ [\mathbb{D}, K_a] = -K_a \text{ and } [K_a, P_b] = 2\eta_{ab}\mathbb{D} + 2M_{ab} \end{aligned} \tag{5.5}$$

as the fundamental non-zero Lie brackets. By inspection, the Poincare algebra is a subalgebra of the conformal algebra. The extra generators, $\mathbb{D}$ and K_a, represent dilatations and special conformal transformations respectively - the new symmetries present in a conformal field theory which are lacking in a merely Lorentz invariant theory.
Therefore, in conformal field theory, it makes sense to use "conformal covariant derivatives[4],"

$$D_a = e_a{}^m \partial_m + \frac{1}{2}\hat{\omega}_{abc}M^{bc} - \mathfrak{f}_a{}^b K_b - \mathfrak{b}_a \mathbb{D}\,, \tag{5.6}$$

for some connection coefficients, $\hat{\omega}_{abc}$, $\mathfrak{f}_a{}^b$ and $\mathfrak{b}_a$. All three are again determined by requiring D_a to be torsion free and requiring $D_a T$ to transform covariantly under local Lorentz, dilatation and special conformal transformations.

While it is more convenient to work with D_a instead of ∇_a when dealing with conformally invariant equations, it would be even more advantageous if one could transition between the two derivatives, e.g. do the calculations with D_a, but present the final result in terms of ∇_a to compare with other work. Since the Poincare group is a subgroup of the conformal group, going from D_a to ∇_a amounts to picking a gauge - or "degauging" - within the conformal group. At this point, I will hasten the discussion and present some results from [23] without proof. From [23], it can be shown that

- When acting on a tensor, T, which transforms as $T' = \mathrm{e}^{n\sigma}T$ under a Weyl transformation, $e'_a{}^m = \mathrm{e}^{\sigma} e_a{}^m$, $\mathbb{D}T = nT$ and $K_a T = 0$.
- For any element, $X \in \mathfrak{io}(4,2)/\mathrm{span}(\{P_a\})$, X's commutation relation/Lie bracket with D_a is the same as its commutation relation/Lie bracket with P_a.

[4]This is not necessarily standard terminology or notation, but it will work for my purposes.

- Upon choosing the gauge $\mathfrak{b}_\mathfrak{a} = 0$, $\hat{\omega}_{abc}$ and $\mathfrak{f}_\mathfrak{a}{}^\mathfrak{b}$ are uniquely determined and lead to $D_a = \nabla_a - \frac{1}{2}P_{ab}K^b$ where $P_{ab} = \frac{1}{2}R_{ab} - \frac{1}{12}\eta_{ab}R$ is the Schouten tensor.
- In the $\mathfrak{b}_\mathfrak{a} = 0$ gauge, $[D_a, D_b] = \frac{1}{2}C_{abcd}M^{cd} + \frac{1}{2}\nabla^d(C_{abcd})K^c$.

These properties are sufficient to translate from D_a to ∇_a. Rather than presenting abstract justification of this assertion, in this case I think it will be better to present the following lemma as an illustrative example.

Lemma 5.1. *In the $\mathfrak{b}_\mathfrak{a} = 0$ gauge, $D^{(2)}$ from theorem 4.7 can be re-written as*

$$\begin{aligned} D^{(2)} = \xi^{\alpha\beta\dot{\alpha}\dot{\beta}}D_{\alpha\dot{\alpha}}D_{\beta\dot{\beta}} + \frac{2}{3}D^{(\alpha}{}_{\dot{\beta}}\xi^{\beta\gamma)\dot{\alpha}\dot{\beta}}D_{\alpha\dot{\alpha}}M_{\beta\gamma} + \frac{2}{3}D_\beta{}^{(\dot{\alpha}}\xi^{\alpha\beta\dot{\beta}\dot{\gamma})}D_{\alpha\dot{\alpha}}\overline{M}_{\dot{\beta}\dot{\gamma}} + \frac{8}{9}D_{\beta\dot{\beta}}(\xi^{\alpha\beta\dot{\alpha}\dot{\beta}})D_{\alpha\dot{\alpha}} \\ + \frac{2}{9}D^{(\alpha}{}_{\dot{\alpha}}D_{\gamma\dot{\beta}}\xi^{\beta)\gamma\dot{\alpha}\dot{\beta}}M_{\alpha\beta} + \frac{2}{9}D_\alpha{}^{(\dot{\alpha}}D_{\beta\dot{\gamma}}\xi^{\alpha\beta\dot{\beta})\dot{\gamma}}\overline{M}_{\dot{\alpha}\dot{\beta}} + \frac{2}{15}D_{\alpha\dot{\alpha}}D_{\beta\dot{\beta}}(\xi^{\alpha\beta\dot{\alpha}\dot{\beta}}) \,. \end{aligned} \tag{5.7}$$

Proof. I will have to convert some of the results above to spinor notation along the way.

$$\begin{aligned} D_{\alpha\dot{\alpha}} &= (\sigma^a)_{\alpha\dot{\alpha}}(\nabla_a - \frac{1}{2}P_{ab}K^b) \\ &= \nabla_{\alpha\dot{\alpha}} - \frac{1}{2}P_{\alpha\dot{\alpha}b}K^b \\ &= \nabla_{\alpha\dot{\alpha}} - \frac{1}{2}P_{\alpha\dot{\alpha}\beta\dot{\beta}}K_{\gamma\dot{\gamma}}\Big(-\frac{1}{2}(\tilde{\sigma}_b)^{\dot{\beta}\beta}\Big)\Big(-\frac{1}{2}(\tilde{\sigma}^b)^{\dot{\gamma}\gamma}\Big) \\ &= \nabla_{\alpha\dot{\alpha}} + \frac{1}{4}P_{\alpha\dot{\alpha}\beta\dot{\beta}}K^{\beta\dot{\beta}} \end{aligned} \tag{5.8}$$

$$\begin{aligned} P_{\alpha\dot{\alpha}\beta\dot{\beta}} &= (\sigma^a)_{\alpha\dot{\alpha}}(\sigma^b)_{\beta\dot{\beta}}\Big(\frac{1}{2}R_{ab} - \frac{1}{12}\eta_{ab}R\Big) \\ &= (\sigma^a)_{\alpha\dot{\alpha}}(\sigma^b)_{\beta\dot{\beta}}\Big(\frac{1}{2}R_{ab} - \frac{1}{8}\eta_{ab}R + \frac{1}{24}\eta_{ab}R\Big) \\ &= E_{\alpha\beta\dot{\alpha}\dot{\beta}} + \frac{1}{24}(\sigma^a)_{\alpha\dot{\alpha}}(\sigma_a)_{\beta\dot{\beta}}R \\ &= E_{\alpha\beta\dot{\alpha}\dot{\beta}} - \frac{1}{12}\varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}}R \\ &= E_{\alpha\beta\dot{\alpha}\dot{\beta}} - \varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}}F \end{aligned} \tag{5.9}$$

Likewise, since D_a has the same commutation relations as P_a (except for $[D_a, D_b]$),

$$\begin{aligned} [K_{\alpha\dot{\alpha}}, D_{\beta\dot{\beta}}] &= (\sigma^a)_{\alpha\dot{\alpha}}(\sigma^b)_{\beta\dot{\beta}}(2\eta_{ab}\mathbb{D} + 2M_{ab}) \\ &= 2(\sigma^a)_{\alpha\dot{\alpha}}(\sigma_a)_{\beta\dot{\beta}}\mathbb{D} + 2(\sigma^a)_{\alpha\dot{\alpha}}(\sigma^b)_{\beta\dot{\beta}}M_{ab} \\ &= -4\varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}}\mathbb{D} + 2(\sigma^a)_{\alpha\dot{\alpha}}(\sigma^b)_{\beta\dot{\beta}}M_{ab} \text{ and} \end{aligned} \tag{5.10}$$

$$\begin{aligned} (\sigma^a)_{\alpha\dot{\alpha}}(\sigma^b)_{\beta\dot{\beta}}M_{ab} &= (\sigma^a)_{\alpha\dot{\alpha}}(\sigma^b)_{\beta\dot{\beta}}((\sigma_{ab})_{\mu\nu}M^{\mu\nu} - (\tilde{\sigma}_{ab})_{\dot{\mu}\dot{\nu}}\overline{M}^{\dot{\mu}\dot{\nu}}) \\ &= -\frac{1}{4}\varepsilon_{\nu\rho}(\sigma^a)_{\alpha\dot{\alpha}}(\sigma^b)_{\beta\dot{\beta}}((\sigma_a)_{\mu\dot{\gamma}}(\tilde{\sigma}_b)^{\dot{\gamma}\rho} - (\sigma_b)_{\mu\dot{\gamma}}(\tilde{\sigma}_a)^{\dot{\gamma}\rho})M^{\mu\nu} \\ &\quad + \frac{1}{4}\varepsilon_{\dot{\mu}\dot{\rho}}(\sigma^a)_{\alpha\dot{\alpha}}(\sigma^b)_{\beta\dot{\beta}}((\tilde{\sigma}_a)^{\dot{\rho}\gamma}(\sigma_b)_{\gamma\dot{\nu}} - (\tilde{\sigma}_b)^{\dot{\rho}\gamma}(\sigma_a)_{\gamma\dot{\nu}})\overline{M}^{\dot{\mu}\dot{\nu}} \\ &= -\varepsilon_{\nu\rho}\varepsilon_{\mu\alpha}\varepsilon_{\dot{\gamma}\dot{\alpha}}\delta^\rho{}_\beta\delta^{\dot{\gamma}}{}_{\dot{\beta}}M^{\mu\nu} + \varepsilon_{\nu\rho}\varepsilon_{\mu\beta}\varepsilon_{\dot{\gamma}\dot{\beta}}\delta^\rho{}_\alpha\delta^{\dot{\gamma}}{}_{\dot{\alpha}}M^{\mu\nu} + \varepsilon_{\dot{\mu}\dot{\rho}}\delta^\gamma{}_\alpha\delta^{\dot{\rho}}{}_{\dot{\alpha}}\varepsilon_{\gamma\beta}\varepsilon_{\dot{\nu}\dot{\beta}}\overline{M}^{\dot{\mu}\dot{\nu}} \\ &\quad - \varepsilon_{\dot{\mu}\dot{\rho}}\delta^\gamma{}_\beta\delta^{\dot{\rho}}{}_{\dot{\beta}}\varepsilon_{\gamma\alpha}\varepsilon_{\dot{\nu}\dot{\alpha}}\overline{M}^{\dot{\mu}\dot{\nu}} \\ &= 2\varepsilon_{\dot{\alpha}\dot{\beta}}M_{\alpha\beta} + 2\varepsilon_{\alpha\beta}\overline{M}_{\dot{\alpha}\dot{\beta}} \,. \end{aligned} \tag{5.11}$$

In summary,

$$[K_{\alpha\dot\alpha}, D_{\beta\dot\beta}] = -4\varepsilon_{\alpha\beta}\varepsilon_{\dot\alpha\dot\beta}\mathbb{D} + 4\varepsilon_{\dot\alpha\dot\beta}M_{\alpha\beta} + 4\varepsilon_{\alpha\beta}\overline{M}_{\dot\alpha\dot\beta}\,. \tag{5.12}$$

It is now time to convert the expression claimed in the lemma to the ∇_a formalism term by term. Since $\xi'^{\alpha\beta\dot\alpha\dot\beta} = \mathrm{e}^{-2\sigma}\xi^{\alpha\beta\dot\alpha\dot\beta}$ and $\mathbf{\Psi}' = \mathrm{e}^{3\sigma/2}\mathbf{\Psi}$ upon $e'_a{}^m = \mathrm{e}^{\sigma}e_a{}^m$, by one one of the properties above, $K_{\gamma\dot\gamma}\xi^{\alpha\beta\dot\alpha\dot\beta} = 0$, $K_{\alpha\dot\alpha}\mathbf{\Psi} = 0$, $\mathbb{D}\xi^{\alpha\beta\dot\alpha\dot\beta} = -2\xi^{\alpha\beta\dot\alpha\dot\beta}$ and $\mathbb{D}\mathbf{\Psi} = \frac{3}{2}\mathbf{\Psi}$.
Using these properties, in the $\mathfrak{b}_{\mathfrak{a}} = 0$ gauge,

$$\begin{aligned}
&\xi^{\alpha\beta\dot\alpha\dot\beta}D_{\alpha\dot\alpha}D_{\beta\dot\beta}\mathbf{\Psi}\\
&= \xi^{\alpha\beta\dot\alpha\dot\beta}\left(\nabla_{\alpha\dot\alpha} + \frac{1}{4}E_{\alpha\gamma\dot\alpha\dot\gamma}K^{\gamma\dot\gamma} - \frac{1}{4}FK_{\alpha\dot\alpha}\right)D_{\beta\dot\beta}\mathbf{\Psi}\\
&= \xi^{\alpha\beta\dot\alpha\dot\beta}\nabla_{\alpha\dot\alpha}D_{\beta\dot\beta}\mathbf{\Psi} + \frac{1}{4}\xi^{\alpha\beta\dot\alpha\dot\beta}E_{\alpha\gamma\dot\alpha\dot\gamma}[K^{\gamma\dot\gamma}, D_{\beta\dot\beta}]\mathbf{\Psi} - \frac{1}{4}\xi^{\alpha\beta\dot\alpha\dot\beta}F[K_{\alpha\dot\alpha}, D_{\beta\dot\beta}]\mathbf{\Psi}\\
&= \xi^{\alpha\beta\dot\alpha\dot\beta}\nabla_{\alpha\dot\alpha}\left(\nabla_{\beta\dot\beta} + \frac{1}{4}E_{\beta\gamma\dot\beta\dot\gamma}K^{\gamma\dot\gamma} - \frac{1}{4}FK_{\beta\dot\beta}\right)\mathbf{\Psi} + \frac{1}{4}\xi^{\alpha\beta\dot\alpha\dot\beta}E_{\alpha\gamma\dot\alpha\dot\gamma}(-4\delta^\gamma{}_\beta\delta^{\dot\gamma}{}_{\dot\beta}\mathbb{D} + 4\delta^\gamma{}_\beta\overline{M}^{\dot\gamma}{}_{\dot\beta}\\
&\quad + 4\delta^{\dot\gamma}{}_{\dot\beta}M^\gamma{}_\beta)\mathbf{\Psi} - \frac{1}{4}\xi^{\alpha\beta\dot\alpha\dot\beta}F(-4\varepsilon_{\alpha\beta}\varepsilon_{\dot\alpha\dot\beta}\mathbb{D} + 4\varepsilon_{\dot\alpha\dot\beta}M_{\alpha\beta} + 4\varepsilon_{\alpha\beta}\overline{M}_{\dot\alpha\dot\beta})\mathbf{\Psi}\\
&= \xi^{\alpha\beta\dot\alpha\dot\beta}\nabla_{\alpha\dot\alpha}\nabla_{\beta\dot\beta}\mathbf{\Psi} - \frac{3}{2}\xi^{\alpha\beta\dot\alpha\dot\beta}E_{\alpha\beta\dot\alpha\dot\beta}\mathbf{\Psi} - \xi^{\alpha\beta\dot\gamma\dot\beta}E_{\alpha\beta\dot\gamma}{}^{\dot\alpha}\overline{M}_{\dot\alpha\dot\beta}\mathbf{\Psi} - \xi^{\gamma\beta\dot\alpha\dot\beta}E^{\alpha}{}_{\gamma\dot\alpha\dot\beta}M_{\alpha\beta}\mathbf{\Psi}\\
&= \xi^{\alpha\beta\dot\alpha\dot\beta}\nabla_{\alpha\dot\alpha}\nabla_{\beta\dot\beta}\mathbf{\Psi} - \frac{3}{2}\xi^{\alpha\beta\dot\alpha\dot\beta}E_{\alpha\beta\dot\alpha\dot\beta}\mathbf{\Psi} - E_{\alpha\beta\dot\gamma}{}^{(\dot\alpha}\xi^{\alpha\beta\dot\beta)\dot\gamma}\overline{M}_{\dot\alpha\dot\beta}\mathbf{\Psi} - E^{(\alpha}{}_{\gamma\dot\alpha\dot\beta}\xi^{\beta)\gamma\dot\alpha\dot\beta}M_{\alpha\beta}\mathbf{\Psi} \quad\text{and}
\end{aligned} \tag{5.13}$$

$$\begin{aligned}
&D^{(\alpha}{}_{\dot\beta}\xi^{\beta\gamma)\dot\alpha\dot\beta}D_{\alpha\dot\alpha}M_{\beta\gamma}\mathbf{\Psi}\\
&= \left(\nabla^{(\alpha}{}_{\dot\beta} + \frac{1}{4}E^{(\alpha}{}_{\mu\dot\beta\dot\mu}K^{|\mu|\dot\mu} - \frac{1}{4}FK^{(\alpha}{}_{\dot\beta}\right)(\xi^{\beta\gamma)\dot\alpha\dot\beta})\left(\nabla_{\alpha\dot\alpha} + \frac{1}{4}E_{\alpha\nu\dot\alpha\dot\nu}K^{\nu\dot\nu} - \frac{1}{4}FK_{\alpha\dot\alpha}\right)(M_{\beta\gamma}\mathbf{\Psi})\\
&= \nabla^{(\alpha}{}_{\dot\beta}\xi^{\beta\gamma)\dot\alpha\dot\beta}\left(\nabla_{\alpha\dot\alpha}M_{\beta\gamma}\mathbf{\Psi} + \frac{1}{4}E_{\alpha\nu\dot\alpha\dot\nu}[K^{\nu\dot\nu}, M_{\beta\gamma}]\mathbf{\Psi} - \frac{1}{4}F[K_{\alpha\dot\alpha}, M_{\beta\gamma}]\mathbf{\Psi}\right)\\
&= \nabla^{(\alpha}{}_{\dot\beta}\xi^{\beta\gamma)\dot\alpha\dot\beta}\nabla_{\alpha\dot\alpha}M_{\beta\gamma}\mathbf{\Psi}
\end{aligned} \tag{5.14}$$

since $[K_a, M_{bc}] = 2\eta_{a[b}K_{c]}$ and $K_c\mathbf{\Psi} = 0$ anyway.
Likewise, $D_\beta{}^{(\dot\alpha}\xi^{\alpha\beta\dot\beta\dot\gamma)}D_{\alpha\dot\alpha}\overline{M}_{\dot\beta\dot\gamma}\mathbf{\Psi} = \nabla_\beta{}^{(\dot\alpha}\xi^{\alpha\beta\dot\beta\dot\gamma)}\nabla_{\alpha\dot\alpha}\overline{M}_{\dot\beta\dot\gamma}\mathbf{\Psi}$ by an analogous calculation. Next,

$$\begin{aligned}
D_{\beta\dot\beta}(\xi^{\alpha\beta\dot\alpha\dot\beta})D_{\alpha\dot\alpha}\mathbf{\Psi} &= \left(\nabla_{\beta\dot\beta} + \frac{1}{4}P_{\beta\dot\beta\gamma\dot\gamma}K^{\gamma\dot\gamma}\right)(\xi^{\alpha\beta\dot\alpha\dot\beta})\left(\nabla_{\alpha\dot\alpha} + \frac{1}{4}P_{\alpha\dot\alpha\mu\dot\mu}K^{\mu\dot\mu}\right)(\mathbf{\Psi})\\
&= \nabla_{\beta\dot\beta}(\xi^{\alpha\beta\dot\alpha\dot\beta})\nabla_{\alpha\dot\alpha}\mathbf{\Psi}\,.
\end{aligned} \tag{5.15}$$

$$\begin{aligned}
&D_\alpha{}^{(\dot\alpha}D_{\beta\dot\gamma}\xi^{\alpha\beta\dot\beta)\dot\gamma}\\
&= \left(\nabla_\alpha{}^{(\dot\alpha} + \frac{1}{4}E_{\alpha\mu\dot\mu}{}^{(\dot\alpha}K^{\mu|\dot\mu|} - \frac{1}{4}FK_\alpha{}^{(\dot\alpha}\right)(D_{\beta\dot\gamma}\xi^{\alpha\beta\dot\beta)\dot\gamma})\\
&= \nabla_\alpha{}^{(\dot\alpha}D_{\beta\dot\gamma}\xi^{\alpha\beta\dot\beta)\dot\gamma} + \frac{1}{4}E_{\alpha\mu\dot\mu}{}^{(\dot\alpha}[K^{\mu|\dot\mu|}, D_{\beta\dot\gamma}]\xi^{\alpha\beta\dot\beta)\dot\gamma} - \frac{1}{4}F[K_\alpha{}^{(\dot\alpha}, D_{\beta\dot\gamma}]\xi^{\alpha\beta\dot\beta)\dot\gamma}\\
&= \nabla_\alpha{}^{(\dot\alpha}\left(\nabla_{\beta\dot\gamma} + \frac{1}{4}P_{\beta\dot\gamma\mu\dot\mu}K^{\mu\dot\mu}\right)\xi^{\alpha\beta\dot\beta)\dot\gamma} + E_{\alpha\mu\dot\mu}{}^{(\dot\alpha}(-\delta^\mu{}_\beta\delta^{|\dot\mu|}{}_{\dot\gamma}\mathbb{D} + \delta^{|\dot\mu|}{}_{\dot\gamma}M^\mu{}_\beta + \delta^\mu{}_\beta\overline{M}^{|\dot\mu|}{}_{\dot\gamma})\xi^{\alpha\beta\dot\beta)\dot\gamma}\\
&\quad - F(-\varepsilon_{\alpha\beta}\delta^{(\dot\alpha}{}_{\dot\gamma}\mathbb{D} + \varepsilon_{\alpha\beta}\overline{M}^{(\dot\alpha}{}_{\dot\gamma} + \delta^{(\dot\alpha}{}_{\dot\gamma}M_{\alpha\beta})\xi^{\alpha\beta\dot\beta)\dot\gamma}\\
&= \nabla_\alpha{}^{(\dot\alpha}\nabla_{\beta\dot\gamma}\xi^{\alpha\beta\dot\beta)\dot\gamma} + 2E_{\alpha\beta\dot\gamma}{}^{(\dot\alpha}\xi^{\alpha\beta\dot\beta)\dot\gamma} + E_{\alpha\mu\dot\gamma}{}^{(\dot\alpha}M^\mu{}_\beta\xi^{\alpha\beta\dot\beta)\dot\gamma} + E_{\alpha\beta\dot\mu}{}^{(\dot\alpha}\overline{M}^{|\dot\mu|}{}_{\dot\gamma}\xi^{\alpha\beta\dot\beta)\dot\gamma}\\
&\quad - FM_{\alpha\beta}\xi^{\alpha\beta\dot\alpha\dot\beta}
\end{aligned} \tag{5.16}$$

$$
\begin{aligned}
&D_\alpha{}^{(\dot\alpha} D_{\beta\dot\gamma} \xi^{\alpha\beta\dot\beta)\dot\gamma} \\
&= \nabla_\alpha{}^{(\dot\alpha} \nabla_{\beta\dot\gamma} \xi^{\alpha\beta\dot\beta)\dot\gamma} + 2E_{\alpha\beta\dot\gamma}{}^{(\dot\alpha} \xi^{\alpha\beta\dot\beta)\dot\gamma} + \frac{1}{2} E_{\alpha\mu\dot\gamma}{}^{(\dot\alpha} (\varepsilon^{\mu\alpha} \xi_\beta{}^{\beta\dot\beta)\dot\gamma} + \delta^\alpha{}_\beta \xi^{\mu\beta\dot\beta)\dot\gamma} + \varepsilon^{\mu\beta} \xi^\alpha{}_\beta{}^{\dot\beta)\dot\gamma} + \delta^\beta{}_\beta \xi^{\alpha\mu\dot\beta)\dot\gamma}) \\
&\quad + \frac{1}{2} E_{\alpha\beta\dot\mu}{}^{(\dot\alpha} (-\varepsilon^{\dot\beta)\dot\mu} \xi^{\alpha\beta}{}_{\dot\gamma}{}^{\dot\gamma} + \delta^{\dot\beta)}{}_{\dot\gamma} \xi^{\alpha\beta\dot\mu\dot\gamma} + \varepsilon^{|\dot\mu\dot\gamma|} \xi^{\alpha\beta\dot\beta)}{}_{\dot\gamma} + \delta^{|\dot\gamma|}{}_{\dot\gamma} \xi^{\alpha\beta\dot\beta)\dot\mu}) \\
&\quad - F(\delta^\alpha{}_\alpha \xi_\beta{}^{\beta\dot\alpha\dot\beta} + \delta^\alpha{}_\beta \xi_\alpha{}^{\beta\dot\alpha\dot\beta} + \delta^\beta{}_\alpha \xi^\alpha{}_\beta{}^{\dot\alpha\dot\beta} + \delta^\beta{}_\beta \xi^\alpha{}_\alpha{}^{\dot\alpha\dot\beta}) \\
&= \nabla_\alpha{}^{(\dot\alpha} \nabla_{\beta\dot\gamma} \xi^{\alpha\beta\dot\beta)\dot\gamma} + 6E_{\alpha\beta\dot\gamma}{}^{(\dot\alpha} \xi^{\alpha\beta\dot\beta)\dot\gamma} \,.
\end{aligned}
\tag{5.17}
$$

By a completely analogous calculation, $D^{(\alpha}{}_{\dot\alpha} D_{\beta\dot\gamma} \xi^{\beta)\gamma\dot\alpha\dot\beta} = \nabla^{(\alpha}{}_{\dot\alpha} \nabla_{\beta\dot\gamma} \xi^{\beta)\gamma\dot\alpha\dot\beta} + 6E^{(\alpha}{}_{\gamma\dot\alpha\dot\beta} \xi^{\beta)\gamma\dot\alpha\dot\beta}$.

$$
\begin{aligned}
&D_{\alpha\dot\alpha} D_{\beta\dot\beta} \xi^{\alpha\beta\dot\alpha\dot\beta} \\
&= \left(\nabla_{\alpha\dot\alpha} + \frac{1}{4} E_{\alpha\beta\dot\alpha\dot\beta} K^{\beta\dot\beta} - \frac{1}{4} F K_{\alpha\dot\alpha} \right) D_{\beta\dot\beta} \xi^{\alpha\beta\dot\alpha\dot\beta} \\
&= \nabla_{\alpha\dot\alpha} D_{\beta\dot\beta} \xi^{\alpha\beta\dot\alpha\dot\beta} + \frac{1}{4} E_{\alpha\gamma\dot\alpha\dot\gamma} [K^{\gamma\dot\gamma}, D_{\beta\dot\beta}] \xi^{\alpha\beta\dot\alpha\dot\beta} - \frac{1}{4} F [K_{\alpha\dot\alpha}, D_{\beta\dot\beta}] \xi^{\alpha\beta\dot\alpha\dot\beta} \\
&= \nabla_{\alpha\dot\alpha} D_{\beta\dot\beta} \xi^{\alpha\beta\dot\alpha\dot\beta} + \frac{1}{4} E_{\alpha\gamma\dot\alpha\dot\gamma} (-4\delta^\gamma{}_\beta \delta^{\dot\gamma}{}_{\dot\beta} \mathbb{D} + 4\delta^\gamma{}_\beta \overline{M}^{\dot\gamma}{}_{\dot\beta} + 4\delta^{\dot\gamma}{}_{\dot\beta} M^\gamma{}_\beta) \xi^{\alpha\beta\dot\alpha\dot\beta} \\
&\quad - \frac{1}{4} F(-4\varepsilon_{\alpha\beta} \varepsilon_{\dot\alpha\dot\beta} \mathbb{D} + 4\varepsilon_{\dot\alpha\dot\beta} M_{\alpha\beta} + 4\varepsilon_{\alpha\beta} \overline{M}_{\dot\alpha\dot\beta}) \xi^{\alpha\beta\dot\alpha\dot\beta} \\
&= \nabla_{\alpha\dot\alpha} \left(\nabla_{\beta\dot\beta} + \frac{1}{4} P_{\beta\dot\beta\gamma\dot\gamma} K^{\gamma\dot\gamma} \right) \xi^{\alpha\beta\dot\alpha\dot\beta} + 2E_{\alpha\beta\dot\alpha\dot\beta} \xi^{\alpha\beta\dot\alpha\dot\beta} \\
&\quad + \frac{1}{2} E_{\alpha\beta\dot\alpha\dot\gamma} (\varepsilon^{\dot\gamma\dot\alpha} \xi^{\alpha\beta}{}_{\dot\beta}{}^{\dot\beta} + \delta^{\dot\alpha}{}_{\dot\beta} \xi^{\alpha\beta\dot\gamma\dot\beta} + \varepsilon^{\dot\gamma\dot\beta} \xi^{\alpha\beta\dot\alpha}{}_{\dot\beta} + \delta^{\dot\beta}{}_{\dot\beta} \xi^{\alpha\beta\dot\alpha\dot\gamma}) \\
&\quad + \frac{1}{2} E_{\alpha\gamma\dot\alpha\dot\beta} (\varepsilon^{\gamma\alpha} \xi_\beta{}^{\beta\dot\alpha\dot\beta} + \delta^\alpha{}_\beta \xi^{\gamma\beta\dot\alpha\dot\beta} + \varepsilon^{\gamma\beta} \xi^\alpha{}_\beta{}^{\dot\alpha\dot\beta} + \delta^\beta{}_\beta \xi^{\alpha\gamma\dot\alpha\dot\beta}) - 0 \\
&= \nabla_{\alpha\dot\alpha} \nabla_{\beta\dot\beta} \xi^{\alpha\beta\dot\alpha\dot\beta} + 6E_{\alpha\beta\dot\alpha\dot\beta} \xi^{\alpha\beta\dot\alpha\dot\beta}
\end{aligned}
\tag{5.18}
$$

Finally, putting all these terms together,

$$
\begin{aligned}
&\Big(\xi^{\alpha\beta\dot\alpha\dot\beta} D_{\alpha\dot\alpha} D_{\beta\dot\beta} + \frac{2}{3} D^{(\alpha}{}_{\dot\beta} \xi^{\beta\gamma)\dot\alpha\dot\beta} D_{\alpha\dot\alpha} M_{\beta\gamma} + \frac{2}{3} D_\beta{}^{(\dot\alpha} \xi^{\alpha\beta\dot\beta\dot\gamma)} D_{\alpha\dot\alpha} \overline{M}_{\dot\beta\dot\gamma} + \frac{8}{9} D_{\beta\dot\beta} (\xi^{\alpha\beta\dot\alpha\dot\beta}) D_{\alpha\dot\alpha} \\
&+ \frac{2}{9} D^{(\alpha}{}_{\dot\alpha} D_{\gamma\dot\beta} \xi^{\beta)\gamma\dot\alpha\dot\beta} M_{\alpha\beta} + \frac{2}{9} D_\alpha{}^{(\dot\alpha} D_{\beta\dot\gamma} \xi^{\alpha\beta\dot\beta)\dot\gamma} \overline{M}_{\dot\alpha\dot\beta} + \frac{2}{15} D_{\alpha\dot\alpha} D_{\beta\dot\beta} (\xi^{\alpha\beta\dot\alpha\dot\beta}) \Big) \mathbf{\Psi} \\
&= \xi^{\alpha\beta\dot\alpha\dot\beta} \nabla_{\alpha\dot\alpha} \nabla_{\beta\dot\beta} \mathbf{\Psi} - \frac{3}{2} \xi^{\alpha\beta\dot\alpha\dot\beta} E_{\alpha\beta\dot\alpha\dot\beta} \mathbf{\Psi} - E_{\alpha\beta\dot\gamma}{}^{(\dot\alpha} \xi^{\alpha\beta\dot\beta)\dot\gamma} \overline{M}_{\dot\alpha\dot\beta} \mathbf{\Psi} - E^{(\alpha}{}_{\gamma\dot\alpha\dot\beta} \xi^{\beta)\gamma\dot\alpha\dot\beta} M_{\alpha\beta} \mathbf{\Psi} \\
&\quad + \frac{2}{3} \nabla^{(\alpha}{}_{\dot\beta} \xi^{\beta\gamma)\dot\alpha\dot\beta} \nabla_{\alpha\dot\alpha} M_{\beta\gamma} \mathbf{\Psi} + \frac{2}{3} \nabla_\beta{}^{(\dot\alpha} \xi^{\alpha\beta\dot\beta\dot\gamma)} \nabla_{\alpha\dot\alpha} \overline{M}_{\dot\beta\dot\gamma} \mathbf{\Psi} + \frac{8}{9} \nabla_{\beta\dot\beta} (\xi^{\alpha\beta\dot\alpha\dot\beta}) \nabla_{\alpha\dot\alpha} \mathbf{\Psi} \\
&\quad + \frac{2}{9} (\nabla^{(\alpha}{}_{\dot\alpha} \nabla_{\beta\dot\gamma} \xi^{\beta)\gamma\dot\alpha\dot\beta} + 6E^{(\alpha}{}_{\gamma\dot\alpha\dot\beta} \xi^{\beta)\gamma\dot\alpha\dot\beta}) M_{\dot\alpha\dot\beta} \mathbf{\Psi} + \frac{2}{9} (\nabla_\alpha{}^{(\dot\alpha} \nabla_{\beta\dot\gamma} \xi^{\alpha\beta\dot\beta)\dot\gamma} + 6E_{\alpha\beta\dot\gamma}{}^{(\dot\alpha} \xi^{\alpha\beta\dot\beta)\dot\gamma}) \overline{M}_{\dot\alpha\dot\beta} \mathbf{\Psi} \\
&\quad + \frac{2}{15} (\nabla_{\alpha\dot\alpha} \nabla_{\beta\dot\beta} (\xi^{\alpha\beta\dot\alpha\dot\beta}) + 6E_{\alpha\beta\dot\alpha\dot\beta} \xi^{\alpha\beta\dot\alpha\dot\beta}) \mathbf{\Psi} \\
&= \Big(\xi^{\alpha\beta\dot\alpha\dot\beta} \nabla_{\alpha\dot\alpha} \nabla_{\beta\dot\beta} + \frac{2}{3} \nabla^{(\alpha}{}_{\dot\beta} \xi^{\beta\gamma)\dot\alpha\dot\beta} \nabla_{\alpha\dot\alpha} M_{\beta\gamma} + \frac{2}{3} \nabla_\beta{}^{(\dot\alpha} \xi^{\alpha\beta\dot\beta\dot\gamma)} \nabla_{\alpha\dot\alpha} \overline{M}_{\dot\beta\dot\gamma} + \frac{8}{9} \nabla_{\beta\dot\beta} (\xi^{\alpha\beta\dot\alpha\dot\beta}) \nabla_{\alpha\dot\alpha} \\
&\quad + \left(\frac{2}{9} \nabla^{(\alpha}{}_{\dot\alpha} \nabla_{\gamma\dot\beta} \xi^{\beta)\gamma\dot\alpha\dot\beta} + \frac{1}{3} E^{(\alpha}{}_{\gamma\dot\alpha\dot\beta} \xi^{\beta)\gamma\dot\alpha\dot\beta} \right) M_{\alpha\beta} + \left(\frac{2}{9} \nabla_\alpha{}^{(\dot\alpha} \nabla_{\beta\dot\gamma} \xi^{\alpha\beta\dot\beta)\dot\gamma} + \frac{1}{3} E_{\alpha\beta\dot\gamma}{}^{(\dot\alpha} \xi^{\alpha\beta\dot\beta)\dot\gamma} \right) \overline{M}_{\dot\alpha\dot\beta} \\
&\quad + \frac{2}{15} \nabla_{\alpha\dot\alpha} \nabla_{\beta\dot\beta} (\xi^{\alpha\beta\dot\alpha\dot\beta}) - \frac{7}{10} E_{\alpha\beta\dot\alpha\dot\beta} \xi^{\alpha\beta\dot\alpha\dot\beta} \Big) \mathbf{\Psi} \\
&= D^{(2)} \mathbf{\Psi} \,.
\end{aligned}
\tag{5.19}
$$

□

There are several advantages to writing $D^{(2)}$ this way. Most saliently, the $E_{\alpha\beta\dot{\alpha}\dot{\beta}}$ terms have disappeared when transferring from $\nabla_{\alpha\dot{\alpha}}$ to $D_{\alpha\dot{\alpha}}$.
Therefore, simplifying $\gamma^a \nabla_a D^{(2)} \mathbf{\Psi}$ in the conformal geometry formalism is identical to simplifying the expression in flat space while keeping track of commutators; the contribution of $E_{\alpha\beta\dot{\alpha}\dot{\beta}}$ terms is no longer an extra complication in curved space.
The commutators themselves are also more convenient in conformal geometry. In the proofs of theorems 3.11 and 4.7, I spent many pages re-writing Riemann tensor descendant contributions in terms of the Weyl tensor. However, in conformal geometry,
$[D_a, D_b] = \frac{1}{2} C_{abcd} M^{cd} + \frac{1}{2} \nabla^d (C_{abcd}) K^c$ in the $\mathfrak{b}_{\mathfrak{a}}$ gauge; the curvature factors are already written in terms of the Weyl tensor.

More generally, if $D^{(n)}$ is a higher symmetry candidate of $\mathfrak{D}$ and both $\mathfrak{D}$ & $D^{(n)}$ are written in the conformal geometry formalism, then proving $\mathfrak{D} D^{(n)} T = 0$ (given $\mathfrak{D} T = 0$) on flat space automatically lifts the result to conformally flat spaces.
e.g. If Eastwood's Δ higher symmetry candidates [17] were re-written in the conformal geometry formalism[5], then his flat space proof that his candidate operators really are higher symmetries of Δ is automatically lifted to conformally flat spaces[6]. This seems to have been implicitly done already in [20].

The process of actually finding candidate symmetry operators, $D^{(n)}$, is slightly different with conformal covariant derivatives. Consider $\mathbf{\Psi}$ and $\gamma^a \nabla_a$ for example. In this approach,
$D'^{(n)} \mathbf{\Psi}' = e^{3\sigma/2} D^{(n)} \mathbf{\Psi}$ upon ${e'}_a{}^m = e^{\sigma} e_a{}^m$ becomes two equations, namely $\mathbb{D} D^{(n)} \mathbf{\Psi} = \frac{3}{2} D^{(n)} \mathbf{\Psi}$ and $K_a D^{(n)} \mathbf{\Psi} = 0$. While the number of equations has doubled, there are fewer terms in $D^{(n)}$ itself because the $E_{\alpha\beta\dot{\alpha}\dot{\beta}}$ "compensating terms" are no longer required. I think the two effects roughly cancel in terms of calculation time saved or lost. Each of $\mathbb{D} D^{(n)} \mathbf{\Psi} = \frac{3}{2} D^{(n)} \mathbf{\Psi}$ and $K_a D^{(n)} \mathbf{\Psi} = 0$ is analysed by pushing (via commutators) $\mathbb{D}$ and K_a towards $\mathbf{\Psi}$ and $\xi^{\alpha_1 \cdots \alpha_n \dot{\alpha}_1 \cdots \dot{\alpha}_n}$ where (assuming $\xi^{\alpha_1 \cdots \alpha_n \dot{\alpha}_1 \cdots \dot{\alpha}_n}$ is conformal Killing - although I did not prove this for $n \geq 3$) $\mathbb{D} \mathbf{\Psi} = \frac{3}{2} \mathbf{\Psi}$, $\mathbb{D} \xi^{\alpha_1 \cdots \alpha_n \dot{\alpha}_1 \cdots \dot{\alpha}_n} = -n \xi^{\alpha_1 \cdots \alpha_n \dot{\alpha}_1 \cdots \dot{\alpha}_n}$, $K_a \mathbf{\Psi} = 0$ and $K_a \xi^{\alpha_1 \cdots \alpha_n \dot{\alpha}_1 \cdots \dot{\alpha}_n} = 0$. An analogous procedure applies for Δ and φ. I find the more pressing issue is that the "ad hoc" approach to guessing terms which may comprise $D^{(n)}$ is invariant upon the variation in formalism.

[5] I suspect this should be possible because his tractor calculus approach is known to be related to the approach I am describing in this section.
[6] Of course, Δ itself would also need to be re-written in the conformal geometry formalism.

Chapter 6

Conclusion

In this thesis I considered higher symmetries of the relativistic wave equations for spin-0 and spin-1/2 massless particles in curved space. I computed the higher symmetries using spinor methods, conformal Killing vectors/tensors and the Weyl transformation properties of matter fields in a conformal field theory. The main results I derived were the following.

- The conformal d'Alembertian, $\Delta = \Box - \frac{1}{6}R$, has a unique 1st order higher symmetry,

  $$D^{(1)} = \xi^a \nabla_a + \frac{1}{4}\nabla_a(\xi^a) + \xi, \tag{6.1}$$

  where $\xi^a(x)$ is an arbitrary conformal Killing vector of the manifold and ξ is any constant.

- At the 2nd order, Δ has a unique (up to the addition of 1st order symmetries) physically admissible higher symmetry candidate,

  $$D^{(2)} = \xi^{ab}\nabla_a\nabla_b + \frac{2}{3}\nabla_b(\xi^{ab})\nabla_a + \frac{1}{15}\nabla_a\nabla_b(\xi^{ab}) - \frac{3}{10}R_{ab}\xi^{ab}, \tag{6.2}$$

  where $\xi^{ab}(x)$ is an arbitrary conformal Killing tensor of the manifold. However, $D^{(2)}$ may not be a symmetry in general. Instead,

  $$\begin{aligned}\Delta D^{(2)}\varphi = &\left(\frac{4}{15}C^a{}_{bcd}\nabla^c(\xi^{bd}) + \frac{4}{5}\nabla^d(C^a{}_{bcd})\xi^{bc}\right)\nabla_a(\varphi)\\ &+\left(\frac{2}{15}C_{abcd}\nabla^a\nabla^c(\xi^{bd}) + \frac{2}{5}\nabla^c\nabla_d(C^d{}_{abc})\xi^{ab} + \frac{4}{15}\nabla_d(C^d{}_{abc})\nabla^c(\xi^{ab})\right)\varphi\,. \end{aligned}\tag{6.3}$$

- The massless Dirac operator, $\gamma^a\nabla_a$, has a unique 1st order symmetry,

  $$D^{(1)} = \xi^{\alpha\dot\alpha}\nabla_{\alpha\dot\alpha} + \frac{1}{2}\nabla^{(\alpha}{}_{\dot\alpha}\xi^{\beta)\dot\alpha}M_{\alpha\beta} + \frac{1}{2}\nabla_{\alpha}{}^{(\dot\alpha}\xi^{\alpha\dot\beta)}\overline{M}_{\dot\alpha\dot\beta} + \frac{3}{8}\nabla_{\alpha\dot\alpha}(\xi^{\alpha\dot\alpha}) + \xi, \tag{6.4}$$

  where $\xi^{\alpha\dot\alpha}(x)$ is an arbitrary conformal Killing vector of the manifold and ξ is an arbitrary constant.

- At the 2nd order, $\gamma^a\nabla_a$ has a unique (up to the addition of 1st order symmetries) physically admissible higher symmetry candidate,

  $$\begin{aligned}D^{(2)} = &\,\xi^{\alpha\beta\dot\alpha\dot\beta}\nabla_{\alpha\dot\alpha}\nabla_{\beta\dot\beta} + \frac{2}{3}\nabla^{(\alpha}{}_{\dot\beta}\xi^{\beta\gamma)\dot\alpha\dot\beta}\nabla_{\alpha\dot\alpha}M_{\beta\gamma} + \frac{2}{3}\nabla_{\beta}{}^{(\dot\alpha}\xi^{\alpha\beta\dot\beta\dot\gamma)}\nabla_{\alpha\dot\alpha}\overline{M}_{\dot\beta\dot\gamma}\\ &+\frac{8}{9}\nabla_{\beta\dot\beta}(\xi^{\alpha\beta\dot\alpha\dot\beta})\nabla_{\alpha\dot\alpha} + \left(\frac{2}{9}\nabla^{(\alpha}{}_{\dot\alpha}\nabla_{\gamma\dot\beta}\xi^{\beta)\gamma\dot\alpha\dot\beta} + \frac{1}{3}E^{(\alpha}{}_{\gamma\dot\alpha\dot\beta}\xi^{\beta)\gamma\dot\alpha\dot\beta}\right)M_{\alpha\beta}\\ &+\left(\frac{2}{9}\nabla_{\alpha}{}^{(\dot\alpha}\nabla_{\beta\dot\gamma}\xi^{\alpha\beta\dot\beta)\dot\gamma} + \frac{1}{3}E_{\alpha\beta\dot\gamma}{}^{(\dot\alpha}\xi^{\alpha\beta\dot\beta)\dot\gamma}\right)\overline{M}_{\dot\alpha\dot\beta} + \frac{2}{15}\nabla_{\alpha\dot\alpha}\nabla_{\beta\dot\beta}(\xi^{\alpha\beta\dot\alpha\dot\beta})\\ &-\frac{7}{10}E_{\alpha\beta\dot\alpha\dot\beta}\xi^{\alpha\beta\dot\alpha\dot\beta},\end{aligned}\tag{6.5}$$

where $\xi^{\alpha\beta\dot{\alpha}\dot{\beta}}$ is an arbitrary conformal Killing tensor of the manifold. However, $D^{(2)}$ may not be a symmetry in general. Instead,

$$
\begin{aligned}
&\gamma^a \nabla_a D^{(2)} \mathbf{\Psi} \\
&= \gamma^a \nabla_a D^{(2)} \begin{bmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{bmatrix} \\
&= \Bigg[\frac{1}{3} (\overline{C}_{\dot{\alpha}\dot{\beta}}{}^{\dot{\gamma}\dot{\mu}} \xi_{\alpha\beta\dot{\gamma}\dot{\mu}} - C_{\alpha\beta}{}^{\gamma\mu} \xi_{\gamma\mu\dot{\alpha}\dot{\beta}}) \nabla^{\beta\dot{\beta}} \bar{\chi}^{\dot{\alpha}} + \Bigg(\frac{4}{15} C^{\mu\gamma\beta}{}_{\alpha} \nabla_{(\beta}{}^{\dot{\beta}} \xi_{\gamma\mu)\dot{\alpha}\dot{\beta}} \\
&\quad - \frac{1}{15} \overline{C}_{\dot{\alpha}}{}^{\dot{\beta}\dot{\gamma}\dot{\mu}} \nabla^{\beta}{}_{(\dot{\mu}} \xi_{\alpha\beta\dot{\gamma}\dot{\beta})} - \frac{2}{15} \xi^{\gamma\beta\dot{\gamma}}{}_{\dot{\alpha}} \nabla^{\mu}{}_{\dot{\gamma}} (C_{\alpha\beta\gamma\mu}) - \frac{7}{15} \xi_{\alpha}{}^{\gamma\dot{\beta}\dot{\gamma}} \nabla_{\gamma}{}^{\dot{\mu}} (\overline{C}_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\mu}}) \Bigg) \bar{\chi}^{\dot{\alpha}}, \\
&\quad \frac{1}{3} (C^{\alpha\beta}{}_{\gamma\mu} \xi^{\gamma\mu\dot{\alpha}\dot{\beta}} - \overline{C}^{\dot{\alpha}\dot{\beta}}{}_{\dot{\gamma}\dot{\mu}} \xi^{\alpha\beta\dot{\gamma}\dot{\mu}}) \nabla_{\beta\dot{\beta}} \psi_\alpha + \Bigg(\frac{4}{15} \overline{C}_{\dot{\mu}\dot{\gamma}\dot{\beta}}{}^{\dot{\alpha}} \nabla_{\beta}{}^{(\dot{\beta}} \xi^{\alpha\beta\dot{\gamma}\dot{\mu})} \\
&\quad - \frac{1}{15} C^{\alpha}{}_{\beta\gamma\mu} \nabla^{(\mu}{}_{\dot{\beta}} \xi^{\gamma\beta)\dot{\alpha}\dot{\beta}} - \frac{2}{15} \xi^{\alpha}{}_{\gamma\dot{\beta}\dot{\gamma}} \nabla^{\gamma}{}_{\dot{\mu}} (\overline{C}^{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\mu}}) - \frac{7}{15} \xi_{\gamma\beta\dot{\gamma}}{}^{\dot{\alpha}} \nabla_{\mu}{}^{\dot{\gamma}} (C^{\alpha\beta\gamma\mu}) \Bigg) \psi_\alpha \Bigg]^T . \qquad (6.6)
\end{aligned}
$$

- Therefore, for both Δ and $\gamma^a \nabla_a$, while 2nd order symmetries definitely exist (just compose two 1st order symmetries), not every rank two, conformal Killing tensor leads to a symmetry. Conformally flat spaces (where the Weyl tensor is zero) are an exception[1]. There, more general higher symmetries are possible - not just compositions of lower order symmetries.

- Actually finding the conformal Killing vectors/tensors of a given manifold is beyond the scope of this thesis.

There are a number of unanswered questions - and therefore future research directions - left at the end of this project. I have shown that conformal flatness is a sufficient condition to have higher symmetries at the 2nd order, but I have not considered necessary conditions - see [20] for further discussion on that subject. The greater unknowns though are the generalisations to arbitrary orders, $D^{(n)}$. I envisage that endeavour requires a more systematic approach than the one I have presented here. It will certainly require a generalised method of constructing $D^{(n)}$ and simplifying $D^{(n)}\mathfrak{D}T$ and I believe I have presented compelling evidence to suggest conformal geometry would ease both tasks. An immediate step towards these overarching goals would be to calculate third order higher symmetries of the conformal d'Alembertian and the massless Dirac operator. As far as I am aware, neither task has been accomplished in general in the literature, but should be readily achievable using the methods I have described in this thesis.

[1] Conformal flatness is a sufficient, but perhaps not necessary, condition.

Appendix A

Proof of theorem 2.10

I have to prove that under infinitesimal general coordinate, local Lorentz and Weyl transformations, covariant derivatives are changed as follows.

$$\delta\nabla_a = \left[\xi^b\nabla_b + \frac{1}{2}K^{bc}M_{bc}, \nabla_a\right] + \sigma\nabla_a - \nabla^b(\sigma)M_{ab} \tag{A.1}$$

and for a conformal Killing vector, $\xi^a(x)$,

$$\delta\nabla_a = \left[\xi^b\nabla_b + \frac{1}{2}K^{bc}(\xi)M_{bc}, \nabla_a\right] + \sigma(\xi)\nabla_a - \nabla^b(\sigma(\xi))M_{ab} = 0$$
$$\text{for } K^{bc}(\xi) = \frac{1}{2}(\nabla^b\xi^c - \nabla^c\xi^b) \text{ and } \sigma(\xi) = \frac{1}{4}\nabla_a\xi^a\,. \tag{A.2}$$

Let T be an arbitrary tensor (as T is arbitrary, there is no use in writing its indices).

$$\delta\nabla_a T = \nabla'_a T - \nabla_a T \tag{A.3}$$

Because I am considering infinitesimal transformations, the three transformations (general, local Lorentz and Weyl) can be considered separately and added together as there cannot be any "cross terms" in the infinitesimal case.
For the Weyl transformation[1],

$$\begin{aligned}\delta\nabla_a T &= \nabla'_a T - \nabla_a T \\ &= (\nabla_a + \sigma\nabla_a - \nabla^b(\sigma)M_{ab})T - \nabla_a T \\ &= (\sigma\nabla_a - \nabla^b(\sigma)M_{ab})T\,.\end{aligned} \tag{A.4}$$

Next, for the local Lorentz and general coordinate parts of the proof, there is in some sense nothing to prove depending on one's choice of definitions. A tensor can be defined as an object transforming as

$$T' = \mathrm{e}^{\xi^m(x)\partial_m + \frac{1}{2}K^{bc}(x)M_{bc}}T \tag{A.5}$$

when exponentiating the infinitesimal $\xi'^m = x^m - \xi^m(x)$ and $e'_a{}^m(x) = e_a{}^m(x) + K_a{}^b(x)e_b{}^m(x)$ of a general coordinate and local Lorentz transformation. The fact that an antisymmetric matrix, K_{ab}, defines an infinitesimal (local) Lorentz transformation follows from the form of elements

[1] I will essentially devolve this part of the proof to equation F.2.

in the Lie algebra, $\mathfrak{o}(3,1) \cong \mathfrak{sl}(2,\mathbb{C})$. ∇_a can then be defined as a derivative preserving this covariant transformation property. Therefore,

$$\begin{aligned} \nabla'_a T' &= \mathrm{e}^{\xi^m(x)\partial_m + \frac{1}{2}K^{bc}(x)M_{bc}} \nabla_a T \\ &= \mathrm{e}^{\xi^m(x)\partial_m + \frac{1}{2}K^{bc}(x)M_{bc}} \nabla_a \mathrm{e}^{-\xi^m(x)\partial_m - \frac{1}{2}K^{bc}(x)M_{bc}} T' \end{aligned} \tag{A.6}$$

$$\begin{aligned} \implies \nabla'_a T &= \mathrm{e}^{\xi^m(x)\partial_m + \frac{1}{2}K^{bc}(x)M_{bc}} \nabla_a \mathrm{e}^{-\xi^m(x)\partial_m - \frac{1}{2}K^{bc}(x)M_{bc}} T \\ &= \mathrm{e}^{\xi^b(x)\partial_b + \frac{1}{2}K^{bc}(x)M_{bc}} \nabla_a \mathrm{e}^{-\xi^b(x)\partial_b - \frac{1}{2}K^{bc}(x)M_{bc}} T \\ &= \mathrm{e}^{\xi^b(x)\nabla_b - \frac{1}{2}\xi^d(x)\omega(x)_d{}^{bc}M_{bc} + \frac{1}{2}K^{bc}(x)M_{bc}} \nabla_a \mathrm{e}^{-\xi^b(x)\nabla_b + \frac{1}{2}\xi^d(x)\omega(x)_d{}^{bc}M_{bc} - \frac{1}{2}K^{bc}(x)M_{bc}} T \\ &= \mathrm{e}^{\xi^b(x)\nabla_b + \frac{1}{2}\tilde{K}^{bc}(x)M_{bc}} \nabla_a \mathrm{e}^{-\xi^b(x)\nabla_b - \frac{1}{2}\tilde{K}^{bc}(x)M_{bc}} T \end{aligned} \tag{A.7}$$

where $\tilde{K}^{bc} = K^{bc} - \xi^d \omega_d{}^{bc}$. Then, taking the infinitesimal of the last equation and renaming $\tilde{K}^{bc} \to K^{bc}$,

$$\begin{aligned} \nabla'_a T &= \Big(I + \xi^b(x)\nabla_b + \frac{1}{2}K^{bc}(x)M_{bc}\Big)\nabla_a\Big(I - \xi^b(x)\nabla_b - \frac{1}{2}K^{bc}(x)M_{bc}\Big)T \\ &= \nabla_a T + \Big(\xi^b(x)\nabla_b + \frac{1}{2}K^{bc}(x)M_{bc}\Big)\nabla_a T - \nabla_a\Big(\xi^b(x)\nabla_b + \frac{1}{2}K^{bc}(x)M_{bc}\Big)T \end{aligned} \tag{A.8}$$

Therefore,

$$\delta\nabla_a T = \Big[\xi^b(x)\nabla_b + \frac{1}{2}K^{bc}(x)M_{bc}, \nabla_a\Big]T \tag{A.9}$$

which completes the proof of the first half of the theorem.

Rather than postulating the transformation property of $\nabla_a T$, the alternative is to simply define ∇_a as $e_a{}^m\partial_m + \frac{1}{2}\omega_{abc}M^{bc}$ for $\omega_{abc} = \frac{1}{2}(C_{bca} + C_{acb} - C_{abc})$ and $C_{ab}{}^c = (e_a{}^n\partial_n(e_b{}^m) - e_b{}^n\partial_n(e_a{}^m))e_m{}^c$. This way, ∇_a is just an operator constructed out of the vierbein and its transformation is determined by that of the vierbein[2]. For completeness, I will prove the $\delta\nabla_a$ transformation from this perspective as well. First consider the local Lorentz transformation. The result I am trying to get in this case is

$$\begin{aligned} &\frac{1}{2}[K^{bc}M_{bc}, \nabla_a]T \\ &= \frac{1}{2}(K^{bc}M_{bc}(\nabla_a(T)) - \nabla_a(K^{bc}M_{bc}(T)) \\ &= \frac{1}{2}(K^{bc}(\eta_{ab}\nabla_c T - \eta_{ac}\nabla_b T + \nabla_a M_{bc}T) - K^{bc}\nabla_a M_{bc}T - \nabla_a(K^{bc})M_{bc}T) \\ &= K_a{}^b\nabla_b T - \frac{1}{2}\nabla_a(K_{bc})M^{bc}T \\ &= K_a{}^b e_b{}^m\partial_m T + \frac{1}{2}K_a{}^b\omega_{bcd}M^{cd}T - \frac{1}{2}e_a{}^m\partial_m(K_{bc})M^{bc}T - \frac{1}{4}\omega_{ade}M^{de}(K_{bc})M^{bc}T \\ &= K_a{}^b e_b{}^m\partial_m T + \frac{1}{2}K_a{}^d\omega_{dbc}M^{bc}T - \frac{1}{2}e_a{}^m\partial_m(K_{bc})M^{bc}T \\ &\quad - \frac{1}{4}\omega_{ade}(\delta^d{}_b K^e{}_c - \delta^e{}_b K^d{}_c + \delta^d{}_c K_b{}^e - \delta^e{}_c K_b{}^d)M^{bc}T \\ &= K_a{}^b e_b{}^m\partial_m T + \frac{1}{2}K_a{}^d\omega_{dbc}M^{bc}T - \frac{1}{2}e_a{}^m\partial_m(K_{bc})M^{bc}T \\ &\quad - \frac{1}{4}(\omega_{abe}K^e{}_c - \omega_{adb}K^d{}_c + \omega_{ace}K_b{}^e - \omega_{adc}K_b{}^d)M^{bc}T \\ &= K_a{}^b e_b{}^m\partial_m T + \frac{1}{2}(K_a{}^d\omega_{dbc} - e_a{}^m\partial_m(K_{bc}) + K_c{}^d\omega_{abd} + K_b{}^d\omega_{adc})M^{bc}T\,. \end{aligned} \tag{A.10}$$

[2] Of course, ∇_a is defined this way so that $\nabla_a T$ transforms covariantly.

On the other hand,

$$\begin{aligned}
\delta\nabla_a T &= \nabla'_a T - \nabla_a T \\
&= e'_a{}^m \partial_m T + \frac{1}{2}\omega'_{abc} M^{bc} T - e_a{}^m \partial_m T - \frac{1}{2}\omega_{abc} M^{bc} T \\
&= K_a{}^b e_b{}^m \partial_m T + \frac{1}{2}(\omega'_{abc} - \omega_{abc}) M^{bc} T \\
&= K_a{}^b e_b{}^m \partial_m T + \frac{1}{4}(C'_{bca} + C'_{acb} - C'_{abc} - C_{bca} - C_{acb} + C_{abc}) M^{bc} T \qquad \text{(A.11)}
\end{aligned}$$

$$\begin{aligned}
C'_{abc} - C_{abc} &= \eta_{cd}(C'_{ab}{}^d - C_{ab}{}^d) \\
&= \eta_{cd}((e'_a{}^n \partial_n(e'_b{}^m) - e'_b{}^n \partial_n(e'_a{}^m)) e'_m{}^d - (e_a{}^n \partial_n(e_b{}^m) - e_b{}^n \partial_n(e_a{}^m)) e_m{}^d) \\
&= \eta_{cd}(K_a{}^e e_e{}^n \partial_n(e_b{}^m) + e_a{}^n \partial_n(K_b{}^e e_e{}^m) - K_b{}^e e_e{}^n \partial_n(e_a{}^m) - e_b{}^n \partial_n(K_a{}^e e_e{}^m)) e_m{}^d \\
&\quad + \eta_{cd}(e_a{}^n \partial_n(e_b{}^m) - e_b{}^n \partial_n(e_a{}^m)) K^d{}_e e_m{}^e \\
&= \eta_{cd} K_a{}^e (e_e{}^n \partial_n(e_b{}^m) - e_b{}^n \partial_n(e_e{}^m)) e_m{}^d - \eta_{cd} K_b{}^e (e_e{}^n \partial_n(e_a{}^m) - e_a{}^n \partial_n(e_e{}^m)) e_m{}^d \\
&\quad + \eta_{cd} e_a{}^n e_e{}^m e_m{}^d \partial_n(K_b{}^e) - \eta_{cd} e_b{}^n e_e{}^m e_m{}^d \partial_n(K_a{}^e) \\
&\quad + (e_a{}^n \partial_n(e_b{}^m) - e_b{}^n \partial_n(e_a{}^m)) K_{cd} e_m{}^d \\
&= \eta_{cd} K_a{}^e C_{eb}{}^d - \eta_{cd} K_b{}^e C_{ea}{}^d + \eta_{cd} e_a{}^n \delta^d{}_e \partial_n(K_b{}^e) - \eta_{cd} e_b{}^n \delta^d{}_e \partial_n(K_a{}^e) \\
&\quad + C_{ab}{}^d K_{cd} \\
&= K_a{}^d C_{dbc} - K_b{}^d C_{dac} + K_c{}^d C_{abd} + e_a{}^m \partial_m(K_{bc}) - e_b{}^m \partial_m(K_{ac}) \qquad \text{(A.12)}
\end{aligned}$$

Putting these back into the expression for $\delta\nabla_a T$,

$$\begin{aligned}
\delta\nabla_a T &= K_a{}^b e_b{}^m \partial_m T + \frac{1}{4}\Big(K_b{}^d C_{dca} - K_c{}^d C_{dba} + K_a{}^d C_{bcd} + e_b{}^m \partial_m(K_{ca}) - e_c{}^m \partial_m(K_{ba}) \\
&\quad + K_a{}^d C_{dcb} - K_c{}^d C_{dab} + K_b{}^d C_{acd} + e_a{}^m \partial_m(K_{cb}) - e_c{}^m \partial_m(K_{ab}) \\
&\quad - K_a{}^d C_{dbc} + K_b{}^d C_{dac} - K_c{}^d C_{abd} - e_a{}^m \partial_m(K_{bc}) + e_b{}^m \partial_m(K_{ac}) \Big) M^{bc} T \\
&= K_a{}^b e_b{}^m \partial_m T + \frac{1}{4} K_a{}^d (C_{bcd} + C_{dcb} - C_{dbc}) M^{bc} T + \frac{1}{4} K_b{}^d (C_{dca} + C_{acd} - C_{adc}) M^{bc} T \\
&\quad - \frac{1}{4} K_c{}^d (C_{dba} + C_{abd} - C_{adb}) M^{bc} T - \frac{1}{2} e_a{}^m \partial_m(K_{bc}) M^{bc} T \\
&= K_a{}^b e_b{}^m \partial_m T + \frac{1}{2}(K_a{}^d \omega_{dbc} + K_b{}^d \omega_{adc} - K_c{}^d \omega_{adb} - e_a{}^m \partial_m(K_{bc})) M^{bc} T \\
&= \frac{1}{2}[K^{bc} M_{bc}, \nabla_a] T \qquad \text{(A.13)}
\end{aligned}$$

by comparing with equation A.10.

That leaves ∇_a's change under general coordinate transformations in the 2nd approach. Let $x'^m = x^m - \xi^m(x)$ be an infinitesimal general coordinate transformation. First, consider the transformation of a scalar, $\varphi(x)$, under a general coordinate transformation. By the definition of a scalar, $\varphi'(x') = \varphi(x)$. Therefore,

$$\begin{aligned}
\varphi'(x) &= \varphi(x' + \xi) \\
&= \varphi(x + \xi) \\
&= \varphi(x) + \xi^m \partial_m(\varphi)|_x \qquad \text{(A.14)}
\end{aligned}$$

and hence $\delta\varphi = \xi^m\partial_m\varphi$. Meanwhile, this expression can be re-written by

$$\begin{aligned}[\xi^m\partial_m, \varphi]T &= \xi^m\partial_m(\varphi T) - \varphi\xi^m\partial_m(T) \\ &= \xi^m\partial_m(\varphi)T + \xi^m\partial_m(T)\varphi - \varphi\xi^m\partial_m(T) \\ &= \xi^m\partial_m(\varphi)T\,. \end{aligned} \tag{A.15}$$

Hence, since ∇_a acts like a scalar under general coordinate transformations,

$$\begin{aligned}\delta\nabla_a T &= [\xi^m\partial_m, \nabla_a]T \\ &= [e_b{}^m\xi^b\partial_m, \nabla_a]T \\ &= \left[\xi^b\nabla_b - \frac{1}{2}\xi^b\omega_{bcd}M^{cd}, \nabla_a\right]T \\ &= \left[\xi^b\nabla_b + \frac{1}{2}K^{bc}M_{bc}.\nabla_a\right]T \end{aligned} \tag{A.16}$$

where $K^{bc} = -\xi_b\omega^{bcd}$. Hence, the general coordinate transformation can be written in the required form.

Either way, putting the three types of transformations together,

$$\delta\nabla_a = \left[\xi^b\nabla_b + \frac{1}{2}K^{bc}M_{bc}, \nabla_a\right] + \sigma\nabla_a - \nabla^b(\sigma)M_{ab} \tag{A.17}$$

for some ξ, K^{bc} and σ, thereby proving the first part of the theorem.

Having established $\delta\nabla_a = \left[\xi^b\nabla_b + \frac{1}{2}K^{bc}M_{bc}, \nabla_a\right] + \sigma\nabla_a - \nabla^b(\sigma)M_{ab}$ in two different ways, the next task is to find the conditions when $\delta\nabla_a = 0$.

$$\begin{aligned}\delta\nabla_a T &= \left(\left[\xi^b\nabla_b + \frac{1}{2}K^{bc}M_{bc}, \nabla_a\right] + \sigma\nabla_a - \nabla^b(\sigma)M_{ab}\right)T \\ &= \xi^b\nabla_b\nabla_a T + \frac{1}{2}K^{bc}M_{bc}(\nabla_a T) - \nabla_a(\xi^b\nabla_b T) - \frac{1}{2}\nabla_a(K^{bc}M_{bc}T) \\ &\quad + \sigma\nabla_a T - \nabla^b(\sigma)M_{ab}T \\ &= \xi^b[\nabla_b, \nabla_a]T - \nabla_a(\xi^b)\nabla_b(T) + \frac{1}{2}K^{bc}(\eta_{ab}\nabla_c T - \eta_{ac}\nabla_b T) + \frac{1}{2}K^{bc}\nabla_a(M_{bc}T) \\ &\quad - \frac{1}{2}\nabla_a(K^{bc})M_{bc}T - \frac{1}{2}K^{bc}\nabla_a(M_{bc}T) + \sigma\nabla_a T - \nabla^b(\sigma)M_{ab}T \\ &= \frac{1}{2}R_{bacd}\xi^b M^{cd}T - \nabla_a(\xi^b)\nabla_b(T) + K_a{}^b\nabla_b T - \frac{1}{2}\nabla_a(K^{bc})M_{bc}T \\ &\quad + \sigma\nabla_a T - \nabla^b(\sigma)M_{ab}T \\ &= (K_a{}^b - \nabla_a(\xi^b) + \delta^b{}_a\sigma)\nabla_b(T) + \left(\frac{1}{2}R_{da}{}^{bc}\xi^d - \frac{1}{2}\nabla_a(K^{bc}) + \delta^c{}_a\nabla^b(\sigma)\right)M_{bc}(T) \end{aligned} \tag{A.18}$$

I am looking for $\delta\nabla_a T = 0$ for arbitrary T, so I can freely choose T to be a scalar. In that case, $M_{bc}T = 0$. Thus,

$$(K_a{}^b - \nabla_a(\xi^b) + \delta^b{}_a\sigma)\nabla_b(T) = 0 \tag{A.19}$$

on its own. Consequently, one must have

$$\left(\frac{1}{2}R_{da}{}^{bc}\xi^d - \frac{1}{2}\nabla_a(K^{bc}) + \delta^c{}_a\nabla^b(\sigma)\right)M_{bc}(T) = 0 \tag{A.20}$$

on its own too. Therefore, since T is still arbitrary, one must have

$$0 = K_a{}^b - \nabla_a(\xi^b) + \delta^b{}_a\sigma \text{ and} \tag{A.21}$$

$$0 = \frac{1}{2}R_{da}{}^{bc}\xi^d - \frac{1}{2}\nabla_a(K^{bc}) + \delta^{[c}{}_a\nabla^{b]}(\sigma) \tag{A.22}$$

with the antisymmetrisation in $\delta^{[c}{}_a\nabla^{b]}(\sigma)$ forced by the antisymmetry of M_{bc}. Next, since K_{ab} is antisymmetric, $K_a{}^a = \eta^{ab}K_{ab} = 0$. Hence,

$$0 = K_a{}^a - \nabla_a(\xi^a) + \delta^a{}_a\sigma$$

$$\implies \sigma = \frac{1}{4}\nabla_a\xi^a \tag{A.23}$$

$$\implies K_{ab} = \nabla_a\xi_b - \eta_{ab}\sigma = \nabla_a\xi_b - \frac{1}{4}\eta_{ab}\nabla_c\xi^c \tag{A.24}$$

Again, since K_{ab} is antisymmetric,

$$\nabla_a\xi_b - \frac{1}{4}\eta_{ab}\nabla_c\xi^c = -\nabla_b\xi_a + \frac{1}{4}\eta_{ba}\nabla_c\xi^c$$

$$\iff \nabla_a\xi_b + \nabla_b\xi_a = \frac{1}{2}\eta_{ab}\nabla_c\xi^c\,. \tag{A.25}$$

Therefore,

$$K_{ab} = \nabla_a\xi_b - \frac{1}{4}\eta_{ab}\nabla_c\xi^c = \frac{1}{2}(\nabla_a\xi_b - \nabla_b\xi_a)\,, \tag{A.26}$$

which makes the antisymmetry clear. In summary, imposing $\delta\nabla_a = 0 \implies \sigma = \frac{1}{4}\nabla_a\xi^a$, $K^{bc} = \frac{1}{2}(\nabla^a\xi^b - \nabla^b\xi^a)$ and $\nabla_a\xi_b + \nabla_b\xi_a = \frac{1}{2}\eta_{ab}\nabla_c\xi^c$. The last of these conditions is equivalent to saying ξ^a is a conformal Killing vector. All that is left to do to check the converse, i.e. check whether choosing $\sigma = \frac{1}{4}\nabla_a\xi^a$, $K^{bc} = \frac{1}{2}(\nabla^a\xi^b - \nabla^b\xi^a)$ and $\nabla_a\xi_b + \nabla_b\xi_a = \frac{1}{2}\eta_{ab}\nabla_c\xi^c \implies \delta\nabla_a = 0$. For that, all that is left to do given the above work is to check whether equation A.22 holds for these particular choices of ξ^a, K_{ab} and σ.

$$\begin{aligned}
&R_{dabc}\xi^d - \nabla_a(K_{bc}) + 2\eta_{[ca}\nabla_{b]}(\sigma)\\
&= R_{dabc}\xi^d - \nabla_a(K_{bc}) + \eta_{ca}\nabla_b(\sigma) - \eta_{ba}\nabla_c(\sigma)\\
&= -[\nabla_b,\nabla_c]\xi_a - \frac{1}{2}\nabla_a(\nabla_b\xi_c - \nabla_c\xi_b) + \frac{1}{4}\eta_{ca}\nabla_b\nabla_d\xi^d - \frac{1}{4}\eta_{ba}\nabla_c\nabla_d\xi^d\\
&= -\nabla_b\nabla_c\xi_a + \nabla_c\nabla_b\xi_a - \frac{1}{2}\nabla_a\nabla_b\xi_c + \frac{1}{2}\nabla_a\nabla_c\xi_b + \nabla_b\left(\frac{1}{2}\nabla_c\xi_a + \frac{1}{2}\nabla_a\xi_c\right)\\
&\quad - \nabla_c\left(\frac{1}{2}\nabla_b\xi_a + \frac{1}{2}\nabla_a\xi_b\right)\\
&= \frac{1}{2}(-\nabla_b\nabla_c\xi_a + \nabla_c\nabla_b\xi_a - \nabla_a\nabla_b\xi_c + \nabla_a\nabla_c\xi_b + \nabla_b\nabla_a\xi_c - \nabla_c\nabla_a\xi_b)\\
&= \frac{1}{2}([\nabla_c,\nabla_b]\xi_a + [\nabla_b,\nabla_a]\xi_c + [\nabla_a,\nabla_c]\xi_b)\\
&= -\frac{1}{2}(R_{dacb} + R_{dcba} + R_{dbac})\xi^d\\
&= 0
\end{aligned} \tag{A.27}$$

Raising indices,

$$0 = \frac{1}{2}R_{da}{}^{bc}\xi^d - \frac{1}{2}\nabla_aK^{bc} + \delta^{[c}{}_a\nabla^{b]}\sigma\,, \tag{A.28}$$

which is indeed equation A.22.

Appendix B

Proof of theorem 3.11

I have to show that $\Delta\varphi = 0$ and

$$D^{(2)} = \xi^{ab}\nabla_a\nabla_b + \frac{2}{3}\nabla_b(\xi^{ab})\nabla_a + \frac{1}{15}\nabla_a\nabla_b(\xi^{ab}) - \frac{3}{10}R_{ab}\xi^{ab} \tag{B.1}$$

$$\begin{aligned}\implies \Delta D^{(2)}\varphi = &\left(\frac{4}{15}C^a{}_{bcd}\nabla^c(\xi^{bd}) + \frac{4}{5}\nabla^d(C^a{}_{bcd})\xi^{bc}\right)\nabla_a(\varphi)\\ &+\left(\frac{2}{15}C_{abcd}\nabla^a\nabla^c(\xi^{bd}) + \frac{2}{5}\nabla^c\nabla_d(C^d{}_{abc})\xi^{ab} + \frac{4}{15}\nabla_d(C^d{}_{abc})\nabla^c(\xi^{ab})\right)\varphi\,,\end{aligned} \tag{B.2}$$

provided ξ^{ab} is a conformal Killing tensor. Therefore,

$$\begin{aligned}&\nabla^c\xi^{ab} + \nabla^a\xi^{bc} + \nabla^b\xi^{ca} = \frac{1}{3}\left(\eta^{ab}\nabla_d\xi^{cd} + \eta^{bc}\nabla_d\xi^{ad} + \eta^{ca}\nabla_d\xi^{bd}\right),\\ &\xi^{ab} = \xi^{ba} \text{ and } \xi^a{}_a = 0\end{aligned} \tag{B.3}$$

by imposing that ξ^{ab} is symmetric and traceless and that the symmetric and traceless part of $\nabla^c\xi^{ab}$ is zero.

$$\Delta D^{(2)}\varphi = \left(\Box - \frac{1}{6}R\right)\left(D^{(2)} = \xi^{ab}\nabla_a\nabla_b + \frac{2}{3}\nabla_b(\xi^{ab})\nabla_a + \frac{1}{15}\nabla_a\nabla_b(\xi^{ab}) - \frac{3}{10}R_{ab}\xi^{ab}\right)\varphi \tag{B.4}$$

I will now expand this term by term and use $\Delta\varphi = 0$ and the conformal Killing conditions, equation B.3, to reduce the number of derivatives on φ.

$$\Box(\xi^{ab}\nabla_a\nabla_b\varphi) = \Box(\xi^{ab})\nabla_a\nabla_b\varphi + 2\nabla^c(\xi^{ab})\nabla_c\nabla_a\nabla_b(\varphi) + \xi^{ab}\Box\nabla_a\nabla_b\varphi \tag{B.5}$$

I will move the $\Box$ to the front in the last term. Expanding commutators in the standard way,

$$\begin{aligned}&[\Box, \nabla_a\nabla_b] = \nabla^c\nabla_a[\nabla_c, \nabla_b] + \nabla^c[\nabla_c, \nabla_a]\nabla_b + \nabla_a[\nabla_c, \nabla_b]\nabla^c + [\nabla_c, \nabla_a]\nabla_b\nabla^c\\ \implies &\xi^{ab}\Box\nabla_a\nabla_b\varphi = \xi^{ab}\nabla_a\nabla_b\Box\varphi + \xi^{ab}[\Box, \nabla_a\nabla_b]\varphi\\ &\qquad = \xi^{ab}(\nabla_a\nabla_b(R\varphi/6) + \nabla^c\nabla_a[\nabla_c, \nabla_b]\varphi + \nabla^c[\nabla_c, \nabla_a]\nabla_b\varphi\\ &\qquad\quad + \nabla_a[\nabla_c, \nabla_b]\nabla^c\varphi + [\nabla_c, \nabla_a]\nabla_b\nabla^c\varphi)\,.\end{aligned} \tag{B.6}$$

$[\nabla_c, \nabla_b]\varphi = 0$ as φ is a scalar, $[\nabla_c, \nabla_a]\nabla_b\varphi = R_{cabd}\nabla^d\varphi$, $[\nabla_c, \nabla_b]\nabla^c\varphi = R_{cb}{}^c{}_d\nabla^d\varphi = R_{bc}\nabla^c\varphi$ and $[\nabla_c, \nabla_a]\nabla_b\nabla^c\varphi = R_{cabd}\nabla^d\nabla^c\varphi + R_{ca}{}^c{}_d\nabla_b\nabla^d\varphi = R_{cabd}\nabla^d\nabla^c\varphi + R_{ac}\nabla_b\nabla^c\varphi$. Thus,

$$\begin{aligned}\xi^{ab}\Box\nabla_a\nabla_b\varphi = &\frac{1}{6}\xi^{ab}\nabla_a\nabla_b(R\varphi) + \xi^{ab}\nabla^c(R_{cabd}\nabla^d\varphi) + \xi^{ab}\nabla_a(R_{bc}\nabla^c\varphi)\\ &+ \xi^{ab}R_{cabd}\nabla^d\nabla^c\varphi + \xi^{ab}R_{ac}\nabla_b\nabla^c\varphi\\ = &\frac{1}{6}\xi^{ab}\nabla_a\nabla_b(R\varphi) + 2\xi^{ab}R_{cabd}\nabla^c\nabla^d\varphi + 2\xi^{ab}R_{bc}\nabla_a\nabla^c\varphi\\ &+ \xi^{ab}\nabla^c(R_{cabd})\nabla^d\varphi + \xi^{ab}\nabla_a(R_{bc})\nabla^c\varphi\,.\end{aligned} \tag{B.7}$$

The next term to simplify is $\nabla^c(\xi^{ab})\nabla_c\nabla_a\nabla_b(\varphi)$. Using the conformal Killing conditions, equation B.3, and some of the previously calculated commutators,

$$\begin{aligned}
&\nabla^c(\xi^{ab})\nabla_c\nabla_a\nabla_b(\varphi)\\
&=\frac{1}{3}\nabla^c(\xi^{ab})(\nabla_c\nabla_a\nabla_b+\nabla_a\nabla_c\nabla_b+\nabla_b\nabla_c\nabla_a+[\nabla_c,\nabla_a]\nabla_b+[\nabla_c,\nabla_b]\nabla_a)\varphi\\
&=\frac{1}{3}(\nabla^c(\xi^{ab})+\nabla^a(\xi^{bc})+\nabla^b(\xi^{ca}))\nabla_c\nabla_a\nabla_b(\varphi)\\
&\quad+\frac{1}{3}\nabla^c(\xi^{ab})([\nabla_c,\nabla_a]\nabla_b\varphi+[\nabla_c,\nabla_b]\nabla_a\varphi)\\
&=\frac{1}{3}(\nabla^c(\xi^{ab})+\nabla^a(\xi^{bc})+\nabla^b(\xi^{ca}))\nabla_c\nabla_a\nabla_b(\varphi)+\frac{2}{3}\nabla^c(\xi^{ab})[\nabla_c,\nabla_a]\nabla_b\varphi\\
&=\frac{1}{9}(\eta^{ab}\nabla_d(\xi^{cd})+\eta^{bc}\nabla_d(\xi^{ad})+\eta^{ca}\nabla_d(\xi^{bd}))\nabla_c\nabla_a\nabla_b(\varphi)+\frac{2}{3}\nabla^c(\xi^{ab})[\nabla_c,\nabla_a]\nabla_b\varphi\\
&=\frac{1}{9}\nabla_a(\xi^{ab})\nabla_b\Box\varphi+\frac{1}{9}\nabla_b(\xi^{ab})\nabla_c\nabla_a\nabla^c\varphi+\frac{1}{9}\nabla_b(\xi^{ab})\Box\nabla_a\varphi+\frac{2}{3}\nabla^c(\xi^{ab})R_{cabd}\nabla^d\varphi\\
&=\frac{1}{3}\nabla_a(\xi^{ab})\nabla_b\Box\varphi+\frac{1}{9}\nabla_a(\xi^{ab})[\nabla_c,\nabla_b]\nabla^c\varphi+\frac{1}{9}\nabla_a(\xi^{ab})[\Box,\nabla_b]\varphi+\frac{2}{3}\nabla^c(\xi^{ab})R_{cabd}\nabla^d\varphi\\
&=\frac{1}{18}\nabla_a(\xi^{ab})\nabla_b(R\varphi)+\frac{2}{9}\nabla_a(\xi^{ab})[\nabla_c,\nabla_b]\nabla^c\varphi+\frac{2}{3}\nabla^c(\xi^{ab})R_{cabd}\nabla^d\varphi\\
&=\frac{1}{18}\nabla_a(\xi^{ab})\nabla_b(R\varphi)+\frac{2}{9}\nabla_a(\xi^{ab})R_{bc}\nabla^c\varphi+\frac{2}{3}\nabla^c(\xi^{ab})R_{cabd}\nabla^d\varphi\,. \qquad \text{(B.8)}
\end{aligned}$$

The only other term in $\Delta D^{(2)}\varphi$ that can lead to more than two derivatives on φ is

$$\begin{aligned}
&\Box(\nabla_b(\xi^{ab})\nabla_a\varphi)\\
&=\Box\nabla_b(\xi^{ab})\nabla_a(\varphi)+2\nabla^c\nabla_b(\xi^{ab})\nabla_c\nabla_a(\varphi)+\nabla_b(\xi^{ab})\Box\nabla_a(\varphi)\\
&=\Box\nabla_b(\xi^{ab})\nabla_a(\varphi)+2\nabla^c\nabla_b(\xi^{ab})\nabla_c\nabla_a(\varphi)+\nabla_b(\xi^{ab})\nabla_a\Box(\varphi)+\nabla_b(\xi^{ab})[\Box,\nabla_a]\varphi\\
&=\Box\nabla_b(\xi^{ab})\nabla_a(\varphi)+2\nabla^c\nabla_b(\xi^{ab})\nabla_c\nabla_a(\varphi)+\frac{1}{6}\nabla_b(\xi^{ab})\nabla_a(R\varphi)+\nabla_b(\xi^{ab})R_{ac}\nabla^c\varphi\,. \qquad \text{(B.9)}
\end{aligned}$$

No commutators are required for the remaining terms in $\Delta D^{(2)}\varphi$. Hence, substituting the previous page of expansions into equation B.4 gives

$$\begin{aligned}
\Delta D^{(2)}\varphi=&\Box(\xi^{ab})\nabla_a\nabla_b\varphi+\frac{1}{9}\nabla_a(\xi^{ab})\nabla_b(R\varphi)+\frac{4}{9}\nabla_a(\xi^{ab})R_{bc}\nabla^c\varphi+\frac{4}{3}\nabla^c(\xi^{ab})R_{cabd}\nabla^d\varphi\\
&+\frac{1}{6}\xi^{ab}\nabla_a\nabla_b(R\varphi)+2\xi^{ab}R_{cabd}\nabla^c\nabla^d\varphi+2\xi^{ab}R_{bc}\nabla_a\nabla^c\varphi+\xi^{ab}\nabla^c(R_{cabd})\nabla^d\varphi\\
&+\xi^{ab}\nabla_a(R_{bc})\nabla^c\varphi+\frac{2}{3}\Box\nabla_b(\xi^{ab})\nabla_a(\varphi)+\frac{4}{3}\nabla^c\nabla_b(\xi^{ab})\nabla_c\nabla_a(\varphi)+\frac{1}{9}\nabla_b(\xi^{ab})\nabla_a(R\varphi)\\
&+\frac{2}{3}\nabla_b(\xi^{ab})R_{ac}\nabla^c\varphi+\frac{1}{15}\Box\nabla_a\nabla_b(\xi^{ab})\varphi+\frac{2}{15}\nabla^c\nabla_a\nabla_b(\xi^{ab})\nabla_c\varphi+\frac{1}{15}\nabla_a\nabla_b(\xi^{ab})\Box(\varphi)\\
&-\frac{3}{10}\Box(R_{ab}\xi^{ab})\varphi-\frac{3}{5}\nabla^c(R_{ab}\xi^{ab})\nabla_c(\varphi)-\frac{3}{10}R_{ab}\xi^{ab}\Box\varphi-\frac{1}{6}R\xi^{ab}\nabla_a\nabla_b\varphi\\
&-\frac{1}{9}R\nabla_b(\xi^{ab})\nabla_a(\varphi)-\frac{1}{90}R\nabla_a\nabla_b(\xi^{ab})\varphi+\frac{1}{20}RR_{ab}\xi^{ab}\varphi\,. \qquad \text{(B.10)}
\end{aligned}$$

Then, using $\Box\varphi = R\varphi/6$, expanding some $\nabla(\cdot)$ by the product rule and packaging together all terms with equal number of derivatives on φ,

$$\begin{aligned}
\Delta D^{(2)}\varphi = &\left\{\Box(\xi^{ab}) + 2\xi^{cd}R^{a}{}_{cd}{}^{b} + 2\xi^{ac}R_{c}{}^{b} + \frac{4}{3}\nabla^b\nabla_c(\xi^{ac})\right\}\nabla_a\nabla_b(\varphi) \\
&+\left\{\frac{4}{9}\nabla_c(\xi^{cb})R_b{}^a + \frac{4}{3}\nabla^c(\xi^{db})R_{cdb}{}^a + \frac{1}{3}\xi^{ab}\nabla_b(R) + \xi^{db}\nabla^c(R_{cdb}{}^a) + \xi^{cb}\nabla_c(R_b{}^a)\right. \\
&\quad + \frac{2}{3}\Box\nabla_b(\xi^{ab}) + \frac{1}{9}R\nabla_b(\xi^{ab}) + \frac{2}{3}\nabla_b(\xi^{cb})R_c{}^a + \frac{2}{15}\nabla^a\nabla_c\nabla_b(\xi^{cb}) \\
&\quad\left. - \frac{3}{5}\nabla^a(R_{bc}\xi^{bc})\right\}\nabla_a(\varphi) \\
&+\left\{\frac{1}{9}\nabla_a(\xi^{ab})\nabla_b(R) + \frac{1}{6}\xi^{ab}\nabla_a\nabla_b(R) + \frac{1}{9}\nabla_b(\xi^{ab})\nabla_a(R) + \frac{1}{15}\Box\nabla_a\nabla_b(\xi^{ab})\right. \\
&\quad\left. - \frac{3}{10}\Box(R_{ab}\xi^{ab})\right\}\varphi\,.
\end{aligned} \tag{B.11}$$

Then, collating some terms,

$$\begin{aligned}
\Delta D^{(2)}\varphi = &\left\{\Box(\xi^{ab}) + 2\xi^{cd}R^{a}{}_{cd}{}^{b} + 2\xi^{ac}R_{c}{}^{b} + \frac{4}{3}\nabla^b\nabla_c(\xi^{ac})\right\}\nabla_a\nabla_b(\varphi) \\
&+\left\{\frac{10}{9}\nabla_c(\xi^{cb})R_b{}^a + \frac{4}{3}\nabla^c(\xi^{db})R_{cdb}{}^a + \frac{1}{3}\xi^{ab}\nabla_b(R) + \xi^{db}\nabla^c(R_{cdb}{}^a) + \xi^{cb}\nabla_c(R_b{}^a)\right. \\
&\quad\left. + \frac{2}{3}\Box\nabla_b(\xi^{ab}) + \frac{1}{9}R\nabla_b(\xi^{ab}) + \frac{2}{15}\nabla^a\nabla_c\nabla_b(\xi^{cb}) - \frac{3}{5}\nabla^a(R_{bc}\xi^{bc})\right\}\nabla_a(\varphi) \\
&+\left\{\frac{2}{9}\nabla_a(\xi^{ab})\nabla_b(R) + \frac{1}{6}\xi^{ab}\nabla_a\nabla_b(R) + \frac{1}{15}\Box\nabla_a\nabla_b(\xi^{ab}) - \frac{3}{10}\Box(R_{ab}\xi^{ab})\right\}\varphi\,.
\end{aligned} \tag{B.12}$$

Let $\{i\}$ denote the coefficient of the term with i derivatives of φ. On each $\{i\}$, my strategy will be to use commutators and the conformal Killing equation to first cancel out all terms without curvature factors.

I will start with $\{2\}$. The conformal Killing condition is

$$\nabla^c\xi^{ab} + \nabla^a\xi^{bc} + \nabla^b\xi^{ca} = \frac{1}{3}\left(\eta^{ab}\nabla_d\xi^{cd} + \eta^{bc}\nabla_d\xi^{ad} + \eta^{ca}\nabla_d\xi^{bd}\right) \tag{B.13}$$

Therefore,

$$\begin{aligned}
&\Box\xi^{ab} + \nabla_c\nabla^a\xi^{bc} + \nabla_c\nabla^b\xi^{ca} = \frac{1}{3}\left(\eta^{ab}\nabla_c\nabla_d\xi^{cd} + \nabla^b\nabla_d\xi^{ad} + \nabla^a\nabla_d\xi^{bd}\right) \\
\iff\ &(\Box\xi^{ab} + 2\nabla_c\nabla^b\xi^{ac})\nabla_a\nabla_b(\varphi) = \frac{1}{3}\nabla_a\nabla_b(\xi^{ab})\Box(\varphi) + \frac{2}{3}\nabla^b\nabla_c(\xi^{ac})\nabla_a\nabla_b(\varphi)
\end{aligned} \tag{B.14}$$

Re-arranging,

$$\begin{aligned}
&(\Box\xi^{ab} + 2\nabla^b\nabla_c\xi^{ac} + 2[\nabla_c,\nabla^b]\xi^{ac})\nabla_a\nabla_b(\varphi) \\
&\quad = \frac{1}{18}R\nabla_a\nabla_b(\xi^{ab})\varphi + \frac{2}{3}\nabla^b\nabla_c(\xi^{ac})\nabla_a\nabla_b(\varphi)\,,
\end{aligned} \tag{B.15}$$

and thus

$$\Box(\xi^{ab})\nabla_a\nabla_b(\varphi) = -\frac{4}{3}\nabla^b\nabla_c(\xi^{ac})\nabla_a\nabla_b(\varphi) - 2[\nabla_c,\nabla^b](\xi^{ac})\nabla_a\nabla_b(\varphi) + \frac{1}{18}R\nabla_a\nabla_b(\xi^{ab})\varphi\,. \tag{B.16}$$

$$\begin{aligned}[\nabla_c, \nabla^b]\xi^{ac} &= R_c{}^{ba}{}_d\xi^{dc} + R_c{}^{bc}{}_d\xi^{ad} \\ &= R^a{}_{dc}{}^b\xi^{cd} + \xi^{ac}R_c{}^b \end{aligned} \tag{B.17}$$

$$\begin{aligned}\implies \Box(\xi^{ab})\nabla_a\nabla_b(\varphi) &= -\frac{4}{3}\nabla^b\nabla_c(\xi^{ac})\nabla_a\nabla_b(\varphi) - 2(R^a{}_{dc}{}^b\xi^{cd} + \xi^{ac}R_c{}^b)\nabla_a\nabla_b(\varphi) \\ &\quad + \frac{1}{18}R\nabla_a\nabla_b(\xi^{ab})\varphi \end{aligned} \tag{B.18}$$

Hence, $\{2\}\nabla_a\nabla_b(\varphi) = \frac{1}{18}R\nabla_a\nabla_b(\xi^{ab})\varphi$ and $\{2\}$ can be absorbed into $\{0\}$. That leaves

$$\begin{aligned}\Delta D^{(2)}\varphi = &\left\{\frac{10}{9}\nabla_c(\xi^{cb})R_b{}^a + \frac{4}{3}\nabla^c(\xi^{db})R_{cdb}{}^a + \frac{1}{3}\xi^{ab}\nabla_b(R) + \xi^{db}\nabla^c(R_{cdb}{}^a) + \xi^{cb}\nabla_c(R_b{}^a)\right. \\ &\left. + \frac{2}{3}\Box\nabla_b(\xi^{ab}) + \frac{1}{9}R\nabla_b(\xi^{ab}) + \frac{2}{15}\nabla^a\nabla_c\nabla_b(\xi^{cb}) - \frac{3}{5}\nabla^a(R_{bc}\xi^{bc})\right\}\nabla_a(\varphi) \\ &+ \left\{\frac{2}{9}\nabla_a(\xi^{ab})\nabla_b(R) + \frac{1}{6}\xi^{ab}\nabla_a\nabla_b(R) + \frac{1}{15}\Box\nabla_a\nabla_b(\xi^{ab}) - \frac{3}{10}\Box(R_{ab}\xi^{ab})\right. \\ &\left. + \frac{1}{18}R\nabla_a\nabla_b(\xi^{ab})\right\}\varphi\,. \end{aligned} \tag{B.19}$$

Like before, in simplifying $\{1\}$ I will try to cancel $\Box\nabla_b(\xi^{ab})$ with $\nabla^a\nabla_c\nabla_b(\xi^{cb})$ at the expense of curvature terms.

$$\Box\nabla_b\xi^{ab} = \nabla_b\Box\xi^{ab} + [\Box, \nabla_b]\xi^{ab} \tag{B.20}$$

$$\begin{aligned}[\Box, \nabla_b]\xi^{ab} &= [\nabla^c\nabla_c, \nabla_b]\xi^{ab} \\ &= \nabla^c[\nabla_c, \nabla_b]\xi^{ab} + [\nabla_c, \nabla_b]\nabla^c\xi^{ab} \\ &= \nabla^c(R_{cb}{}^a{}_d\xi^{db} + R_{cb}{}^b{}_d\xi^{ad}) + R_{cb}{}^c{}_d\nabla^d\xi^{ab} + R_{cb}{}^a{}_d\nabla^c\xi^{db} + R_{cb}{}^b{}_d\nabla^c\xi^{ad} \\ &= \nabla^c(R^a{}_{dcb}\xi^{db} - R_{cd}\xi^{ad}) + R_{bd}\nabla^d\xi^{ab} + R^a{}_{dcb}\nabla^c\xi^{db} - R_{cd}\nabla^c\xi^{ad} \\ &= \nabla^c(R^a{}_{dcb}\xi^{db} - R_{cd}\xi^{ad}) + R^a{}_{dcb}\nabla^c\xi^{db} \\ &= 2R^a{}_{dcb}\nabla^c\xi^{db} + \xi^{db}\nabla^c(R^a{}_{dcb}) - \nabla^c(R_{cd})\xi^{ad} - R_{cd}\nabla^c(\xi^{ad}) \\ &= 2R^a{}_{dcb}\nabla^c\xi^{db} + \xi^{db}\nabla^c(R^a{}_{dcb}) - \frac{1}{2}\nabla_b(R)\xi^{ab} - R_{cb}\nabla^c(\xi^{ab}) \end{aligned} \tag{B.21}$$

Then, by equation B.14,

$$\begin{aligned}\nabla_b\Box\xi^{ab} &= \nabla_b\Big(-\nabla_c\nabla^a\xi^{bc} - \nabla_c\nabla^b\xi^{ca} + \frac{1}{3}\Big(\eta^{ab}\nabla_c\nabla_d\xi^{cd} + \nabla^b\nabla_d\xi^{ad} + \nabla^a\nabla_d\xi^{bd}\Big)\Big) \\ &= -\nabla_b\nabla_c\nabla^a\xi^{bc} - \nabla_b\nabla_c\nabla^b\xi^{ca} + \frac{1}{3}\nabla^a\nabla_c\nabla_d\xi^{cd} + \frac{1}{3}\Box\nabla_b\xi^{ab} + \frac{1}{3}\nabla_b\nabla^a\nabla_d\xi^{bd} \\ &= -\frac{1}{3}\nabla^a\nabla_c\nabla_b\xi^{cb} - [\nabla_c\nabla_b, \nabla^a]\xi^{cb} + \frac{1}{3}[\nabla_c, \nabla^a]\nabla_b\xi^{cb} - \frac{2}{3}\Box\nabla_b\xi^{ab} \\ &\quad - \nabla_c[\nabla_b, \nabla^c]\xi^{ab} \\ &= -\frac{1}{3}\nabla^a\nabla_c\nabla_b\xi^{cb} - \nabla_c[\nabla_b, \nabla^a]\xi^{cb} - \frac{2}{3}[\nabla_c, \nabla^a]\nabla_b\xi^{cb} - \frac{2}{3}\Box\nabla_b\xi^{ab} \\ &\quad - \nabla_c[\nabla_b, \nabla^c]\xi^{ab}\,. \end{aligned} \tag{B.22}$$

$$\begin{aligned}\nabla_c[\nabla_b, \nabla^a]\xi^{cb} &= \nabla_c(R_b{}^{ac}{}_d\xi^{db} + R_b{}^{ab}{}_d\xi^{cd}) \\ &= -\nabla^c(R^a{}_{bcd}\xi^{db}) + \nabla_c(R_b{}^a\xi^{cb}) \\ &= -\nabla^c(R^a{}_{bcd})\xi^{db} - R^a{}_{bcd}\nabla^c(\xi^{db}) + \nabla_c(R_b{}^a)\xi^{cb} + R_b{}^a\nabla_c(\xi^{cb}) \end{aligned} \tag{B.23}$$

$$[\nabla_c, \nabla^a]\nabla_b\xi^{cb} = R_c{}^{ac}{}_d\nabla_b\xi^{db} = R_b{}^a\nabla_c\xi^{bc} \tag{B.24}$$

$$
\begin{aligned}
\nabla_c[\nabla_b, \nabla^c]\xi^{ab} &= \nabla_c({R_b}^{ca}{}_d\xi^{db} + {R_b}^{cb}{}_d\xi^{ad}) \\
&= \nabla^c({R^a}_{bdc}\xi^{db}) + \nabla_c({R^c}_b\xi^{ab}) \\
&= -\nabla^c({R^a}_{bcd})\xi^{db} - {R^a}_{bcd}\nabla^c\xi^{db} + \frac{1}{2}\nabla_b(R)\xi^{ab} + R_{cb}\nabla^c\xi^{ab} \qquad \text{(B.25)}
\end{aligned}
$$

Substituting back and re-arranging along the way,

$$
\begin{aligned}
\nabla_b\Box\xi^{ab} &= -\frac{1}{3}\nabla^a\nabla_c\nabla_b\xi^{cb} - \frac{2}{3}\Box\nabla_b\xi^{ab} + \nabla^c({R^a}_{bcd})\xi^{db} + {R^a}_{bcd}\nabla^c(\xi^{db}) - \nabla_c({R_b}^a)\xi^{cb} \\
&\quad - {R_b}^a\nabla_c(\xi^{cb}) - \frac{2}{3}{R_b}^a\nabla_c\xi^{bc} + \nabla^c({R^a}_{bcd})\xi^{db} + {R^a}_{bcd}\nabla^c\xi^{db} - \frac{1}{2}\nabla_b(R)\xi^{ab} \\
&\quad - R_{cb}\nabla^c\xi^{ab} \\
&= -\frac{1}{3}\nabla^a\nabla_c\nabla_b\xi^{cb} - \frac{2}{3}\Box\nabla_b\xi^{ab} + 2\nabla^c({R^a}_{bcd})\xi^{db} + 2{R^a}_{bcd}\nabla^c(\xi^{db}) - \nabla_c({R_b}^a)\xi^{cb} \\
&\quad - \frac{5}{3}{R_b}^a\nabla_c\xi^{bc} - \frac{1}{2}\nabla_b(R)\xi^{ab} - R_{cb}\nabla^c\xi^{ab} \qquad \text{(B.26)}
\end{aligned}
$$

$$
\begin{aligned}
\implies \Box\nabla_b\xi^{ab} &= \nabla_b\Box\xi^{ab} + [\Box, \nabla_b]\xi^{ab} \\
&= -\frac{1}{3}\nabla^a\nabla_c\nabla_b\xi^{cb} - \frac{2}{3}\Box\nabla_b\xi^{ab} + 2\nabla^c({R^a}_{bcd})\xi^{db} + 2{R^a}_{bcd}\nabla^c(\xi^{db}) - \nabla_c({R_b}^a)\xi^{cb} \\
&\quad - \frac{5}{3}{R_b}^a\nabla_c\xi^{bc} - \frac{1}{2}\nabla_b(R)\xi^{ab} - R_{cb}\nabla^c\xi^{ab} + 2{R^a}_{dcb}\nabla^c\xi^{db} + \xi^{db}\nabla^c({R^a}_{dcb}) \\
&\quad - \frac{1}{2}\nabla_b(R)\xi^{ab} - R_{cb}\nabla^c(\xi^{ab}) \qquad \text{(B.27)}
\end{aligned}
$$

$$
\begin{aligned}
\frac{5}{3}\Box\nabla_b\xi^{ab} &= -\frac{1}{3}\nabla^a\nabla_c\nabla_b\xi^{cb} + 3\nabla^c({R^a}_{bcd})\xi^{db} + 4{R^a}_{bcd}\nabla^c(\xi^{db}) - \nabla_c({R_b}^a)\xi^{cb} \\
&\quad - \frac{5}{3}{R_b}^a\nabla_c\xi^{bc} - \nabla_b(R)\xi^{ab} - 2R_{cb}\nabla^c\xi^{ab} \qquad \text{(B.28)}
\end{aligned}
$$

$$
\begin{aligned}
\frac{2}{3}\Box\nabla_b\xi^{ab} &= -\frac{2}{15}\nabla^a\nabla_c\nabla_b\xi^{cb} + \frac{6}{5}\nabla^c({R^a}_{bcd})\xi^{db} + \frac{8}{5}{R^a}_{bcd}\nabla^c(\xi^{db}) - \frac{2}{5}\nabla_c({R_b}^a)\xi^{cb} \\
&\quad - \frac{2}{3}{R_b}^a\nabla_c\xi^{bc} - \frac{2}{5}\nabla_b(R)\xi^{ab} - \frac{4}{5}R_{cb}\nabla^c\xi^{ab}\,. \qquad \text{(B.29)}
\end{aligned}
$$

With this equation, $\{1\}$ can be re-written as

$$
\begin{aligned}
\{1\} &= \frac{10}{9}\nabla_c(\xi^{cb}){R_b}^a + \frac{4}{3}\nabla^c(\xi^{db}){R_{cdb}}^a + \frac{1}{3}\xi^{ab}\nabla_b(R) + \xi^{db}\nabla^c({R_{cdb}}^a) + \xi^{cb}\nabla_c({R_b}^a) + \frac{1}{9}R\nabla_b\xi^{ab} \\
&\quad - \frac{3}{5}\nabla^a(R_{cb}\xi^{cb}) + \frac{6}{5}\nabla^c({R^a}_{bcd})\xi^{bd} + \frac{8}{5}{R^a}_{bcd}\nabla^c\xi^{bd} - \frac{2}{5}\nabla_c({R_b}^a)\xi^{bc} - \frac{2}{3}{R_b}^a\nabla_c(\xi^{bc}) \\
&\quad - \frac{2}{5}\nabla_b(R)\xi^{ab} - \frac{4}{5}R_{cb}\nabla^c\xi^{ab} \\
&= \frac{4}{9}{R_b}^a\nabla_c(\xi^{cb}) + \frac{4}{15}{R^a}_{bcd}\nabla^c(\xi^{bd}) - \frac{1}{15}\nabla_b(R)\xi^{ab} + \frac{1}{5}\nabla^c({R^a}_{bcd})\xi^{bd} + \frac{3}{5}\nabla_c({R_b}^a)\xi^{bc} \\
&\quad + \frac{1}{9}R\nabla_b(\xi^{ab}) - \frac{3}{5}\nabla^a(R_{cb}\xi^{cb}) - \frac{4}{5}R_{cb}\nabla^c\xi^{ab}\,. \qquad \text{(B.30)}
\end{aligned}
$$

Then, via $\nabla^c({R^a}_{bcd}) = \nabla^c({R_{cd}}^a{}_b) = -\nabla^a({R_{cdb}}^c) - \nabla_b({R_{cd}}^{ca}) = \nabla^a(R_{bd}) - \nabla_b({R_d}^a)$,

$$
\begin{aligned}
\{1\} &= \frac{4}{9}{R_b}^a\nabla_c(\xi^{cb}) + \frac{4}{15}{R^a}_{bcd}\nabla^c(\xi^{bd}) - \frac{1}{15}\nabla_b(R)\xi^{ab} + \frac{2}{5}\nabla_c({R_b}^a)\xi^{bc} + \frac{1}{9}R\nabla_b(\xi^{ab}) \\
&\quad - \frac{2}{5}\nabla^a(R_{cb})\xi^{cb} - \frac{3}{5}R_{bc}\nabla^a(\xi^{bc}) - \frac{4}{5}R_{cb}\nabla^c(\xi^{ab})\,. \qquad \text{(B.31)}
\end{aligned}
$$

Having removed all "curvature-free" terms from $\{1\}$, I will leave its analysis here for now and try manipulate $\{0\}$ in the same way.

From equation B.19, in $\{0\}$ the only term without a curvature factor is $\Box\nabla_a\nabla_b\xi^{ab}$.

$$\Box\nabla_a\nabla_b\xi^{ab} = \nabla_a\nabla_b\Box\xi^{ab} + [\Box, \nabla_a\nabla_b]\xi^{ab} \tag{B.32}$$

$$\begin{aligned}[\Box, \nabla_a\nabla_b]\xi^{ab} &= \nabla^c[\nabla_c, \nabla_a\nabla_b]\xi^{ab} + [\nabla_c, \nabla_a\nabla_b]\nabla^c\xi^{ab} \\ &= \nabla^c\nabla_a[\nabla_c, \nabla_b]\xi^{ab} + \nabla^c[\nabla_c, \nabla_a]\nabla_b\xi^{ab} + \nabla_a[\nabla_c, \nabla_b]\nabla^c\xi^{ab} \\ &\quad + [\nabla_c, \nabla_a]\nabla_b\nabla^c\xi^{ab}\end{aligned} \tag{B.33}$$

Again, using the conformal Killing equation,

$$\begin{aligned}\nabla_a\nabla_b\Box\xi^{ab} &= \nabla_a\nabla_b\Big(- \nabla_c\nabla^a\xi^{bc} - \nabla_c\nabla^b\xi^{ca} \\ &\quad + \frac{1}{3}\Big(\eta^{ab}\nabla_c\nabla_d\xi^{cd} + \eta^{bc}\nabla_c\nabla_d\xi^{ad} + \eta^{ca}\nabla_c\nabla_d\xi^{bd}\Big)\Big) \\ &= -\nabla_a\nabla_b\nabla_c\nabla^a\xi^{bc} - \nabla_a\nabla_b\nabla_c\nabla^b\xi^{ca} + \frac{1}{3}\Box\nabla_a\nabla_b\xi^{ab} + \frac{1}{3}\nabla_a\Box\nabla_b\xi^{ab} \\ &\quad + \frac{1}{3}\nabla_a\nabla_b\nabla^a\nabla_c\xi^{bc}.\end{aligned} \tag{B.34}$$

$$\begin{aligned}\nabla_a\nabla_b\nabla_c\nabla^a\xi^{bc} &= \Box\nabla_a\nabla_b\xi^{ab} + \nabla_a[\nabla_b\nabla_c, \nabla^a]\xi^{bc} \\ &= \Box\nabla_a\nabla_b\xi^{ab} + \nabla_a\nabla_b[\nabla_c, \nabla^a]\xi^{bc} + \nabla_a[\nabla_b, \nabla^a]\nabla_c\xi^{bc}\end{aligned} \tag{B.35}$$

$$\begin{aligned}\nabla_a\nabla_b\nabla_c\nabla^b\xi^{ca} &= \nabla_b\nabla_a\nabla_c\nabla^b\xi^{ca} + [\nabla_a, \nabla_b]\nabla_c\nabla^b\xi^{ca} \\ &= \Box\nabla_a\nabla_b\xi^{ab} + \nabla_b[\nabla_a\nabla_c, \nabla^b]\xi^{ca} + [\nabla_a, \nabla_b]\nabla_c\nabla^b\xi^{ca} \\ &= \Box\nabla_a\nabla_b\xi^{ab} + \nabla_b\nabla_a[\nabla_c, \nabla^b]\xi^{ca} + \nabla_b[\nabla_a, \nabla^b]\nabla_c\xi^{ca} + [\nabla_a, \nabla_b]\nabla_c\nabla^b\xi^{ca}\end{aligned} \tag{B.36}$$

$$\begin{aligned}\nabla_a\Box\nabla_b\xi^{ab} &= \Box\nabla_a\nabla_b\xi^{ab} + [\nabla_a, \Box]\nabla_b\xi^{ab} \\ &= \Box\nabla_a\nabla_b\xi^{ab} + \nabla^c[\nabla_a, \nabla_c]\nabla_b\xi^{ab} + [\nabla_a, \nabla_c]\nabla^c\nabla_b\xi^{ab}\end{aligned} \tag{B.37}$$

$$\nabla_a\nabla_b\nabla^a\nabla_c\xi^{bc} = \Box\nabla_a\nabla_b\xi^{ab} + \nabla_a[\nabla_b, \nabla^a]\nabla_c\xi^{bc} \tag{B.38}$$

Substituting back,

$$\begin{aligned}\nabla_a\nabla_b\Box\xi^{ab} &= -\Box\nabla_a\nabla_b\xi^{ab} - \nabla_a\nabla_b[\nabla_c, \nabla^a]\xi^{bc} - \nabla_a[\nabla_b, \nabla^a]\nabla_c\xi^{bc} - \nabla_b\nabla_a[\nabla_c, \nabla^b]\xi^{ca} \\ &\quad - \nabla_b[\nabla_a, \nabla^b]\nabla_c\xi^{ca} - [\nabla_a, \nabla_b]\nabla_c\nabla^b\xi^{ca} + \frac{1}{3}\nabla^c[\nabla_a, \nabla_c]\nabla_b\xi^{ab} \\ &\quad + \frac{1}{3}[\nabla_a, \nabla_c]\nabla^c\nabla_b\xi^{ab} + \frac{1}{3}\nabla_a[\nabla_b, \nabla^a]\nabla_c\xi^{bc} \\ &= -\Box\nabla_a\nabla_b\xi^{ab} - 2\nabla_a\nabla_b[\nabla_c, \nabla^a]\xi^{bc} - \frac{4}{3}\nabla^c[\nabla_b, \nabla_c]\nabla_a\xi^{ab} - [\nabla_a, \nabla_b]\nabla_c\nabla^b\xi^{ca} \\ &\quad + \frac{1}{3}[\nabla_a, \nabla_c]\nabla^c\nabla_b\xi^{ab}.\end{aligned} \tag{B.39}$$

Therefore,

$$\begin{aligned}\Box\nabla_a\nabla_b\xi^{ab} &= \nabla_a\nabla_b\Box\xi^{ab} + [\Box, \nabla_a\nabla_b]\xi^{ab} \\ &= -\Box\nabla_a\nabla_b\xi^{ab} - 2\nabla_a\nabla_b[\nabla_c, \nabla^a]\xi^{bc} - \frac{4}{3}\nabla^c[\nabla_b, \nabla_c]\nabla_a\xi^{ab} - [\nabla_a, \nabla_b]\nabla_c\nabla^b\xi^{ca} \\ &\quad + \frac{1}{3}[\nabla_a, \nabla_c]\nabla^c\nabla_b\xi^{ab} + \nabla^c\nabla_a[\nabla_c, \nabla_b]\xi^{ab} + \nabla^c[\nabla_c, \nabla_a]\nabla_b\xi^{ab} \\ &\quad + \nabla_a[\nabla_c, \nabla_b]\nabla^c\xi^{ab} + [\nabla_c, \nabla_a]\nabla_b\nabla^c\xi^{ab} \\ &= -\frac{3}{2}\nabla_c\nabla_b[\nabla_a, \nabla^c]\xi^{ab} - \frac{7}{6}\nabla^c[\nabla_b, \nabla_c]\nabla_a\xi^{ab} + [\nabla_c, \nabla_a]\nabla_b\nabla^c\xi^{ab} \\ &\quad + \frac{1}{6}[\nabla_a, \nabla_c]\nabla^c\nabla_b\xi^{ab} + \frac{1}{2}\nabla_a[\nabla_c, \nabla_b]\nabla^c\xi^{ab}.\end{aligned} \tag{B.40}$$

It is now time to evaluate each of these commutators[1].

$$\begin{aligned}
\nabla_c\nabla_b[\nabla_a,\nabla^c]\xi^{ab} &= \nabla_c\nabla_b(R_a{}^{ca}{}_d\xi^{db} + R_a{}^{cb}{}_d\xi^{ad}) \\
&= \nabla_a\nabla_c(R_b{}^a\xi^{bc}) + \nabla^c\nabla^d(R_{acdb}\xi^{ad}) && \text{(B.41)} \\
\nabla^c[\nabla_b,\nabla_c]\nabla_a\xi^{ab} &= \nabla^c(R_{bc}{}^b{}_d\nabla_a\xi^{ad}) \\
&= \nabla^c(R_{cb}\nabla_a\xi^{ab}) && \text{(B.42)} \\
[\nabla_c,\nabla_a]\nabla_b\nabla^c\xi^{ab} &= R_{ca}{}^c{}_d\nabla_b\nabla^d\xi^{ab} + R_{ca}{}^a{}_d\nabla_b\nabla^c\xi^{db} \\
&= R_{ad}\nabla_b\nabla^d\xi^{ab} - R_{cd}\nabla_b\nabla^c\xi^{db} \\
&= 0 && \text{(B.43)} \\
[\nabla_a,\nabla_c]\nabla^c\nabla_b\xi^{ab} &= R_{ac}{}^c{}_d\nabla^d\nabla_b\xi^{ab}R_{ac}{}^a{}_d\nabla^c\nabla_b\xi^{db} \\
&= -R_{ad}\nabla^d\nabla_b\xi^{ab} + R_{cd}\nabla^c\nabla_b\xi^{db} \\
&= 0 && \text{(B.44)} \\
\nabla_a[\nabla_c,\nabla_b]\nabla^c\xi^{ab} &= \nabla_a(R_{cb}{}^c{}_d\nabla^d\xi^{ab} + R_{cb}{}^a{}_d\nabla^c\xi^{db} + R_{cb}{}^b{}_d\nabla^c\xi^{ad}) \\
&= \nabla_a(R_{bd}\nabla^d\xi^{ab} + R^a{}_{dcb}\nabla^c\xi^{db} - R_{cd}\nabla^c\xi^{ad}) \\
&= \nabla_a(R^a{}_{dcb}\nabla^c\xi^{db}) && \text{(B.45)}
\end{aligned}$$

Altogether,

$$\begin{aligned}
\Box\nabla_a\nabla_b\xi^{ab} = &-\frac{3}{2}\nabla_a\nabla_c(R_b{}^a\xi^{bc}) - \frac{3}{2}\nabla^c\nabla^d(R_{acdb}\xi^{ad}) - \frac{7}{6}\nabla^c(R_{cb}\nabla_a\xi^{ab}) \\
&+ \frac{1}{2}\nabla_a(R^a{}_{dcb}\nabla^c\xi^{db})\,. && \text{(B.46)}
\end{aligned}$$

Putting this expression back into equation B.19,

$$\begin{aligned}
\{0\} = &\frac{2}{9}\nabla_a(\xi^{ab})\nabla_b(R) + \frac{1}{6}\xi^{ab}\nabla_a\nabla_b(R) - \frac{3}{10}\Box(R_{ab}\xi^{ab}) + \frac{1}{18}R\nabla_a\nabla_b(\xi^{ab}) - \frac{1}{10}\nabla_a\nabla_c(R_b{}^a\xi^{bc}) \\
&- \frac{1}{10}\nabla^c\nabla^d(R_{acdb}\xi^{ab}) - \frac{7}{90}\nabla^c(R_{cb}\nabla_a\xi^{ab}) + \frac{1}{30}\nabla_a(R^a{}_{dcb}\nabla^c\xi^{db})\,. && \text{(B.47)}
\end{aligned}$$

To progress further, I will need to expand each of the terms coming from the commutators above.

$$\begin{aligned}
\nabla_a\nabla_c(R_b{}^a\xi^{bc}) &= \nabla_a(\nabla_c(R_b{}^a)\xi^{bc} + R_b{}^a\nabla_c(\xi^{bc})) \\
&= \nabla_a\nabla_c(R_b{}^a)\xi^{bc} + R_b{}^a\nabla_a\nabla_c(\xi^{bc}) + \nabla_c(R_{ab})\nabla^a(\xi^{bc}) + \frac{1}{2}\nabla_b(R)\nabla_a(\xi^{ab}) && \text{(B.48)}
\end{aligned}$$

$$\begin{aligned}
\nabla^d R_{acdb} &= \nabla^d R_{dbac} \\
&= -\nabla_a R_{dbc}{}^d - \nabla_c R_{db}{}^d{}_a \\
&= \nabla_a R_{bc} - \nabla_c R_{ba} && \text{(B.49)} \\
\nabla^c(R_{acdb}) &= -\nabla_d(R_{acb}{}^c) - \nabla_b(R_{ac}{}^c{}_d) = -\nabla_d(R_{ab}) + \nabla_b(R_{ad}) && \text{(B.50)} \\
\implies \nabla^c\nabla^d(R_{acdb}\xi^{ab}) &= \nabla^c(\nabla^d(R_{acdb})\xi^{ab} + R_{acdb}\nabla^d(\xi^{ab})) \\
&= \nabla^c(\nabla_a(R_{bc})\xi^{ab} - \nabla_c(R_{ab})\xi^{ab} + R_{acdb}\nabla^d(\xi^{ab})) \\
&= \nabla^c\nabla_a(R_{bc})\xi^{ab} + \nabla_a(R_{bc})\nabla^c(\xi^{ab}) - \Box(R_{ab})\xi^{ab} - \nabla_c(R_{ab})\nabla^c(\xi^{ab}) \\
&\quad + \nabla^c(R_{acdb})\nabla^d(\xi^{ab}) + R_{acdb}\nabla^c\nabla^d(\xi^{ab}) \\
&= R_{acdb}\nabla^c\nabla^d(\xi^{ab}) + \nabla^c\nabla_a(R_{bc})\xi^{ab} - \Box(R_{ab})\xi^{ab} + 2\nabla_a(R_{bc})\nabla^c(\xi^{ab}) \\
&\quad - 2\nabla_c(R_{ab})\nabla^c(\xi^{ab}) && \text{(B.51)}
\end{aligned}$$

[1]Technically, the curvature-free terms are already gone since all commutators generate Riemann tensors and their descendants.

$$\nabla^c(R_{cb}\nabla_a\xi^{ab}) = \frac{1}{2}\nabla_b(R)\nabla_a\xi^{ab} + R_{cb}\nabla^c\nabla_a\xi^{ab} \tag{B.52}$$

$$\begin{aligned}\nabla_a(R^a{}_{dcb}\nabla^c\xi^{db}) &= \nabla_a(R^a{}_{dcb})\nabla^c\xi^{db} + R^a{}_{dcb}\nabla_a\nabla^c\xi^{db} \\ &= -\nabla_c(R^a{}_{dba})\nabla^c\xi^{bd} - \nabla_b(R^a{}_{dac})\nabla^c\xi^{bd} + R_{abcd}\nabla^a\nabla^c\xi^{bd} \\ &= \nabla_a(R_{bc})\nabla^a\xi^{bc} - \nabla_a(R_{bc})\nabla^c\xi^{ab} + R_{abcd}\nabla^a\nabla^c\xi^{bd}\end{aligned} \tag{B.53}$$

With these expressions,

$$\begin{aligned}\{0\} &= \frac{2}{9}\nabla_a(\xi^{ab})\nabla_b(R) + \frac{1}{6}\xi^{ab}\nabla_a\nabla_b(R) - \frac{3}{10}\Box(R_{ab}\xi^{ab}) + \frac{1}{18}R\nabla_a\nabla_b(\xi^{ab}) - \frac{1}{10}\nabla_a\nabla_c(R_b{}^a)\xi^{bc} \\ &\quad - \frac{1}{10}R_b{}^a\nabla_a\nabla_c(\xi^{bc}) - \frac{1}{10}\nabla_c(R_{ab})\nabla^a(\xi^{bc}) - \frac{1}{20}\nabla_b(R)\nabla_a(\xi^{ab}) \\ &\quad - \frac{1}{10}R_{acdb}\nabla^c\nabla^d(\xi^{ab}) - \frac{1}{10}\nabla^c\nabla_a(R_{bc})\xi^{ab} + \frac{1}{10}\Box(R_{ab})\xi^{ab} - \frac{1}{5}\nabla_a(R_{bc})\nabla^c(\xi^{ab}) \\ &\quad + \frac{1}{5}\nabla_c(R_{ab})\nabla^c(\xi^{ab}) - \frac{7}{180}\nabla_b(R)\nabla_a(\xi^{ab}) - \frac{7}{90}R_{cb}\nabla^c\nabla_a\xi^{ab} + \frac{1}{30}\nabla_a(R_{bc})\nabla^a\xi^{bc} \\ &\quad - \frac{1}{30}\nabla_a(R_{bc})\nabla^c\xi^{ab} + \frac{1}{30}R_{abcd}\nabla^a\nabla^c\xi^{bd} \\ &= \frac{2}{15}\nabla_a(\xi^{ab})\nabla_b(R) + \frac{1}{6}\xi^{ab}\nabla_a\nabla_b(R) - \frac{1}{5}\Box(R_{ab})\xi^{ab} - \frac{11}{30}\nabla_c(R_{ab})\nabla^c(\xi^{ab}) - \frac{3}{10}R_{ab}\Box\xi^{ab} \\ &\quad + \frac{1}{18}R\nabla_a\nabla_b(\xi^{ab}) - \frac{1}{5}\nabla^c\nabla_a(R_{bc})\xi^{ab} - \frac{8}{45}R_{cb}\nabla^c\nabla_a(\xi^{ab}) - \frac{1}{3}\nabla_a(R_{bc})\nabla^c(\xi^{ab}) \\ &\quad + \frac{2}{15}R_{abcd}\nabla^a\nabla^c\xi^{bd}\,.\end{aligned} \tag{B.54}$$

Hence, so far I have

$$\begin{aligned}\Delta D^{(2)}\varphi &= \left\{\frac{4}{9}R_b{}^a\nabla_c(\xi^{cb}) + \frac{4}{15}R^a{}_{bcd}\nabla^c(\xi^{bd}) - \frac{1}{15}\nabla_b(R)\xi^{ab} + \frac{2}{5}\nabla_c(R_b{}^a)\xi^{bc} + \frac{1}{9}R\nabla_b(\xi^{ab})\right. \\ &\qquad \left. - \frac{2}{5}\nabla^a(R_{cb})\xi^{cb} - \frac{3}{5}R_{bc}\nabla^a(\xi^{bc}) - \frac{4}{5}R_{cb}\nabla^c(\xi^{ab})\right\}\nabla_a(\varphi) \\ &\quad + \left\{\frac{2}{15}\nabla_a(\xi^{ab})\nabla_b(R) + \frac{1}{6}\xi^{ab}\nabla_a\nabla_b(R) - \frac{1}{5}\Box(R_{ab})\xi^{ab} - \frac{11}{30}\nabla_c(R_{ab})\nabla^c(\xi^{ab})\right. \\ &\qquad - \frac{3}{10}R_{ab}\Box\xi^{ab} + \frac{1}{18}R\nabla_a\nabla_b(\xi^{ab}) - \frac{1}{5}\nabla^c\nabla_a(R_{bc})\xi^{ab} - \frac{8}{45}R_{cb}\nabla^c\nabla_a(\xi^{ab}) \\ &\qquad \left. - \frac{1}{3}\nabla_a(R_{bc})\nabla^c(\xi^{ab}) + \frac{2}{15}R_{abcd}\nabla^a\nabla^c\xi^{bd}\right\}\varphi\,.\end{aligned} \tag{B.55}$$

It is not possible to use the conformal Killing equation on terms contracted to the Riemann tensor because the conformal Killing equation requires symmetrising indices; the Riemann tensor's antisymmetries then give zero. To work around this issue, I will re-write all Riemann tensors in terms of the Weyl tensor, i.e. via

$$R_{abcd} = C_{abcd} + \frac{1}{2}\eta_{ac}R_{bd} + \frac{1}{2}\eta_{bd}R_{ac} - \frac{1}{2}\eta_{ad}R_{bc} - \frac{1}{2}\eta_{bc}R_{ad} + \frac{1}{6}R\eta_{bc}\eta_{ad} - \frac{1}{6}R\eta_{ac}\eta_{bd}\,. \tag{B.56}$$

In $\{1\}$ the only term with a Riemann tensor is $\frac{4}{15}R^a{}_{bcd}\nabla^c(\xi^{bd})$.

$$\begin{aligned}
R^a{}_{bcd}\nabla^c\xi^{bd} &= C^a{}_{bcd}\nabla^c\xi^{bd} + \frac{1}{2}\delta^a{}_c R_{bd}\nabla^c\xi^{bd} + \frac{1}{2}\eta_{bd}R^a{}_c\nabla^c\xi^{bd} - \frac{1}{2}\delta^a{}_d R_{bc}\nabla^c\xi^{bd} - \frac{1}{2}\eta_{bc}R^a{}_d\nabla^c\xi^{bd} \\
&\quad + \frac{1}{6}R\eta_{bc}\delta^a{}_d\nabla^c\xi^{bd} - \frac{1}{6}R\delta^a{}_c\eta_{bd}\nabla^c\xi^{bd} \\
&= C^a{}_{bcd}\nabla^c\xi^{bd} + \frac{1}{2}R_{bd}\nabla^a\xi^{bd} + \frac{1}{2}R^a{}_c\nabla^c\xi^b{}_b - \frac{1}{2}R_{bc}\nabla^c\xi^{ba} - \frac{1}{2}R^a{}_d\nabla_b\xi^{bd} \\
&\quad + \frac{1}{6}R\nabla_b\xi^{ba} - \frac{1}{6}R\nabla^a\xi^b{}_b \\
&= C^a{}_{bcd}\nabla^c\xi^{bd} + \frac{1}{2}R_{bc}\nabla^a\xi^{bc} - \frac{1}{2}R_{bc}\nabla^c\xi^{ab} - \frac{1}{2}R_b{}^a\nabla_c\xi^{bc} + \frac{1}{6}R\nabla_b\xi^{ab} \qquad \text{(B.57)}
\end{aligned}$$

Substituting this back into $\{1\}$ gives

$$\begin{aligned}
\{1\} &= \frac{14}{45}R_b{}^a\nabla_c\xi^{bc} - \frac{1}{15}\nabla_b(R)\xi^{ab} + \frac{2}{5}\nabla_c(R_b{}^a)\xi^{bc} + \frac{7}{45}R\nabla_b\xi^{ab} - \frac{2}{5}\nabla^a(R_{bc})\xi^{bc} - \frac{7}{15}R_{bc}\nabla^a\xi^{bc} \\
&\quad - \frac{14}{45}R_{bc}\nabla^c\xi^{ab} + \frac{4}{15}C^a{}_{bcd}\nabla^c\xi^{bd}. \qquad \text{(B.58)}
\end{aligned}$$

Then, by the conformal Killing equation,

$$\begin{aligned}
R_{bc}\nabla^a\xi^{bc} &= R_{bc}\left(-\nabla^b\xi^{ca} - \nabla^c\xi^{ab} + \frac{1}{3}\left(\eta^{ab}\nabla_d\xi^{cd} + \eta^{bc}\nabla_d\xi^{ad} + \eta^{ca}\nabla_d\xi^{bd}\right)\right) \\
&= -2R_{bc}\nabla^c\xi^{ab} + \frac{2}{3}R_b{}^a\nabla_c\xi^{bc} + \frac{1}{3}R\nabla_b\xi^{ab} \qquad \text{(B.59)}
\end{aligned}$$

Therefore,

$$\{1\} = -\frac{1}{15}\nabla_b(R)\xi^{ab} + \frac{2}{5}\nabla_c(R_b{}^a)\xi^{bc} - \frac{2}{5}\nabla^a(R_{bc})\xi^{bc} + \frac{4}{15}C^a{}_{bcd}\nabla^c\xi^{bd}. \qquad \text{(B.60)}$$

Next, the Bianchi identity in terms of the Weyl tensor is

$$\begin{aligned}
0 &= \nabla_a R^d{}_{ebc} + \nabla_b R^d{}_{eca} + \nabla_c R^d{}_{eab} \\
&= \nabla_a C^d{}_{ebc} + \nabla_b C^d{}_{eca} + \nabla_c C^d{}_{eab} \\
&\quad + \nabla_a\left(\frac{1}{2}\delta^d{}_b R_{ec} + \frac{1}{2}\eta_{ec}R^d{}_b - \frac{1}{2}\delta^d{}_c R_{eb} - \frac{1}{2}\eta_{eb}R^d{}_c + \frac{1}{6}R\eta_{eb}\delta^d{}_c - \frac{1}{6}R\delta^d{}_b\eta_{ec}\right) \\
&\quad + \nabla_b\left(\frac{1}{2}\delta^d{}_c R_{ea} + \frac{1}{2}\eta_{ea}R^d{}_c - \frac{1}{2}\delta^d{}_a R_{ec} - \frac{1}{2}\eta_{ec}R^d{}_a + \frac{1}{6}R\eta_{ec}\delta^d{}_a - \frac{1}{6}R\delta^d{}_c\eta_{ea}\right) \\
&\quad + \nabla_c\left(\frac{1}{2}\delta^d{}_a R_{eb} + \frac{1}{2}\eta_{eb}R^d{}_a - \frac{1}{2}\delta^d{}_b R_{ea} - \frac{1}{2}\eta_{ea}R^d{}_b + \frac{1}{6}R\eta_{ea}\delta^d{}_b - \frac{1}{6}R\delta^d{}_a\eta_{eb}\right) \qquad \text{(B.61)} \\
\implies 0 &= \nabla_a C^a{}_{ebc} + \nabla_b C^a{}_{eca} + \nabla_c C^a{}_{eab} \\
&\quad + \nabla_a\left(\frac{1}{2}\delta^a{}_b R_{ec} + \frac{1}{2}\eta_{ec}R^a{}_b - \frac{1}{2}\delta^a{}_c R_{eb} - \frac{1}{2}\eta_{eb}R^a{}_c + \frac{1}{6}R\eta_{eb}\delta^a{}_c - \frac{1}{6}R\delta^a{}_b\eta_{ec}\right) \\
&\quad + \nabla_b\left(\frac{1}{2}\delta^a{}_c R_{ea} + \frac{1}{2}\eta_{ea}R^a{}_c - \frac{1}{2}\delta^a{}_a R_{ec} - \frac{1}{2}\eta_{ec}R^a{}_a + \frac{1}{6}R\eta_{ec}\delta^a{}_a - \frac{1}{6}R\delta^a{}_c\eta_{ea}\right) \\
&\quad + \nabla_c\left(\frac{1}{2}\delta^a{}_a R_{eb} + \frac{1}{2}\eta_{eb}R^a{}_a - \frac{1}{2}\delta^a{}_b R_{ea} - \frac{1}{2}\eta_{ea}R^a{}_b + \frac{1}{6}R\eta_{ea}\delta^a{}_b - \frac{1}{6}R\delta^a{}_a\eta_{eb}\right) \\
&= \nabla_a C^a{}_{ebc} - \frac{1}{2}\nabla_b R_{ec} + \frac{1}{12}\eta_{ec}\nabla_b R + \frac{1}{2}\nabla_c R_{eb} - \frac{1}{12}\eta_{eb}\nabla_c R \qquad \text{(B.62)} \\
\iff 0 &= \nabla_b R_{ac} + \frac{1}{6}\eta_{ab}\nabla_c R - \nabla_c R_{ab} - \frac{1}{6}\eta_{ac}\nabla_b R - 2\nabla_d C^d{}_{abc}. \qquad \text{(B.63)}
\end{aligned}$$

Then applying the last equation to {1},

$$
\begin{aligned}
0 &= -\frac{2}{5}\bigg(\nabla^a(R_{bc}) + \frac{1}{6}\delta^a{}_b\nabla_c(R) - \nabla_c(R_b{}^a) - \frac{1}{6}\eta_{bc}\nabla^a(R) - 2\nabla_d(C^d{}_b{}^a{}_c)\bigg)\xi^{bc} \\
&= -\frac{2}{5}\nabla^a(R_{bc})\xi^{bc} - \frac{1}{15}\nabla_b(R)\xi^{ab} + \frac{2}{5}\nabla_c(R_b{}^a)\xi^{bc} - \frac{4}{5}\nabla^d(C^a{}_{bcd})\xi^{bc}\,. \qquad \text{(B.64)}
\end{aligned}
$$

That finally results in

$$
\{1\} = \frac{4}{15}C^a{}_{bcd}\nabla^c\xi^{bd} + \frac{4}{5}\nabla^d(C^a{}_{bcd})\xi^{bc}\,, \qquad \text{(B.65)}
$$

which finishes the required manipulation for {1}.

Next I have to deal with {0} where the only term with a Riemann tensor in it is $\frac{2}{15}R_{abcd}\nabla^a\nabla^c\xi^{bd}$. Writing it in terms of the Weyl tensor,

$$
\begin{aligned}
R_{abcd}\nabla^a\nabla^c\xi^{bd} &= \bigg(C_{abcd} + \frac{1}{2}\eta_{ac}R_{bd} + \frac{1}{2}\eta_{bd}R_{ac} - \frac{1}{2}\eta_{ad}R_{bc} - \frac{1}{2}\eta_{bc}R_{ad} \\
&\quad + \frac{1}{6}R\eta_{bc}\eta_{ad} - \frac{1}{6}R\eta_{ac}\eta_{bd}\bigg)\nabla^a\nabla^c\xi^{bd} \\
&= C_{abcd}\nabla^a\nabla^c\xi^{bd} + \frac{1}{2}R_{bd}\Box\xi^{bd} + \frac{1}{2}R_{ac}\nabla^a\nabla^c\xi^b{}_b - \frac{1}{2}R_{bc}\nabla_d\nabla^c\xi^{bd} - \frac{1}{2}R_{ad}\nabla^a\nabla_b\xi^{bd} \\
&\quad + \frac{1}{6}R\nabla_d\nabla_b\xi^{bd} - \frac{1}{6}R\Box\xi^b{}_b \\
&= C_{abcd}\nabla^a\nabla^c\xi^{bd} + \frac{1}{2}R_{ab}\Box\xi^{ab} - \frac{1}{2}R_{bc}\nabla_a\nabla^c\xi^{ab} - \frac{1}{2}R_{bc}\nabla^c\nabla_a\xi^{ab} + \frac{1}{6}R\nabla_a\nabla_b\xi^{ab}\,.
\end{aligned}
\qquad \text{(B.66)}
$$

Substituting this back into {0},

$$
\begin{aligned}
\{0\} &= \frac{2}{15}\nabla_b(R)\nabla_a\xi^{ab} + \frac{1}{6}\xi^{ab}\nabla_a\nabla_b(R) - \frac{1}{5}\Box(R_{ab})\xi^{ab} - \frac{11}{30}\nabla_c(R_{ab})\nabla^c\xi^{ab} - \frac{7}{30}R_{ab}\Box\xi^{ab} \\
&\quad + \frac{7}{90}R\nabla_a\nabla_b\xi^{ab} - \frac{1}{5}\nabla^c\nabla_a(R_{bc})\xi^{ab} - \frac{11}{45}R_{bc}\nabla^c\nabla_a\xi^{ab} - \frac{1}{3}\nabla_a(R_{bc})\nabla^c\xi^{ab} \\
&\quad - \frac{1}{15}R_{bc}\nabla_a\nabla^c\xi^{ab} + \frac{2}{15}C_{abcd}\nabla^a\nabla^c\xi^{bd} \qquad \text{(B.67)}
\end{aligned}
$$

From equation B.63,

$$
\begin{aligned}
0 &= \nabla^b\bigg(\nabla_b(R_{ac}) + \frac{1}{6}\eta_{ab}\nabla_c(R) - \nabla_c(R_{ab}) - \frac{1}{6}\eta_{ac}\nabla_b(R) - 2\nabla_d(C^d{}_{abc})\bigg)\xi^{ac} \\
&= \Box(R_{ab})\xi^{ab} + \frac{1}{6}\nabla_a\nabla_b(R)\xi^{ab} - \nabla^c\nabla_a(R_{bc})\xi^{ab} + 2\nabla^c\nabla_d(C^d{}_{abc})\xi^{ab} \qquad \text{(B.68)}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\{0\} &= \frac{2}{15}\nabla_b(R)\nabla_a\xi^{ab} - \frac{6}{5}\Box(R_{ab})\xi^{ab} - \frac{11}{30}\nabla_c(R_{ab})\nabla^c\xi^{ab} - \frac{7}{30}R_{ab}\Box\xi^{ab} \\
&\quad + \frac{7}{90}R\nabla_a\nabla_b\xi^{ab} + \frac{4}{5}\nabla^c\nabla_a(R_{bc})\xi^{ab} - \frac{11}{45}R_{bc}\nabla^c\nabla_a\xi^{ab} - \frac{1}{3}\nabla_a(R_{bc})\nabla^c\xi^{ab} \\
&\quad - \frac{1}{15}R_{bc}\nabla_a\nabla^c\xi^{ab} + \frac{2}{15}C_{abcd}\nabla^a\nabla^c\xi^{bd} - 2\nabla^c\nabla_d(C^d{}_{abc})\xi^{ab}\,. \qquad \text{(B.69)}
\end{aligned}
$$

$$
\begin{aligned}
R_{ab}\Box\xi^{ab} &= R_{ab}\nabla_c\bigg(-\nabla^a\xi^{bc} - \nabla^b\xi^{ca} + \frac{1}{3}\Big(\eta^{ab}\nabla_d\xi^{cd} + \eta^{bc}\nabla_d\xi^{ad} + \eta^{ca}\nabla_d\xi^{bd}\Big)\bigg) \\
&= -2R_{bc}\nabla_a\nabla^c\xi^{ab} + \frac{1}{3}R\nabla_a\nabla_b\xi^{ab} + \frac{2}{3}R_{bc}\nabla^c\nabla_a\xi^{ab} \qquad \text{(B.70)} \\
\implies \{0\} &= \frac{2}{15}\nabla_b(R)\nabla_a\xi^{ab} - \frac{6}{5}\Box(R_{ab})\xi^{ab} - \frac{11}{30}\nabla_c(R_{ab})\nabla^c\xi^{ab} + \frac{4}{5}\nabla^c\nabla_a(R_{bc})\xi^{ab} \\
&\quad - \frac{2}{5}R_{bc}\nabla^c\nabla_a\xi^{ab} - \frac{1}{3}\nabla_a(R_{bc})\nabla^c\xi^{ab} + \frac{2}{5}R_{bc}\nabla_a\nabla^c\xi^{ab} + \frac{2}{15}C_{abcd}\nabla^a\nabla^c\xi^{bd} \\
&\quad - 2\nabla^c\nabla_d(C^d{}_{abc})\xi^{ab} \qquad \text{(B.71)}
\end{aligned}
$$

$$
\begin{aligned}
\nabla_c(R_{ab})\nabla^c(\xi^{ab}) &= \nabla_c(R_{ab})\bigg(-\nabla^a(\xi^{bc}) - \nabla^b(\xi^{ca}) \\
&\quad + \frac{1}{3}\Big(\eta^{ab}\nabla_d(\xi^{cd}) + \eta^{bc}\nabla_d(\xi^{ad}) + \eta^{ca}\nabla_d(\xi^{bd})\Big)\bigg) \\
&= -2\nabla_c(R_{ab})\nabla^a(\xi^{bc}) + \frac{1}{3}\nabla_c(R)\nabla_d(\xi^{cd}) + \frac{1}{3}\nabla^b(R_{ab})\nabla_d(\xi^{ad}) \\
&\quad + \frac{1}{3}\nabla^a(R_{ab})\nabla_d(\xi^{bd}) \\
&= -2\nabla_c(R_{ab})\nabla^a(\xi^{bc}) + \frac{2}{3}\nabla_b(R)\nabla_a(\xi^{ab}) \qquad \text{(B.72)}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\{0\} &= -\frac{1}{9}\nabla_b(R)\nabla_a\xi^{ab} - \frac{6}{5}\Box(R_{ab})\xi^{ab} + \frac{4}{5}\nabla^c\nabla_a(R_{bc})\xi^{ab} - \frac{2}{5}R_{bc}\nabla^c\nabla_a\xi^{ab} + \frac{2}{5}\nabla_c(R_{ab})\nabla^a\xi^{bc} \\
&\quad + \frac{2}{5}R_{bc}\nabla_a\nabla^c\xi^{ab} + \frac{2}{15}C_{abcd}\nabla^a\nabla^c\xi^{bd} - 2\nabla^c\nabla_d(C^d{}_{abc})\xi^{ab} \\
&= -\frac{1}{9}\nabla_b(R)\nabla_a\xi^{ab} - \frac{6}{5}\Box(R_{ab})\xi^{ab} + \frac{4}{5}\nabla^c\nabla_a(R_{bc})\xi^{ab} + \frac{2}{5}\nabla_c(R_{ab})\nabla^a\xi^{bc} \\
&\quad + \frac{2}{5}R_{bc}[\nabla_a, \nabla^c]\xi^{ab} + \frac{2}{15}C_{abcd}\nabla^a\nabla^c\xi^{bd} - 2\nabla^c\nabla_d(C^d{}_{abc})\xi^{ab}\,. \qquad \text{(B.73)}
\end{aligned}
$$

Then, from equation B.63,

$$
\begin{aligned}
0 &= \bigg(\nabla_b(R_{ac}) + \frac{1}{6}\eta_{ab}\nabla_c(R) - \nabla_c(R_{ab}) - \frac{1}{6}\eta_{ac}\nabla_b(R) - 2\nabla_d(C^d{}_{abc})\bigg)\nabla^b(\xi^{ac}) \\
&= \nabla_b(R_{ac})\nabla^b(\xi^{ac}) + \frac{1}{6}\nabla_c(R)\nabla_a(\xi^{ac}) - \nabla_c(R_{ab})\nabla^b(\xi^{ac}) - 0 - 2\nabla_d(C^d{}_{abc})\nabla^b(\xi^{ac}) \\
&= \nabla_c(R_{ab})\nabla^c(\xi^{ab}) + \frac{1}{6}\nabla_b(R)\nabla_a(\xi^{ab}) - \nabla_c(R_{ab})\nabla^a(\xi^{bc}) - 2\nabla_d(C^d{}_{abc})\nabla^b(\xi^{ac}) \\
&= \frac{5}{6}\nabla_b(R)\nabla_a(\xi^{ab}) - 3\nabla_c(R_{ab})\nabla^a(\xi^{bc}) - 2\nabla_d(C^d{}_{abc})\nabla^b(\xi^{ac})\,, \qquad \text{(B.74)}
\end{aligned}
$$

where the last line was derived using the expression for $\nabla_c(R_{ab})\nabla^c(\xi^{ab})$ from above. Thus,

$$
\begin{aligned}
\{0\} &= -\frac{6}{5}\Box(R_{ab})\xi^{ab} + \frac{4}{5}\nabla^c\nabla_a(R_{bc})\xi^{ab} + \frac{2}{5}R_{bc}[\nabla_a, \nabla^c]\xi^{ab} \\
&\quad + \frac{2}{15}C_{abcd}\nabla^a\nabla^c\xi^{bd} - 2\nabla^c\nabla_d(C^d{}_{abc})\xi^{ab} - \frac{4}{15}\nabla_d(C^d{}_{abc})\nabla^b(\xi^{ac}) \qquad \text{(B.75)} \\
R_{bc}[\nabla_a, \nabla^c]\xi^{ab} &= R_{bc}R_a{}^{ca}{}_d\xi^{db} + R_{bc}R_a{}^{cb}{}_d\xi^{ad} \\
&= R_{bc}R^c{}_d\xi^{db} + R_{bc}R_a{}^{cb}{}_d\xi^{ad} \qquad \text{(B.76)}
\end{aligned}
$$

Again, I will re-write the Riemann tensor in terms of the Weyl tensor.

$$\begin{aligned}
R_a{}^{cb}{}_d\xi^{ad} &= \bigg(C_a{}^{cb}{}_d + \frac{1}{2}\delta^b{}_a R^c{}_d + \frac{1}{2}\delta^c{}_d R^b{}_a - \frac{1}{2}\eta_{ad}R^{bc} - \frac{1}{2}\eta^{bc}R_{ad} \\
&\quad + \frac{1}{6}R\eta^{bc}\eta_{ad} - \frac{1}{6}R\delta^b{}_a\delta^c{}_d\bigg)\xi^{ad} \\
&= C_a{}^{cb}{}_d\xi^{ad} + \frac{1}{2}R^c{}_d\xi^{bd} + \frac{1}{2}R^b{}_a\xi^{ac} - \frac{1}{2}R^{bc}\xi^a{}_a - \frac{1}{2}R_{ad}\xi^{ad}\eta^{bc} \\
&\quad + \frac{1}{6}R\eta^{bc}\xi^a{}_a - \frac{1}{6}R\xi^{bc} \\
&= C_a{}^{cb}{}_d\xi^{ad} + \frac{1}{2}R^c{}_d\xi^{bd} + \frac{1}{2}R^b{}_d\xi^{cd} - \frac{1}{2}\eta^{bc}R_{ad}\xi^{ad} - \frac{1}{6}R\xi^{bc}
\end{aligned} \tag{B.77}$$

Substituting that back,

$$\begin{aligned}
R_{bc}[\nabla_a, \nabla^c]\xi^{ab} &= R_{bc}R^c{}_d\xi^{db} + R_{bc}C_a{}^{cb}{}_d\xi^{ad} + \frac{1}{2}R_{bc}R^c{}_d\xi^{bd} + \frac{1}{2}R_{bc}R^b{}_d\xi^{cd} \\
&\quad - \frac{1}{2}R_{bc}\eta^{bc}R_{ad}\xi^{ad} - \frac{1}{6}R_{bc}R\xi^{bc} \\
&= R^{bc}C_{abcd}\xi^{ad} + 2R_{ac}R^c{}_b\xi^{ab} - \frac{2}{3}RR_{ab}\xi^{ab}\,.
\end{aligned} \tag{B.78}$$

With the benefit of hindsight, another term in equation B.75 that should be re-written is

$$\begin{aligned}
\nabla^c\nabla_a(R_{bc}) &= \nabla_a\nabla^c(R_{bc}) + [\nabla^c, \nabla_a]R_{bc} \\
&= \frac{1}{2}\nabla_a\nabla_b(R) + R^c{}_{abd}R^d{}_c + R^c{}_{acd}R_b{}^d \\
&= \frac{1}{2}\nabla_a\nabla_b(R) + R_{cabd}R^{cd} + R_{ac}R^c{}_b \\
&= \frac{1}{2}\nabla_a\nabla_b(R) + R_{ac}R^c{}_b + \bigg(C_{cabd} + \frac{1}{2}\eta_{cb}R_{ad} + \frac{1}{2}\eta_{ad}R_{bc} - \frac{1}{2}\eta_{cd}R_{ab} \\
&\quad - \frac{1}{2}\eta_{ab}R_{cd} + \frac{1}{6}R\eta_{ab}\eta_{cd} - \frac{1}{6}R\eta_{cb}\eta_{ad}\bigg)R^{cd} \\
&= \frac{1}{2}\nabla_a\nabla_b(R) + R_{ac}R^c{}_b + C_{cabd}R^{cd} + \frac{1}{2}R_{ac}R^c{}_b + \frac{1}{2}R_{bc}R^c{}_a - \frac{1}{2}R_{ab}R \\
&\quad - \frac{1}{2}\eta_{ab}R_{cd}R^{cd} + \frac{1}{6}R^2\eta_{ab} - \frac{1}{6}RR_{ab} \\
&= \frac{1}{2}\nabla_a\nabla_b(R) + \frac{3}{2}R_{ac}R^c{}_b + C_{cabd}R^{cd} + \frac{1}{2}R_{bc}R^c{}_a - \frac{2}{3}RR_{ab} \\
&\quad - \frac{1}{2}\eta_{ab}R_{cd}R^{cd} + \frac{1}{6}R^2\eta_{ab}\,.
\end{aligned} \tag{B.79}$$

Therefore,

$$\nabla^c\nabla_a(R_{bc})\xi^{ab} = \frac{1}{2}\nabla_a\nabla_b(R)\xi^{ab} + 2R_{ac}R^c{}_b\xi^{ab} - \frac{2}{3}RR_{ab}\xi^{ab} + C_{cabd}R^{cd}\xi^{ab} \tag{B.80}$$

Putting the previous two parts together,

$$\begin{aligned}
&\frac{4}{5}\nabla^c\nabla_a(R_{bc})\xi^{ab} + \frac{2}{5}R_{bc}[\nabla_a, \nabla^c]\xi^{ab} \\
&= \frac{12}{5}R_{ac}R^c{}_b\xi^{ab} - \frac{4}{5}RR_{ab}\xi^{ab} + \frac{2}{5}\nabla_a\nabla_b(R) + \frac{6}{5}C_{abcd}R^{bc}\xi^{ad}
\end{aligned} \tag{B.81}$$

Then, using equations B.68, B.80 and B.81 (in that order),

$$
\begin{aligned}
0 &= -\frac{6}{5}\Box(R_{ab})\xi^{ab} - \frac{1}{5}\nabla_a\nabla_b(R)\xi^{ab} + \frac{6}{5}\nabla^c\nabla_a(R_{bc}) - \frac{12}{5}\nabla^c\nabla_d(C^d{}_{abc})\xi^{ab} \\
&= -\frac{6}{5}\Box(R_{ab})\xi^{ab} - \frac{1}{5}\nabla_a\nabla_b(R)\xi^{ab} + \frac{3}{5}\nabla_a\nabla_b(R)\xi^{ab} + \frac{12}{5}R_{ac}R^c{}_b\xi^{ab} - \frac{4}{5}RR_{ab}\xi^{ab} \\
&\quad + \frac{6}{5}C_{cabd}R^{cd}\xi^{ab} - \frac{12}{5}\nabla^c\nabla_d(C^d{}_{abc})\xi^{ab} \\
&= -\frac{6}{5}\Box(R_{ab})\xi^{ab} + \frac{2}{5}\nabla_a\nabla_b(R)\xi^{ab} + \frac{12}{5}R_{ac}R^c{}_b\xi^{ab} - \frac{4}{5}RR_{ab}\xi^{ab} \\
&\quad + \frac{6}{5}C_{cabd}R^{cd}\xi^{ab} - \frac{12}{5}\nabla^c\nabla_d(C^d{}_{abc})\xi^{ab} \\
&= -\frac{6}{5}\Box(R_{ab})\xi^{ab} + \frac{4}{5}\nabla^c\nabla_a(R_{bc})\xi^{ab} + \frac{2}{5}R_{bc}[\nabla_a, \nabla^c]\xi^{ab} - \frac{12}{5}\nabla^c\nabla_d(C^d{}_{abc})\xi^{ab}\,.
\end{aligned}
\tag{B.82}
$$

Substituting this expression back into equation B.75,

$$
\{0\} = \frac{2}{15}C_{abcd}\nabla^a\nabla^c\xi^{bd} + \frac{2}{5}\nabla^c\nabla_d(C^d{}_{abc})\xi^{ab} - \frac{4}{15}\nabla_d(C^d{}_{abc})\nabla^b(\xi^{ac})\,. \tag{B.83}
$$

Using the last equation together with equation B.65, one finally has

$$
\begin{aligned}
\Delta D^{(2)}\varphi &= \left\{\frac{4}{15}C^a{}_{bcd}\nabla^c(\xi^{bd}) + \frac{4}{5}\nabla^d(C^a{}_{bcd})\xi^{bc}\right\}\nabla_a(\varphi) \\
&\quad + \left\{\frac{2}{15}C_{abcd}\nabla^a\nabla^c(\xi^{bd}) + \frac{2}{5}\nabla^c\nabla_d(C^d{}_{abc})\xi^{ab} - \frac{4}{15}\nabla_d(C^d{}_{abc})\nabla^b(\xi^{ac})\right\}\varphi\,,
\end{aligned}
\tag{B.84}
$$

which proves the theorem.

Appendix C

Proof of theorem 4.7

I have to show that the only candidate for a physically admissible, 2nd order higher symmetry of the massless Dirac operator such that $D'^{(2)}\mathbf{\Psi}' = \mathrm{e}^{3\sigma/2}D^{(2)}\mathbf{\Psi}$ under a Weyl transformation is

$$\begin{aligned} D^{(2)} &= \xi^{\alpha\beta\dot{\alpha}\dot{\beta}}\nabla_{\alpha\dot{\alpha}}\nabla_{\beta\dot{\beta}} + \frac{2}{3}\nabla^{(\alpha}{}_{\dot{\beta}}\xi^{\beta\gamma)\dot{\alpha}\dot{\beta}}\nabla_{\alpha\dot{\alpha}}M_{\beta\gamma} + \frac{2}{3}\nabla_{\beta}{}^{(\dot{\alpha}}\xi^{\alpha\beta\dot{\beta}\dot{\gamma})}\nabla_{\alpha\dot{\alpha}}\overline{M}_{\dot{\beta}\dot{\gamma}} + \frac{8}{9}\nabla_{\beta\dot{\beta}}(\xi^{\alpha\beta\dot{\alpha}\dot{\beta}})\nabla_{\alpha\dot{\alpha}} \\ &+ \left(\frac{2}{9}\nabla^{(\alpha}{}_{\dot{\alpha}}\nabla_{\gamma\dot{\beta}}\xi^{\beta)\gamma\dot{\alpha}\dot{\beta}} + \frac{1}{3}E^{(\alpha}{}_{\gamma\dot{\alpha}\dot{\beta}}\xi^{\beta)\gamma\dot{\alpha}\dot{\beta}}\right)M_{\alpha\beta} \\ &+ \left(\frac{2}{9}\nabla_{\alpha}{}^{(\dot{\alpha}}\nabla_{\beta\dot{\gamma}}\xi^{\alpha\beta\dot{\beta})\dot{\gamma}} + \frac{1}{3}E_{\alpha\beta\dot{\gamma}}{}^{(\dot{\alpha}}\xi^{\alpha\beta\dot{\beta})\dot{\gamma}}\right)\overline{M}_{\dot{\alpha}\dot{\beta}} + \frac{2}{15}\nabla_{\alpha\dot{\alpha}}\nabla_{\beta\dot{\beta}}(\xi^{\alpha\beta\dot{\alpha}\dot{\beta}}) - \frac{7}{10}E_{\alpha\beta\dot{\alpha}\dot{\beta}}\xi^{\alpha\beta\dot{\alpha}\dot{\beta}}\,, \end{aligned} \tag{C.1}$$

and subsequently

$$\begin{aligned} \gamma^a\nabla_a D^{(2)}\mathbf{\Psi} = \Bigg[&\frac{1}{3}(\overline{C}_{\dot{\alpha}\dot{\beta}}{}^{\dot{\gamma}\dot{\mu}}\xi_{\alpha\beta\dot{\gamma}\dot{\mu}} - C_{\alpha\beta}{}^{\gamma\mu}\xi_{\gamma\mu\dot{\alpha}\dot{\beta}})\nabla^{\beta\dot{\beta}}\bar{\chi}^{\dot{\alpha}} + \Bigg(\frac{4}{15}C^{\mu\gamma\beta}{}_{\alpha}\nabla_{(\beta}{}^{\dot{\beta}}\xi_{\gamma\mu)\dot{\alpha}\dot{\beta}} \\ &- \frac{1}{15}\overline{C}_{\dot{\alpha}}{}^{\dot{\beta}\dot{\gamma}\dot{\mu}}\nabla^{\beta}{}_{(\dot{\mu}}\xi_{\alpha\beta\dot{\gamma}\dot{\beta})} - \frac{2}{15}\xi^{\gamma\beta\dot{\gamma}}{}_{\dot{\alpha}}\nabla^{\mu}{}_{\dot{\gamma}}(C_{\alpha\beta\gamma\mu}) - \frac{7}{15}\xi_{\alpha}{}^{\gamma\dot{\beta}\dot{\gamma}}\nabla_{\gamma}{}^{\dot{\mu}}(\overline{C}_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\mu}})\Bigg)\bar{\chi}^{\dot{\alpha}}, \\ &\frac{1}{3}(C^{\alpha\beta}{}_{\gamma\mu}\xi^{\gamma\mu\dot{\alpha}\dot{\beta}} - \overline{C}^{\dot{\alpha}\dot{\beta}}{}_{\dot{\gamma}\dot{\mu}}\xi^{\alpha\beta\dot{\gamma}\dot{\mu}})\nabla_{\beta\dot{\beta}}\psi_{\alpha} + \Bigg(\frac{4}{15}\overline{C}_{\dot{\mu}\dot{\gamma}\dot{\beta}}{}^{\dot{\alpha}}\nabla_{\beta}{}^{(\dot{\beta}}\xi^{\alpha\beta\dot{\gamma}\dot{\mu})} \\ &- \frac{1}{15}C^{\alpha}{}_{\beta\gamma\mu}\nabla^{(\mu}{}_{\dot{\beta}}\xi^{\gamma\beta)\dot{\alpha}\dot{\beta}} - \frac{2}{15}\xi^{\alpha}{}_{\gamma\dot{\beta}\dot{\gamma}}\nabla^{\gamma}{}_{\dot{\mu}}(\overline{C}^{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\mu}}) - \frac{7}{15}\xi_{\gamma\beta\dot{\gamma}}{}^{\dot{\alpha}}\nabla_{\mu}{}^{\dot{\gamma}}(C^{\alpha\beta\gamma\mu})\Bigg)\psi_{\alpha}\Bigg]^{T}. \end{aligned} \tag{C.2}$$

To start off with, I only have equation 4.47, namely

$$D^{(2)} = \xi^{\alpha\beta\dot{\alpha}\dot{\beta}}\nabla_{\alpha\dot{\alpha}}\nabla_{\beta\dot{\beta}} + \xi^{\alpha\beta\gamma\dot{\alpha}}\nabla_{\alpha\dot{\alpha}}M_{\beta\gamma} + \xi^{\alpha\dot{\alpha}\dot{\beta}\dot{\gamma}}\nabla_{\alpha\dot{\alpha}}\overline{M}_{\dot{\beta}\dot{\gamma}} + \xi^{\alpha\dot{\alpha}}\nabla_{\alpha\dot{\alpha}} + \xi^{\alpha\beta}M_{\alpha\beta} + \xi^{\dot{\alpha}\dot{\beta}}\overline{M}_{\dot{\alpha}\dot{\beta}} + \xi\,, \tag{C.3}$$

with $\xi^{\alpha\beta\dot{\alpha}\dot{\beta}} = \xi^{(\alpha\beta)(\dot{\alpha}\dot{\beta})}$, $\xi^{\alpha\beta\gamma\dot{\alpha}} = \xi^{(\alpha\beta\gamma)\dot{\alpha}}$, $\xi^{\alpha\dot{\alpha}\dot{\beta}\dot{\gamma}} = \xi^{\alpha(\dot{\alpha}\dot{\beta}\dot{\gamma})}$, $\xi^{\alpha\beta} = \xi^{(\alpha\beta)}$ and $\xi^{\dot{\alpha}\dot{\beta}} = \xi^{(\dot{\alpha}\dot{\beta})}$. From here, I have to constrain the lower order coefficients in terms of the top component, thereby deriving the claimed form of $D^{(2)}$. In analogy with lemma 4.4, I will begin by evaluating

$\gamma^a \nabla_a D^{(2)} \mathbf{\Psi}$ (albeit term by term because the whole expression is too long).

$$
\begin{aligned}
\gamma^a \nabla_a (\xi^{\alpha\beta\dot\alpha\dot\beta} \nabla_{\alpha\dot\alpha} \nabla_{\beta\dot\beta} \mathbf{\Psi}) &= \begin{bmatrix} 0 & (\sigma^a)_{\alpha\dot\alpha} \\ (\tilde\sigma^a)^{\dot\alpha\alpha} & 0 \end{bmatrix} \begin{bmatrix} \nabla_a (\xi^{\beta\gamma\dot\beta\dot\gamma} \nabla_{\beta\dot\beta} \nabla_{\gamma\dot\gamma} \psi_\alpha) \\ \nabla_a (\xi^{\beta\gamma\dot\beta\dot\gamma} \nabla_{\beta\dot\beta} \nabla_{\gamma\dot\gamma} \bar\chi^{\dot\alpha}) \end{bmatrix} \\
&= \begin{bmatrix} \nabla_{\alpha\dot\alpha} (\xi^{\beta\gamma\dot\beta\dot\gamma} \nabla_{\beta\dot\beta} \nabla_{\gamma\dot\gamma} \bar\chi^{\dot\alpha}) \\ \nabla^{\alpha\dot\alpha} (\xi^{\beta\gamma\dot\beta\dot\gamma} \nabla_{\beta\dot\beta} \nabla_{\gamma\dot\gamma} \psi_\alpha) \end{bmatrix} \\
&= \begin{bmatrix} \nabla_{\alpha\dot\alpha} (\xi^{\beta\gamma\dot\beta\dot\gamma}) \nabla_{\beta\dot\beta} \nabla_{\gamma\dot\gamma} \bar\chi^{\dot\alpha} + \xi^{\beta\gamma\dot\beta\dot\gamma} \nabla_{\alpha\dot\alpha} \nabla_{\beta\dot\beta} \nabla_{\gamma\dot\gamma} \bar\chi^{\dot\alpha} \\ \nabla^{\alpha\dot\alpha} (\xi^{\beta\gamma\dot\beta\dot\gamma}) \nabla_{\beta\dot\beta} \nabla_{\gamma\dot\gamma} \psi_\alpha + \xi^{\beta\gamma\dot\beta\dot\gamma} \nabla^{\alpha\dot\alpha} \nabla_{\beta\dot\beta} \nabla_{\gamma\dot\gamma} \psi_\alpha \end{bmatrix} \\
&= \begin{bmatrix} \nabla_{\alpha\dot\alpha} (\xi^{\beta\gamma\dot\beta\dot\gamma}) \nabla_{\beta\dot\beta} \nabla_{\gamma\dot\gamma} \bar\chi^{\dot\alpha} + \xi^{\beta\gamma\dot\beta\dot\gamma} [\nabla_{\alpha\dot\alpha}, \nabla_{\beta\dot\beta} \nabla_{\gamma\dot\gamma}] \bar\chi^{\dot\alpha} \\ \nabla^{\alpha\dot\alpha} (\xi^{\beta\gamma\dot\beta\dot\gamma}) \nabla_{\beta\dot\beta} \nabla_{\gamma\dot\gamma} \psi_\alpha + \xi^{\beta\gamma\dot\beta\dot\gamma} [\nabla^{\alpha\dot\alpha}, \nabla_{\beta\dot\beta} \nabla_{\gamma\dot\gamma}] \psi_\alpha \end{bmatrix}
\end{aligned} \tag{C.4}
$$

as $\nabla_{\alpha\dot\alpha} \bar\chi^{\dot\alpha} = 0$ and $\nabla^{\alpha\dot\alpha} \psi_\alpha = 0$.
Since a commutator reduces the number of derivatives by 2 and each of the remaining terms in $D^{(2)}$ has at most 1 derivative, $\gamma^a \nabla_a D^{(2)} \mathbf{\Psi}$ can have at most two derivatives on $\mathbf{\Psi}$. $\nabla_{\alpha\dot\alpha} (\xi^{\beta\gamma\dot\beta\dot\gamma}) \nabla_{\beta\dot\beta} \nabla_{\gamma\dot\gamma} \bar\chi^{\dot\alpha}$ and $\nabla^{\alpha\dot\alpha} (\xi^{\beta\gamma\dot\beta\dot\gamma}) \nabla_{\beta\dot\beta} \nabla_{\gamma\dot\gamma} \psi_\alpha$ are two such terms. Since $\xi^{\beta\gamma\dot\beta\dot\gamma} = \xi^{(\beta\gamma)(\dot\beta\dot\gamma)}$, by equation 3.28,

$$
\begin{aligned}
&\nabla^{\alpha\dot\alpha} (\xi^{\beta\gamma\dot\beta\dot\gamma}) \nabla_{\beta\dot\beta} \nabla_{\gamma\dot\gamma} \psi_\alpha \\
&= \Big(\nabla^{(\alpha(\dot\alpha} \xi^{\beta\gamma)\dot\beta\dot\gamma)} + \frac{1}{3} (\varepsilon^{\alpha\beta} \nabla_\mu{}^{(\dot\alpha} \xi^{\gamma\mu\dot\beta\dot\gamma)} + \varepsilon^{\alpha\gamma} \nabla_\mu{}^{(\dot\alpha} \xi^{\beta\mu\dot\beta\dot\gamma)} + \varepsilon^{\dot\alpha\dot\beta} \nabla^{(\alpha}{}_{\dot\mu} \xi^{\beta\gamma)\dot\gamma\dot\mu} + \varepsilon^{\dot\alpha\dot\gamma} \nabla^{(\alpha}{}_{\dot\mu} \xi^{\beta\gamma)\dot\beta\dot\mu}) \\
&\quad + \frac{1}{9} (\varepsilon^{\alpha\beta} \varepsilon^{\dot\alpha\dot\beta} \nabla_{\mu\dot\mu} \xi^{\gamma\mu\dot\gamma\dot\mu} + \varepsilon^{\alpha\beta} \varepsilon^{\dot\alpha\dot\gamma} \nabla_{\mu\dot\mu} \xi^{\gamma\mu\dot\beta\dot\mu} + \varepsilon^{\alpha\gamma} \varepsilon^{\dot\alpha\dot\beta} \nabla_{\mu\dot\mu} \xi^{\beta\mu\dot\gamma\dot\mu} + \varepsilon^{\alpha\gamma} \varepsilon^{\dot\alpha\dot\gamma} \nabla_{\mu\dot\mu} \xi^{\beta\mu\dot\beta\dot\mu}) \Big) \\
&\quad \times \nabla_{\beta\dot\beta} \nabla_{\gamma\dot\gamma} \psi_\alpha \, .
\end{aligned} \tag{C.5}
$$

$\varepsilon^{\alpha\gamma} \nabla_{\beta\dot\beta} \nabla_{\gamma\dot\gamma} \psi_\alpha = \nabla_{\beta\dot\beta} \nabla^\alpha{}_{\dot\gamma} \psi_\alpha = 0$ and $\varepsilon^{\alpha\beta} \nabla_{\beta\dot\beta} \nabla_{\gamma\dot\gamma} \psi_\alpha = \nabla^\alpha{}_{\dot\beta} \nabla_{\gamma\dot\gamma} \psi_\alpha$. The latter simplifies to

$$
\begin{aligned}
\nabla^\alpha{}_{\dot\beta} \nabla_{\gamma\dot\gamma} \psi_\alpha &= \nabla_{\gamma\dot\gamma} \nabla^\alpha{}_{\dot\beta} \psi_\alpha + [\nabla^\alpha{}_{\dot\beta}, \nabla_{\gamma\dot\gamma}] \psi_\alpha \\
&= 0 + (R^\alpha{}_{\dot\beta\gamma\dot\gamma}{}^{\mu\nu} M_{\mu\nu} + \overline{R}^\alpha{}_{\dot\beta\gamma\dot\gamma}{}^{\dot\mu\dot\nu} \overline{M}_{\dot\mu\dot\nu}) \psi_\alpha \\
&= -R_{\beta\dot\beta\gamma\dot\gamma}{}^{\beta\alpha} \psi_\alpha \\
&= -(\varepsilon_{\dot\beta\dot\gamma} C_{\beta\gamma}{}^{\beta\alpha} + \varepsilon_{\beta\gamma} E^{\beta\alpha}{}_{\dot\beta\dot\gamma} + \varepsilon_{\dot\beta\dot\gamma} (\delta^\beta{}_\beta \delta^\alpha{}_\gamma + \delta^\beta{}_\gamma \delta^\alpha{}_\beta) F) \psi_\alpha \\
&= E^\alpha{}_{\gamma\dot\beta\dot\gamma} \psi_\alpha + 3\varepsilon_{\dot\gamma\dot\beta} F \psi_\gamma \, .
\end{aligned} \tag{C.6}
$$

Plugging this back into the expression above,

$$
\begin{aligned}
&\nabla^{\alpha\dot\alpha} (\xi^{\beta\gamma\dot\beta\dot\gamma}) \nabla_{\beta\dot\beta} \nabla_{\gamma\dot\gamma} \psi_\alpha \\
&= \Big(\nabla^{(\alpha(\dot\alpha} \xi^{\beta\gamma)\dot\beta\dot\gamma)} + \frac{1}{3} (\varepsilon^{\dot\alpha\dot\beta} \nabla^{(\alpha}{}_{\dot\mu} \xi^{\beta\gamma)\dot\gamma\dot\mu} + \varepsilon^{\dot\alpha\dot\gamma} \nabla^{(\alpha}{}_{\dot\mu} \xi^{\beta\gamma)\dot\beta\dot\mu}) \Big) \nabla_{\beta\dot\beta} \nabla_{\gamma\dot\gamma} \psi_\alpha \\
&\quad + \frac{1}{3} (E^\alpha{}_{\gamma\dot\beta\dot\gamma} \psi_\alpha + 3\varepsilon_{\dot\gamma\dot\beta} F \psi_\gamma) \nabla_\mu{}^{(\dot\alpha} \xi^{\gamma\mu\dot\beta\dot\gamma)} \\
&\quad + \frac{1}{9} (\varepsilon^{\dot\alpha\dot\beta} \nabla_{\mu\dot\mu} \xi^{\gamma\mu\dot\gamma\dot\mu} + \varepsilon^{\dot\alpha\dot\gamma} \nabla_{\mu\dot\mu} \xi^{\gamma\mu\dot\beta\dot\mu}) (E^\alpha{}_{\gamma\dot\beta\dot\gamma} \psi_\alpha + 3\varepsilon_{\dot\gamma\dot\beta} F \psi_\gamma) \\
&= \Big(\nabla^{(\alpha(\dot\alpha} \xi^{\beta\gamma)\dot\beta\dot\gamma)} + \frac{1}{3} (\varepsilon^{\dot\alpha\dot\beta} \nabla^{(\alpha}{}_{\dot\mu} \xi^{\beta\gamma)\dot\gamma\dot\mu} + \varepsilon^{\dot\alpha\dot\gamma} \nabla^{(\alpha}{}_{\dot\mu} \xi^{\beta\gamma)\dot\beta\dot\mu}) \Big) \nabla_{\beta\dot\beta} \nabla_{\gamma\dot\gamma} \psi_\alpha \\
&\quad + \frac{1}{3} E^\alpha{}_{\gamma\dot\beta\dot\gamma} \nabla_\beta{}^{(\dot\alpha} \xi^{\beta\gamma\dot\beta\dot\gamma)} \psi_\alpha + \frac{2}{9} E^{\alpha\beta\dot\alpha\dot\beta} \nabla_{\gamma\dot\gamma} (\xi^{\beta\gamma\dot\beta\dot\gamma}) \psi_\alpha \, .
\end{aligned} \tag{C.7}
$$

Some further simplification is possible because

$$\varepsilon^{\dot\alpha\dot\gamma}\nabla^{(\alpha}{}_{\dot\mu}\xi^{\beta\gamma)\dot\beta\dot\mu}\nabla_{\beta\dot\beta}\nabla_{\gamma\dot\gamma}\psi_\alpha = \varepsilon^{\dot\alpha\dot\beta}\nabla^{(\alpha}{}_{\dot\mu}\xi^{\beta\gamma)\dot\gamma\dot\mu}\nabla_{\gamma\dot\gamma}\nabla_{\beta\dot\beta}\psi_\alpha$$
$$= \varepsilon^{\dot\alpha\dot\beta}\nabla^{(\alpha}{}_{\dot\mu}\xi^{\beta\gamma)\dot\gamma\dot\mu}\nabla_{\beta\dot\beta}\nabla_{\gamma\dot\gamma}\psi_\alpha + \varepsilon^{\dot\alpha\dot\beta}\nabla^{(\alpha}{}_{\dot\mu}\xi^{\beta\gamma)\dot\gamma\dot\mu}[\nabla_{\gamma\dot\gamma},\nabla_{\beta\dot\beta}]\psi_\alpha \qquad \text{(C.8)}$$

and the commutator term can be decomposed to

$$\begin{aligned}
&\varepsilon^{\dot\alpha\dot\beta}\nabla^{(\alpha}{}_{\dot\mu}\xi^{\beta\gamma)\dot\gamma\dot\mu}[\nabla_{\gamma\dot\gamma},\nabla_{\beta\dot\beta}]\psi_\alpha\\
&= \nabla^{(\alpha}{}_{\dot\beta}\xi^{\beta\gamma)\dot\beta\dot\gamma}[\nabla_{\gamma\dot\gamma},\nabla_\beta{}^{\dot\alpha}]\psi_\alpha\\
&= \nabla^{(\alpha}{}_{\dot\beta}\xi^{\beta\gamma)\dot\beta\dot\gamma}(R_{\gamma\dot\gamma\beta}{}^{\dot\alpha\mu\nu}M_{\mu\nu} + \bar R_{\gamma\dot\gamma\beta}{}^{\dot\alpha\dot\mu\dot\nu}\bar M_{\dot\mu\dot\nu})\psi_\alpha\\
&= \nabla^{(\alpha}{}_{\dot\beta}\xi^{\beta\gamma)\dot\beta\dot\gamma}R_{\gamma\dot\gamma\beta}{}^{\dot\alpha}{}_\alpha{}^\mu\psi_\mu\\
&= \nabla^{(\alpha}{}_{\dot\beta}\xi^{\beta\gamma)\dot\beta\dot\gamma}(-\delta^{\dot\alpha}{}_{\dot\gamma}C_{\gamma\beta\alpha}{}^\mu + \varepsilon_{\gamma\beta}E_\alpha{}^\mu{}_{\dot\gamma}{}^{\dot\alpha} - \delta^{\dot\alpha}{}_{\dot\gamma}(-\delta^\mu{}_\beta\varepsilon_{\gamma\alpha} - \delta^\mu{}_\gamma\varepsilon_{\beta\alpha})F)\psi_\mu\\
&= \nabla^{(\alpha}{}_{\dot\beta}\xi^{\beta\gamma)\dot\alpha\dot\beta}C_{\alpha\beta\gamma\mu}\psi^\mu\,. \qquad \text{(C.9)}
\end{aligned}$$

Thus finally,

$$\nabla^{\alpha\dot\alpha}(\xi^{\beta\gamma\dot\beta\dot\gamma})\nabla_{\beta\dot\beta}\nabla_{\gamma\dot\gamma}\psi_\alpha = \left(\nabla^{(\alpha(\dot\alpha}\xi^{\beta\gamma)\dot\beta\dot\gamma)} + \frac{2}{3}\varepsilon^{\dot\alpha\dot\beta}\nabla^{(\alpha}{}_{\dot\mu}\xi^{\beta\gamma)\dot\gamma\dot\mu}\right)\nabla_{\beta\dot\beta}\nabla_{\gamma\dot\gamma}\psi_\alpha + \frac{1}{3}\nabla^{(\alpha}{}_{\dot\beta}\xi^{\beta\gamma)\dot\alpha\dot\beta}C_{\alpha\beta\gamma\mu}\psi^\mu$$
$$+ \frac{1}{3}E^\alpha{}_{\gamma\dot\beta\dot\gamma}\nabla_\beta{}^{(\dot\alpha}\xi^{\beta\gamma\dot\beta\dot\gamma)}\psi_\alpha + \frac{2}{9}E^{\alpha\beta\dot\alpha\dot\beta}\nabla_{\gamma\dot\gamma}(\xi^{\beta\gamma\dot\beta\dot\gamma})\psi_\alpha\,. \qquad \text{(C.10)}$$

Similarly, for the other term in $\gamma^a\nabla_a(\xi^{\alpha\beta\dot\alpha\dot\beta}\nabla_{\alpha\dot\alpha}\nabla_{\beta\dot\beta}\boldsymbol{\Psi})$ with two derivatives,

$$\begin{aligned}
&\nabla_{\alpha\dot\alpha}(\xi^{\beta\gamma\dot\beta\dot\gamma})\nabla_{\beta\dot\beta}\nabla_{\gamma\dot\gamma}\bar\chi^{\dot\alpha}\\
&= \nabla_{\alpha\dot\alpha}(\xi_{\beta\gamma\dot\beta\dot\gamma})\nabla^{\beta\dot\beta}\nabla^{\gamma\dot\gamma}\bar\chi^{\dot\alpha}\\
&= \Big(\nabla_{(\alpha(\dot\alpha}\xi_{\beta\gamma)\dot\beta\dot\gamma)} + \frac{1}{3}(\varepsilon_{\alpha\beta}\nabla^\mu{}_{(\dot\alpha}\xi_{\gamma\mu\dot\beta\dot\gamma)} + \varepsilon_{\alpha\gamma}\nabla^\mu{}_{(\dot\alpha}\xi_{\beta\mu\dot\beta\dot\gamma)} + \varepsilon_{\dot\alpha\dot\beta}\nabla_{(\alpha}{}^{\dot\mu}\xi_{\beta\gamma)\dot\gamma\dot\mu} + \varepsilon_{\dot\alpha\dot\gamma}\nabla_{(\alpha}{}^{\dot\mu}\xi_{\beta\gamma)\dot\beta\dot\mu})\\
&\quad + \frac{1}{9}(\varepsilon_{\alpha\beta}\varepsilon_{\dot\alpha\dot\beta}\nabla^{\mu\dot\mu}\xi_{\gamma\mu\dot\gamma\dot\mu} + \varepsilon_{\alpha\beta}\varepsilon_{\dot\alpha\dot\gamma}\nabla^{\mu\dot\mu}\xi_{\gamma\mu\dot\beta\dot\mu} + \varepsilon_{\alpha\gamma}\varepsilon_{\dot\alpha\dot\beta}\nabla^{\mu\dot\mu}\xi_{\beta\mu\dot\gamma\dot\mu} + \varepsilon_{\alpha\gamma}\varepsilon_{\dot\alpha\dot\gamma}\nabla^{\mu\dot\mu}\xi_{\beta\mu\dot\beta\dot\mu})\Big)\\
&\quad \times \nabla^{\beta\dot\beta}\nabla^{\gamma\dot\gamma}\bar\chi^{\dot\alpha}\,. \qquad \text{(C.11)}
\end{aligned}$$

Again, $\varepsilon_{\dot\alpha\dot\gamma}\nabla^{\beta\dot\beta}\nabla^{\gamma\dot\gamma}\bar\chi^{\dot\alpha} = \nabla^{\beta\dot\beta}\nabla^\gamma{}_{\dot\alpha}\bar\chi^{\dot\alpha} = 0$, $\varepsilon_{\dot\alpha\dot\beta}\nabla^{\beta\dot\beta}\nabla^{\gamma\dot\gamma}\bar\chi^{\dot\alpha} = \nabla^\beta{}_{\dot\alpha}\nabla^{\gamma\dot\gamma}\bar\chi^{\dot\alpha}$ and

$$\begin{aligned}
\nabla^\beta{}_{\dot\alpha}\nabla^{\gamma\dot\gamma}\bar\chi^{\dot\alpha} &= \nabla^{\gamma\dot\gamma}\nabla^\beta{}_{\dot\alpha}\bar\chi^{\dot\alpha} + [\nabla^\beta{}_{\dot\alpha},\nabla^{\gamma\dot\gamma}]\bar\chi^{\dot\alpha}\\
&= 0 + (R^\beta{}_{\dot\alpha}{}^{\gamma\dot\gamma\mu\nu}M_{\mu\nu} + \bar R^\beta{}_{\dot\alpha}{}^{\gamma\dot\gamma\dot\mu\dot\nu}M_{\dot\mu\dot\nu})\bar\chi^{\dot\alpha}\\
&= \bar R^{\beta\dot\alpha\gamma\dot\gamma}{}_{\dot\alpha\dot\beta}\bar\chi^{\dot\beta}\\
&= (\varepsilon^{\gamma\beta}\bar C^{\dot\alpha\dot\gamma}{}_{\dot\alpha\dot\beta} + \varepsilon^{\dot\gamma\dot\alpha}E^{\beta\gamma}{}_{\dot\alpha\dot\beta} + \varepsilon^{\gamma\beta}(\delta^{\dot\alpha}{}_{\dot\alpha}\delta^{\dot\gamma}{}_{\dot\beta} + \delta^{\dot\alpha}{}_{\dot\beta}\delta^{\dot\gamma}{}_{\dot\alpha})F)\bar\chi^{\dot\beta}\\
&= E^{\gamma\beta\dot\gamma}{}_{\dot\beta}\bar\chi^{\dot\beta} + 3\varepsilon^{\gamma\beta}F\bar\chi^{\dot\gamma}\,. \qquad \text{(C.12)}
\end{aligned}$$

Again, plugging this back into the expression above,

$$\begin{aligned}
&\nabla_{\alpha\dot\alpha}(\xi^{\beta\gamma\dot\beta\dot\gamma})\nabla^{\beta\dot\beta}\nabla^{\gamma\dot\gamma}\bar\chi^{\dot\alpha}\\
&=\left(\nabla_{(\alpha(\dot\alpha}\xi_{\beta\gamma)\dot\beta\dot\gamma)}+\frac{1}{3}(\varepsilon_{\alpha\beta}\nabla^{\mu}{}_{(\dot\alpha}\xi_{\gamma\mu\dot\beta\dot\gamma)}+\varepsilon_{\alpha\gamma}\nabla^{\mu}{}_{(\dot\alpha}\xi_{\beta\mu\dot\beta\dot\gamma)})\right)\nabla^{\beta\dot\beta}\nabla^{\gamma\dot\gamma}\bar\chi^{\dot\alpha}\\
&\quad+\frac{1}{3}(E^{\gamma\beta\dot\gamma}{}_{\dot\beta}\bar\chi^{\dot\beta}+3\varepsilon^{\gamma\beta}F\bar\chi^{\dot\gamma})\nabla_{(\alpha}{}^{\dot\mu}\xi_{\beta\gamma)\dot\gamma\dot\mu}\\
&\quad+\frac{1}{9}(\varepsilon_{\alpha\beta}\nabla^{\mu\dot\mu}\xi_{\gamma\mu\dot\gamma\dot\mu}+\varepsilon_{\alpha\gamma}\nabla^{\mu\dot\mu}\xi_{\beta\mu\dot\gamma\dot\mu})(E^{\gamma\beta\dot\gamma}{}_{\dot\beta}\bar\chi^{\dot\beta}+3\varepsilon^{\gamma\beta}F\bar\chi^{\dot\gamma})\\
&=\left(\nabla_{(\alpha(\dot\alpha}\xi_{\beta\gamma)\dot\beta\dot\gamma)}+\frac{1}{3}(\varepsilon_{\alpha\beta}\nabla^{\mu}{}_{(\dot\alpha}\xi_{\gamma\mu\dot\beta\dot\gamma)}+\varepsilon_{\alpha\gamma}\nabla^{\mu}{}_{(\dot\alpha}\xi_{\beta\mu\dot\beta\dot\gamma)})\right)\nabla^{\beta\dot\beta}\nabla^{\gamma\dot\gamma}\bar\chi^{\dot\alpha}\\
&\quad+\frac{1}{3}E^{\beta\gamma\dot\gamma}{}_{\dot\alpha}\nabla_{(\alpha}{}^{\dot\beta}\xi_{\beta\gamma)\dot\beta\dot\gamma}\bar\chi^{\dot\alpha}+\frac{2}{9}E_{\alpha\beta\dot\alpha\dot\beta}\nabla_{\gamma\dot\gamma}(\xi^{\beta\gamma\dot\beta\dot\gamma})\bar\chi^{\dot\alpha}\,. \qquad \text{(C.13)}
\end{aligned}$$

As before,

$$\begin{aligned}
\varepsilon_{\alpha\gamma}\nabla^{\mu}{}_{(\dot\alpha}\xi_{\beta\mu\dot\beta\dot\gamma)}\nabla^{\beta\dot\beta}\nabla^{\gamma\dot\gamma}\bar\chi^{\dot\alpha}&=\varepsilon_{\alpha\beta}\nabla^{\mu}{}_{(\dot\alpha}\xi_{\gamma\mu\dot\beta\dot\gamma)}\nabla^{\gamma\dot\gamma}\nabla^{\beta\dot\beta}\bar\chi^{\dot\alpha}\\
&=\varepsilon_{\alpha\beta}\nabla^{\mu}{}_{(\dot\alpha}\xi_{\gamma\mu\dot\beta\dot\gamma)}\nabla^{\beta\dot\beta}\nabla^{\gamma\dot\gamma}\bar\chi^{\dot\alpha}+\varepsilon_{\alpha\beta}\nabla^{\mu}{}_{(\dot\alpha}\xi_{\gamma\mu\dot\beta\dot\gamma)}[\nabla^{\gamma\dot\gamma},\nabla^{\beta\dot\beta}]\bar\chi^{\dot\alpha} \qquad \text{(C.14)}\\
\varepsilon_{\alpha\beta}\nabla^{\mu}{}_{(\dot\alpha}\xi_{\gamma\mu\dot\beta\dot\gamma)}[\nabla^{\gamma\dot\gamma},\nabla^{\beta\dot\beta}]\bar\chi^{\dot\alpha}&=\nabla^{\beta}{}_{(\dot\alpha}\xi_{\beta\gamma\dot\beta\dot\gamma)}[\nabla^{\gamma\dot\gamma},\nabla_{\alpha}{}^{\dot\beta}]\bar\chi^{\dot\alpha}\\
&=\nabla^{\beta}{}_{(\dot\alpha}\xi_{\beta\gamma\dot\beta\dot\gamma)}(R^{\gamma\dot\gamma}{}_{\alpha}{}^{\dot\beta\mu\nu}M_{\mu\nu}+\bar R^{\gamma\dot\gamma}{}_{\alpha}{}^{\dot\beta\dot\mu\dot\nu}\overline{M}_{\mu\nu})\bar\chi^{\dot\alpha}\\
&=\nabla^{\beta}{}_{(\dot\alpha}\xi_{\beta\gamma\dot\beta\dot\gamma)}\bar R^{\gamma\dot\gamma}{}_{\alpha}{}^{\dot\beta\dot\alpha\dot\mu}\bar\chi_{\dot\mu}\\
&=\nabla^{\beta}{}_{(\dot\alpha}\xi_{\beta\gamma\dot\beta\dot\gamma)}(\delta^{\gamma}{}_{\alpha}\bar C^{\dot\gamma\dot\beta\dot\alpha\dot\mu}+\varepsilon^{\dot\beta\dot\gamma}E^{\gamma}{}_{\alpha}{}^{\dot\alpha\dot\mu}\\
&\quad+\delta^{\gamma}{}_{\alpha}(\varepsilon^{\dot\alpha\dot\gamma}\varepsilon^{\dot\mu\dot\beta}+\varepsilon^{\dot\alpha\dot\beta}\varepsilon^{\dot\mu\dot\gamma})F)\bar\chi_{\dot\mu}\\
&=\nabla^{\beta}{}_{(\dot\alpha}\xi_{\alpha\beta\dot\beta\dot\gamma)}\bar C^{\dot\alpha\dot\beta\dot\gamma\dot\mu}\bar\chi_{\dot\mu}\,. \qquad \text{(C.15)}
\end{aligned}$$

Thus finally,

$$\begin{aligned}
\nabla_{\alpha\dot\alpha}(\xi^{\beta\gamma\dot\beta\dot\gamma})\nabla^{\beta\dot\beta}\nabla^{\gamma\dot\gamma}\bar\chi^{\dot\alpha}&=\left(\nabla_{(\alpha(\dot\alpha}\xi_{\beta\gamma)\dot\beta\dot\gamma)}+\frac{2}{3}\varepsilon_{\alpha\beta}\nabla^{\mu}{}_{(\dot\alpha}\xi_{\gamma\mu\dot\beta\dot\gamma)}\right)\nabla^{\beta\dot\beta}\nabla^{\gamma\dot\gamma}\bar\chi^{\dot\alpha}+\frac{1}{3}\nabla^{\beta}{}_{(\dot\alpha}\xi_{\alpha\beta\dot\beta\dot\gamma)}\bar C^{\dot\alpha\dot\beta\dot\gamma\dot\mu}\bar\chi_{\dot\mu}\\
&\quad+\frac{1}{3}E^{\beta\gamma\dot\gamma}{}_{\dot\alpha}\nabla_{(\alpha}{}^{\dot\beta}\xi_{\beta\gamma)\dot\beta\dot\gamma}\bar\chi^{\dot\alpha}+\frac{2}{9}E_{\alpha\beta\dot\alpha\dot\beta}\nabla_{\gamma\dot\gamma}(\xi^{\beta\gamma\dot\beta\dot\gamma})\bar\chi^{\dot\alpha}\,. \qquad \text{(C.16)}
\end{aligned}$$

Next, another term of $\gamma^a\nabla_a D^{(2)}\mathbf{\Psi}$ which could lead to two derivatives on $\mathbf{\Psi}$ is

$$\begin{aligned}
&\gamma^a\nabla_a((\xi^{\alpha\beta\gamma\dot\alpha}\nabla_{\alpha\dot\alpha}M_{\beta\gamma}+\xi^{\alpha\dot\alpha\dot\beta\dot\gamma}\nabla_{\alpha\dot\alpha}\overline{M}_{\dot\beta\dot\gamma})\mathbf{\Psi})\\
&=\begin{bmatrix}0 & (\sigma^a)_{\alpha\dot\alpha}\\ (\tilde\sigma^a)^{\dot\alpha\alpha} & 0\end{bmatrix}\begin{bmatrix}\nabla_a((\xi^{\beta\gamma\mu\dot\beta}\nabla_{\beta\dot\beta}M_{\gamma\mu}+\xi^{\beta\dot\beta\dot\gamma\dot\mu}\nabla_{\beta\dot\beta}\overline{M}_{\dot\gamma\dot\mu})\psi_\alpha)\\ \nabla_a((\xi^{\beta\gamma\mu\dot\beta}\nabla_{\beta\dot\beta}M_{\gamma\mu}+\xi^{\beta\dot\beta\dot\gamma\dot\mu}\nabla_{\beta\dot\beta}\overline{M}_{\dot\gamma\dot\mu})\bar\chi^{\dot\alpha})\end{bmatrix}\\
&=\frac{1}{2}\begin{bmatrix}\nabla_{\alpha\dot\alpha}(\xi^{\beta\dot\beta\dot\gamma\dot\mu}\nabla_{\beta\dot\beta}(\delta^{\dot\alpha}{}_{\dot\gamma}\bar\chi_{\dot\mu}+\delta^{\dot\alpha}{}_{\dot\mu}\bar\chi_{\dot\gamma}))\\ \nabla^{\alpha\dot\alpha}(\xi^{\beta\gamma\mu\dot\beta}\nabla_{\beta\dot\beta}(\varepsilon_{\alpha\gamma}\psi_\mu+\varepsilon_{\alpha\mu}\psi_\gamma))\end{bmatrix}\\
&=\begin{bmatrix}\nabla_{\alpha\dot\alpha}(\xi^{\beta\dot\alpha\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}\bar\chi_{\dot\gamma})\\ \nabla^{\alpha\dot\alpha}(\xi_{\alpha}{}^{\beta\gamma\dot\beta}\nabla_{\beta\dot\beta}\psi_\gamma)\end{bmatrix}\\
&=\begin{bmatrix}\nabla_{\alpha\dot\alpha}(\xi^{\beta\dot\alpha\dot\beta\dot\gamma})\nabla_{\beta\dot\beta}\bar\chi_{\dot\gamma}+\xi^{\beta\dot\alpha\dot\beta\dot\gamma}\nabla_{\alpha\dot\alpha}\nabla_{\beta\dot\beta}\bar\chi_{\dot\gamma}\\ \nabla^{\alpha\dot\alpha}(\xi_{\alpha}{}^{\beta\gamma\dot\beta})\nabla_{\beta\dot\beta}\psi_\gamma+\xi_{\alpha}{}^{\beta\gamma\dot\beta}\nabla^{\alpha\dot\alpha}\nabla_{\beta\dot\beta}\psi_\gamma\end{bmatrix}\\
&=\begin{bmatrix}\nabla_{\alpha\dot\alpha}(\xi^{\beta\dot\alpha\dot\beta\dot\gamma})\nabla_{\beta\dot\beta}\bar\chi_{\dot\gamma}+\varepsilon_{\alpha\beta}\xi_{\gamma\dot\alpha\dot\beta\dot\gamma}\nabla^{\beta\dot\beta}\nabla^{\gamma\dot\gamma}\bar\chi^{\dot\alpha}\\ \nabla^{\alpha\dot\alpha}(\xi_{\alpha}{}^{\beta\gamma\dot\beta})\nabla_{\beta\dot\beta}\psi_\gamma-\varepsilon^{\dot\alpha\dot\beta}\xi^{\alpha\beta\gamma\dot\gamma}\nabla_{\beta\dot\beta}\nabla_{\gamma\dot\gamma}\psi_\alpha\,,\end{bmatrix} \qquad \text{(C.17)}
\end{aligned}$$

which has two terms with two derivatives on $\mathbf{\Psi}$. The only other term of $\gamma^a\nabla_a D^{(2)}\mathbf{\Psi}$ which could lead to two derivatives on $\mathbf{\Psi}$ is

$$\begin{aligned}
\gamma^a\nabla_a(\xi^{\alpha\dot{\alpha}}\nabla_{\alpha\dot{\alpha}}\mathbf{\Psi}) &= \begin{bmatrix} 0 & (\sigma^a)_{\alpha\dot{\alpha}} \\ (\tilde{\sigma}^a)^{\dot{\alpha}\alpha} & 0 \end{bmatrix} \begin{bmatrix} \nabla_a(\xi^{\beta\dot{\beta}}\nabla_{\beta\dot{\beta}}\psi_\alpha) \\ \nabla_a(\xi^{\beta\dot{\beta}}\nabla_{\beta\dot{\beta}}\bar{\chi}^{\dot{\alpha}}) \end{bmatrix} \\
&= \begin{bmatrix} \nabla_{\alpha\dot{\alpha}}(\xi^{\beta\dot{\beta}}\nabla_{\beta\dot{\beta}}\bar{\chi}^{\dot{\alpha}}) \\ \nabla^{\alpha\dot{\alpha}}(\xi^{\beta\dot{\beta}}\nabla_{\beta\dot{\beta}}\psi_\alpha) \end{bmatrix} \\
&= \begin{bmatrix} \nabla_{\alpha\dot{\alpha}}(\xi^{\beta\dot{\beta}})\nabla_{\beta\dot{\beta}}\bar{\chi}^{\dot{\alpha}} + \xi^{\beta\dot{\beta}}\nabla_{\alpha\dot{\alpha}}\nabla_{\beta\dot{\beta}}\bar{\chi}^{\dot{\alpha}} \\ \nabla^{\alpha\dot{\alpha}}(\xi^{\beta\dot{\beta}})\nabla_{\beta\dot{\beta}}\psi_\alpha + \xi^{\beta\dot{\beta}}\nabla^{\alpha\dot{\alpha}}\nabla_{\beta\dot{\beta}}\psi_\alpha \end{bmatrix} \\
&= \begin{bmatrix} \nabla_{\alpha\dot{\alpha}}(\xi^{\beta\dot{\beta}})\nabla_{\beta\dot{\beta}}\bar{\chi}^{\dot{\alpha}} + \xi^{\beta\dot{\beta}}[\nabla_{\alpha\dot{\alpha}},\nabla_{\beta\dot{\beta}}]\bar{\chi}^{\dot{\alpha}} \\ \nabla^{\alpha\dot{\alpha}}(\xi^{\beta\dot{\beta}})\nabla_{\beta\dot{\beta}}\psi_\alpha + \xi^{\beta\dot{\beta}}[\nabla^{\alpha\dot{\alpha}},\nabla_{\beta\dot{\beta}}]\psi_\alpha \end{bmatrix}, \qquad \text{(C.18)}
\end{aligned}$$

which does not actually have any terms with two derivatives on $\mathbf{\Psi}$.

Collating the results of the last few pages,

$$\begin{aligned}
&\gamma^a\nabla_a D^{(2)}\mathbf{\Psi} \\
&= \Bigg[\left(\nabla_{(\alpha(\dot{\alpha}}\xi_{\beta\gamma)\dot{\beta}\dot{\gamma})} + \varepsilon_{\alpha\beta}\left(\frac{2}{3}\nabla^{\mu}{}_{(\dot{\alpha}}\xi_{\gamma\mu\dot{\beta}\dot{\gamma})} + \xi_{\gamma\dot{\alpha}\dot{\beta}\dot{\gamma}}\right)\right)\nabla^{\beta\dot{\beta}}\nabla^{\gamma\dot{\gamma}}\bar{\chi}^{\dot{\alpha}} + \frac{1}{3}\nabla^{\beta}{}_{(\dot{\alpha}}\xi_{\alpha\beta\dot{\beta}\dot{\gamma})}\bar{C}^{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\mu}}\bar{\chi}_{\dot{\mu}} \\
&\quad + \frac{1}{3}E^{\beta\gamma\dot{\gamma}}{}_{\dot{\alpha}}\nabla_{(\alpha}{}^{\dot{\beta}}\xi_{\beta\gamma)\dot{\beta}\dot{\gamma}}\bar{\chi}^{\dot{\alpha}} + \frac{2}{9}E_{\alpha\beta\dot{\alpha}\dot{\beta}}\nabla_{\gamma\dot{\gamma}}(\xi^{\beta\gamma\dot{\beta}\dot{\gamma}})\bar{\chi}^{\dot{\alpha}} + \xi^{\beta\gamma\dot{\beta}\dot{\gamma}}[\nabla_{\alpha\dot{\alpha}},\nabla_{\beta\dot{\beta}}\nabla_{\gamma\dot{\gamma}}]\bar{\chi}^{\dot{\alpha}} \\
&\quad + \nabla_{\alpha\dot{\alpha}}(\xi^{\beta\dot{\alpha}\dot{\beta}\dot{\gamma}})\nabla_{\beta\dot{\beta}}\bar{\chi}_{\dot{\gamma}} + \nabla_{\alpha\dot{\alpha}}(\xi^{\beta\dot{\beta}})\nabla_{\beta\dot{\beta}}\bar{\chi}^{\dot{\alpha}} + \xi^{\beta\dot{\beta}}[\nabla_{\alpha\dot{\alpha}},\nabla_{\beta\dot{\beta}}]\bar{\chi}^{\dot{\alpha}} + \nabla_{\alpha\dot{\alpha}}(\xi^{\dot{\beta}\dot{\gamma}}\overline{M}_{\dot{\beta}\dot{\gamma}}\bar{\chi}^{\dot{\alpha}} + \xi\bar{\chi}^{\dot{\alpha}}), \\
&\quad \left(\nabla^{(\alpha(\dot{\alpha}}\xi^{\beta\gamma)\dot{\beta}\dot{\gamma})} + \varepsilon^{\dot{\alpha}\dot{\beta}}\left(\frac{2}{3}\nabla^{(\alpha}{}_{\dot{\mu}}\xi^{\beta\gamma)\dot{\gamma}\dot{\mu}} - \xi^{\alpha\beta\gamma\dot{\gamma}}\right)\right)\nabla_{\beta\dot{\beta}}\nabla_{\gamma\dot{\gamma}}\psi_\alpha + \frac{1}{3}\nabla^{(\alpha}{}_{\dot{\beta}}\xi^{\beta\gamma)\dot{\alpha}\dot{\beta}}C_{\alpha\beta\gamma\mu}\psi^{\mu} \\
&\quad + \frac{1}{3}E^{\alpha}{}_{\gamma\dot{\beta}\dot{\gamma}}\nabla_{\beta}{}^{(\dot{\alpha}}\xi^{\beta\gamma\dot{\beta}\dot{\gamma})}\psi_\alpha + \frac{2}{9}E^{\alpha\beta\dot{\alpha}\dot{\beta}}\nabla_{\gamma\dot{\gamma}}(\xi^{\beta\gamma\dot{\beta}\dot{\gamma}})\psi_\alpha + \xi^{\beta\gamma\dot{\beta}\dot{\gamma}}[\nabla^{\alpha\dot{\alpha}},\nabla_{\beta\dot{\beta}}\nabla\gamma\dot{\gamma}]\psi_\alpha \\
&\quad + \nabla^{\alpha\dot{\alpha}}(\xi_{\alpha}{}^{\beta\gamma\dot{\beta}})\nabla_{\beta\dot{\beta}}\psi_\gamma + \nabla^{\alpha\dot{\alpha}}(\xi^{\beta\dot{\beta}})\nabla_{\beta\dot{\beta}}\psi_\alpha + \xi^{\beta\dot{\beta}}[\nabla^{\alpha\dot{\alpha}},\nabla_{\beta\dot{\beta}}]\psi_\alpha + \nabla^{\alpha\dot{\alpha}}(\xi^{\beta\gamma}M_{\beta\gamma}\psi_\alpha + \xi\psi_\alpha)\Bigg]^T. \qquad \text{(C.19)}
\end{aligned}$$

The terms with two derivatives (the maximum) cannot be simplified further (at least in terms of reducing the number of derivatives) since the coefficients of $\nabla^{\beta\dot{\beta}}\nabla^{\gamma\dot{\gamma}}\bar{\chi}^{\dot{\alpha}}$ and $\nabla_{\beta\dot{\beta}}\nabla_{\gamma\dot{\gamma}}\psi_\alpha$ are symmetric in $\dot{\alpha}$, $\dot{\beta}$ & $\dot{\gamma}$ and α, β & γ respectively (thereby preventing the creation of $\nabla_{\alpha\dot{\alpha}}\bar{\chi}^{\dot{\alpha}}$ or $\nabla^{\alpha\dot{\alpha}}\psi_\alpha$ like terms).
Since $\mathbf{\Psi}$ is an arbitrary solutions of $\gamma^a\nabla_a\mathbf{\Psi} = 0$, the ony way $\gamma^a\nabla_a D^{(2)}\mathbf{\Psi}$ can equal zero is if

$$\begin{aligned}
&\nabla_{(\alpha(\dot{\alpha}}\xi_{\beta\gamma)\dot{\beta}\dot{\gamma})} + \varepsilon_{\alpha\beta}\left(\frac{2}{3}\nabla^{\mu}{}_{(\dot{\alpha}}\xi_{\gamma\mu\dot{\beta}\dot{\gamma})} + \xi_{\gamma\dot{\alpha}\dot{\beta}\dot{\gamma}}\right) = 0 \quad \text{and} \\
&\nabla^{(\alpha(\dot{\alpha}}\xi^{\beta\gamma)\dot{\beta}\dot{\gamma})} + \varepsilon^{\dot{\alpha}\dot{\beta}}\left(\frac{2}{3}\nabla^{(\alpha}{}_{\dot{\mu}}\xi^{\beta\gamma)\dot{\gamma}\dot{\mu}} - \xi^{\alpha\beta\gamma\dot{\gamma}}\right) = 0\,. \qquad \text{(C.20)}
\end{aligned}$$

However, $\nabla_{(\alpha(\dot{\alpha}}\xi_{\beta\gamma)\dot{\beta}\dot{\gamma})}$ is symmetric in α and β while the $\varepsilon_{\alpha\beta}$ term is antisymmetric in those indices. Hence, they must vanish individually. Applying a similar logic to the other equation as well, it follows that

$$\begin{aligned}
&0 = \nabla_{(\alpha(\dot{\alpha}}\xi_{\beta\gamma)\dot{\beta}\dot{\gamma})} \iff 0 = \nabla^{(\alpha(\dot{\alpha}}\xi^{\beta\gamma)\dot{\beta}\dot{\gamma})}, \\
&0 = \frac{2}{3}\nabla^{\mu}{}_{(\dot{\alpha}}\xi_{\gamma\mu\dot{\beta}\dot{\gamma})} + \xi_{\gamma\dot{\alpha}\dot{\beta}\dot{\gamma}} \implies \xi^{\alpha\dot{\alpha}\dot{\beta}\dot{\gamma}} = \frac{2}{3}\nabla_{\beta}{}^{(\dot{\alpha}}\xi^{\alpha\beta\dot{\beta}\dot{\gamma})} \quad \text{and} \\
&0 = \frac{2}{3}\nabla^{(\alpha}{}_{\dot{\mu}}\xi^{\beta\gamma)\dot{\gamma}\dot{\mu}} - \xi^{\alpha\beta\gamma\dot{\gamma}} \implies \xi^{\alpha\beta\gamma\dot{\alpha}} = \frac{2}{3}\nabla^{(\alpha}{}_{\dot{\beta}}\xi^{\beta\gamma)\dot{\alpha}\dot{\beta}}\,. \qquad \text{(C.21)}
\end{aligned}$$

By lemma 3.3.1, the 1st equation implies $\xi^{\alpha\beta\dot{\alpha}\dot{\beta}}$ is conformal Killing.
Thus, so far, I have shown that the only 2nd order higher symmetry candidate is

$$\begin{aligned}D^{(2)} = \xi^{\alpha\beta\dot{\alpha}\dot{\beta}}\nabla_{\alpha\dot{\alpha}}\nabla_{\beta\dot{\beta}} + \frac{2}{3}\nabla^{(\alpha}{}_{\dot{\beta}}\xi^{\beta\gamma)\dot{\alpha}\dot{\beta}}\nabla_{\alpha\dot{\alpha}}M_{\beta\gamma} + \frac{2}{3}\nabla_{\beta}{}^{(\dot{\alpha}}\xi^{\alpha\beta\dot{\beta}\dot{\gamma})}\nabla_{\alpha\dot{\alpha}}\overline{M}_{\dot{\beta}\dot{\gamma}} + \xi^{\alpha\dot{\alpha}}\nabla_{\alpha\dot{\alpha}} \\ + \xi^{\alpha\beta}M_{\alpha\beta} + \xi^{\dot{\alpha}\dot{\beta}}\overline{M}_{\dot{\alpha}\dot{\beta}} + \xi \end{aligned} \tag{C.22}$$

for a conformal Killing tensor, $\xi^{\alpha\beta\dot{\alpha}\dot{\beta}}$.

Next, as I did in section 4.3, I will fix the remaining components, $\xi^{\alpha\dot{\alpha}}$, $\xi^{\alpha\beta}$, $\xi^{\dot{\alpha}\dot{\beta}}$ and ξ, by ensuring that $D'^{(2)}\mathbf{\Psi}' = \mathrm{e}^{3\sigma/2}D^{(2)}\mathbf{\Psi}$ upon a Weyl transformation, $e'_a{}^m = \mathrm{e}^{\sigma}e_a{}^m$.
By lemma 3.5, $\xi'^{\alpha\beta\dot{\alpha}\dot{\beta}} = (1-2\sigma)\xi^{\alpha\beta\dot{\alpha}\dot{\beta}}$ under an infinitesimal Weyl transformation. Each term in $D^{(2)}$ must have a net conformal weight[1] of zero for $D^{(2)}\mathbf{\Psi}$ to have the same conformal weight as $\mathbf{\Psi}$, namely 3/2. By equation 4.61, $\nabla_{\alpha\dot{\alpha}}$ has a conformal weight of 1. The Lorentz generators are unchanged under a Weyl transformation.
Therefore, $\xi^{\alpha\dot{\alpha}}$ has a net conformal weight of -1 and $\xi^{\alpha\beta}$, $\xi^{\dot{\alpha}\dot{\beta}}$ & ξ all have a net conformal weight of 0.
All four tensors must be constructed out of $\xi^{\alpha\beta\dot{\alpha}\dot{\beta}}$ and tensors constructed fully out of the vierbien. Furthermore, on physical grounds[2], each coefficient can have at most one factor of $\xi^{\alpha\beta\dot{\alpha}\dot{\beta}}$; no products or contractions of $\xi^{\alpha\beta\dot{\alpha}\dot{\beta}}$ are allowed.
Since all the Riemann tensor descendants have conformal weight, -2, the only possible expression for $\xi^{\alpha\dot{\alpha}}$ is $A\nabla_{\beta\dot{\beta}}\xi^{\alpha\beta\dot{\alpha}\dot{\beta}}$, for some constant, A.
For ξ, I can have two covariant derivatives to compensate for $\xi^{\alpha\beta\dot{\alpha}\dot{\beta}}$'s -2 conformal weight or a single Riemann tensor descendant.
Since ξ is a scalar, the only possibility is $\xi = B\nabla_{\alpha\dot{\alpha}}\nabla_{\beta\dot{\beta}}\xi^{\alpha\beta\dot{\alpha}\dot{\beta}} + CE_{\alpha\beta\dot{\alpha}\dot{\beta}}\xi^{\alpha\beta\dot{\alpha}\dot{\beta}}$ for some constants, B and C.
The same logic as ξ applies for $\xi^{\alpha\beta}$, except this time I have to create a different index structure and ensure $\xi^{\alpha\beta} = \xi^{\beta\alpha}$. On the surface it would seem that I have four possible terms to work with, $\nabla^{(\alpha}{}_{\dot{\alpha}}\nabla_{\gamma\dot{\beta}}\xi^{\beta)\gamma\dot{\alpha}\dot{\beta}}$, $\nabla_{\gamma\dot{\beta}}\nabla^{(\alpha}{}_{\dot{\alpha}}\xi^{\beta)\gamma\dot{\alpha}\dot{\beta}}$, $\nabla^{\gamma}{}_{\dot{\alpha}}\nabla_{\gamma\dot{\beta}}\xi^{\alpha\beta\dot{\alpha}\dot{\beta}}$ and $E^{(\alpha}{}_{\gamma\dot{\alpha}\dot{\beta}}\xi^{\beta)\gamma\dot{\alpha}\dot{\beta}}$. However,

$$\begin{aligned}
&\nabla_{\gamma\dot{\beta}}\nabla^{(\alpha}{}_{\dot{\alpha}}\xi^{\beta)\gamma\dot{\alpha}\dot{\beta}} \\
&= \nabla^{(\alpha}{}_{\dot{\alpha}}\nabla_{\gamma\dot{\beta}}\xi^{\beta)\gamma\dot{\alpha}\dot{\beta}} + [\nabla_{\gamma\dot{\beta}}, \nabla^{(\alpha}{}_{\dot{\alpha}}]\xi^{\beta)\gamma\dot{\alpha}\dot{\beta}} \\
&= \nabla^{(\alpha}{}_{\dot{\alpha}}\nabla_{\gamma\dot{\beta}}\xi^{\beta)\gamma\dot{\alpha}\dot{\beta}} + (R_{\gamma\dot{\beta}}{}^{(\alpha}{}_{\dot{\alpha}}{}^{|\mu\nu|}M_{\mu\nu} + \overline{R}_{\gamma\dot{\beta}}{}^{(\alpha}{}_{\dot{\alpha}}{}^{\dot{\mu}\dot{\nu}}\overline{M}_{\dot{\mu}\dot{\nu}})\xi^{\beta)\gamma\dot{\alpha}\dot{\beta}} \\
&= \nabla^{(\alpha}{}_{\dot{\alpha}}\nabla_{\gamma\dot{\beta}}\xi^{\beta)\gamma\dot{\alpha}\dot{\beta}} - R_{\gamma\dot{\beta}}{}^{(\alpha}{}_{\dot{\alpha}}{}^{\beta)}{}_{\mu}\xi^{\mu\gamma\dot{\alpha}\dot{\beta}} - R_{\gamma\dot{\beta}}{}^{(\alpha}{}_{\dot{\alpha}}{}^{|\gamma|}{}_{\mu}\xi^{\beta)\mu\dot{\alpha}\dot{\beta}} - 2\overline{R}_{\gamma\dot{\beta}}{}^{(\alpha}{}_{\dot{\alpha}}{}^{(\dot{\alpha}}{}_{\dot{\mu}}\xi^{\beta)\gamma\dot{\beta})\dot{\mu}} \\
&= \nabla^{(\alpha}{}_{\dot{\alpha}}\nabla_{\gamma\dot{\beta}}\xi^{\beta)\gamma\dot{\alpha}\dot{\beta}} - (\varepsilon_{\dot{\beta}\dot{\alpha}}C_{\gamma}{}^{(\alpha\beta)}{}_{\mu} - \delta^{(\alpha}{}_{\gamma}E^{\beta)}{}_{\mu\dot{\beta}\dot{\alpha}} + \varepsilon_{\dot{\beta}\dot{\alpha}}(-\delta^{(\beta}{}_{\gamma}\delta^{\alpha)}{}_{\mu} - \varepsilon^{(\beta\alpha)}\varepsilon_{\gamma\mu})F)\xi^{\mu\gamma\dot{\alpha}\dot{\beta}} \\
&\quad - (\varepsilon_{\dot{\beta}\dot{\alpha}}C_{\gamma}{}^{(\alpha|\gamma|}{}_{\mu} - \delta^{(\alpha}{}_{\gamma}E^{|\gamma|}{}_{\mu\dot{\beta}\dot{\alpha}} + \varepsilon_{\dot{\beta}\dot{\alpha}}(-\delta^{\gamma}{}_{\gamma}\delta^{(\alpha}{}_{\mu} - \varepsilon^{\gamma(\alpha}\varepsilon_{\gamma\mu})F)\xi^{\beta)\mu\dot{\alpha}\dot{\beta}} \\
&\quad - 2(-\delta^{(\alpha}{}_{\gamma}\overline{C}_{\dot{\beta}\dot{\alpha}}{}^{(\dot{\alpha}}{}_{\dot{\mu}} + \varepsilon_{\dot{\beta}\dot{\alpha}}E_{\gamma}{}^{(\alpha(\dot{\alpha}}{}_{\dot{\mu}} - \delta^{(\alpha}{}_{\gamma}(-\delta^{(\dot{\alpha}}{}_{\dot{\beta}}\varepsilon_{\dot{\alpha}\dot{\mu}} - \delta^{(\dot{\alpha}}{}_{\dot{\alpha}}\varepsilon_{\dot{\beta}\dot{\mu}})F)\xi^{\beta)\gamma\dot{\beta})\dot{\mu}} \\
&= \nabla^{(\alpha}{}_{\dot{\alpha}}\nabla_{\gamma\dot{\beta}}\xi^{\beta)\gamma\dot{\alpha}\dot{\beta}} + 2E^{(\alpha}{}_{\gamma\dot{\alpha}\dot{\beta}}\xi^{\beta)\gamma\dot{\alpha}\dot{\beta}}.
\end{aligned} \tag{C.23}$$

[1]If, upon a Weyl transformation, a tensor, T, transforms as $T' = n\sigma T +$ other terms, then T is said to have a conformal weight of n.

[2]By physical grounds I mean the previously discussed interpretation of $x'^m = x^m + \xi^m(x)$ as an infinitesimal conformal isometry.

Therefore, contributions from $\nabla_{\gamma\dot{\beta}}\nabla^{(\alpha}{}_{\dot{\alpha}}\xi^{\beta)\gamma\dot{\alpha}\dot{\beta}}$ can be absorbed into contributions from $\nabla^{(\alpha}{}_{\dot{\alpha}}\nabla_{\gamma\dot{\beta}}\xi^{\beta)\gamma\dot{\alpha}\dot{\beta}}$ and $E^{\alpha}{}_{\gamma\dot{\alpha}\dot{\beta}}\xi^{\beta)\gamma\dot{\alpha}\dot{\beta}}$. Likewise,

$$
\begin{aligned}
\nabla^{\gamma}{}_{\dot{\alpha}}\nabla_{\gamma\dot{\beta}}\xi^{\alpha\beta\dot{\alpha}\dot{\beta}} &= \nabla^{\gamma}{}_{\dot{\beta}}\nabla_{\gamma\dot{\alpha}}\xi^{\alpha\beta\dot{\alpha}\dot{\beta}} \\
&= -\nabla_{\gamma\dot{\beta}}\nabla^{\gamma}{}_{\dot{\alpha}}\xi^{\alpha\beta\dot{\alpha}\dot{\beta}} \\
\implies 2\nabla^{\gamma}{}_{\dot{\alpha}}\nabla_{\gamma\dot{\beta}}\xi^{\alpha\beta\dot{\alpha}\dot{\beta}} &= [\nabla^{\gamma}{}_{\dot{\alpha}}, \nabla_{\gamma\dot{\beta}}]\xi^{\alpha\beta\dot{\alpha}\dot{\beta}} \\
&= (R^{\gamma}{}_{\dot{\alpha}\gamma\dot{\beta}}{}^{\mu\nu}M_{\mu\nu} + \overline{R}^{\gamma}{}_{\dot{\alpha}\gamma\dot{\beta}}{}^{\dot{\mu}\dot{\nu}}\overline{M}_{\dot{\mu}\dot{\nu}})\xi^{\alpha\beta\dot{\alpha}\dot{\beta}} \\
&= -R^{\gamma}{}_{\dot{\alpha}\gamma\dot{\beta}}{}^{(\alpha}{}_{\mu}\xi^{\beta)\mu\dot{\alpha}\dot{\beta}} - \overline{R}^{\gamma}{}_{\dot{\alpha}\gamma\dot{\beta}}{}^{(\dot{\alpha}}{}_{\dot{\mu}}\xi^{\alpha\beta\dot{\beta})\dot{\mu}} \\
&= -(\varepsilon_{\dot{\alpha}\dot{\beta}}C^{\gamma}{}_{\gamma}{}^{(\alpha}{}_{\mu} + \delta^{\gamma}{}_{\gamma}E^{(\alpha}{}_{\mu\dot{\alpha}\dot{\beta}} + \varepsilon_{\dot{\alpha}\dot{\beta}}(-\varepsilon^{\gamma(\alpha}\varepsilon_{\gamma\mu} - \delta^{(\alpha}{}_{\gamma}\delta^{|\gamma|}{}_{\mu})F)\xi^{\beta)\mu\dot{\alpha}\dot{\beta}} \\
&\quad - (\delta^{\gamma}{}_{\gamma}\overline{C}_{\dot{\alpha}\dot{\beta}}{}^{(\dot{\alpha}}{}_{\dot{\mu}} + \varepsilon_{\dot{\alpha}\dot{\beta}}E^{\gamma}{}_{\gamma}{}^{(\dot{\alpha}}{}_{\dot{\mu}} + \delta^{\gamma}{}_{\gamma}(-\delta^{(\dot{\alpha}}{}_{\dot{\alpha}}\varepsilon_{\dot{\beta}\dot{\mu}} - \delta^{(\dot{\alpha}}{}_{\dot{\beta}}\varepsilon_{\dot{\alpha}\dot{\mu}})F)\xi^{\alpha\beta\dot{\beta})\dot{\mu}} \\
\implies \nabla^{\gamma}{}_{\dot{\alpha}}\nabla_{\gamma\dot{\beta}}\xi^{\alpha\beta\dot{\alpha}\dot{\beta}} &= -E^{(\alpha}{}_{\gamma\dot{\alpha}\dot{\beta}}\xi^{\beta)\gamma\dot{\alpha}\dot{\beta}} \qquad \text{(C.24)}
\end{aligned}
$$

and thus $\nabla^{\gamma}{}_{\dot{\alpha}}\nabla_{\gamma\dot{\beta}}\xi^{\alpha\beta\dot{\alpha}\dot{\beta}}$ can be absorbed into $E^{(\alpha}{}_{\gamma\dot{\alpha}\dot{\beta}}\xi^{\beta)\gamma\dot{\alpha}\dot{\beta}}$ too.
Hence, it suffices to let $\xi^{\alpha\beta} = D\nabla^{(\alpha}{}_{\dot{\alpha}}\nabla_{\gamma\dot{\beta}}\xi^{\beta)\gamma\dot{\alpha}\dot{\beta}} + GE^{\alpha}{}_{\gamma\dot{\alpha}\dot{\beta}}\xi^{\beta)\gamma\dot{\alpha}\dot{\beta}}$ for some constants, D and G.
Similarly, $\xi^{\dot{\alpha}\dot{\beta}} = H\nabla_{\alpha}{}^{(\dot{\alpha}}\nabla_{\beta\dot{\gamma}}\xi^{\alpha\beta\dot{\beta})\dot{\gamma}} + IE_{\alpha\beta\dot{\gamma}}{}^{(\dot{\alpha}}\xi^{\alpha\beta\dot{\beta})\dot{\gamma}}$ for some constants, H and I.
With these results, so far I have

$$
\begin{aligned}
D^{(2)} = {}& \xi^{\alpha\beta\dot{\alpha}\dot{\beta}}\nabla_{\alpha\dot{\alpha}}\nabla_{\beta\dot{\beta}} + \frac{2}{3}\nabla^{(\alpha}{}_{\dot{\beta}}\xi^{\beta\gamma)\dot{\alpha}\dot{\beta}}\nabla_{\alpha\dot{\alpha}}M_{\beta\gamma} + \frac{2}{3}\nabla_{\beta}{}^{(\dot{\alpha}}\xi^{\alpha\beta\dot{\beta}\dot{\gamma})}\nabla_{\alpha\dot{\alpha}}\overline{M}_{\dot{\beta}\dot{\gamma}} + A\nabla_{\beta\dot{\beta}}(\xi^{\alpha\beta\dot{\alpha}\dot{\beta}})\nabla_{\alpha\dot{\alpha}} \\
&+ (D\nabla^{(\alpha}{}_{\dot{\alpha}}\nabla_{\gamma\dot{\beta}}\xi^{\beta)\gamma\dot{\alpha}\dot{\beta}} + GE^{(\alpha}{}_{\gamma\dot{\alpha}\dot{\beta}}\xi^{\beta)\gamma\dot{\alpha}\dot{\beta}})M_{\alpha\beta} \\
&+ (H\nabla_{\alpha}{}^{(\dot{\alpha}}\nabla_{\beta\dot{\gamma}}\xi^{\alpha\beta\dot{\beta})\dot{\gamma}} + IE_{\alpha\beta\dot{\gamma}}{}^{(\dot{\alpha}}\xi^{\alpha\beta\dot{\beta})\dot{\gamma}})\overline{M}_{\dot{\alpha}\dot{\beta}} + B\nabla_{\alpha\dot{\alpha}}\nabla_{\beta\dot{\beta}}(\xi^{\alpha\beta\dot{\alpha}\dot{\beta}}) + CE_{\alpha\beta\dot{\alpha}\dot{\beta}}\xi^{\alpha\beta\dot{\alpha}\dot{\beta}} \qquad \text{(C.25)}
\end{aligned}
$$

and I have to find A, B, C, D, G, H and I from the Weyl transformation of $D^{(2)}\mathbf{\Psi}$. I shall embark on that task by evaluating the Weyl transformations term by term.

$$
\begin{aligned}
&\xi'^{\alpha\beta\dot{\alpha}\dot{\beta}}\nabla'_{\alpha\dot{\alpha}}\nabla'_{\beta\dot{\beta}}\mathbf{\Psi}' \\
&= (1-2\sigma)\xi^{\alpha\beta\dot{\alpha}\dot{\beta}}((1+\sigma)\nabla_{\alpha\dot{\alpha}} + \nabla^{\gamma}{}_{\dot{\alpha}}(\sigma)M_{\alpha\gamma} + \nabla_{\alpha}{}^{\dot{\gamma}}\overline{M}_{\dot{\alpha}\dot{\gamma}})((1+\sigma)\nabla_{\beta\dot{\beta}} + \nabla^{\rho}{}_{\dot{\beta}}(\sigma)M_{\beta\rho} \\
&\quad + \nabla_{\beta}{}^{\dot{\rho}}(\sigma)\overline{M}_{\dot{\beta}\dot{\rho}})\left(\left(1+\frac{3}{2}\sigma\right)\mathbf{\Psi}\right) \\
&= \xi^{\alpha\beta\dot{\alpha}\dot{\beta}}\nabla_{\alpha\dot{\alpha}}\nabla_{\beta\dot{\beta}}\mathbf{\Psi} - 2\sigma\xi^{\alpha\beta\dot{\alpha}\dot{\beta}}\nabla_{\alpha\dot{\alpha}}\nabla_{\beta\dot{\beta}}\mathbf{\Psi} + \xi^{\alpha\beta\dot{\alpha}\dot{\beta}}\sigma\nabla_{\alpha\dot{\alpha}}\nabla_{\beta\dot{\beta}}\mathbf{\Psi} + \xi^{\alpha\beta\dot{\alpha}\dot{\beta}}\nabla^{\gamma}{}_{\dot{\alpha}}(\sigma)M_{\alpha\gamma}(\nabla_{\beta\dot{\beta}}\mathbf{\Psi}) \\
&\quad + \xi^{\alpha\beta\dot{\alpha}\dot{\beta}}\nabla_{\alpha}{}^{\dot{\gamma}}(\sigma)\overline{M}_{\dot{\alpha}\dot{\gamma}}(\nabla_{\beta\dot{\beta}}\mathbf{\Psi}) + \xi^{\alpha\beta\dot{\alpha}\dot{\beta}}\nabla_{\alpha\dot{\alpha}}(\sigma\nabla_{\beta\dot{\beta}}\mathbf{\Psi}) + \xi^{\alpha\beta\dot{\alpha}\dot{\beta}}\nabla_{\alpha\dot{\alpha}}(\nabla^{\rho}{}_{\dot{\beta}}(\sigma)M_{\beta\rho}\mathbf{\Psi}) \\
&\quad + \xi^{\alpha\beta\dot{\alpha}\dot{\beta}}\nabla_{\alpha\dot{\alpha}}(\nabla_{\beta}{}^{\dot{\rho}}(\sigma)\overline{M}_{\dot{\beta}\dot{\rho}}\mathbf{\Psi}) + \frac{3}{2}\xi^{\alpha\beta\dot{\alpha}\dot{\beta}}\nabla_{\alpha\dot{\alpha}}\nabla_{\beta\dot{\beta}}(\sigma\mathbf{\Psi}) \\
&= \left(1+\frac{3}{2}\sigma\right)\xi^{\alpha\beta\dot{\alpha}\dot{\beta}}\nabla_{\alpha\dot{\alpha}}\nabla_{\beta\dot{\beta}}\mathbf{\Psi} + \frac{1}{2}\xi^{\alpha\beta\dot{\alpha}\dot{\beta}}\nabla^{\gamma}{}_{\dot{\alpha}}(\sigma)(\varepsilon_{\beta\alpha}\nabla_{\gamma\dot{\beta}}\mathbf{\Psi} + \varepsilon_{\beta\gamma}\nabla_{\alpha\dot{\beta}}\mathbf{\Psi}) \\
&\quad + \xi^{\alpha\beta\dot{\alpha}\dot{\beta}}\nabla^{\gamma}{}_{\dot{\alpha}}(\sigma)\nabla_{\beta\dot{\beta}}M_{\alpha\gamma}\mathbf{\Psi} + \frac{1}{2}\xi^{\alpha\beta\dot{\alpha}\dot{\beta}}\nabla_{\alpha}{}^{\dot{\gamma}}(\sigma)(\varepsilon_{\dot{\beta}\dot{\alpha}}\nabla_{\beta\dot{\gamma}}\mathbf{\Psi} + \varepsilon_{\dot{\beta}\dot{\gamma}}\nabla_{\beta\dot{\alpha}}\mathbf{\Psi}) \\
&\quad + \xi^{\alpha\beta\dot{\alpha}\dot{\beta}}\nabla_{\alpha}{}^{\dot{\gamma}}(\sigma)\nabla_{\beta\dot{\beta}}\overline{M}_{\dot{\alpha}\dot{\gamma}}\mathbf{\Psi} + \xi^{\alpha\beta\dot{\alpha}\dot{\beta}}\nabla_{\alpha\dot{\alpha}}(\sigma)\nabla_{\beta\dot{\beta}}\mathbf{\Psi} + \xi^{\alpha\beta\dot{\alpha}\dot{\beta}}\nabla_{\alpha\dot{\alpha}}(\nabla^{\rho}{}_{\dot{\beta}}(\sigma)M_{\beta\rho}\mathbf{\Psi}) \\
&\quad + \xi^{\alpha\beta\dot{\alpha}\dot{\beta}}\nabla_{\alpha\dot{\alpha}}(\nabla_{\beta}{}^{\dot{\rho}}(\sigma)\overline{M}_{\dot{\beta}\dot{\rho}}\mathbf{\Psi}) + \frac{3}{2}\xi^{\alpha\beta\dot{\alpha}\dot{\beta}}\nabla_{\alpha\dot{\alpha}}\nabla_{\beta\dot{\beta}}(\sigma)\mathbf{\Psi} + 3\xi^{\alpha\beta\dot{\alpha}\dot{\beta}}\nabla_{\alpha\dot{\alpha}}(\sigma)\nabla_{\beta\dot{\beta}}\mathbf{\Psi} \qquad \text{(C.26)}
\end{aligned}
$$

$$
\begin{aligned}
&\xi'^{\beta\gamma\dot\beta\dot\gamma}\nabla'_{\beta\dot\beta}\nabla'_{\gamma\dot\gamma}\mathbf{\Psi}' \\
&= \left(1+\frac{3}{2}\sigma\right)\xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}\nabla_{\gamma\dot\gamma}\mathbf{\Psi} + 5\xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}(\sigma)\nabla_{\gamma\dot\gamma}\mathbf{\Psi} + \xi^{\beta\gamma\dot\beta\dot\gamma}\nabla^{\mu}{}_{\dot\beta}(\sigma)\nabla_{\gamma\dot\gamma}M_{\beta\mu}\mathbf{\Psi} \\
&\quad + \xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\beta}{}^{\dot\mu}(\sigma)\nabla_{\gamma\dot\gamma}\overline{M}_{\dot\beta\dot\mu}\mathbf{\Psi} + \xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}(\nabla^{\mu}{}_{\dot\gamma}(\sigma)M_{\gamma\mu}\mathbf{\Psi}) + \xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}(\nabla_{\gamma}{}^{\dot\mu}(\sigma)\overline{M}_{\dot\gamma\dot\mu}\mathbf{\Psi}) \\
&\quad + \frac{3}{2}\xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}\nabla_{\gamma\dot\gamma}(\sigma)\mathbf{\Psi}
\end{aligned} \tag{C.27}
$$

$$
\begin{aligned}
&\xi^{\beta\gamma\dot\beta\dot\gamma}\nabla^{\mu}{}_{\dot\beta}(\sigma)\nabla_{\gamma\dot\gamma}M_{\beta\mu}\mathbf{\Psi} + \xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\beta}{}^{\dot\mu}(\sigma)\nabla_{\gamma\dot\gamma}\overline{M}_{\dot\beta\dot\mu}\mathbf{\Psi} \\
&= \begin{bmatrix} \xi^{\beta\gamma\dot\beta\dot\gamma}\nabla^{\mu}{}_{\dot\beta}(\sigma)\nabla_{\gamma\dot\gamma}M_{\beta\mu}\psi_\alpha \\ \xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\beta}{}^{\dot\mu}(\sigma)\nabla_{\gamma\dot\gamma}\overline{M}_{\dot\beta\dot\mu}\bar\chi^{\dot\alpha} \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} \xi^{\beta\gamma\dot\beta\dot\gamma}\nabla^{\mu}{}_{\dot\beta}(\sigma)\nabla_{\gamma\dot\gamma}(\varepsilon_{\alpha\beta}\psi_\mu + \varepsilon_{\alpha\mu}\psi_\beta) \\ \xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\beta}{}^{\dot\mu}(\sigma)\nabla_{\gamma\dot\gamma}(\delta^{\dot\alpha}{}_{\dot\beta}\bar\chi_{\dot\mu} + \delta^{\dot\alpha}{}_{\dot\mu}\bar\chi_{\dot\beta}) \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} \xi_{\alpha}{}^{\gamma\dot\beta\dot\gamma}\nabla^{\beta}{}_{\dot\beta}(\sigma)\nabla_{\gamma\dot\gamma}\psi_\beta + \xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\alpha\dot\beta}(\sigma)\nabla_{\gamma\dot\gamma}\psi_\beta \\ \xi^{\beta\gamma\dot\alpha\dot\gamma}\nabla_{\beta}{}^{\dot\beta}(\sigma)\nabla_{\gamma\dot\gamma}\bar\chi_{\dot\beta} + \xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\beta}{}^{\dot\alpha}(\sigma)\nabla_{\gamma\dot\gamma}\bar\chi_{\dot\beta} \end{bmatrix}
\end{aligned} \tag{C.28}
$$

$$
\begin{aligned}
&\xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}(\nabla^{\mu}{}_{\dot\gamma}(\sigma)M_{\gamma\mu}\mathbf{\Psi}) + \xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}(\nabla_{\gamma}{}^{\dot\mu}(\sigma)\overline{M}_{\dot\gamma\dot\mu}\mathbf{\Psi}) \\
&= \begin{bmatrix} \xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}(\nabla^{\mu}{}_{\dot\gamma}(\sigma)M_{\gamma\mu}\psi_\alpha) \\ \xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}(\nabla_{\gamma}{}^{\dot\mu}(\sigma)\overline{M}_{\dot\gamma\dot\mu}\bar\chi^{\dot\alpha}) \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} \xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}(\nabla^{\mu}{}_{\dot\gamma}(\sigma)(\varepsilon_{\alpha\gamma}\psi_\mu + \varepsilon_{\alpha\mu}\psi_\gamma)) \\ \xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}(\nabla_{\gamma}{}^{\dot\mu}(\sigma)(\delta^{\dot\alpha}{}_{\dot\gamma}\bar\chi_{\dot\mu} + \delta^{\dot\alpha}{}_{\dot\mu}\bar\chi_{\dot\gamma})) \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} \xi^{\beta}{}_{\alpha}{}^{\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}(\nabla^{\gamma}{}_{\dot\gamma}(\sigma)\psi_\gamma) + \xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}(\nabla_{\alpha\dot\gamma}(\sigma)\psi_\gamma) \\ \xi^{\beta\gamma\dot\beta\dot\alpha}\nabla_{\beta\dot\beta}(\nabla_{\gamma}{}^{\dot\gamma}(\sigma)\bar\chi_{\dot\gamma}) + \xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}(\nabla_{\gamma}{}^{\dot\alpha}(\sigma)\bar\chi_{\dot\gamma}) \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} \xi_{\alpha}{}^{\gamma\dot\beta\dot\gamma}\nabla^{\beta}{}_{\dot\beta}(\sigma)\nabla_{\gamma\dot\gamma}\psi_\beta + \xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\alpha\dot\beta}(\sigma)\nabla_{\gamma\dot\gamma}\psi_\beta + \xi^{\beta}{}_{\alpha}{}^{\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}\nabla^{\gamma}{}_{\dot\gamma}(\sigma)\psi_\gamma + \xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}\nabla_{\alpha\dot\gamma}(\sigma)\psi_\gamma \\ \xi^{\beta\gamma\dot\alpha\dot\gamma}\nabla_{\beta}{}^{\dot\beta}(\sigma)\nabla_{\gamma\dot\gamma}\bar\chi_{\dot\beta} + \xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\beta}{}^{\dot\alpha}(\sigma)\nabla_{\gamma\dot\gamma}\bar\chi_{\dot\beta} + \xi^{\beta\gamma\dot\beta\dot\alpha}\nabla_{\beta\dot\beta}\nabla_{\gamma}{}^{\dot\gamma}(\sigma)\bar\chi_{\dot\gamma} + \xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}\nabla_{\gamma}{}^{\dot\alpha}(\sigma)\bar\chi_{\dot\gamma} \end{bmatrix}
\end{aligned} \tag{C.29}
$$

Putting these expressions back,

$$
\begin{aligned}
&\xi'^{\beta\gamma\dot\beta\dot\gamma}\nabla'_{\beta\dot\beta}\nabla'_{\gamma\dot\gamma}\mathbf{\Psi}' \\
&= \left(1+\frac{3}{2}\sigma\right)\xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}\nabla_{\gamma\dot\gamma}\mathbf{\Psi} + 5\xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}(\sigma)\nabla_{\gamma\dot\gamma}\mathbf{\Psi} + \frac{3}{2}\xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}\nabla_{\gamma\dot\gamma}(\sigma)\mathbf{\Psi} \\
&\quad + \Bigg[\xi_{\alpha}{}^{\gamma\dot\beta\dot\gamma}\nabla^{\beta}{}_{\dot\beta}(\sigma)\nabla_{\gamma\dot\gamma}\psi_\beta + \xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\alpha\dot\beta}(\sigma)\nabla_{\gamma\dot\gamma}\psi_\beta + \frac{1}{2}\xi^{\beta}{}_{\alpha}{}^{\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}\nabla^{\gamma}{}_{\dot\gamma}(\sigma)\psi_\gamma \\
&\qquad + \frac{1}{2}\xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}\nabla_{\alpha\dot\gamma}(\sigma)\psi_\gamma, \\
&\qquad \xi^{\beta\gamma\dot\alpha\dot\gamma}\nabla_{\beta}{}^{\dot\beta}(\sigma)\nabla_{\gamma\dot\gamma}\bar\chi_{\dot\beta} + \xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\beta}{}^{\dot\alpha}(\sigma)\nabla_{\gamma\dot\gamma}\bar\chi_{\dot\beta} + \frac{1}{2}\xi^{\beta\gamma\dot\beta\dot\alpha}\nabla_{\beta\dot\beta}\nabla_{\gamma}{}^{\dot\gamma}(\sigma)\bar\chi_{\dot\gamma} \\
&\qquad + \frac{1}{2}\xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}\nabla_{\gamma}{}^{\dot\alpha}(\sigma)\bar\chi_{\dot\gamma}\Bigg]^T
\end{aligned} \tag{C.30}
$$

$$
\begin{aligned}
\nabla'^{(\alpha}{}_{\dot\beta}\xi'^{\beta\gamma)\dot\alpha\dot\beta} &= ((1+\sigma)\nabla^{(\alpha}{}_{\dot\beta} + \nabla^{\mu}{}_{\dot\beta}(\sigma)M^{(\alpha}{}_{\mu} + \nabla^{(\alpha\dot\gamma}(\sigma)\overline{M}_{\dot\beta\dot\gamma})((1-2\sigma)\xi^{\beta\gamma)\dot\alpha\dot\beta}) \\
&= (1-\sigma)\nabla^{(\alpha}{}_{\dot\beta}\xi^{\beta\gamma)\dot\alpha\dot\beta} - 2\nabla^{(\alpha}{}_{\dot\beta}(\sigma)\xi^{\beta\gamma)\dot\alpha\dot\beta} + \nabla^{\mu}{}_{\dot\beta}(\sigma)M^{(\alpha}{}_{\mu}\xi^{\beta\gamma)\dot\alpha\dot\beta} \\
&\quad + \nabla^{(\alpha\dot\gamma}(\sigma)\overline{M}_{\dot\beta\dot\gamma}\xi^{\beta\gamma)\dot\alpha\dot\beta}
\end{aligned} \tag{C.31}
$$

$$\begin{aligned}
\nabla'^{(\alpha}{}_{\dot\beta}\xi'^{\beta\gamma)\dot\alpha\dot\beta} &= (1-\sigma)\nabla^{(\alpha}{}_{\dot\beta}\xi^{\beta\gamma)\dot\alpha\dot\beta} - 2\nabla^{(\alpha}{}_{\dot\beta}(\sigma)\xi^{\beta\gamma)\dot\alpha\dot\beta} + \frac{1}{2}\nabla^{\mu}{}_{\dot\beta}(\sigma)\big(\varepsilon^{(\alpha\beta}\xi_{\mu}{}^{\gamma)\dot\alpha\dot\beta} + \delta^{(\beta}{}_{\mu}\xi^{\alpha\gamma)\dot\alpha\dot\beta} \\
&\quad + \varepsilon^{(\alpha\gamma}\xi^{\beta)}{}_{\mu}{}^{\dot\alpha\dot\beta} + \delta^{(\gamma}{}_{\mu}\xi^{\beta\alpha)\dot\alpha\dot\beta}\big) + \frac{1}{2}\nabla^{(\alpha\dot\gamma}(\sigma)\big(\delta^{\dot\alpha}{}_{\dot\beta}\xi^{\beta\gamma)}{}_{\dot\gamma}{}^{\dot\beta} + \delta^{\dot\alpha}{}_{\dot\gamma}\xi^{\beta\gamma)}{}_{\dot\beta}{}^{\dot\beta} \\
&\quad + \delta^{\dot\beta}{}_{\dot\beta}\xi^{\beta\gamma)\dot\alpha}{}_{\dot\gamma} + \delta^{\dot\beta}{}_{\dot\gamma}\xi^{\beta\gamma)\dot\alpha}{}_{\dot\beta}\big) \\
&= (1-\sigma)\nabla^{(\alpha}{}_{\dot\beta}\xi^{\beta\gamma)\dot\alpha\dot\beta} - 3\nabla^{(\alpha}{}_{\dot\beta}(\sigma)\xi^{\beta\gamma)\dot\alpha\dot\beta} && \text{(C.32)} \\
\nabla'_{\beta}{}^{(\dot\alpha}\xi'^{\alpha\beta\dot\beta\dot\gamma)} &= ((1+\sigma)\nabla_{\beta}{}^{(\dot\alpha} + \nabla^{\gamma(\dot\alpha}(\sigma)M_{\beta\gamma} + \nabla_{\beta}{}^{\dot\mu}(\sigma)\overline{M}^{(\dot\alpha}{}_{\dot\mu})((1-2\sigma)\xi^{\alpha\beta\dot\beta\dot\gamma)}) \\
&= (1-\sigma)\nabla_{\beta}{}^{(\dot\alpha}\xi^{\alpha\beta\dot\beta\dot\gamma)} - 2\nabla_{\beta}{}^{(\dot\alpha}(\sigma)\xi^{\alpha\beta\dot\beta\dot\gamma)} + \frac{1}{2}\nabla^{\gamma(\dot\alpha}(\sigma)\big(\delta^{\alpha}{}_{\beta}\xi_{\gamma}{}^{\beta\dot\beta\dot\gamma)} + \delta^{\alpha}{}_{\gamma}\xi_{\beta}{}^{\beta\dot\beta\dot\gamma)} \\
&\quad + \delta^{\beta}{}_{\beta}\xi^{\alpha}{}_{\gamma}{}^{\dot\beta\dot\gamma)} + \delta^{\beta}{}_{\gamma}\xi^{\alpha}{}_{\beta}{}^{\dot\beta\dot\gamma)}\big) + \frac{1}{2}\nabla_{\beta}{}^{\dot\mu}(\sigma)\big(\varepsilon^{(\dot\alpha\dot\beta}\xi^{\alpha\beta}{}_{\dot\mu}{}^{\dot\gamma)} + \delta^{(\dot\beta}{}_{\dot\mu}\xi^{\alpha\beta\dot\alpha\dot\gamma)} \\
&\quad + \varepsilon^{(\dot\alpha\dot\gamma}\xi^{\alpha\beta\dot\beta}{}_{\dot\mu}{}^{)} + \delta^{(\dot\gamma}{}_{\dot\mu}\xi^{\alpha\beta\dot\beta\dot\alpha)}\big) \\
&= (1-\sigma)\nabla_{\beta}{}^{(\dot\alpha}\xi^{\alpha\beta\dot\beta\dot\gamma)} - 3\nabla_{\beta}{}^{(\dot\alpha}(\sigma)\xi^{\alpha\beta\dot\beta\dot\gamma)} && \text{(C.33)}
\end{aligned}$$

$$\begin{aligned}
&\implies (\nabla'^{(\beta}{}_{\dot\gamma}\xi'^{\gamma\mu)\dot\beta\dot\gamma}\nabla'_{\beta\dot\beta}M_{\gamma\mu} + \nabla'_{\gamma}{}^{(\dot\beta}\xi'^{\beta\gamma\dot\gamma\dot\mu)}\nabla'_{\beta\dot\beta}\overline{M}_{\dot\gamma\dot\mu})\mathbf{\Psi}' \\
&= (((1-\sigma)\nabla^{(\beta}{}_{\dot\gamma}\xi^{\gamma\mu)\dot\beta\dot\gamma} - 3\nabla^{(\beta}{}_{\dot\gamma}(\sigma)\xi^{\gamma\mu)\dot\beta\dot\gamma})((1+\sigma)\nabla_{\beta\dot\beta} + \nabla^{\nu}{}_{\dot\beta}(\sigma)M_{\beta\nu} + \nabla_{\beta}{}^{\dot\nu}(\sigma)\overline{M}_{\dot\beta\dot\nu})M_{\gamma\mu} \\
&\quad + ((1-\sigma)\nabla_{\gamma}{}^{(\dot\beta}\xi^{\beta\gamma\dot\gamma\dot\mu)} - 3\nabla_{\gamma}{}^{(\dot\beta}(\sigma)\xi^{\beta\gamma\dot\gamma\dot\mu)})((1+\sigma)\nabla_{\beta\dot\beta} + \nabla^{\nu}{}_{\dot\beta}(\sigma)M_{\beta\nu} + \nabla_{\beta}{}^{\dot\nu}(\sigma)\overline{M}_{\dot\beta\dot\nu})\overline{M}_{\dot\gamma\dot\mu}) \\
&\quad \times \left(\left(1+\frac{3}{2}\sigma\right)\mathbf{\Psi}\right) \\
&= \left(1+\frac{3}{2}\sigma\right)(\nabla^{(\beta}{}_{\dot\gamma}\xi^{\gamma\mu)\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}M_{\gamma\mu} + \nabla_{\gamma}{}^{(\dot\beta}\xi^{\beta\gamma\dot\gamma\dot\mu)}\nabla_{\beta\dot\beta}\overline{M}_{\dot\gamma\dot\mu})\mathbf{\Psi} - 3(\nabla^{(\beta}{}_{\dot\gamma}(\sigma)\xi^{\gamma\mu)\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}M_{\gamma\mu} \\
&\quad + \nabla_{\gamma}{}^{(\dot\beta}(\sigma)\xi^{\beta\gamma\dot\gamma\dot\mu)}\nabla_{\beta\dot\beta}\overline{M}_{\dot\gamma\dot\mu})\mathbf{\Psi} + (\nabla^{(\beta}{}_{\dot\gamma}\xi^{\gamma\mu)\dot\beta\dot\gamma}\nabla^{\nu}{}_{\dot\beta}(\sigma)M_{\beta\nu}M_{\gamma\mu} \\
&\quad + \nabla_{\gamma}{}^{(\dot\beta}\xi^{\beta\gamma\dot\gamma\dot\mu)}\nabla_{\beta}{}^{\dot\nu}(\sigma)\overline{M}_{\dot\beta\dot\nu}\overline{M}_{\dot\gamma\dot\mu})\mathbf{\Psi} \\
&\quad + \frac{3}{2}(\nabla^{(\beta}{}_{\dot\gamma}\xi^{\gamma\mu)\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}(\sigma)M_{\gamma\mu} + \nabla_{\gamma}{}^{(\dot\beta}\xi^{\beta\gamma\dot\gamma\dot\mu)}\nabla_{\beta\dot\beta}(\sigma)\overline{M}_{\dot\gamma\dot\mu})\mathbf{\Psi} \\
&= \left(1+\frac{3}{2}\sigma\right)(\nabla^{(\beta}{}_{\dot\gamma}\xi^{\gamma\mu)\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}M_{\gamma\mu} + \nabla_{\gamma}{}^{(\dot\beta}\xi^{\beta\gamma\dot\gamma\dot\mu)}\nabla_{\beta\dot\beta}\overline{M}_{\dot\gamma\dot\mu})\mathbf{\Psi} \\
&\quad + \Big[-3\nabla^{(\beta}{}_{\dot\gamma}(\sigma)\xi^{\gamma\mu)\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}M_{\gamma\mu}\psi_{\alpha} + \nabla^{(\beta}{}_{\dot\gamma}\xi^{\gamma\mu)\dot\beta\dot\gamma}\nabla^{\nu}{}_{\dot\beta}(\sigma)M_{\beta\nu}M_{\gamma\mu}\psi_{\alpha} \\
&\qquad + \frac{3}{2}\nabla^{(\beta}{}_{\dot\gamma}\xi^{\gamma\mu)\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}(\sigma)M_{\gamma\mu}\psi_{\alpha}, \\
&\qquad -3\nabla_{\gamma}{}^{(\dot\beta}(\sigma)\xi^{\beta\gamma\dot\gamma\dot\mu)}\nabla_{\beta\dot\beta}\overline{M}_{\dot\gamma\dot\mu}\bar\chi^{\dot\alpha} + \nabla_{\gamma}{}^{(\dot\beta}\xi^{\beta\gamma\dot\gamma\dot\mu)}\nabla_{\beta}{}^{\dot\nu}(\sigma)\overline{M}_{\dot\beta\dot\nu}\overline{M}_{\dot\gamma\dot\mu}\bar\chi^{\dot\alpha} \\
&\qquad + \frac{3}{2}\nabla_{\gamma}{}^{(\dot\beta}\xi^{\beta\gamma\dot\gamma\dot\mu)}\nabla_{\beta\dot\beta}(\sigma)\overline{M}_{\dot\gamma\dot\mu}\bar\chi^{\dot\alpha}\Big]^{T} \\
&= \left(1+\frac{3}{2}\sigma\right)(\nabla^{(\beta}{}_{\dot\gamma}\xi^{\gamma\mu)\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}M_{\gamma\mu} + \nabla_{\gamma}{}^{(\dot\beta}\xi^{\beta\gamma\dot\gamma\dot\mu)}\nabla_{\beta\dot\beta}\overline{M}_{\dot\gamma\dot\mu})\mathbf{\Psi} \\
&\quad + \Big[-3\varepsilon_{\alpha\mu}\nabla^{(\beta}{}_{\dot\gamma}(\sigma)\xi^{\gamma\mu)\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}\psi_{\gamma} + \nabla^{(\beta}{}_{\dot\gamma}\xi^{\gamma\mu)\dot\beta\dot\gamma}\nabla^{\nu}{}_{\dot\beta}(\sigma)M_{\beta\nu}(\varepsilon_{\alpha\mu}\psi_{\gamma}) \\
&\qquad + \frac{3}{2}\varepsilon_{\alpha\mu}\nabla^{(\beta}{}_{\dot\gamma}\xi^{\gamma\mu)\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}(\sigma)\psi_{\gamma}, \\
&\qquad -3\nabla_{\gamma}{}^{(\dot\alpha}(\sigma)\xi^{\beta\gamma\dot\beta\dot\gamma)}\nabla_{\beta\dot\beta}\bar\chi_{\dot\gamma} + \nabla_{\gamma}{}^{(\dot\alpha}\xi^{\beta\gamma\dot\beta\dot\gamma)}\nabla_{\beta}{}^{\dot\nu}(\sigma)\overline{M}_{\dot\beta\dot\nu}\bar\chi_{\dot\gamma} + \frac{3}{2}\nabla_{\gamma}{}^{(\dot\alpha}\xi^{\beta\gamma\dot\beta\dot\gamma)}\nabla_{\beta\dot\beta}(\sigma)\bar\chi_{\dot\gamma}\Big]^{T}
\end{aligned} \tag{C.34}$$

$$
\begin{aligned}
&(\nabla'^{(\beta}{}_{\dot\gamma}\xi'^{\gamma\mu)\dot\beta\dot\gamma}\nabla'_{\beta\dot\beta}M_{\gamma\mu} + \nabla'_{\gamma}{}^{(\dot\beta}\xi'^{\beta\gamma\dot\gamma\dot\mu)}\nabla'_{\beta\dot\beta}\overline{M}_{\dot\gamma\dot\mu})\mathbf{\Psi}' \\
&= \left(1+\frac{3}{2}\sigma\right)(\nabla^{(\beta}{}_{\dot\gamma}\xi^{\gamma\mu)\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}M_{\gamma\mu} + \nabla_{\gamma}{}^{(\dot\beta}\xi^{\beta\gamma\dot\gamma\dot\mu)}\nabla_{\beta\dot\beta}\overline{M}_{\dot\gamma\dot\mu})\mathbf{\Psi} \\
&\quad + \begin{bmatrix} \nabla_{(\alpha}{}^{\dot\gamma}\xi_{\beta\gamma)\dot\beta\dot\gamma}\nabla^{\nu\dot\beta}(\sigma)M^{\beta}{}_{\nu}\psi^{\gamma} - 3\nabla_{(\alpha}{}^{\dot\gamma}(\sigma)\xi_{\beta\gamma)\dot\beta\dot\gamma}\nabla^{\beta\dot\beta}\psi^{\gamma} + \frac{3}{2}\nabla_{(\alpha}{}^{\dot\gamma}\xi_{\beta\gamma)\dot\beta\dot\gamma}\nabla^{\beta\dot\beta}(\sigma)\psi^{\gamma} \\ \nabla_{\gamma}{}^{(\dot\alpha}\xi^{\beta\gamma\dot\beta\dot\gamma)}\nabla_{\beta}{}^{\dot\nu}(\sigma)\overline{M}_{\dot\beta\dot\nu}\bar\chi_{\dot\gamma} - 3\nabla_{\gamma}{}^{(\dot\alpha}(\sigma)\xi^{\beta\gamma\dot\beta\dot\gamma)}\nabla_{\beta\dot\beta}\bar\chi_{\dot\gamma} + \frac{3}{2}\nabla_{\gamma}{}^{(\dot\alpha}\xi^{\beta\gamma\dot\beta\dot\gamma)}\nabla_{\beta\dot\beta}(\sigma)\bar\chi_{\dot\gamma} \end{bmatrix} \\
&= \left(1+\frac{3}{2}\sigma\right)(\nabla^{(\beta}{}_{\dot\gamma}\xi^{\gamma\mu)\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}M_{\gamma\mu} + \nabla_{\gamma}{}^{(\dot\beta}\xi^{\beta\gamma\dot\gamma\dot\mu)}\nabla_{\beta\dot\beta}\overline{M}_{\dot\gamma\dot\mu})\mathbf{\Psi} \\
&\quad + \begin{bmatrix} \frac{1}{2}\nabla_{(\alpha}{}^{\dot\gamma}\xi_{\beta\gamma)\dot\beta\dot\gamma}\nabla^{\nu\dot\beta}(\sigma)(\varepsilon^{\beta\gamma}\psi_{\nu} + \delta^{\gamma}{}_{\nu}\psi^{\beta}) - 3\nabla_{(\alpha}{}^{\dot\gamma}(\sigma)\xi_{\beta\gamma)\dot\beta\dot\gamma}\nabla^{\beta\dot\beta}\psi^{\gamma} + \frac{3}{2}\nabla_{(\alpha}{}^{\dot\gamma}\xi_{\beta\gamma)\dot\beta\dot\gamma}\nabla^{\beta\dot\beta}(\sigma)\psi^{\gamma} \\ \frac{1}{2}\nabla_{\gamma}{}^{(\dot\alpha}\xi^{\beta\gamma\dot\beta\dot\gamma)}\nabla_{\beta}{}^{\dot\nu}(\sigma)(\varepsilon_{\dot\gamma\dot\beta}\bar\chi_{\dot\nu} + \varepsilon_{\dot\gamma\dot\nu}\bar\chi_{\dot\beta}) - 3\nabla_{\gamma}{}^{(\dot\alpha}(\sigma)\xi^{\beta\gamma\dot\beta\dot\gamma)}\nabla_{\beta\dot\beta}\bar\chi_{\dot\gamma} + \frac{3}{2}\nabla_{\gamma}{}^{(\dot\alpha}\xi^{\beta\gamma\dot\beta\dot\gamma)}\nabla_{\beta\dot\beta}(\sigma)\bar\chi_{\dot\gamma} \end{bmatrix} \\
&= \left(1+\frac{3}{2}\sigma\right)(\nabla^{(\beta}{}_{\dot\gamma}\xi^{\gamma\mu)\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}M_{\gamma\mu} + \nabla_{\gamma}{}^{(\dot\beta}\xi^{\beta\gamma\dot\gamma\dot\mu)}\nabla_{\beta\dot\beta}\overline{M}_{\dot\gamma\dot\mu})\mathbf{\Psi} \\
&\quad + \begin{bmatrix} 2\nabla_{(\alpha}{}^{\dot\gamma}\xi_{\beta\gamma)\dot\beta\dot\gamma}\nabla^{\beta\dot\beta}(\sigma)\psi^{\gamma} - 3\nabla_{(\alpha}{}^{\dot\gamma}(\sigma)\xi_{\beta\gamma)\dot\beta\dot\gamma}\nabla^{\beta\dot\beta}\psi^{\gamma} \\ 2\nabla_{\gamma}{}^{(\dot\alpha}\xi^{\beta\gamma\dot\beta\dot\gamma)}\nabla_{\beta\dot\beta}(\sigma)\bar\chi_{\dot\gamma} - 3\nabla_{\gamma}{}^{(\dot\alpha}(\sigma)\xi^{\beta\gamma\dot\beta\dot\gamma)}\nabla_{\beta\dot\beta}\bar\chi_{\dot\gamma} \end{bmatrix}
\end{aligned}
\tag{C.35}
$$

$$
\begin{aligned}
&\nabla'_{\gamma\dot\gamma}(\xi'^{\beta\gamma\dot\beta\dot\gamma})\nabla'_{\beta\dot\beta}\mathbf{\Psi}' \\
&= ((1+\sigma)\nabla_{\gamma\dot\gamma} + \nabla^{\mu}{}_{\dot\gamma}(\sigma)M_{\gamma\mu} + \nabla_{\gamma}{}^{\dot\mu}(\sigma)\overline{M}_{\dot\gamma\dot\mu})((1-2\sigma)\xi^{\beta\gamma\dot\beta\dot\gamma})((1+\sigma)\nabla_{\beta\dot\beta} \\
&\qquad + \nabla^{\nu}{}_{\dot\beta}(\sigma)M_{\beta\nu} + \nabla_{\beta}{}^{\dot\nu}(\sigma)\overline{M}_{\dot\beta\dot\nu})\left(\left(1+\frac{3}{2}\sigma\right)\mathbf{\Psi}\right) \\
&= \left(1+\frac{3}{2}\sigma\right)\nabla_{\gamma\dot\gamma}(\xi^{\beta\gamma\dot\beta\dot\gamma})\nabla_{\beta\dot\beta}\mathbf{\Psi} - 2\nabla_{\gamma\dot\gamma}(\sigma)\xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}\mathbf{\Psi} + \nabla^{\mu}{}_{\dot\gamma}(\sigma)M_{\gamma\mu}(\xi^{\beta\gamma\dot\beta\dot\gamma})\nabla_{\beta\dot\beta}\mathbf{\Psi} \\
&\qquad + \nabla_{\gamma}{}^{\dot\mu}(\sigma)\overline{M}_{\dot\gamma\dot\mu}(\xi^{\beta\gamma\dot\beta\dot\gamma})\nabla_{\beta\dot\beta}\mathbf{\Psi} + \nabla_{\gamma\dot\gamma}(\xi^{\beta\gamma\dot\beta\dot\gamma})\nabla^{\nu}{}_{\dot\beta}(\sigma)M_{\beta\nu}\mathbf{\Psi} + \nabla_{\gamma\dot\gamma}(\xi^{\beta\gamma\dot\beta\dot\gamma})\nabla_{\beta}{}^{\dot\nu}(\sigma)\overline{M}_{\dot\beta\dot\nu}\mathbf{\Psi} \\
&\qquad + \frac{3}{2}\nabla_{\gamma\dot\gamma}(\xi^{\beta\gamma\dot\beta\dot\gamma})\nabla_{\beta\dot\beta}(\sigma)\mathbf{\Psi} \\
&= \left(1+\frac{3}{2}\sigma\right)\nabla_{\gamma\dot\gamma}(\xi^{\beta\gamma\dot\beta\dot\gamma})\nabla_{\beta\dot\beta}\mathbf{\Psi} - 2\nabla_{\gamma\dot\gamma}(\sigma)\xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}\mathbf{\Psi} + \frac{3}{2}\nabla_{\gamma\dot\gamma}(\xi^{\beta\gamma\dot\beta\dot\gamma})\nabla_{\beta\dot\beta}(\sigma)\mathbf{\Psi} \\
&\qquad + \frac{1}{2}(\nabla^{\mu}{}_{\dot\gamma}(\sigma)(\delta^{\beta}{}_{\gamma}\xi_{\mu}{}^{\gamma\dot\beta\dot\gamma} + \delta^{\beta}{}_{\mu}\xi_{\gamma}{}^{\gamma\dot\beta\dot\gamma} + \delta^{\gamma}{}_{\gamma}\xi^{\beta}{}_{\mu}{}^{\dot\beta\dot\gamma} + \delta^{\gamma}{}_{\mu}\xi^{\beta}{}_{\gamma}{}^{\dot\beta\dot\gamma}) \\
&\qquad + \nabla_{\gamma}{}^{\dot\mu}(\sigma)(\delta^{\dot\beta}{}_{\dot\gamma}\xi^{\beta\gamma}{}_{\dot\mu}{}^{\dot\gamma} + \delta^{\dot\beta}{}_{\dot\mu}\xi^{\beta\gamma}{}_{\dot\gamma}{}^{\dot\gamma} + \delta^{\dot\gamma}{}_{\dot\gamma}\xi^{\beta\gamma\dot\beta}{}_{\dot\mu} + \delta^{\dot\gamma}{}_{\dot\mu}\xi^{\beta\gamma\dot\beta}{}_{\dot\gamma}))\nabla_{\beta\dot\beta}\mathbf{\Psi} \\
&\qquad + \begin{bmatrix} \nabla_{\gamma\dot\gamma}(\xi^{\beta\gamma\dot\beta\dot\gamma})\nabla^{\nu}{}_{\dot\beta}(\sigma)M_{\beta\nu}\psi_{\alpha} \\ \nabla_{\gamma\dot\gamma}(\xi^{\beta\gamma\dot\beta\dot\gamma})\nabla_{\beta}{}^{\dot\nu}(\sigma)\overline{M}_{\dot\beta\dot\nu}\bar\chi^{\dot\alpha} \end{bmatrix} \\
&= \left(1+\frac{3}{2}\sigma\right)\nabla_{\gamma\dot\gamma}(\xi^{\beta\gamma\dot\beta\dot\gamma})\nabla_{\beta\dot\beta}\mathbf{\Psi} - 6\nabla_{\gamma\dot\gamma}(\sigma)\xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}\mathbf{\Psi} + \frac{3}{2}\nabla_{\gamma\dot\gamma}(\xi^{\beta\gamma\dot\beta\dot\gamma})\nabla_{\beta\dot\beta}(\sigma)\mathbf{\Psi} \\
&\qquad + \frac{1}{2}\begin{bmatrix} \nabla_{\gamma\dot\gamma}(\xi^{\beta\gamma\dot\beta\dot\gamma})\nabla^{\nu}{}_{\dot\beta}(\sigma)(\varepsilon_{\alpha\beta}\psi_{\nu} + \varepsilon_{\alpha\nu}\psi_{\beta}) \\ \nabla_{\gamma\dot\gamma}(\xi^{\beta\gamma\dot\beta\dot\gamma})\nabla_{\beta}{}^{\dot\nu}(\sigma)(\delta^{\dot\alpha}{}_{\dot\beta}\bar\chi_{\dot\nu} + \delta^{\dot\alpha}{}_{\dot\nu}\bar\chi_{\dot\beta}) \end{bmatrix} \\
&= \left(1+\frac{3}{2}\sigma\right)\nabla_{\gamma\dot\gamma}(\xi^{\beta\gamma\dot\beta\dot\gamma})\nabla_{\beta\dot\beta}\mathbf{\Psi} - 6\nabla_{\gamma\dot\gamma}(\sigma)\xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}\mathbf{\Psi} + \frac{3}{2}\nabla_{\gamma\dot\gamma}(\xi^{\beta\gamma\dot\beta\dot\gamma})\nabla_{\beta\dot\beta}(\sigma)\mathbf{\Psi} \\
&\qquad + \frac{1}{2}\begin{bmatrix} \nabla_{\gamma\dot\gamma}(\xi^{\beta\gamma\dot\beta\dot\gamma})\nabla_{\alpha\dot\beta}(\sigma)\psi_{\beta} - \nabla^{\gamma\dot\gamma}(\xi_{\alpha\gamma\dot\beta\dot\gamma})\nabla^{\beta\dot\beta}(\sigma)\psi_{\beta} \\ \nabla^{\gamma\dot\gamma}(\xi_{\beta\gamma\dot\beta\dot\gamma})\nabla^{\beta\dot\alpha}(\sigma)\bar\chi^{\dot\beta} - \nabla_{\gamma\dot\gamma}(\xi^{\beta\gamma\dot\alpha\dot\gamma})\nabla_{\beta\dot\beta}(\sigma)\bar\chi^{\dot\beta} \end{bmatrix}
\end{aligned}
\tag{C.36}
$$

From above, $\nabla'_{\gamma\dot{\beta}}\xi'^{\beta\gamma\dot{\alpha}\dot{\beta}} = (1-\sigma)\nabla_{\gamma\dot{\beta}}\xi^{\beta\gamma\dot{\alpha}\dot{\beta}} - 6\nabla_{\gamma\dot{\beta}}(\sigma)\xi^{\beta\gamma\dot{\alpha}\dot{\beta}}$. Therefore,

$$\begin{aligned}
&\nabla'^{(\alpha}{}_{\dot{\alpha}}\nabla'_{\gamma\dot{\beta}}\xi'^{\beta)\gamma\dot{\alpha}\dot{\beta}} \\
&= ((1+\sigma)\nabla^{(\alpha}{}_{\dot{\alpha}} + \nabla^{\mu}{}_{\dot{\alpha}}(\sigma)M^{(\alpha}{}_{\mu} + \nabla^{(\alpha\dot{\mu}}(\sigma)\overline{M}_{\dot{\alpha}\dot{\mu}})((1-\sigma)\nabla_{\gamma\dot{\beta}}\xi^{\beta)\gamma\dot{\alpha}\dot{\beta}} - 6\nabla_{\gamma\dot{\beta}}(\sigma)\xi^{\beta)\gamma\dot{\alpha}\dot{\beta}}) \\
&= \nabla^{(\alpha}{}_{\dot{\alpha}}\nabla_{\gamma\dot{\beta}}\xi^{\beta)\gamma\dot{\alpha}\dot{\beta}} - \nabla^{(\alpha}{}_{\dot{\alpha}}(\sigma)\nabla_{\gamma\dot{\beta}}\xi^{\beta)\gamma\dot{\alpha}\dot{\beta}} - 6\nabla^{(\alpha}{}_{\dot{\alpha}}\nabla_{\gamma\dot{\beta}}(\sigma)\xi^{\beta)\gamma\dot{\alpha}\dot{\beta}} - 6\nabla_{\gamma\dot{\beta}}(\sigma)\nabla^{(\alpha}{}_{\dot{\alpha}}\xi^{\beta)\gamma\dot{\alpha}\dot{\beta}} \\
&\quad + \nabla^{\mu}{}_{\dot{\alpha}}(\sigma)M^{(\alpha}{}_{\mu}\nabla_{\gamma\dot{\beta}}\xi^{\beta)\gamma\dot{\alpha}\dot{\beta}} + \nabla^{(\alpha\dot{\mu}}(\sigma)\overline{M}_{\dot{\alpha}\dot{\mu}}\nabla_{\gamma\dot{\beta}}\xi^{\beta)\gamma\dot{\alpha}\dot{\beta}} \\
&= \nabla^{(\alpha}{}_{\dot{\alpha}}\nabla_{\gamma\dot{\beta}}\xi^{\beta)\gamma\dot{\alpha}\dot{\beta}} - \nabla^{(\alpha}{}_{\dot{\alpha}}(\sigma)\nabla_{\gamma\dot{\beta}}\xi^{\beta)\gamma\dot{\alpha}\dot{\beta}} - 6\nabla^{(\alpha}{}_{\dot{\alpha}}\nabla_{\gamma\dot{\beta}}(\sigma)\xi^{\beta)\gamma\dot{\alpha}\dot{\beta}} - 6\nabla_{\gamma\dot{\beta}}(\sigma)\nabla^{(\alpha}{}_{\dot{\alpha}}\xi^{\beta)\gamma\dot{\alpha}\dot{\beta}} \\
&\quad + \frac{1}{2}\nabla^{\mu}{}_{\dot{\alpha}}(\sigma)(\varepsilon^{(\alpha\beta)}\nabla_{\gamma\dot{\beta}}\xi_{\mu}{}^{\gamma\dot{\alpha}\dot{\beta}} + \delta^{(\beta}{}_{\mu}\nabla_{\gamma\dot{\beta}}\xi^{\alpha)\gamma\dot{\alpha}\dot{\beta}}) \\
&\quad + \frac{1}{2}\nabla^{(\alpha\dot{\mu}}(\sigma)(\delta^{\dot{\alpha}}{}_{\dot{\alpha}}\nabla_{\gamma\dot{\beta}}\xi^{\beta)\gamma}{}_{\dot{\mu}}{}^{\dot{\beta}} + \nabla^{(\alpha\dot{\mu}}(\sigma)\delta^{\dot{\alpha}}{}_{\dot{\mu}}\nabla_{\gamma\dot{\beta}}\xi^{\beta)\gamma}{}_{\dot{\alpha}}{}^{\dot{\beta}}) \\
&= \nabla^{(\alpha}{}_{\dot{\alpha}}\nabla_{\gamma\dot{\beta}}\xi^{\beta)\gamma\dot{\alpha}\dot{\beta}} - 2\nabla^{(\alpha}{}_{\dot{\alpha}}(\sigma)\nabla_{\gamma\dot{\beta}}\xi^{\beta)\gamma\dot{\alpha}\dot{\beta}} - 6\nabla^{(\alpha}{}_{\dot{\alpha}}\nabla_{\gamma\dot{\beta}}(\sigma)\xi^{\beta)\gamma\dot{\alpha}\dot{\beta}} - 6\nabla_{\gamma\dot{\beta}}(\sigma)\nabla^{(\alpha}{}_{\dot{\alpha}}\xi^{\beta)\gamma\dot{\alpha}\dot{\beta}}\,. \quad \text{(C.37)}
\end{aligned}$$

Likewise, $\nabla'_{\beta\dot{\gamma}}\xi'^{\alpha\beta\dot{\beta}\dot{\gamma}} = (1-\sigma)\nabla_{\beta\dot{\gamma}}\xi^{\alpha\beta\dot{\beta}\dot{\gamma}} - 6\nabla_{\beta\dot{\gamma}}(\sigma)\xi^{\alpha\beta\dot{\beta}\dot{\gamma}}$ and thus

$$\begin{aligned}
&\nabla'_{\alpha}{}^{(\dot{\alpha}}\nabla'_{\beta\dot{\gamma}}\xi'^{\alpha\beta\dot{\beta})\dot{\gamma}} \\
&= ((1+\sigma)\nabla_{\alpha}{}^{(\dot{\alpha}} + \nabla^{\mu(\dot{\alpha}}(\sigma)M_{\alpha\mu} + \nabla_{\alpha}{}^{\dot{\mu}}(\sigma)\overline{M}^{(\dot{\alpha}}{}_{\dot{\mu}})((1-\sigma)\nabla_{\beta\dot{\gamma}}\xi^{\alpha\beta\dot{\beta})\dot{\gamma}} - 6\nabla_{\beta\dot{\gamma}}(\sigma)\xi^{\alpha\beta\dot{\beta})\dot{\gamma}}) \\
&= \nabla_{\alpha}{}^{(\dot{\alpha}}\nabla_{\beta\dot{\gamma}}\xi^{\alpha\beta\dot{\beta})\dot{\gamma}} - \nabla_{\alpha}{}^{(\dot{\alpha}}(\sigma)\nabla_{\beta\dot{\gamma}}\xi^{\alpha\beta\dot{\beta})\dot{\gamma}} - 6\nabla_{\alpha}{}^{(\dot{\alpha}}\nabla_{\beta\dot{\gamma}}(\sigma)\xi^{\alpha\beta\dot{\beta})\dot{\gamma}} - 6\nabla_{\beta\dot{\gamma}}(\sigma)\nabla_{\alpha}{}^{(\dot{\alpha}}\xi^{\alpha\beta\dot{\beta})\dot{\gamma}} \\
&\quad + \nabla^{\mu(\dot{\alpha}}(\sigma)M_{\alpha\mu}\nabla_{\beta\dot{\gamma}}\xi^{\alpha\beta\dot{\beta})\dot{\gamma}} + \nabla_{\alpha}{}^{\dot{\mu}}(\sigma)\overline{M}^{(\dot{\alpha}}{}_{\dot{\mu}}\nabla_{\beta\dot{\gamma}}\xi^{\alpha\beta\dot{\beta})\dot{\gamma}} \\
&= \nabla_{\alpha}{}^{(\dot{\alpha}}\nabla_{\beta\dot{\gamma}}\xi^{\alpha\beta\dot{\beta})\dot{\gamma}} - \nabla_{\alpha}{}^{(\dot{\alpha}}(\sigma)\nabla_{\beta\dot{\gamma}}\xi^{\alpha\beta\dot{\beta})\dot{\gamma}} - 6\nabla_{\alpha}{}^{(\dot{\alpha}}\nabla_{\beta\dot{\gamma}}(\sigma)\xi^{\alpha\beta\dot{\beta})\dot{\gamma}} - 6\nabla_{\beta\dot{\gamma}}(\sigma)\nabla_{\alpha}{}^{(\dot{\alpha}}\xi^{\alpha\beta\dot{\beta})\dot{\gamma}} \\
&\quad + \frac{1}{2}\nabla^{\mu(\dot{\alpha}}(\sigma)(\delta^{\alpha}{}_{\alpha}\nabla_{\beta\dot{\gamma}}\xi_{\mu}{}^{\beta\dot{\beta})\dot{\gamma}} + \delta^{\alpha}{}_{\mu}\nabla_{\beta\dot{\gamma}}\xi_{\alpha}{}^{\beta\dot{\beta})\dot{\gamma}}) \\
&\quad + \frac{1}{2}\nabla_{\alpha}{}^{\dot{\mu}}(\sigma)(\varepsilon^{(\dot{\alpha}\dot{\beta})}\nabla_{\beta\dot{\gamma}}\xi^{\alpha\beta}{}_{\dot{\mu}}{}^{\dot{\gamma}} + \delta^{(\dot{\beta}}{}_{\dot{\mu}}\nabla_{\beta\dot{\gamma}}\xi^{\alpha\beta\dot{\alpha})\dot{\gamma}}) \\
&= \nabla_{\alpha}{}^{(\dot{\alpha}}\nabla_{\beta\dot{\gamma}}\xi^{\alpha\beta\dot{\beta})\dot{\gamma}} - 2\nabla_{\alpha}{}^{(\dot{\alpha}}(\sigma)\nabla_{\beta\dot{\gamma}}\xi^{\alpha\beta\dot{\beta})\dot{\gamma}} - 6\nabla_{\alpha}{}^{(\dot{\alpha}}\nabla_{\beta\dot{\gamma}}(\sigma)\xi^{\alpha\beta\dot{\beta})\dot{\gamma}} - 6\nabla_{\beta\dot{\gamma}}(\sigma)\nabla_{\alpha}{}^{(\dot{\alpha}}\xi^{\alpha\beta\dot{\beta})\dot{\gamma}}\,. \quad \text{(C.38)}
\end{aligned}$$

Combining some of these results,

$$\begin{aligned}
&(D\nabla'^{(\beta}{}_{\dot{\beta}}\nabla'_{\mu\dot{\gamma}}\xi'^{\gamma)\mu\dot{\beta}\dot{\gamma}}M_{\beta\gamma} + H\nabla'_{\beta}{}^{(\dot{\beta}}\nabla'_{\gamma\dot{\mu}}\xi'^{\beta\gamma\dot{\gamma})\dot{\mu}}\overline{M}_{\dot{\beta}\dot{\gamma}})\mathbf{\Psi}' \\
&= \left(1+\frac{3}{2}\sigma\right)(D\nabla^{(\beta}{}_{\dot{\beta}}\nabla_{\mu\dot{\gamma}}\xi^{\gamma)\mu\dot{\beta}\dot{\gamma}}M_{\beta\gamma} + H\nabla_{\beta}{}^{(\dot{\beta}}\nabla_{\gamma\dot{\mu}}\xi^{\beta\gamma\dot{\gamma})\dot{\mu}}\overline{M}_{\dot{\beta}\dot{\gamma}})\mathbf{\Psi} \\
&\quad - 2\begin{bmatrix} D(\nabla^{(\beta}{}_{\dot{\beta}}(\sigma)\nabla_{\mu\dot{\gamma}}\xi^{\gamma)\mu\dot{\beta}\dot{\gamma}} + 3\nabla^{(\beta}{}_{\dot{\beta}}\nabla_{\mu\dot{\gamma}}(\sigma)\xi^{\gamma)\mu\dot{\beta}\dot{\gamma}} + 3\nabla_{\mu\dot{\gamma}}(\sigma)\nabla^{(\beta}{}_{\dot{\beta}}\xi^{\gamma)\mu\dot{\beta}\dot{\gamma}})M_{\beta\gamma}\psi_{\alpha} \\ H(\nabla_{\beta}{}^{(\dot{\beta}}(\sigma)\nabla_{\gamma\dot{\mu}}\xi^{\beta\gamma\dot{\gamma})\dot{\mu}} + 3\nabla_{\beta}{}^{(\dot{\beta}}\nabla_{\gamma\dot{\mu}}(\sigma)\xi^{\beta\gamma\dot{\gamma})\dot{\mu}} + 3\nabla_{\gamma\dot{\mu}}(\sigma)\nabla_{\beta}{}^{(\dot{\beta}}\xi^{\beta\gamma\dot{\gamma})\dot{\mu}})\overline{M}_{\dot{\beta}\dot{\gamma}}\bar{\chi}^{\dot{\alpha}} \end{bmatrix} \\
&= \left(1+\frac{3}{2}\sigma\right)(D\nabla^{(\beta}{}_{\dot{\beta}}\nabla_{\mu\dot{\gamma}}\xi^{\gamma)\mu\dot{\beta}\dot{\gamma}}M_{\beta\gamma} + H\nabla_{\beta}{}^{(\dot{\beta}}\nabla_{\gamma\dot{\mu}}\xi^{\beta\gamma\dot{\gamma})\dot{\mu}}\overline{M}_{\dot{\beta}\dot{\gamma}})\mathbf{\Psi} \\
&\quad - 2\begin{bmatrix} D(\nabla_{(\alpha}{}^{\dot{\beta}}(\sigma)\nabla^{\gamma\dot{\gamma}}\xi_{\beta)\gamma\dot{\beta}\dot{\gamma}} + 3\nabla_{(\alpha}{}^{\dot{\beta}}\nabla^{\gamma\dot{\gamma}}(\sigma)\xi_{\beta)\gamma\dot{\beta}\dot{\gamma}} + 3\nabla^{\gamma\dot{\gamma}}(\sigma)\nabla_{(\alpha}{}^{\dot{\beta}}\xi_{\beta)\gamma\dot{\beta}\dot{\gamma}})\psi^{\beta} \\ H(\nabla_{\beta}{}^{(\dot{\alpha}}(\sigma)\nabla_{\gamma\dot{\gamma}}\xi^{\beta\gamma\dot{\beta})\dot{\gamma}} + 3\nabla_{\beta}{}^{(\dot{\alpha}}\nabla_{\gamma\dot{\gamma}}(\sigma)\xi^{\beta\gamma\dot{\beta})\dot{\gamma}} + 3\nabla_{\gamma\dot{\gamma}}(\sigma)\nabla_{\beta}{}^{(\dot{\alpha}}\xi^{\beta\gamma\dot{\beta})\dot{\gamma}})\bar{\chi}_{\dot{\beta}} \end{bmatrix}. \quad \text{(C.39)}
\end{aligned}$$

$$\begin{aligned}
E'_{\alpha\beta\dot{\alpha}\dot{\beta}} &= \frac{1}{2}(\sigma^a)_{\alpha\dot{\alpha}}(\sigma^b)_{\beta\dot{\beta}}\Big(R'_{ab} - \frac{1}{4}\eta_{ab}R'\Big) \\
&= \frac{1}{2}(\sigma^a)_{\alpha\dot{\alpha}}(\sigma^b)_{\beta\dot{\beta}}\Big((1+2\sigma)R'_{ab} + \eta_{ab}\nabla^c\nabla_c(\sigma) + 2\nabla_a\nabla_b(\sigma) \\
&\quad - \frac{1}{4}\eta_{ab}(1+2\sigma)R' - \frac{3}{2}\eta_{ab}\nabla^c\nabla_c(\sigma)\Big) \\
&= (1+2\sigma)E_{\alpha\beta\dot{\alpha}\dot{\beta}} - \frac{1}{4}\varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}}\nabla^{\gamma\dot{\gamma}}\nabla_{\gamma\dot{\gamma}}(\sigma) + \nabla_{\alpha\dot{\alpha}}\nabla_{\beta\dot{\beta}}(\sigma) \quad \text{(C.40)}
\end{aligned}$$

$$
\begin{aligned}
\Longrightarrow\ & (GE'^{(\beta}{}_{\mu\dot\beta\dot\gamma}\xi'^{\gamma)\mu\dot\beta\dot\gamma}M_{\beta\gamma} + IE'_{\beta\gamma\dot\mu}{}^{(\dot\beta}\xi'^{\beta\gamma\dot\gamma)\dot\mu}\overline{M}_{\dot\beta\dot\gamma})\mathbf{\Psi}' \\
&= \bigg(G\bigg((1+2\sigma)E^{(\beta}{}_{\mu\dot\beta\dot\gamma} - \frac{1}{4}\delta^{(\beta}{}_{\mu}\varepsilon_{\dot\beta\dot\gamma}\nabla^{\nu\dot\nu}\nabla_{\nu\dot\nu}(\sigma) + \nabla^{(\beta}{}_{\dot\beta}\nabla_{\mu\dot\gamma}(\sigma)\bigg)((1-2\sigma)\xi^{\gamma)\mu\dot\beta\dot\gamma})M_{\beta\gamma} \\
&\quad + I\bigg((1+2\sigma)E_{\beta\gamma\dot\mu}{}^{(\dot\beta} + \frac{1}{4}\varepsilon_{\beta\gamma}\delta^{(\dot\beta}{}_{\dot\mu}\nabla^{\nu\dot\nu}\nabla_{\nu\dot\nu}(\sigma) + \nabla_{\beta\dot\mu}\nabla_{\gamma}{}^{(\dot\beta}(\sigma)\bigg)((1-2\sigma)\xi^{\beta\gamma\dot\gamma)\dot\mu})\overline{M}_{\dot\beta\dot\gamma}\bigg) \\
&\quad \times \bigg(\bigg(1+\frac{3}{2}\sigma\bigg)\mathbf{\Psi}\bigg) \\
&= \bigg(1+\frac{3}{2}\sigma\bigg)(GE^{(\beta}{}_{\mu\dot\beta\dot\gamma}\xi^{\gamma)\mu\dot\beta\dot\gamma}M_{\beta\gamma} + IE_{\beta\gamma\dot\mu}{}^{(\dot\beta}\xi^{\beta\gamma\dot\gamma)\dot\mu}\overline{M}_{\dot\beta\dot\gamma})\mathbf{\Psi} \\
&\quad + \begin{bmatrix} G\nabla^{(\beta}{}_{\dot\beta}\nabla_{\mu\dot\gamma}(\sigma)\xi^{\gamma)\mu\dot\beta\dot\gamma}M_{\beta\gamma}\psi_\alpha \\ I\nabla_{\beta\dot\mu}\nabla_{\gamma}{}^{(\dot\beta}(\sigma)\xi^{\beta\gamma\dot\gamma)\dot\mu}\overline{M}_{\dot\beta\dot\gamma}\bar\chi^{\dot\alpha} \end{bmatrix} \\
&= \bigg(1+\frac{3}{2}\sigma\bigg)(GE^{(\beta}{}_{\mu\dot\beta\dot\gamma}\xi^{\gamma)\mu\dot\beta\dot\gamma}M_{\beta\gamma} + IE_{\beta\gamma\dot\mu}{}^{(\dot\beta}\xi^{\beta\gamma\dot\gamma)\dot\mu}\overline{M}_{\dot\beta\dot\gamma})\mathbf{\Psi} \\
&\quad + \begin{bmatrix} G\nabla_{(\alpha}{}^{\dot\beta}\nabla^{\gamma\dot\gamma}(\sigma)\xi_{\beta)\gamma\dot\beta\dot\gamma}\psi^\beta \\ I\nabla_{\beta}{}^{(\dot\alpha}\nabla_{\gamma\dot\gamma}(\sigma)\xi^{\beta\gamma\dot\beta)\dot\gamma}\bar\chi_{\dot\beta} \end{bmatrix}
\end{aligned} \tag{C.41}
$$

$$
\begin{aligned}
\nabla'_{\alpha\dot\alpha}\nabla'_{\beta\dot\beta}\xi'^{\alpha\beta\dot\alpha\dot\beta} &= ((1+\sigma)\nabla_{\alpha\dot\alpha} + \nabla^{\gamma}{}_{\dot\alpha}(\sigma)M_{\alpha\gamma} + \nabla_{\alpha}{}^{\dot\gamma}(\sigma)\overline{M}_{\dot\alpha\dot\gamma}) \\
&\quad \times ((1-\sigma)\nabla_{\beta\dot\beta}\xi^{\alpha\beta\dot\alpha\dot\beta} - 6\nabla_{\beta\dot\beta}(\sigma)\xi^{\alpha\beta\dot\alpha\dot\beta}) \\
&= \nabla_{\alpha\dot\alpha}\nabla_{\beta\dot\beta}\xi^{\alpha\beta\dot\alpha\dot\beta} - \nabla_{\alpha\dot\alpha}(\sigma)\nabla_{\beta\dot\beta}\xi^{\alpha\beta\dot\alpha\dot\beta} + \nabla^{\gamma}{}_{\dot\alpha}(\sigma)M_{\alpha\gamma}\nabla_{\beta\dot\beta}\xi^{\alpha\beta\dot\alpha\dot\beta} \\
&\quad + \nabla_{\alpha}{}^{\dot\gamma}(\sigma)\overline{M}_{\dot\alpha\dot\gamma}\nabla_{\beta\dot\beta}\xi^{\alpha\beta\dot\alpha\dot\beta} - 6\nabla_{\alpha\dot\alpha}\nabla_{\beta\dot\beta}(\sigma)\xi^{\alpha\beta\dot\alpha\dot\beta} - 6\nabla_{\beta\dot\beta}(\sigma)\nabla_{\alpha\dot\alpha}\xi^{\alpha\beta\dot\alpha\dot\beta} \\
&= \nabla_{\alpha\dot\alpha}\nabla_{\beta\dot\beta}\xi^{\alpha\beta\dot\alpha\dot\beta} - \nabla_{\alpha\dot\alpha}(\sigma)\nabla_{\beta\dot\beta}\xi^{\alpha\beta\dot\alpha\dot\beta} + \frac{1}{2}\nabla^{\gamma}{}_{\dot\alpha}(\sigma)(\delta^{\alpha}{}_{\alpha}\nabla_{\beta\dot\beta}\xi_{\gamma}{}^{\beta\dot\alpha\dot\beta} + \delta^{\alpha}{}_{\gamma}\nabla_{\beta\dot\beta}\xi_{\alpha}{}^{\beta\dot\alpha\dot\beta}) \\
&\quad + \nabla_{\alpha}{}^{\dot\gamma}(\sigma)(\delta^{\dot\alpha}{}_{\dot\alpha}\nabla_{\beta\dot\beta}\xi^{\alpha\beta}{}_{\dot\gamma}{}^{\dot\beta} + \delta^{\dot\alpha}{}_{\dot\gamma}\nabla_{\beta\dot\beta}\xi^{\alpha\beta}{}_{\dot\alpha}{}^{\dot\beta}) - 6\nabla_{\alpha\dot\alpha}\nabla_{\beta\dot\beta}(\sigma)\xi^{\alpha\beta\dot\alpha\dot\beta} \\
&\quad - 6\nabla_{\beta\dot\beta}(\sigma)\nabla_{\alpha\dot\alpha}\xi^{\alpha\beta\dot\alpha\dot\beta} \\
&= \nabla_{\alpha\dot\alpha}\nabla_{\beta\dot\beta}\xi^{\alpha\beta\dot\alpha\dot\beta} - 10\nabla_{\alpha\dot\alpha}(\sigma)\nabla_{\beta\dot\beta}\xi^{\alpha\beta\dot\alpha\dot\beta} - 6\nabla_{\alpha\dot\alpha}\nabla_{\beta\dot\beta}(\sigma)\xi^{\alpha\beta\dot\alpha\dot\beta}
\end{aligned} \tag{C.42}
$$

$$
\begin{aligned}
E'_{\alpha\beta\dot\alpha\dot\beta}\xi'^{\alpha\beta\dot\alpha\dot\beta} &= \bigg((1+2\sigma)E_{\alpha\beta\dot\alpha\dot\beta} - \frac{1}{4}\varepsilon_{\alpha\beta}\varepsilon_{\dot\alpha\dot\beta}\nabla^{\gamma\dot\gamma}\nabla_{\gamma\dot\gamma}(\sigma) + \nabla_{\alpha\dot\alpha}\nabla_{\beta\dot\beta}(\sigma)\bigg) \\
&\quad \times ((1-2\sigma)\xi^{\alpha\beta\dot\alpha\dot\beta}) \\
&= E_{\alpha\beta\dot\alpha\dot\beta}\xi^{\alpha\beta\dot\alpha\dot\beta} + \nabla_{\alpha\dot\alpha}\nabla_{\beta\dot\beta}(\sigma)\xi^{\alpha\beta\dot\alpha\dot\beta}
\end{aligned} \tag{C.43}
$$

Finally, putting together equations C.25, C.30, C.35, C.36, C.39, C.41, C.42 and C.43,

$$\begin{aligned}
D'^{(2)}\mathbf{\Psi}' = &\left(1+\frac{3}{2}\sigma\right)D^{(2)}\mathbf{\Psi} + (5-6A)\xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}(\sigma)\nabla_{\gamma\dot\gamma}\mathbf{\Psi} \\
&+\left(\frac{3}{2}-6B+C\right)\xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}\nabla_{\gamma\dot\gamma}(\sigma)\mathbf{\Psi} + \left(\frac{3A}{2}-10B\right)\nabla_{\gamma\dot\gamma}(\xi^{\beta\gamma\dot\beta\dot\gamma})\nabla_{\beta\dot\beta}(\sigma)\mathbf{\Psi} \\
&+\Bigg[\xi_\alpha{}^{\gamma\dot\beta\dot\gamma}\nabla^\beta{}_{\dot\beta}(\sigma)\nabla_{\gamma\dot\gamma}\psi_\beta + \xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\alpha\dot\beta}(\sigma)\nabla_{\gamma\dot\gamma}\psi_\beta + \frac{1}{2}\xi^\beta{}_\alpha{}^{\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}\nabla^\gamma{}_{\dot\gamma}(\sigma)\psi_\gamma \\
&\quad+\frac{1}{2}\xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}\nabla_{\alpha\dot\gamma}(\sigma)\psi_\gamma + \frac{4}{3}\nabla_{(\alpha}{}^{\dot\gamma}\xi_{\beta\gamma)\dot\beta\dot\gamma}\nabla^{\beta\dot\beta}(\sigma)\psi^\gamma - 2\nabla_{(\alpha}{}^{\dot\gamma}(\sigma)\xi_{\beta\gamma)\dot\beta\dot\gamma}\nabla^{\beta\dot\beta}\psi^\gamma \\
&\quad+\frac{A}{2}\nabla_{\gamma\dot\gamma}(\xi^{\beta\gamma\dot\beta\dot\gamma})\nabla_{\alpha\dot\beta}(\sigma)\psi_\beta - \frac{A}{2}\nabla^{\gamma\dot\gamma}(\xi_{\alpha\gamma\dot\beta\dot\gamma})\nabla^{\beta\dot\beta}(\sigma)\psi_\beta - 2D\nabla_{(\alpha}{}^{\dot\beta}(\sigma)\nabla^{\gamma\dot\gamma}\xi_{\beta)\gamma\dot\beta\dot\gamma}\psi^\beta \\
&\quad+(G-6D)\nabla_{(\alpha}{}^{\dot\beta}\nabla^{\gamma\dot\gamma}(\sigma)\xi_{\beta)\gamma\dot\beta\dot\gamma}\psi^\beta - 6D\nabla^{\gamma\dot\gamma}(\sigma)\nabla_{(\alpha}{}^{\dot\beta}\xi_{\beta)\gamma\dot\beta\dot\gamma}\psi^\beta, \\
&\quad\xi^{\beta\gamma\dot\alpha\dot\gamma}\nabla_\beta{}^{\dot\beta}(\sigma)\nabla_{\gamma\dot\gamma}\bar\chi_{\dot\beta} + \xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_\beta{}^{\dot\alpha}(\sigma)\nabla_{\gamma\dot\gamma}\bar\chi_{\dot\beta} + \frac{1}{2}\xi^{\beta\gamma\dot\beta\dot\alpha}\nabla_{\beta\dot\beta}\nabla_\gamma{}^{\dot\gamma}(\sigma)\bar\chi_{\dot\gamma} \\
&\quad+\frac{1}{2}\xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}\nabla_\gamma{}^{\dot\alpha}(\sigma)\bar\chi_{\dot\gamma} + \frac{4}{3}\nabla_\gamma{}^{(\dot\alpha}\xi^{\beta\gamma\dot\beta\dot\gamma)}\nabla_{\beta\dot\beta}(\sigma)\bar\chi_{\dot\gamma} - 2\nabla_\gamma{}^{(\dot\alpha}(\sigma)\xi^{\beta\gamma\dot\beta\dot\gamma)}\nabla_{\beta\dot\beta}\bar\chi_{\dot\gamma} \\
&\quad+\frac{A}{2}\nabla^{\gamma\dot\gamma}(\xi_{\beta\gamma\dot\beta\dot\gamma})\nabla^{\beta\dot\alpha}(\sigma)\bar\chi^{\dot\beta} - \frac{A}{2}\nabla_{\gamma\dot\gamma}(\xi^{\beta\gamma\dot\alpha\dot\gamma})\nabla_{\beta\dot\beta}(\sigma)\bar\chi^{\dot\beta} - 2H\nabla_\beta{}^{(\dot\alpha}(\sigma)\nabla_{\gamma\dot\gamma}\xi^{\beta\gamma\dot\beta)\dot\gamma}\bar\chi_{\dot\beta} \\
&\quad+(I-6H)\nabla_\beta{}^{(\dot\alpha}\nabla_{\gamma\dot\gamma}(\sigma)\xi^{\beta\gamma\dot\beta)\dot\gamma}\bar\chi_{\dot\beta} - 6H\nabla_{\gamma\dot\gamma}(\sigma)\nabla_\beta{}^{(\dot\alpha}\xi^{\beta\gamma\dot\beta)\dot\gamma}\bar\chi_{\dot\beta}\Bigg]^T.
\end{aligned} \tag{C.44}$$

Some further simplification is possible using equation 3.25.

$$\begin{aligned}
&(\xi_\alpha{}^{\gamma\dot\beta\dot\gamma}\nabla^\beta{}_{\dot\beta}(\sigma) + \xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\alpha\dot\beta}(\sigma))\nabla_{\gamma\dot\gamma}\psi_\beta - 2\nabla_{(\alpha}{}^{\dot\gamma}(\sigma)\xi_{\beta\gamma)\dot\beta\dot\gamma}\nabla^{\beta\dot\beta}\psi^\gamma \\
&= (\nabla_\beta{}^{\dot\beta}(\sigma)\xi_{\alpha\gamma\dot\beta\dot\gamma} + \nabla_\alpha{}^{\dot\beta}(\sigma)\xi_{\beta\gamma\dot\beta\dot\gamma} - 2\nabla_{(\alpha}{}^{\dot\beta}(\sigma)\xi_{\beta\gamma)\dot\beta\dot\gamma})\nabla^{\gamma\dot\gamma}\psi^\beta \\
&= \frac{1}{3}(\varepsilon_{\beta\alpha}\nabla^{\mu\dot\beta}(\sigma)\xi_{\mu\gamma\dot\beta\dot\gamma} + \varepsilon_{\beta\gamma}\nabla^{\mu\dot\beta}(\sigma)\xi_{\alpha\mu\dot\beta\dot\gamma} + \varepsilon_{\alpha\beta}\nabla^{\mu\dot\beta}(\sigma)\xi_{\mu\gamma\dot\beta\dot\gamma} + \varepsilon_{\alpha\gamma}\nabla^{\mu\dot\beta}(\sigma)\xi_{\beta\mu\dot\beta\dot\gamma})\nabla^{\gamma\dot\gamma}\psi^\beta \\
&= \frac{1}{3}\nabla^{\gamma\dot\beta}(\sigma)\xi_{\beta\gamma\dot\beta\dot\gamma}\nabla_\alpha{}^{\dot\gamma}\psi^\beta \quad \text{since } \nabla_\beta{}^{\dot\gamma}\psi^\beta = 0 \\
&= \frac{1}{3}\xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}(\sigma)\nabla_{\alpha\dot\gamma}\psi_\gamma \\
&= \frac{1}{3}\xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}(\sigma)\nabla_{\gamma\dot\gamma}\psi_\alpha \quad \text{since } \nabla_{\alpha\dot\gamma}\psi_\gamma = \nabla_{(\alpha\dot\gamma}\psi_{\gamma)}
\end{aligned} \tag{C.45}$$

Completely analogously,

$$\begin{aligned}
&(\xi^{\beta\gamma\dot\alpha\dot\gamma}\nabla_\beta{}^{\dot\beta}(\sigma) + \xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_\beta{}^{\dot\alpha}(\sigma))\nabla_{\gamma\dot\gamma}\bar\chi_{\dot\beta} - 2\nabla_\gamma{}^{(\dot\alpha}(\sigma)\xi^{\beta\gamma\dot\beta\dot\gamma)}\nabla_{\beta\dot\beta}\bar\chi_{\dot\gamma} \\
&= (\nabla_\beta{}^{\dot\beta}(\sigma)\xi^{\beta\gamma\dot\alpha\dot\gamma} + \nabla_\beta{}^{\dot\alpha}(\sigma)\xi^{\beta\gamma\dot\beta\dot\gamma} - 2\nabla_\beta{}^{(\dot\alpha}(\sigma)\xi^{\beta\gamma\dot\beta\dot\gamma)})\nabla_{\gamma\dot\gamma}\bar\chi_{\dot\beta} \\
&= \frac{1}{3}(\varepsilon^{\dot\beta\dot\alpha}\nabla_{\beta\dot\mu}(\sigma)\xi^{\beta\gamma\dot\mu\dot\gamma} + \varepsilon^{\dot\beta\dot\gamma}\nabla_{\beta\dot\mu}(\sigma)\xi^{\beta\gamma\dot\alpha\dot\mu} + \varepsilon^{\dot\alpha\dot\beta}\nabla_{\beta\dot\mu}(\sigma)\xi^{\beta\gamma\dot\mu\dot\gamma} + \varepsilon^{\dot\alpha\dot\gamma}\nabla_{\beta\dot\mu}(\sigma)\xi^{\beta\gamma\dot\beta\dot\mu})\nabla_{\gamma\dot\gamma}\bar\chi_{\dot\beta} \\
&= \frac{1}{3}\nabla_{\beta\dot\gamma}(\sigma)\xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_\gamma{}^{\dot\alpha}\bar\chi_{\dot\beta} \\
&= \frac{1}{3}\xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}(\sigma)\nabla_{\gamma\dot\gamma}\bar\chi^{\dot\alpha}
\end{aligned} \tag{C.46}$$

$$\begin{bmatrix}(\xi_\alpha{}^{\gamma\dot\beta\dot\gamma}\nabla^\beta{}_{\dot\beta}(\sigma) + \xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\alpha\dot\beta}(\sigma))\nabla_{\gamma\dot\gamma}\psi_\beta - 2\nabla_{(\alpha}{}^{\dot\gamma}(\sigma)\xi_{\beta\gamma)\dot\beta\dot\gamma}\nabla^{\beta\dot\beta}\psi^\gamma \\ (\xi^{\beta\gamma\dot\alpha\dot\gamma}\nabla_\beta{}^{\dot\beta}(\sigma) + \xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_\beta{}^{\dot\alpha}(\sigma))\nabla_{\gamma\dot\gamma}\bar\chi_{\dot\beta} - 2\nabla_\gamma{}^{(\dot\alpha}(\sigma)\xi^{\beta\gamma\dot\beta\dot\gamma)}\nabla_{\beta\dot\beta}\bar\chi_{\dot\gamma}\end{bmatrix} = \frac{1}{3}\xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}(\sigma)\nabla_{\gamma\dot\gamma}\mathbf{\Psi}\,. \tag{C.47}$$

$$\frac{1}{2}\xi^{\beta}{}_{\alpha}{}^{\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}\nabla^{\gamma}{}_{\dot\gamma}(\sigma)\psi_\gamma+\frac{1}{2}\xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}\nabla_{\alpha\dot\gamma}(\sigma)\psi_\gamma$$
$$=\frac{1}{2}(\xi_{\alpha\beta\dot\beta\dot\gamma}\nabla^{\beta\dot\beta}\nabla_{\gamma}{}^{\dot\gamma}(\sigma)+\xi_{\beta\gamma\dot\beta\dot\gamma}\nabla^{\beta\dot\beta}\nabla_{\alpha}{}^{\dot\gamma}(\sigma))\psi^\gamma$$
$$=\nabla_{(\alpha}{}^{\dot\beta}\nabla^{\gamma\dot\gamma}(\sigma)\xi_{\beta)\gamma\dot\beta\dot\gamma}\psi^\beta \tag{C.48}$$
$$\frac{1}{2}\xi^{\beta\gamma\dot\beta\dot\alpha}\nabla_{\beta\dot\beta}\nabla_{\gamma}{}^{\dot\gamma}(\sigma)\bar\chi_{\dot\gamma}+\frac{1}{2}\xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}\nabla_{\gamma}{}^{\dot\alpha}(\sigma)\bar\chi_{\dot\gamma}$$
$$=\frac{1}{2}\xi^{\beta\gamma\dot\alpha\dot\gamma}\nabla_{\gamma\dot\gamma}\nabla_{\beta}{}^{\dot\beta}(\sigma)\bar\chi_{\dot\beta}+\frac{1}{2}\xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\gamma\dot\gamma}\nabla_{\beta}{}^{\dot\alpha}(\sigma)\bar\chi_{\dot\beta}$$
$$=\nabla_{\beta}{}^{(\dot\alpha}\nabla_{\gamma\dot\gamma}(\sigma)\xi^{\beta\gamma\dot\beta)\dot\gamma}\bar\chi_{\dot\beta} \tag{C.49}$$
$$\frac{A}{2}\nabla_{\gamma\dot\gamma}(\xi^{\beta\gamma\dot\beta\dot\gamma})\nabla_{\alpha\dot\beta}(\sigma)\psi_\beta-\frac{A}{2}\nabla^{\gamma\dot\gamma}(\xi_{\alpha\gamma\dot\beta\dot\gamma})\nabla^{\beta\dot\beta}(\sigma)\psi_\beta$$
$$=\frac{A}{2}\nabla^{\gamma\dot\gamma}(\xi_{\beta\gamma\dot\beta\dot\gamma})\nabla_{\alpha}{}^{\dot\beta}(\sigma)\psi^\beta+\frac{A}{2}\nabla^{\gamma\dot\gamma}(\xi_{\alpha\gamma\dot\beta\dot\gamma})\nabla_{\beta}{}^{\dot\beta}(\sigma)\psi^\beta$$
$$=A\nabla_{(\alpha}{}^{\dot\beta}(\sigma)\nabla^{\gamma\dot\gamma}(\xi_{\beta)\gamma\dot\beta\dot\gamma})\psi^\beta \tag{C.50}$$
$$\frac{A}{2}\nabla^{\gamma\dot\gamma}(\xi_{\beta\gamma\dot\beta\dot\gamma})\nabla^{\beta\dot\alpha}(\sigma)\bar\chi^{\dot\beta}-\frac{A}{2}\nabla_{\gamma\dot\gamma}(\xi^{\beta\gamma\dot\alpha\dot\gamma})\nabla_{\beta\dot\beta}(\sigma)\bar\chi^{\dot\beta}$$
$$=\frac{A}{2}\nabla_{\gamma\dot\gamma}(\xi^{\beta\gamma\dot\beta\dot\gamma})\nabla_{\beta}{}^{\dot\alpha}(\sigma)\bar\chi_{\dot\beta}+\frac{A}{2}\nabla_{\gamma\dot\gamma}(\xi^{\beta\gamma\dot\alpha\dot\gamma})\nabla_{\beta}{}^{\dot\beta}(\sigma)\bar\chi_{\dot\beta}$$
$$=A\nabla_{\beta}{}^{(\dot\alpha}(\sigma)\nabla_{\gamma\dot\gamma}\xi^{\beta\gamma\dot\beta)\dot\gamma}\bar\chi_{\dot\beta} \tag{C.51}$$
$$\nabla^{\gamma\dot\gamma}(\sigma)\nabla_{(\alpha}{}^{\dot\beta}\xi_{\beta)\gamma\dot\beta\dot\gamma}\psi^\beta$$
$$=\left(\nabla_{(\alpha}{}^{\dot\beta}\xi_{\beta\gamma)\dot\beta\dot\gamma}+\frac{1}{3}\varepsilon_{\gamma\alpha}\nabla_{(\mu}{}^{\dot\beta}\xi_{\beta)}{}^{\mu}{}_{\dot\beta\dot\gamma}+\frac{1}{3}\varepsilon_{\gamma\beta}\nabla_{(\alpha}{}^{\dot\beta}\xi_{\mu)}{}^{\mu}{}_{\dot\beta\dot\gamma}\right)\nabla^{\gamma\dot\gamma}(\sigma)\psi^\beta$$
$$=\nabla_{(\alpha}{}^{\dot\gamma}\xi_{\beta\gamma)\dot\beta\dot\gamma}\nabla^{\beta\dot\beta}(\sigma)\psi^\gamma-\frac{1}{6}\nabla_{\gamma}{}^{\dot\beta}\xi_{\beta}{}^{\gamma}{}_{\dot\beta\dot\gamma}\nabla_{\alpha}{}^{\dot\gamma}(\sigma)\psi^\beta-\frac{1}{6}\nabla_{\gamma}{}^{\dot\beta}\xi_{\alpha}{}^{\gamma}{}_{\dot\beta\dot\gamma}\nabla_{\beta}{}^{\dot\gamma}(\sigma)\psi^\beta$$
$$=\nabla_{(\alpha}{}^{\dot\gamma}\xi_{\beta\gamma)\dot\beta\dot\gamma}\nabla^{\beta\dot\beta}(\sigma)\psi^\gamma+\frac{1}{3}\nabla_{(\alpha}{}^{\dot\beta}(\sigma)\nabla^{\gamma\dot\gamma}(\xi_{\beta)\gamma\dot\beta\dot\gamma})\psi^\beta \tag{C.52}$$
$$\nabla_{\gamma\dot\gamma}(\sigma)\nabla_{\beta}{}^{(\dot\alpha}\xi^{\beta\gamma\dot\beta)\dot\gamma}\bar\chi_{\dot\beta}$$
$$=\left(\nabla_{\beta}{}^{(\dot\alpha}\xi^{\beta\gamma\dot\beta\dot\gamma)}+\frac{1}{3}\varepsilon^{\dot\gamma\dot\alpha}\nabla_{\beta}{}^{(\dot\mu}\xi^{\beta\gamma\dot\beta)}{}_{\dot\mu}+\frac{1}{3}\varepsilon^{\dot\gamma\dot\beta}\nabla_{\beta}{}^{(\dot\alpha}\xi^{\beta\gamma\dot\mu)}{}_{\dot\mu}\right)\nabla_{\gamma\dot\gamma}(\sigma)\bar\chi_{\dot\beta}$$
$$=\nabla_{\gamma}{}^{(\dot\alpha}\xi^{\beta\gamma\dot\beta\dot\gamma)}\nabla_{\beta\dot\beta}(\sigma)\bar\chi_{\dot\gamma}-\frac{1}{6}\nabla_{\beta}{}^{\dot\gamma}\xi^{\beta\gamma\dot\beta}{}_{\dot\gamma}\nabla_{\gamma}{}^{\dot\alpha}(\sigma)\bar\chi_{\dot\beta}-\frac{1}{6}\nabla_{\beta}{}^{\dot\gamma}\xi^{\beta\gamma\dot\alpha}{}_{\dot\gamma}\nabla_{\gamma}{}^{\dot\beta}(\sigma)\bar\chi_{\dot\beta}$$
$$=\nabla_{\gamma}{}^{(\dot\alpha}\xi^{\beta\gamma\dot\beta\dot\gamma)}\nabla_{\beta\dot\beta}(\sigma)\bar\chi_{\dot\gamma}+\frac{1}{3}\nabla_{\beta}{}^{(\dot\alpha}(\sigma)\nabla_{\gamma\dot\gamma}\xi^{\beta\gamma\dot\beta)\dot\gamma}\bar\chi_{\dot\beta} \tag{C.53}$$

Putting the past few results back into equation C.44,

$$\begin{aligned}D'^{(2)}\mathbf{\Psi}'=&\Big(1+\frac{3}{2}\sigma\Big)D^{(2)}\mathbf{\Psi}+\Big(\frac{16}{3}-6A\Big)\xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}(\sigma)\nabla_{\gamma\dot\gamma}\mathbf{\Psi}\\&+\Big(\frac{3}{2}-6B+C\Big)\xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}\nabla_{\gamma\dot\gamma}(\sigma)\mathbf{\Psi}+\Big(\frac{3A}{2}-10B\Big)\nabla_{\gamma\dot\gamma}(\xi^{\beta\gamma\dot\beta\dot\gamma})\nabla_{\beta\dot\beta}(\sigma)\mathbf{\Psi}\\&+\Bigg[\Big(\frac{4}{3}-6D\Big)\nabla_{(\alpha}{}^{\dot\gamma}\xi_{\beta\gamma)\dot\beta\dot\gamma}\nabla^{\beta\dot\beta}(\sigma)\psi^\gamma+(A-4D)\nabla_{(\alpha}{}^{\dot\beta}(\sigma)\nabla^{\gamma\dot\gamma}\xi_{\beta)\gamma\dot\beta\dot\gamma}\psi^\beta\\&\quad+(1+G-6D)\nabla_{(\alpha}{}^{\dot\beta}\nabla^{\gamma\dot\gamma}(\sigma)\xi_{\beta)\gamma\dot\beta\dot\gamma}\psi^\beta,\\&\quad\Big(\frac{4}{3}-6H\Big)\nabla_{\gamma}{}^{(\dot\alpha}\xi^{\beta\gamma\dot\beta\dot\gamma)}\nabla_{\beta\dot\beta}(\sigma)\bar\chi_{\dot\gamma}+(A-4H)\nabla_{\beta}{}^{(\dot\alpha}(\sigma)\nabla_{\gamma\dot\gamma}\xi^{\beta\gamma\dot\beta)\dot\gamma}\bar\chi_{\dot\beta}\\&\quad+(1+I-6H)\nabla_{\beta}{}^{(\dot\alpha}\nabla_{\gamma\dot\gamma}(\sigma)\xi^{\beta\gamma\dot\beta)\dot\gamma}\bar\chi_{\dot\beta}\Bigg]^T.\end{aligned} \tag{C.54}$$

Therefore, $D'^{(2)}\mathbf{\Psi}' = \left(1+\frac{3}{2}\sigma\right)D^{(2)}\mathbf{\Psi}$ if and only if $16/3 - 6A = 0$, $3/2 - 6B + C = 0$, $3A/2 - 10B = 0$, $4/3 - 6D = 0$, $A - 4D = 0$, $1 + G - 6D = 0$, $4/3 - 6H = 0$, $A - 4H = 0$ and $1 + I - 6H = 0$.
That is, $D'^{(2)}\mathbf{\Psi}' = \left(1+\frac{3}{2}\sigma\right)D^{(2)}\mathbf{\Psi}$ if and only if $A = 8/9$, $B = 2/15$, $C = -7/10$, $D = 2/9$, $G = 1/3$, $H = 2/9$ and $I = 1/3$.
Then, by equation C.25, the only candidate for a 2nd order symmetry of the Dirac operator is

$$\begin{aligned}D^{(2)} &= \xi^{\alpha\beta\dot{\alpha}\dot{\beta}}\nabla_{\alpha\dot{\alpha}}\nabla_{\beta\dot{\beta}} + \frac{2}{3}\nabla^{(\alpha}{}_{\dot{\beta}}\xi^{\beta\gamma)\dot{\alpha}\dot{\beta}}\nabla_{\alpha\dot{\alpha}}M_{\beta\gamma} + \frac{2}{3}\nabla_{\beta}{}^{(\dot{\alpha}}\xi^{\alpha\beta\dot{\beta}\dot{\gamma})}\nabla_{\alpha\dot{\alpha}}\overline{M}_{\dot{\beta}\dot{\gamma}} + \frac{8}{9}\nabla_{\beta\dot{\beta}}(\xi^{\alpha\beta\dot{\alpha}\dot{\beta}})\nabla_{\alpha\dot{\alpha}}\\ &+ \left(\frac{2}{9}\nabla^{(\alpha}{}_{\dot{\alpha}}\nabla_{\gamma\dot{\beta}}\xi^{\beta)\gamma\dot{\alpha}\dot{\beta}} + \frac{1}{3}E^{(\alpha}{}_{\gamma\dot{\alpha}\dot{\beta}}\xi^{\beta)\gamma\dot{\alpha}\dot{\beta}}\right)M_{\alpha\beta}\\ &+ \left(\frac{2}{9}\nabla_{\alpha}{}^{(\dot{\alpha}}\nabla_{\beta\dot{\gamma}}\xi^{\alpha\beta\dot{\beta})\dot{\gamma}} + \frac{1}{3}E_{\alpha\beta\dot{\gamma}}{}^{(\dot{\alpha}}\xi^{\alpha\beta\dot{\beta})\dot{\gamma}}\right)\overline{M}_{\dot{\alpha}\dot{\beta}} + \frac{2}{15}\nabla_{\alpha\dot{\alpha}}\nabla_{\beta\dot{\beta}}(\xi^{\alpha\beta\dot{\alpha}\dot{\beta}}) - \frac{7}{10}E_{\alpha\beta\dot{\alpha}\dot{\beta}}\xi^{\alpha\beta\dot{\alpha}\dot{\beta}}\,.\end{aligned} \tag{C.55}$$

I can now continue simplifying equation C.19,

$$\begin{aligned}\gamma^a\nabla_a D^{(2)}\mathbf{\Psi} &= \Bigg[\frac{1}{3}\nabla^{\beta}{}_{(\dot{\alpha}}\xi_{\alpha\beta\dot{\beta}\dot{\gamma})}\overline{C}^{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\mu}}\overline{\chi}_{\dot{\mu}} + \frac{1}{3}E^{\beta\gamma\dot{\gamma}}{}_{\dot{\alpha}}\nabla_{(\alpha}{}^{\dot{\beta}}\xi_{\beta\gamma)\dot{\beta}\dot{\gamma}}\overline{\chi}^{\dot{\alpha}} + \frac{2}{9}E_{\alpha\beta\dot{\alpha}\dot{\beta}}\nabla_{\gamma\dot{\gamma}}(\xi^{\beta\gamma\dot{\beta}\dot{\gamma}})\overline{\chi}^{\dot{\alpha}}\\ &+ \xi^{\beta\gamma\dot{\beta}\dot{\gamma}}[\nabla_{\alpha\dot{\alpha}},\nabla_{\beta\dot{\beta}}\nabla_{\gamma\dot{\gamma}}]\overline{\chi}^{\dot{\alpha}} + \nabla_{\alpha\dot{\alpha}}(\xi^{\beta\dot{\alpha}\dot{\beta}\dot{\gamma}})\nabla_{\beta\dot{\beta}}\overline{\chi}_{\dot{\gamma}} + \nabla_{\alpha\dot{\alpha}}(\xi^{\beta\dot{\beta}})\nabla_{\beta\dot{\beta}}\overline{\chi}^{\dot{\alpha}}\\ &+ \xi^{\beta\dot{\beta}}[\nabla_{\alpha\dot{\alpha}},\nabla_{\beta\dot{\beta}}]\overline{\chi}^{\dot{\alpha}} + \nabla_{\alpha\dot{\alpha}}(\xi^{\dot{\beta}\dot{\gamma}}\overline{M}_{\dot{\beta}\dot{\gamma}}\overline{\chi}^{\dot{\alpha}} + \xi\overline{\chi}^{\dot{\alpha}}),\\ &\frac{1}{3}\nabla^{(\alpha}{}_{\dot{\beta}}\xi^{\beta\gamma)\dot{\alpha}\dot{\beta}}C_{\alpha\beta\gamma\mu}\psi^{\mu} + \frac{1}{3}E^{\alpha}{}_{\gamma\dot{\beta}\dot{\gamma}}\nabla_{\beta}{}^{(\dot{\alpha}}\xi^{\beta\gamma\dot{\beta}\dot{\gamma})}\psi_{\alpha} + \frac{2}{9}E^{\alpha\beta\dot{\alpha}\dot{\beta}}\nabla_{\gamma\dot{\gamma}}(\xi^{\beta\gamma\dot{\beta}\dot{\gamma}})\psi_{\alpha}\\ &+ \xi^{\beta\gamma\dot{\beta}\dot{\gamma}}[\nabla^{\alpha\dot{\alpha}},\nabla_{\beta\dot{\beta}}\nabla\gamma\dot{\gamma}]\psi_{\alpha} + \nabla^{\alpha\dot{\alpha}}(\xi_{\alpha}{}^{\beta\gamma\dot{\beta}})\nabla_{\beta\dot{\beta}}\psi_{\gamma} + \nabla^{\alpha\dot{\alpha}}(\xi^{\beta\dot{\beta}})\nabla_{\beta\dot{\beta}}\psi_{\alpha}\\ &+ \xi^{\beta\dot{\beta}}[\nabla^{\alpha\dot{\alpha}},\nabla_{\beta\dot{\beta}}]\psi_{\alpha} + \nabla^{\alpha\dot{\alpha}}(\xi^{\beta\gamma}M_{\beta\gamma}\psi_{\alpha} + \xi\psi_{\alpha})\Bigg]^{T}\\ &= \begin{bmatrix}\psi'_{\alpha}\\ \overline{\chi}'^{\dot{\alpha}}\end{bmatrix}\ \text{say},\end{aligned} \tag{C.56}$$

but with $\xi^{\alpha\beta\gamma\dot{\alpha}}$, $\xi^{\alpha\dot{\alpha}\dot{\beta}\dot{\gamma}}$, $\xi^{\alpha\dot{\alpha}}$, $\xi^{\alpha\beta}$, $\xi^{\dot{\alpha}\dot{\beta}}$ and ξ all determined. Since $D^{(2)}$ and $\mathbf{\Psi}$ both have complete symmetry between dotted and undotted indices, if I simplify the first two components of equation C.56 - or ψ'_{α} as I have called them - to get the first two components of equation C.2, then it automatically follows that the second two components of equation C.56 - or $\overline{\chi}'^{\dot{\alpha}}$ as I have called them - simplify to the second two components of equation C.2.
Therefore, I will continue the proof only for[3] ψ'_{α}.

$$\begin{aligned}&\xi^{\beta\gamma\dot{\beta}\dot{\gamma}}[\nabla_{\alpha\dot{\alpha}},\nabla_{\beta\dot{\beta}}\nabla_{\gamma\dot{\gamma}}]\overline{\chi}^{\dot{\alpha}}\\ &= \xi^{\beta\gamma\dot{\beta}\dot{\gamma}}\nabla_{\beta\dot{\beta}}[\nabla_{\alpha\dot{\alpha}},\nabla_{\gamma\dot{\gamma}}]\overline{\chi}^{\dot{\alpha}} + \xi^{\beta\gamma\dot{\beta}\dot{\gamma}}[\nabla_{\alpha\dot{\alpha}},\nabla_{\beta\dot{\beta}}]\nabla_{\gamma\dot{\gamma}}\overline{\chi}^{\dot{\alpha}}\\ &= \xi^{\beta\gamma\dot{\beta}\dot{\gamma}}\nabla_{\beta\dot{\beta}}(\overline{R}_{\alpha\dot{\alpha}\gamma\dot{\gamma}}{}^{\dot{\mu}\dot{\nu}}\overline{M}_{\dot{\mu}\dot{\nu}}\overline{\chi}^{\dot{\alpha}}) + \xi^{\beta\gamma\dot{\beta}\dot{\gamma}}(R_{\alpha\dot{\alpha}\beta\dot{\beta}}{}^{\mu\nu}M_{\mu\nu} + \overline{R}_{\alpha\dot{\alpha}\beta\dot{\beta}}{}^{\dot{\mu}\dot{\nu}}\overline{M}_{\dot{\mu}\dot{\nu}})\nabla_{\gamma\dot{\gamma}}\overline{\chi}^{\dot{\alpha}}\\ &= \xi^{\beta\gamma\dot{\beta}\dot{\gamma}}(\nabla_{\beta\dot{\beta}}(\overline{R}_{\alpha\dot{\alpha}\gamma\dot{\gamma}}{}^{\dot{\mu}\dot{\alpha}}\overline{\chi}_{\dot{\mu}}) + R_{\alpha\dot{\alpha}\beta\dot{\beta}\gamma}{}^{\mu}\nabla_{\mu\dot{\gamma}}\overline{\chi}^{\dot{\alpha}} + \overline{R}_{\alpha\dot{\alpha}\beta\dot{\beta}\dot{\gamma}}{}^{\dot{\mu}}\nabla_{\gamma\dot{\mu}}\overline{\chi}^{\dot{\alpha}} + \overline{R}_{\alpha\dot{\alpha}\beta\dot{\beta}}{}^{\dot{\alpha}\dot{\mu}}\nabla_{\gamma\dot{\gamma}}\overline{\chi}_{\dot{\mu}})\\ &= \xi^{\beta\gamma\dot{\beta}\dot{\gamma}}(\nabla_{\beta\dot{\beta}}((\varepsilon_{\alpha\gamma}\overline{C}_{\dot{\alpha}\dot{\gamma}}{}^{\dot{\mu}\dot{\alpha}} + \varepsilon_{\dot{\alpha}\dot{\gamma}}E_{\alpha\gamma}{}^{\dot{\mu}\dot{\alpha}} + \varepsilon_{\alpha\gamma}(\delta^{\dot{\mu}}{}_{\dot{\alpha}}\delta^{\dot{\alpha}}{}_{\dot{\gamma}} + \delta^{\dot{\mu}}{}_{\dot{\gamma}}\delta^{\dot{\alpha}}{}_{\dot{\alpha}})F)\overline{\chi}_{\dot{\mu}})\\ &\quad + (\varepsilon_{\dot{\alpha}\dot{\beta}}C_{\alpha\beta\gamma}{}^{\mu} + \varepsilon_{\alpha\beta}E_{\gamma}{}^{\mu}{}_{\dot{\alpha}\dot{\beta}} + \varepsilon_{\dot{\alpha}\dot{\beta}}(-\varepsilon_{\alpha\gamma}\delta^{\mu}{}_{\beta} - \delta^{\mu}{}_{\alpha}\varepsilon_{\beta\gamma})F)\nabla_{\mu\dot{\gamma}}\overline{\chi}^{\dot{\alpha}}\\ &\quad + (\varepsilon_{\alpha\beta}\overline{C}_{\dot{\alpha}\dot{\beta}\dot{\gamma}}{}^{\dot{\mu}} + \varepsilon_{\dot{\alpha}\dot{\beta}}E_{\alpha\beta\dot{\gamma}}{}^{\dot{\mu}} + \varepsilon_{\alpha\beta}(-\delta^{\dot{\mu}}{}_{\dot{\alpha}}\varepsilon_{\dot{\beta}\dot{\gamma}} - \delta^{\dot{\mu}}{}_{\dot{\beta}}\varepsilon_{\dot{\alpha}\dot{\gamma}})F)\nabla_{\gamma\dot{\mu}}\overline{\chi}^{\dot{\alpha}}\\ &\quad + (\varepsilon_{\alpha\beta}\overline{C}_{\dot{\alpha}\dot{\beta}}{}^{\dot{\alpha}\dot{\mu}} + \varepsilon_{\dot{\alpha}\dot{\beta}}E_{\alpha\beta}{}^{\dot{\alpha}\dot{\mu}} + \varepsilon_{\alpha\beta}(\delta^{\dot{\alpha}}{}_{\dot{\alpha}}\delta^{\dot{\mu}}{}_{\dot{\beta}} + \delta^{\dot{\alpha}}{}_{\dot{\beta}}\delta^{\dot{\mu}}{}_{\dot{\alpha}})F)\nabla_{\gamma\dot{\gamma}}\overline{\chi}_{\dot{\mu}})\end{aligned} \tag{C.57}$$

[3]Really, I could have focused on just two components right from the start, rather than carrying all four components. However, I have chosen to be pedantic in waiting until I have explicitly shown that $D^{(2)}$ has full symmetry between dotted and undotted indices.

$$
\begin{aligned}
&\xi^{\beta\gamma\dot\beta\dot\gamma}[\nabla_{\alpha\dot\alpha},\nabla_{\beta\dot\beta}\nabla_{\gamma\dot\gamma}]\bar\chi^{\dot\alpha}\\
&= \xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}(E_{\alpha\gamma\dot\alpha\dot\gamma}\bar\chi^{\dot\alpha}) - 3\xi_{\alpha\beta\dot\alpha\dot\beta}\nabla^{\beta\dot\beta}(F\bar\chi^{\dot\alpha}) + \xi^{\beta\gamma}{}_{\dot\alpha}{}^{\dot\gamma}C_{\alpha\beta\gamma}{}^{\mu}\nabla_{\mu\dot\gamma}\bar\chi^{\dot\alpha} + \xi_{\alpha}{}^{\beta\dot\beta\dot\gamma}E_{\beta}{}^{\gamma}{}_{\dot\alpha\dot\beta}\nabla_{\gamma\dot\gamma}\bar\chi^{\dot\alpha}\\
&\quad - \xi_{\alpha\beta\dot\alpha\dot\beta}F\nabla^{\beta\dot\beta}\bar\chi^{\dot\alpha} + \xi_{\alpha}{}^{\gamma\dot\beta\dot\gamma}\bar C_{\dot\alpha\dot\beta\dot\gamma}{}^{\dot\mu}\nabla_{\gamma\dot\mu}\bar\chi^{\dot\alpha} + \xi^{\beta\gamma}{}_{\dot\alpha}{}^{\dot\gamma}E_{\alpha\beta\dot\gamma}{}^{\dot\mu}\nabla_{\gamma\dot\mu}\bar\chi^{\dot\alpha} - \xi_{\alpha\beta\dot\alpha\dot\beta}F\nabla^{\beta\dot\beta}\bar\chi^{\dot\alpha}\\
&\quad + \xi^{\beta\gamma}{}_{\dot\alpha}{}^{\dot\gamma}E_{\alpha\beta}{}^{\dot\alpha\dot\beta}\nabla_{\gamma\dot\gamma}\bar\chi_{\dot\beta} - 3\xi_{\alpha\beta\dot\alpha\dot\beta}F\nabla^{\gamma\dot\gamma}\bar\chi^{\dot\beta}\\
&= \xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}(E_{\alpha\gamma\dot\alpha\dot\gamma})\bar\chi^{\dot\alpha} + \xi^{\beta\gamma\dot\beta\dot\gamma}E_{\alpha\gamma\dot\alpha\dot\gamma}\nabla_{\beta\dot\beta}\bar\chi^{\dot\alpha} - 3\xi_{\alpha\beta\dot\alpha\dot\beta}\nabla^{\beta\dot\beta}(F)\bar\chi^{\dot\alpha} - 8\xi_{\alpha\beta\dot\alpha\dot\beta}F\nabla^{\beta\dot\beta}\bar\chi^{\dot\alpha}\\
&\quad + \xi^{\beta\gamma}{}_{\dot\alpha}{}^{\dot\gamma}C_{\alpha\beta\gamma}{}^{\mu}\nabla_{\mu\dot\gamma}\bar\chi^{\dot\alpha} + \xi_{\alpha}{}^{\beta\dot\beta\dot\gamma}E_{\beta}{}^{\gamma}{}_{\dot\alpha\dot\beta}\nabla_{\gamma\dot\gamma}\bar\chi^{\dot\alpha} + \xi_{\alpha}{}^{\gamma\dot\beta\dot\gamma}\bar C_{\dot\alpha\dot\beta\dot\gamma}{}^{\dot\mu}\nabla_{\gamma\dot\mu}\bar\chi^{\dot\alpha}\\
&\quad + \xi^{\beta\gamma}{}_{\dot\alpha}{}^{\dot\gamma}E_{\alpha\beta\dot\gamma}{}^{\dot\mu}\nabla_{\gamma\dot\mu}\bar\chi^{\dot\alpha} + \xi^{\beta\gamma}{}_{\dot\alpha}{}^{\dot\gamma}E_{\alpha\beta}{}^{\dot\alpha\dot\beta}\nabla_{\gamma\dot\gamma}\bar\chi_{\dot\beta} \qquad \text{(C.58)}
\end{aligned}
$$

$$
\begin{aligned}
\xi^{\beta\dot\beta}[\nabla_{\alpha\dot\alpha},\nabla_{\beta\dot\beta}]\bar\chi^{\dot\alpha} &= \xi^{\beta\dot\beta}\bar R_{\alpha\dot\alpha\beta\dot\beta}{}^{\dot\mu\dot\nu}\bar M_{\dot\mu\dot\nu}\bar\chi^{\dot\alpha}\\
&= \xi^{\beta\dot\beta}\bar R_{\alpha\dot\alpha\beta\dot\beta}{}^{\dot\alpha\dot\mu}\bar\chi_{\dot\mu}\\
&= \xi^{\beta\dot\beta}(\varepsilon_{\alpha\beta}\bar C_{\dot\alpha\dot\beta}{}^{\dot\alpha\dot\mu} + \varepsilon_{\dot\alpha\dot\beta}E_{\alpha\beta}{}^{\dot\alpha\dot\mu} + \varepsilon_{\alpha\beta}(\delta^{\dot\alpha}{}_{\dot\alpha}\delta^{\dot\mu}{}_{\dot\beta} + \delta^{\dot\alpha}{}_{\dot\beta}\delta^{\dot\mu}{}_{\dot\alpha})F)\bar\chi_{\dot\mu}\\
&= \xi^{\beta\dot\beta}E_{\alpha\beta\dot\alpha\dot\beta}\bar\chi^{\dot\alpha} - 3\xi_{\alpha\dot\alpha}F\bar\chi^{\dot\alpha}\\
&= \frac{8}{9}\nabla_{\gamma\dot\gamma}(\xi^{\beta\gamma\dot\beta\dot\gamma})E_{\alpha\beta\dot\alpha\dot\beta}\bar\chi^{\dot\alpha} - \frac{8}{3}\nabla^{\beta\dot\beta}(\xi_{\alpha\beta\dot\alpha\dot\beta})F\bar\chi^{\dot\alpha} \qquad \text{(C.59)}\\
\nabla_{\alpha\dot\alpha}((\xi^{\dot\beta\dot\gamma}\bar M_{\dot\beta\dot\gamma} + \xi)\bar\chi^{\dot\alpha}) &= \nabla_{\alpha\dot\alpha}(\xi^{\dot\alpha\dot\beta}\bar\chi_{\dot\beta} + \xi\bar\chi^{\dot\alpha})\\
&= \xi^{\dot\alpha\dot\beta}\nabla_{\alpha\dot\alpha}\bar\chi_{\dot\beta} + \nabla_{\alpha\dot\alpha}(\xi^{\dot\alpha\dot\beta})\bar\chi_{\dot\beta} + \nabla_{\alpha\dot\alpha}(\xi)\bar\chi^{\dot\alpha} + \xi\nabla_{\alpha\dot\alpha}\bar\chi^{\dot\alpha}\\
&= \xi^{\dot\alpha\dot\beta}\nabla_{\alpha\dot\alpha}\bar\chi_{\dot\beta} + \nabla_{\alpha\dot\alpha}(\xi^{\dot\alpha\dot\beta})\bar\chi_{\dot\beta} + \nabla_{\alpha\dot\alpha}(\xi)\bar\chi^{\dot\alpha} \qquad \text{(C.60)}
\end{aligned}
$$

Putting the past few results back into equation C.56,

$$
\begin{aligned}
\psi'_\alpha &= \frac{1}{3}\nabla^{\beta}{}_{(\dot\alpha}\xi_{\alpha\beta\dot\beta\dot\gamma)}\bar C^{\dot\alpha\dot\beta\dot\gamma\dot\mu}\bar\chi_{\dot\mu} + \frac{1}{3}E^{\beta\gamma\dot\gamma}{}_{\dot\alpha}\nabla_{(\alpha}{}^{\dot\beta}\xi_{\beta\gamma)\dot\beta\dot\gamma}\bar\chi^{\dot\alpha} + \frac{2}{9}E_{\alpha\beta\dot\alpha\dot\beta}\nabla_{\gamma\dot\gamma}(\xi^{\beta\gamma\dot\beta\dot\gamma})\bar\chi^{\dot\alpha}\\
&\quad + \xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}(E_{\alpha\gamma\dot\alpha\dot\gamma})\bar\chi^{\dot\alpha} + \xi^{\beta\gamma\dot\beta\dot\gamma}E_{\alpha\gamma\dot\alpha\dot\gamma}\nabla_{\beta\dot\beta}\bar\chi^{\dot\alpha} - 3\xi_{\alpha\beta\dot\alpha\dot\beta}\nabla^{\beta\dot\beta}(F)\bar\chi^{\dot\alpha} - 8\xi_{\alpha\beta\dot\alpha\dot\beta}F\nabla^{\beta\dot\beta}\bar\chi^{\dot\alpha}\\
&\quad + \xi^{\beta\gamma}{}_{\dot\alpha}{}^{\dot\gamma}C_{\alpha\beta\gamma}{}^{\mu}\nabla_{\mu\dot\gamma}\bar\chi^{\dot\alpha} + \xi_{\alpha}{}^{\beta\dot\beta\dot\gamma}E_{\beta}{}^{\gamma}{}_{\dot\alpha\dot\beta}\nabla_{\gamma\dot\gamma}\bar\chi^{\dot\alpha} + \xi_{\alpha}{}^{\gamma\dot\beta\dot\gamma}\bar C_{\dot\alpha\dot\beta\dot\gamma}{}^{\dot\mu}\nabla_{\gamma\dot\mu}\bar\chi^{\dot\alpha} + \xi^{\beta\gamma}{}_{\dot\alpha}{}^{\dot\gamma}E_{\alpha\beta\dot\gamma}{}^{\dot\mu}\nabla_{\gamma\dot\mu}\bar\chi^{\dot\alpha}\\
&\quad + \xi^{\beta\gamma}{}_{\dot\alpha}{}^{\dot\gamma}E_{\alpha\beta}{}^{\dot\alpha\dot\beta}\nabla_{\gamma\dot\gamma}\bar\chi_{\dot\beta} + \nabla_{\alpha\dot\alpha}(\xi^{\beta\dot\alpha\dot\beta\dot\gamma})\nabla_{\beta\dot\beta}\bar\chi_{\dot\gamma} + \nabla_{\alpha\dot\alpha}(\xi^{\beta\dot\beta})\nabla_{\beta\dot\beta}\bar\chi^{\dot\alpha} + \frac{8}{9}\nabla_{\gamma\dot\gamma}(\xi^{\beta\gamma\dot\beta\dot\gamma})E_{\alpha\beta\dot\alpha\dot\beta}\bar\chi^{\dot\alpha}\\
&\quad - \frac{8}{3}\nabla^{\beta\dot\beta}(\xi_{\alpha\beta\dot\alpha\dot\beta})F\bar\chi^{\dot\alpha} + \xi^{\dot\alpha\dot\beta}\nabla_{\alpha\dot\alpha}\bar\chi_{\dot\beta} + \nabla_{\alpha\dot\alpha}(\xi^{\dot\alpha\dot\beta})\bar\chi_{\dot\beta} + \nabla_{\alpha\dot\alpha}(\xi)\bar\chi^{\dot\alpha}\\
&= \frac{1}{3}\nabla^{\beta}{}_{(\dot\alpha}\xi_{\alpha\beta\dot\beta\dot\gamma)}\bar C^{\dot\alpha\dot\beta\dot\gamma\dot\mu}\bar\chi_{\dot\mu} + \frac{1}{3}E^{\beta\gamma\dot\gamma}{}_{\dot\alpha}\nabla_{(\alpha}{}^{\dot\beta}\xi_{\beta\gamma)\dot\beta\dot\gamma}\bar\chi^{\dot\alpha} + \frac{10}{9}E_{\alpha\beta\dot\alpha\dot\beta}\nabla_{\gamma\dot\gamma}(\xi^{\beta\gamma\dot\beta\dot\gamma})\bar\chi^{\dot\alpha}\\
&\quad + \xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}(E_{\alpha\gamma\dot\alpha\dot\gamma})\bar\chi^{\dot\alpha} + 2\xi^{\beta\gamma\dot\beta\dot\gamma}E_{\alpha\gamma\dot\alpha\dot\gamma}\nabla_{\beta\dot\beta}\bar\chi^{\dot\alpha} - 3\xi_{\alpha\beta\dot\alpha\dot\beta}\nabla^{\beta\dot\beta}(F)\bar\chi^{\dot\alpha} - 8\xi_{\alpha\beta\dot\alpha\dot\beta}F\nabla^{\beta\dot\beta}\bar\chi^{\dot\alpha}\\
&\quad + \xi^{\beta\gamma}{}_{\dot\alpha}{}^{\dot\gamma}C_{\alpha\beta\gamma}{}^{\mu}\nabla_{\mu\dot\gamma}\bar\chi^{\dot\alpha} + \xi_{\alpha}{}^{\beta\dot\beta\dot\gamma}E_{\beta}{}^{\gamma}{}_{\dot\alpha\dot\beta}\nabla_{\gamma\dot\gamma}\bar\chi^{\dot\alpha} + \xi_{\alpha}{}^{\gamma\dot\beta\dot\gamma}\bar C_{\dot\alpha\dot\beta\dot\gamma}{}^{\dot\mu}\nabla_{\gamma\dot\mu}\bar\chi^{\dot\alpha} + \xi^{\beta\gamma}{}_{\dot\alpha}{}^{\dot\gamma}E_{\alpha\beta\dot\gamma}{}^{\dot\mu}\nabla_{\gamma\dot\mu}\bar\chi^{\dot\alpha}\\
&\quad + \nabla_{\alpha\dot\alpha}(\xi^{\beta\dot\alpha\dot\beta\dot\gamma})\nabla_{\beta\dot\beta}\bar\chi_{\dot\gamma} + \nabla_{\alpha\dot\alpha}(\xi^{\beta\dot\beta})\nabla_{\beta\dot\beta}\bar\chi^{\dot\alpha} - \frac{8}{3}\nabla^{\beta\dot\beta}(\xi_{\alpha\beta\dot\alpha\dot\beta})F\bar\chi^{\dot\alpha} + \xi^{\dot\alpha\dot\beta}\nabla_{\alpha\dot\alpha}\bar\chi_{\dot\beta}\\
&\quad + \nabla_{\alpha\dot\alpha}(\xi^{\dot\alpha\dot\beta})\bar\chi_{\dot\beta} + \nabla_{\alpha\dot\alpha}(\xi)\bar\chi^{\dot\alpha}\,. \qquad \text{(C.61)}
\end{aligned}
$$

Exactly as I did in the other long calculations in my thesis, I will proceed by removing/simplifying terms without a curvature factor on them.

$$
\begin{aligned}
\nabla_{\alpha\dot\alpha}(\xi^{\beta\dot\alpha\dot\beta\dot\gamma})\nabla_{\beta\dot\beta}\bar\chi_{\dot\gamma} &= \nabla_{\alpha}{}^{\dot\alpha}(\xi_{\beta\dot\alpha\dot\beta\dot\gamma})\nabla^{\beta\dot\beta}\bar\chi^{\dot\gamma}\\
&= \nabla_{(\alpha}{}^{\dot\alpha}\xi_{\beta)\dot\alpha\dot\beta\dot\gamma}\nabla^{\beta\dot\beta}\bar\chi^{\dot\gamma} + \frac{1}{2}\varepsilon_{\alpha\beta}\nabla^{\gamma\dot\alpha}(\xi_{\gamma\dot\alpha\dot\beta\dot\gamma})\nabla^{\beta\dot\beta}\bar\chi^{\dot\gamma} \qquad \text{(C.62)}
\end{aligned}
$$

$$\begin{aligned}
\nabla_{\alpha\dot\alpha}(\xi^{\beta\dot\beta})\nabla_{\beta\dot\beta}\bar\chi^{\dot\alpha} &= \nabla_{\alpha\dot\alpha}(\xi_{\beta\dot\beta})\nabla^{\beta\dot\beta}\bar\chi^{\dot\alpha} \\
&= \nabla_{\alpha\dot\gamma}(\xi_{\beta\dot\beta})\nabla^{\beta\dot\beta}\bar\chi^{\dot\gamma} \\
&= \Big(\nabla_{(\alpha(\dot\gamma}\xi_{\beta)\dot\beta)} + \frac{1}{2}\varepsilon_{\alpha\beta}\nabla^{\mu}{}_{(\dot\gamma}\xi_{\mu\dot\beta)} + \frac{1}{2}\varepsilon_{\dot\gamma\dot\beta}\nabla_{(\alpha}{}^{\dot\mu}\xi_{\beta)\dot\mu} \\
&\quad + \frac{1}{4}\varepsilon_{\alpha\beta}\varepsilon_{\dot\gamma\dot\beta}\nabla^{\mu\dot\mu}(\xi_{\mu\dot\mu})\Big)\nabla^{\beta\dot\beta}\bar\chi^{\dot\gamma} \\
&= \left(\nabla_{(\alpha(\dot\gamma}\xi_{\beta)\dot\beta)} + \frac{1}{2}\varepsilon_{\alpha\beta}\nabla^{\mu}{}_{(\dot\gamma}\xi_{\mu\dot\beta)}\right)\nabla^{\beta\dot\beta}\bar\chi^{\dot\gamma} \quad \text{as } \nabla^{\beta}{}_{\dot\gamma}\bar\chi^{\dot\gamma} = 0 \qquad \text{(C.63)}
\end{aligned}$$

$$\xi^{\dot\alpha\dot\beta}\nabla_{\alpha\dot\alpha}\bar\chi_{\dot\beta} = \varepsilon_{\alpha\beta}\xi_{\dot\beta\dot\gamma}\nabla^{\beta\dot\beta}\bar\chi_{\dot\gamma} \qquad \text{(C.64)}$$

Together,

$$\begin{aligned}
&\nabla_{\alpha\dot\alpha}(\xi^{\beta\dot\alpha\dot\beta\dot\gamma})\nabla_{\beta\dot\beta}\bar\chi_{\dot\gamma} + \nabla_{\alpha\dot\alpha}(\xi^{\beta\dot\beta})\nabla_{\beta\dot\beta}\bar\chi^{\dot\alpha} + \xi^{\dot\alpha\dot\beta}\nabla_{\alpha\dot\alpha}\bar\chi_{\dot\beta} \\
&= \left(\nabla_{(\alpha}{}^{\dot\alpha}\xi_{\beta)\dot\alpha\dot\beta\dot\gamma} + \nabla_{(\alpha(\dot\gamma}\xi_{\beta)\dot\beta)} + \varepsilon_{\alpha\beta}\left(\frac{1}{2}\nabla^{\gamma\dot\alpha}(\xi_{\gamma\dot\alpha\dot\beta\dot\gamma}) + \frac{1}{2}\nabla^{\mu}{}_{(\dot\gamma}\xi_{\mu\dot\beta)} + \xi_{\dot\beta\dot\gamma}\right)\right)\nabla^{\beta\dot\beta}\bar\chi^{\dot\gamma}\,, \qquad \text{(C.65)}
\end{aligned}$$

thus collating all terms in equation C.61 with a derivative on $\bar\chi$, but no curvature factor.

$$\begin{aligned}
&\frac{1}{2}\nabla^{\gamma\dot\alpha}(\xi_{\gamma\dot\alpha\dot\beta\dot\gamma}) + \frac{1}{2}\nabla^{\mu}{}_{(\dot\gamma}\xi_{\mu\dot\beta)} + \xi_{\dot\beta\dot\gamma} \\
&= -\frac{1}{3}\nabla^{\gamma\dot\alpha}\nabla^{\mu}{}_{(\dot\alpha}\xi_{\gamma\mu\dot\beta\dot\gamma)} + \frac{4}{9}\nabla^{\mu}{}_{(\dot\gamma}\nabla^{\gamma\dot\alpha}\xi_{\mu\gamma\dot\beta)\dot\alpha} - \frac{2}{9}\nabla^{\gamma}{}_{(\dot\beta}\nabla^{\mu\dot\alpha}\xi_{\gamma\mu\dot\gamma)\dot\alpha} - \frac{1}{3}E^{\gamma\mu\dot\mu}{}_{(\dot\beta}\xi_{\gamma\mu\dot\gamma)\dot\mu} \\
&= -\frac{1}{3}\nabla^{\gamma\dot\alpha}\nabla^{\mu}{}_{(\dot\beta}\xi_{\gamma\mu\dot\gamma\dot\alpha)} + \frac{2}{9}\nabla^{\mu}{}_{(\dot\beta}\nabla^{\gamma\dot\alpha}\xi_{\gamma\mu\dot\gamma)\dot\alpha} - \frac{1}{3}E^{\gamma\mu\dot\alpha}{}_{(\dot\beta}\xi_{\gamma\mu\dot\gamma)\dot\alpha} \qquad \text{(C.66)} \\
&[\nabla^{\mu}{}_{\dot\beta}, \nabla^{\gamma\dot\alpha}]\xi_{\gamma\mu\dot\gamma\dot\alpha} \\
&= (R^{\mu}{}_{\dot\beta}{}^{\gamma\dot\alpha\nu\rho}M_{\nu\rho} + \bar R^{\mu}{}_{\dot\beta}{}^{\gamma\dot\alpha\dot\nu\dot\rho}\bar M_{\dot\nu\dot\rho})\xi_{\gamma\mu\dot\gamma\dot\alpha} \\
&= R^{\mu}{}_{\dot\beta}{}^{\gamma\dot\alpha}{}_{\gamma}{}^{\nu}\xi_{\nu\mu\dot\gamma\dot\alpha} + R^{\mu}{}_{\dot\beta}{}^{\gamma\dot\alpha}{}_{\mu}{}^{\nu}\xi_{\gamma\nu\dot\gamma\dot\alpha} + \bar R^{\mu}{}_{\dot\beta}{}^{\gamma\dot\alpha}{}_{\dot\gamma}{}^{\dot\nu}\xi_{\gamma\mu\dot\nu\dot\alpha} + \bar R^{\mu}{}_{\dot\beta}{}^{\gamma\dot\alpha}{}_{\dot\alpha}{}^{\dot\nu}\xi_{\gamma\mu\dot\gamma\dot\nu} \\
&= (-\delta^{\dot\alpha}{}_{\dot\beta}C^{\mu\gamma}{}_{\gamma}{}^{\nu} + \varepsilon^{\gamma\mu}E_{\gamma}{}^{\nu}{}_{\dot\beta}{}^{\dot\alpha} - \delta^{\dot\alpha}{}_{\dot\beta}(\delta^{\mu}{}_{\gamma}\varepsilon^{\nu\gamma} + \delta^{\gamma}{}_{\gamma}\varepsilon^{\nu\mu})F)\xi_{\nu\mu\dot\gamma\dot\alpha} \\
&\quad + (-\delta^{\dot\alpha}{}_{\dot\beta}C^{\mu\gamma}{}_{\mu}{}^{\nu} + \varepsilon^{\gamma\mu}E_{\mu}{}^{\nu}{}_{\dot\beta}{}^{\dot\alpha} - \delta^{\dot\alpha}{}_{\dot\beta}(\delta^{\mu}{}_{\mu}\varepsilon^{\nu\gamma} + \delta^{\gamma}{}_{\mu}\varepsilon^{\nu\mu})F)\xi_{\gamma\nu\dot\gamma\dot\alpha} \\
&\quad + (\varepsilon^{\gamma\mu}\bar C_{\dot\beta}{}^{\dot\alpha}{}_{\dot\gamma}{}^{\dot\nu} - \delta^{\dot\alpha}{}_{\dot\beta}E^{\mu\gamma}{}_{\dot\gamma}{}^{\dot\nu} + \varepsilon^{\gamma\mu}(\varepsilon_{\dot\beta\dot\gamma}\varepsilon^{\dot\nu\dot\alpha} - \delta^{\dot\nu}{}_{\dot\beta}\delta^{\dot\alpha}{}_{\dot\gamma})F)\xi_{\gamma\mu\dot\nu\dot\alpha} \\
&\quad + (\varepsilon^{\gamma\mu}\bar C_{\dot\beta}{}^{\dot\alpha}{}_{\dot\alpha}{}^{\dot\nu} - \delta^{\dot\alpha}{}_{\dot\beta}E^{\mu\gamma}{}_{\dot\alpha}{}^{\dot\nu} + \varepsilon^{\gamma\mu}(\varepsilon_{\dot\beta\dot\alpha}\varepsilon^{\dot\nu\dot\alpha} - \delta^{\dot\nu}{}_{\dot\beta}\delta^{\dot\alpha}{}_{\dot\alpha})F)\xi_{\gamma\mu\dot\gamma\dot\nu} \\
&= -2E^{\gamma\mu\dot\alpha}{}_{(\dot\beta}\xi_{\gamma\mu\dot\gamma)\dot\alpha} \qquad \text{(C.67)}
\end{aligned}$$

Therefore, $[\nabla^{\mu}{}_{(\dot\beta}, \nabla^{\gamma\dot\alpha}]\xi_{\gamma\mu\dot\gamma)\dot\alpha} = -2E^{\gamma\mu\dot\alpha}{}_{(\dot\beta}\xi_{\gamma\mu\dot\gamma)\dot\alpha}$ and

$$\begin{aligned}
&\frac{1}{2}\nabla^{\gamma\dot\alpha}(\xi_{\gamma\dot\alpha\dot\beta\dot\gamma}) + \frac{1}{2}\nabla^{\mu}{}_{(\dot\gamma}\xi_{\mu\dot\beta)} + \xi_{\dot\beta\dot\gamma} \\
&= -\frac{1}{3}\nabla^{\gamma\dot\alpha}\nabla^{\mu}{}_{(\dot\beta}\xi_{\gamma\mu\dot\gamma\dot\alpha)} + \frac{2}{9}\nabla^{\gamma\dot\alpha}\nabla^{\mu}{}_{(\dot\beta}\xi_{\gamma\mu\dot\gamma)\dot\alpha} - \frac{7}{9}E^{\gamma\mu\dot\alpha}{}_{(\dot\beta}\xi_{\gamma\mu\dot\gamma)\dot\alpha}\,. \qquad \text{(C.68)} \\
&\nabla^{\gamma\dot\alpha}\nabla^{\mu}{}_{(\dot\beta}\xi_{\gamma\mu\dot\gamma)\dot\alpha} \\
&= \nabla^{\gamma\dot\alpha}\nabla^{\mu}{}_{(\dot\beta}\xi_{\gamma\mu\dot\gamma\dot\alpha)} + \frac{1}{3}\varepsilon_{\dot\alpha\dot\beta}\nabla^{\gamma\dot\alpha}\nabla^{\mu}{}_{(\dot\mu}\xi_{\gamma\mu\dot\gamma)}{}^{\dot\mu} + \frac{1}{3}\varepsilon_{\dot\alpha\dot\gamma}\nabla^{\gamma\dot\alpha}\nabla^{\mu}{}_{(\dot\beta}\xi_{\gamma\mu\dot\mu)}{}^{\dot\mu} \\
&= \nabla^{\gamma\dot\alpha}\nabla^{\mu}{}_{(\dot\beta}\xi_{\gamma\mu\dot\gamma\dot\alpha)} + \frac{1}{6}\nabla^{\gamma}{}_{\dot\beta}\nabla^{\mu\dot\alpha}\xi_{\gamma\mu\dot\gamma\dot\alpha} + \frac{1}{6}\nabla^{\gamma}{}_{\dot\gamma}\nabla^{\mu\dot\alpha}\xi_{\gamma\mu\dot\beta\dot\alpha} \\
&= \nabla^{\gamma\dot\alpha}\nabla^{\mu}{}_{(\dot\beta}\xi_{\gamma\mu\dot\gamma\dot\alpha)} + \frac{1}{3}\nabla^{\mu}{}_{(\dot\beta}\nabla^{\gamma\dot\alpha}\xi_{\gamma\mu\dot\gamma)\dot\alpha} \\
\implies &\frac{1}{2}\nabla^{\gamma\dot\alpha}(\xi_{\gamma\dot\alpha\dot\beta\dot\gamma}) + \frac{1}{2}\nabla^{\mu}{}_{(\dot\gamma}\xi_{\mu\dot\beta)} + \xi_{\dot\beta\dot\gamma} = -E^{\gamma\mu\dot\alpha}{}_{(\dot\beta}\xi_{\gamma\mu\dot\gamma)\dot\alpha} \qquad \text{(C.69)}
\end{aligned}$$

Next consider $\nabla_{(\alpha}{}^{\dot\alpha}\xi_{\beta)\dot\alpha\dot\beta\dot\gamma} + \nabla_{(\alpha(\dot\gamma}\xi_{\beta)\dot\beta)}$.

$$
\begin{aligned}
&\nabla_{(\alpha}{}^{\dot\alpha}\xi_{\beta)\dot\alpha\dot\beta\dot\gamma} + \nabla_{(\alpha(\dot\gamma}\xi_{\beta)\dot\beta)} \\
&= -\frac{2}{3}\nabla_{(\alpha}{}^{\dot\alpha}\nabla^{\gamma}{}_{(\dot\gamma}\xi_{\beta)\gamma\dot\beta\dot\alpha)} + \frac{8}{9}\nabla_{(\alpha(\dot\gamma}\nabla^{\gamma\dot\alpha}\xi_{\beta)\gamma\dot\beta)\dot\alpha} \\
&= -\frac{2}{3}\nabla_{(\alpha}{}^{\dot\alpha}\nabla^{\gamma}{}_{(\dot\gamma}\xi_{\beta)\gamma\dot\beta\dot\alpha)} + \frac{8}{9}\nabla^{\gamma\dot\alpha}\nabla_{(\alpha(\dot\gamma}\xi_{\beta)\gamma\dot\beta)\dot\alpha} + \frac{8}{9}[\nabla_{(\alpha(\dot\gamma},\nabla^{\gamma\dot\alpha}]\xi_{\beta)\gamma\dot\beta)\dot\alpha}
\end{aligned}
\tag{C.70}
$$

$$
\begin{aligned}
&\nabla^{\gamma\dot\alpha}\nabla_{(\alpha(\dot\gamma}\xi_{\beta)\gamma\dot\beta)\dot\alpha} \\
&= \nabla^{\gamma\dot\alpha}\nabla_{(\alpha(\dot\gamma}\xi_{\beta\gamma)\dot\beta\dot\alpha)} + \frac{1}{3}\varepsilon_{\gamma\alpha}\nabla^{\gamma\dot\alpha}\nabla_{(\mu(\dot\gamma}\xi_{\beta)}{}^{\mu}{}_{\dot\beta\dot\alpha)} + \frac{1}{3}\varepsilon_{\gamma\beta}\nabla^{\gamma\dot\alpha}\nabla_{(\alpha(\dot\gamma}\xi_{\mu)}{}^{\mu}{}_{\dot\beta\dot\alpha)} \\
&\quad + \frac{1}{3}\varepsilon_{\dot\alpha\dot\gamma}\nabla^{\gamma\dot\alpha}\nabla_{(\alpha(\dot\mu}\xi_{\beta\gamma)\dot\beta)}{}^{\dot\mu} + \frac{1}{3}\varepsilon_{\dot\alpha\dot\beta}\nabla^{\gamma\dot\alpha}\nabla_{(\alpha(\dot\gamma}\xi_{\beta\gamma)\dot\mu)}{}^{\dot\mu} + \frac{1}{9}\varepsilon_{\gamma\alpha}\varepsilon_{\dot\alpha\dot\gamma}\nabla^{\gamma\dot\alpha}\nabla_{(\mu(\dot\mu}\xi_{\beta)}{}^{\mu}{}_{\dot\beta)}{}^{\dot\mu} \\
&\quad + \frac{1}{9}\varepsilon_{\gamma\alpha}\varepsilon_{\dot\alpha\dot\beta}\nabla^{\gamma\dot\alpha}\nabla_{(\mu(\dot\gamma}\xi_{\beta)}{}^{\mu}{}_{\dot\mu)}{}^{\dot\mu} + \frac{1}{9}\varepsilon_{\gamma\beta}\varepsilon_{\dot\alpha\dot\gamma}\nabla^{\gamma\dot\alpha}\nabla_{(\alpha(\dot\mu}\xi_{\mu)}{}^{\mu}{}_{\dot\beta)}{}^{\dot\mu} + \frac{1}{9}\varepsilon_{\gamma\beta}\varepsilon_{\dot\alpha\dot\beta}\nabla^{\gamma\dot\alpha}\nabla_{(\alpha(\dot\gamma}\xi_{\mu)}{}^{\mu}{}_{\dot\mu)}{}^{\dot\mu} \\
&= \frac{1}{6}\nabla_{\alpha}{}^{\dot\alpha}\nabla^{\gamma}{}_{(\dot\gamma}\xi_{\beta\gamma\dot\beta\dot\alpha)} + \frac{1}{6}\nabla_{\beta}{}^{\dot\alpha}\nabla^{\gamma}{}_{(\dot\gamma}\xi_{\alpha\gamma\dot\beta\dot\alpha)} + \frac{1}{6}\nabla^{\gamma}{}_{\dot\gamma}\nabla_{(\alpha}{}^{\dot\alpha}\xi_{\beta\gamma)\dot\beta\dot\alpha} + \frac{1}{6}\nabla^{\gamma}{}_{\dot\beta}\nabla_{(\alpha}{}^{\dot\alpha}\xi_{\beta\gamma)\dot\gamma\dot\alpha} \\
&\quad + \frac{1}{36}\nabla_{\alpha\dot\gamma}\nabla^{\gamma\dot\alpha}\xi_{\beta\gamma\dot\beta\dot\alpha} + \frac{1}{36}\nabla_{\alpha\dot\beta}\nabla^{\gamma\dot\alpha}\xi_{\beta\gamma\dot\gamma\dot\alpha} + \frac{1}{36}\nabla_{\beta\dot\gamma}\nabla^{\gamma\dot\alpha}\xi_{\alpha\gamma\dot\beta\dot\alpha} + \frac{1}{36}\nabla_{\beta\dot\beta}\nabla^{\gamma\dot\alpha}\xi_{\alpha\gamma\dot\gamma\dot\alpha} \\
&= \frac{1}{3}\nabla_{(\alpha}{}^{\dot\alpha}\nabla^{\gamma}{}_{(\dot\gamma}\xi_{\beta)\gamma\dot\beta\dot\alpha)} + \frac{1}{3}\nabla^{\gamma}{}_{(\dot\gamma}\nabla_{(\alpha}{}^{\dot\alpha}\xi_{\beta\gamma)\dot\beta)\dot\alpha} + \frac{1}{9}\nabla^{\gamma\dot\alpha}\nabla_{(\alpha(\dot\gamma}\xi_{\beta)\gamma\dot\beta)\dot\alpha} + \frac{1}{9}[\nabla_{(\alpha(\dot\gamma},\nabla^{\gamma\dot\alpha}]\xi_{\beta)\gamma\dot\beta)\dot\alpha}
\end{aligned}
\tag{C.71}
$$

$$
\begin{aligned}
&\implies \nabla_{(\alpha}{}^{\dot\alpha}\xi_{\beta)\dot\alpha\dot\beta\dot\gamma} + \nabla_{(\alpha(\dot\gamma}\xi_{\beta)\dot\beta)} \\
&= -\frac{1}{3}\nabla_{(\alpha}{}^{\dot\alpha}\nabla^{\gamma}{}_{(\dot\gamma}\xi_{\beta)\gamma\dot\beta\dot\alpha)} + \frac{1}{3}\nabla^{\gamma}{}_{(\dot\gamma}\nabla_{(\alpha}{}^{\dot\alpha}\xi_{\beta\gamma)\dot\beta)\dot\alpha} + [\nabla_{(\alpha(\dot\gamma},\nabla^{\gamma\dot\alpha}]\xi_{\beta)\gamma\dot\beta)\dot\alpha}
\end{aligned}
\tag{C.72}
$$

$$
\begin{aligned}
&\nabla^{\gamma}{}_{(\dot\gamma}\nabla_{(\alpha}{}^{\dot\alpha}\xi_{\beta\gamma)\dot\beta)\dot\alpha} \\
&= \nabla^{\gamma}{}_{(\dot\gamma}\nabla_{(\alpha}{}^{\dot\alpha}\xi_{\beta)\gamma\dot\beta)\dot\alpha} - \frac{1}{3}\varepsilon_{\gamma\alpha}\nabla^{\gamma}{}_{(\dot\gamma}\nabla_{(\mu}{}^{\dot\alpha}\xi_{\beta)}{}^{\mu}{}_{\dot\beta)\dot\alpha} - \frac{1}{3}\varepsilon_{\gamma\beta}\nabla^{\gamma}{}_{(\dot\gamma}\nabla_{(\alpha}{}^{\dot\alpha}\xi_{\mu)}{}^{\mu}{}_{\dot\beta)\dot\alpha} \\
&= \nabla^{\gamma}{}_{(\dot\gamma}\nabla_{(\alpha}{}^{\dot\alpha}\xi_{\beta)\gamma\dot\beta\dot\alpha)} + \frac{1}{3}\varepsilon_{\dot\alpha\dot\gamma}\nabla^{\gamma}{}_{(\dot\mu}\nabla_{(\alpha}{}^{\dot\alpha}\xi_{\beta)\gamma\dot\beta)}{}^{\dot\mu} + \frac{1}{3}\varepsilon_{\dot\alpha\dot\beta}\nabla^{\gamma}{}_{(\dot\gamma}\nabla_{(\alpha}{}^{\dot\alpha}\xi_{\beta)\gamma\dot\mu)}{}^{\dot\mu} - \frac{1}{6}\nabla_{\alpha(\dot\gamma}\nabla^{\gamma\dot\alpha}\xi_{\beta\gamma\dot\beta)\dot\alpha} \\
&\quad - \frac{1}{6}\nabla_{\beta(\dot\gamma}\nabla^{\gamma\dot\alpha}\xi_{\alpha\gamma\dot\beta)\dot\alpha} \\
&= \nabla_{(\alpha}{}^{\dot\alpha}\nabla^{\gamma}{}_{(\dot\gamma}\xi_{\beta)\gamma\dot\beta\dot\alpha)} + [\nabla^{\gamma}{}_{(\dot\gamma},\nabla_{(\alpha}{}^{\dot\alpha}]\xi_{\beta)\gamma\dot\beta\dot\alpha)} + \frac{1}{6}\nabla^{\gamma\dot\alpha}\nabla_{(\alpha\dot\gamma}\xi_{\beta)\gamma\dot\beta\dot\alpha} + \frac{1}{6}\nabla^{\gamma\dot\alpha}\nabla_{(\alpha\dot\beta}\xi_{\beta)\gamma\dot\gamma\dot\alpha} \\
&\quad - \frac{1}{3}\nabla_{(\alpha(\dot\gamma}\nabla^{\gamma\dot\alpha}\xi_{\beta)\gamma\dot\beta)\dot\alpha} \\
&= \nabla_{(\alpha}{}^{\dot\alpha}\nabla^{\gamma}{}_{(\dot\gamma}\xi_{\beta)\gamma\dot\beta\dot\alpha)} + [\nabla^{\gamma}{}_{(\dot\gamma},\nabla_{(\alpha}{}^{\dot\alpha}]\xi_{\beta)\gamma\dot\beta\dot\alpha)} + \frac{1}{3}\nabla^{\gamma\dot\alpha}\nabla_{(\alpha(\dot\gamma}\xi_{\beta)\gamma\dot\beta)\dot\alpha} - \frac{1}{3}\nabla_{(\alpha(\dot\gamma}\nabla^{\gamma\dot\alpha}\xi_{\beta)\gamma\dot\beta)\dot\alpha} \\
&= \nabla_{(\alpha}{}^{\dot\alpha}\nabla^{\gamma}{}_{(\dot\gamma}\xi_{\beta)\gamma\dot\beta\dot\alpha)} + [\nabla^{\gamma}{}_{(\dot\gamma},\nabla_{(\alpha}{}^{\dot\alpha}]\xi_{\beta)\gamma\dot\beta\dot\alpha)} - \frac{1}{3}[\nabla_{(\alpha(\dot\gamma},\nabla^{\gamma\dot\alpha}]\xi_{\beta)\gamma\dot\beta)\dot\alpha}
\end{aligned}
\tag{C.73}
$$

$$
\implies \nabla_{(\alpha}{}^{\dot\alpha}\xi_{\beta)\dot\alpha\dot\beta\dot\gamma} + \nabla_{(\alpha(\dot\gamma}\xi_{\beta)\dot\beta)} = \frac{1}{3}[\nabla^{\gamma}{}_{(\dot\gamma},\nabla_{(\alpha}{}^{\dot\alpha}]\xi_{\beta)\gamma\dot\beta\dot\alpha)} + \frac{8}{9}[\nabla_{(\alpha(\dot\gamma},\nabla^{\gamma\dot\alpha}]\xi_{\beta)\gamma\dot\beta)\dot\alpha}
\tag{C.74}
$$

$$
\begin{aligned}
&[\nabla_{\alpha\dot\gamma},\nabla^{\gamma\dot\alpha}]\xi_{\beta\gamma\dot\beta\dot\alpha} \\
&= (R_{\alpha\dot\gamma}{}^{\gamma\dot\alpha\mu\nu}M_{\mu\nu} + \overline{R}_{\alpha\dot\gamma}{}^{\gamma\dot\alpha\dot\mu\dot\nu}\overline{M}_{\dot\mu\dot\nu})\xi_{\beta\gamma\dot\beta\dot\alpha} \\
&= R_{\alpha\dot\gamma}{}^{\gamma\dot\alpha}{}_{\beta}{}^{\mu}\xi_{\mu\gamma\dot\beta\dot\alpha} + R_{\alpha\dot\gamma}{}^{\gamma\dot\alpha}{}_{\gamma}{}^{\mu}\xi_{\beta\mu\dot\beta\dot\alpha} + \overline{R}_{\alpha\dot\gamma}{}^{\gamma\dot\alpha}{}_{\dot\beta}{}^{\dot\mu}\xi_{\beta\gamma\dot\mu\dot\alpha} + \overline{R}_{\alpha\dot\gamma}{}^{\gamma\dot\alpha}{}_{\dot\alpha}{}^{\dot\mu}\xi_{\beta\gamma\dot\beta\dot\mu} \\
&= (-\delta^{\dot\alpha}{}_{\dot\gamma}C_{\alpha}{}^{\gamma}{}_{\beta}{}^{\mu} - \delta^{\gamma}{}_{\alpha}E_{\beta}{}^{\mu}{}_{\dot\gamma}{}^{\dot\alpha} - \delta^{\dot\alpha}{}_{\dot\gamma}(\varepsilon_{\alpha\beta}\varepsilon^{\mu\gamma} - \delta^{\mu}{}_{\alpha}\delta^{\gamma}{}_{\beta})F)\xi_{\mu\gamma\dot\beta\dot\alpha} \\
&\quad + (-\delta^{\dot\alpha}{}_{\dot\gamma}C_{\alpha}{}^{\gamma}{}_{\gamma}{}^{\mu} - \delta^{\gamma}{}_{\alpha}E_{\gamma}{}^{\mu}{}_{\dot\gamma}{}^{\dot\alpha} - \delta^{\dot\alpha}{}_{\dot\gamma}(\varepsilon_{\alpha\gamma}\varepsilon^{\mu\gamma} - \delta^{\mu}{}_{\alpha}\delta^{\gamma}{}_{\gamma})F)\xi_{\beta\mu\dot\beta\dot\alpha} \\
&\quad + (-\delta^{\gamma}{}_{\alpha}\overline{C}_{\dot\gamma}{}^{\dot\alpha}{}_{\dot\beta}{}^{\dot\mu} - \delta^{\dot\alpha}{}_{\dot\gamma}E_{\alpha}{}^{\gamma}{}_{\dot\beta}{}^{\dot\mu} - \delta^{\gamma}{}_{\alpha}(\varepsilon_{\dot\gamma\dot\beta}\varepsilon^{\dot\mu\dot\alpha} - \delta^{\dot\mu}{}_{\dot\gamma}\delta^{\dot\alpha}{}_{\dot\beta})F)\xi_{\beta\gamma\dot\mu\dot\alpha} \\
&\quad + (-\delta^{\gamma}{}_{\alpha}\overline{C}_{\dot\gamma}{}^{\dot\alpha}{}_{\dot\alpha}{}^{\dot\mu} - \delta^{\dot\alpha}{}_{\dot\gamma}E_{\alpha}{}^{\gamma}{}_{\dot\alpha}{}^{\dot\mu} - \delta^{\gamma}{}_{\alpha}(\varepsilon_{\dot\gamma\dot\alpha}\varepsilon^{\dot\mu\dot\alpha} - \delta^{\dot\mu}{}_{\dot\gamma}\delta^{\dot\alpha}{}_{\dot\alpha})F)\xi_{\beta\gamma\dot\beta\dot\mu}
\end{aligned}
\tag{C.75}
$$

$$[\nabla_{\alpha\dot\gamma}, \nabla^{\gamma\dot\alpha}]\xi_{\beta\gamma\dot\beta\dot\alpha}$$

$$= -C_{\alpha\beta}{}^{\gamma\mu}\xi_{\gamma\mu\dot\beta\dot\gamma} - E_\beta{}^\gamma{}_{\dot\gamma}{}^{\dot\alpha}\xi_{\alpha\gamma\dot\beta\dot\alpha} + F\xi_{\alpha\beta\dot\beta\dot\gamma} - E_\alpha{}^\gamma{}_{\dot\gamma}{}^{\dot\alpha}\xi_{\beta\gamma\dot\beta\dot\alpha} + 3F\xi_{\beta\alpha\dot\beta\dot\gamma} - \overline{C}_{\dot\gamma}{}^{\dot\alpha}{}_{\dot\beta}{}^{\dot\mu}\xi_{\beta\alpha\dot\mu\dot\alpha} - E_\alpha{}^\gamma{}_{\dot\beta}{}^{\dot\alpha}\xi_{\beta\gamma\dot\alpha\dot\gamma}$$

$$+ F\xi_{\beta\alpha\dot\gamma\dot\beta} - E_\alpha{}^\gamma{}_{\dot\gamma}{}^{\dot\alpha}\xi_{\beta\gamma\dot\alpha\dot\gamma} + 3F\xi_{\beta\alpha\dot\beta\dot\gamma}$$

$$= -C_{\alpha\beta}{}^{\gamma\mu}\xi_{\gamma\mu\dot\beta\dot\gamma} - \overline{C}_{\dot\gamma}{}^{\dot\alpha}{}_{\dot\beta}{}^{\dot\mu}\xi_{\beta\alpha\dot\mu\dot\alpha} - E_\beta{}^\gamma{}_{\dot\gamma}{}^{\dot\alpha}\xi_{\alpha\gamma\dot\beta\dot\alpha} - E_\alpha{}^\gamma{}_{\dot\gamma}{}^{\dot\alpha}\xi_{\beta\gamma\dot\beta\dot\alpha} - E_\alpha{}^\gamma{}_{\dot\beta}{}^{\dot\alpha}\xi_{\beta\gamma\dot\alpha\dot\gamma}$$

$$- E_\alpha{}^\gamma{}_{\dot\gamma}{}^{\dot\alpha}\xi_{\beta\gamma\dot\alpha\dot\gamma} + 8F\xi_{\alpha\beta\dot\beta\dot\gamma} \tag{C.76}$$

$$\implies [\nabla_{(\alpha(\dot\gamma}, \nabla^{\gamma\dot\alpha}]\xi_{\beta)\gamma\dot\beta)\dot\alpha} = -C_{\alpha\beta}{}^{\gamma\mu}\xi_{\gamma\mu\dot\beta\dot\gamma} - \overline{C}_{\dot\beta}{}^{\dot\alpha}{}_{\dot\gamma}{}^{\dot\mu}\xi_{\alpha\beta\dot\alpha\dot\mu} - 4E_{(\alpha}{}^\gamma{}_{(\dot\beta}{}^{\dot\alpha}\xi_{\beta)\gamma\dot\gamma)\dot\alpha} + 8F\xi_{\alpha\beta\dot\beta\dot\gamma} \tag{C.77}$$

$$[\nabla^\gamma{}_{\dot\beta}, \nabla_\alpha{}^{\dot\alpha}]\xi_{\beta\gamma\dot\alpha\dot\gamma}$$

$$= (R^\gamma{}_{\dot\beta\alpha}{}^{\dot\alpha\mu\nu}M_{\mu\nu} + \overline{R}^\gamma{}_{\dot\beta\alpha}{}^{\dot\alpha\dot\mu\dot\nu}\overline{M}_{\dot\mu\dot\nu})\xi_{\beta\gamma\dot\alpha\dot\gamma}$$

$$= R^\gamma{}_{\dot\beta\alpha}{}^{\dot\alpha}{}_\beta{}^\mu\xi_{\mu\gamma\dot\alpha\dot\gamma} + R^\gamma{}_{\dot\beta\alpha}{}^{\dot\alpha}{}_\gamma{}^\mu\xi_{\beta\mu\dot\alpha\dot\gamma} + \overline{R}^\gamma{}_{\dot\beta\alpha}{}^{\dot\alpha}{}_{\dot\alpha}{}^{\dot\mu}\xi_{\beta\gamma\dot\mu\dot\gamma} + \overline{R}^\gamma{}_{\dot\beta\alpha}{}^{\dot\alpha}{}_{\dot\gamma}{}^{\dot\mu}\xi_{\beta\gamma\dot\alpha\dot\mu}$$

$$= (-\delta^{\dot\alpha}{}_{\dot\beta}C^\gamma{}_{\alpha\beta}{}^\mu + \delta^\gamma{}_\alpha E_\beta{}^\mu{}_{\dot\beta}{}^{\dot\alpha} - \delta^{\dot\alpha}{}_{\dot\beta}(-\delta^\gamma{}_\beta\delta^\mu{}_\alpha + \varepsilon_{\alpha\beta}\varepsilon^{\mu\gamma})F)\xi_{\mu\gamma\dot\alpha\dot\gamma}$$

$$+ (-\delta^{\dot\alpha}{}_{\dot\beta}C^\gamma{}_{\alpha\gamma}{}^\mu + \delta^\gamma{}_\alpha E_\gamma{}^\mu{}_{\dot\beta}{}^{\dot\alpha} - \delta^{\dot\alpha}{}_{\dot\beta}(-\delta^\gamma{}_\gamma\delta^\mu{}_\alpha + \varepsilon_{\alpha\gamma}\varepsilon^{\mu\gamma})F)\xi_{\beta\mu\dot\alpha\dot\gamma}$$

$$+ (\delta^\gamma{}_\alpha\overline{C}_{\dot\beta}{}^{\dot\alpha}{}_{\dot\alpha}{}^{\dot\mu} - \delta^{\dot\alpha}{}_{\dot\beta}E^\gamma{}_{\alpha\dot\alpha}{}^{\dot\mu} + \delta^\gamma{}_\alpha(\varepsilon_{\dot\beta\dot\alpha}\varepsilon^{\dot\mu\dot\alpha} - \delta^{\dot\mu}{}_{\dot\beta}\delta^{\dot\alpha}{}_{\dot\alpha})F)\xi_{\beta\gamma\dot\mu\dot\gamma}$$

$$+ (\delta^\gamma{}_\alpha\overline{C}_{\dot\beta}{}^{\dot\alpha}{}_{\dot\gamma}{}^{\dot\mu} - \delta^{\dot\alpha}{}_{\dot\beta}E^\gamma{}_{\alpha\dot\gamma}{}^{\dot\mu} + \delta^\gamma{}_\alpha(\varepsilon_{\dot\beta\dot\gamma}\varepsilon^{\dot\mu\dot\alpha} - \delta^{\dot\mu}{}_{\dot\beta}\delta^{\dot\alpha}{}_{\dot\gamma})F)\xi_{\beta\gamma\dot\alpha\dot\mu}$$

$$= -C_{\alpha\beta}{}^{\gamma\mu}\xi_{\gamma\mu\dot\beta\dot\gamma} + E_\beta{}^\gamma{}_{\dot\beta}{}^{\dot\alpha}\xi_{\alpha\gamma\dot\alpha\dot\gamma} + F\xi_{\alpha\beta\dot\beta\dot\gamma} + E_\alpha{}^\gamma{}_{\dot\beta}{}^{\dot\alpha}\xi_{\beta\gamma\dot\alpha\dot\gamma} + 3F\xi_{\alpha\beta\dot\beta\dot\gamma} - E_\alpha{}^\gamma{}_{\dot\beta}{}^{\dot\alpha}\xi_{\beta\gamma\dot\alpha\dot\gamma}$$

$$- 3F\xi_{\alpha\beta\dot\beta\dot\gamma} + \overline{C}_{\dot\beta\dot\gamma}{}^{\dot\alpha\dot\mu}\xi_{\alpha\beta\dot\alpha\dot\mu} - E_\alpha{}^\gamma{}_{\dot\gamma}{}^{\dot\alpha}\xi_{\beta\gamma\dot\alpha\dot\beta} - F\xi_{\alpha\beta\dot\beta\dot\gamma}$$

$$= \overline{C}_{\dot\beta\dot\gamma}{}^{\dot\alpha\dot\mu}\xi_{\alpha\beta\dot\alpha\dot\mu} - C_{\alpha\beta}{}^{\gamma\mu}\xi_{\gamma\mu\dot\beta\dot\gamma} + E_\beta{}^\gamma{}_{\dot\beta}{}^{\dot\alpha}\xi_{\alpha\gamma\dot\alpha\dot\gamma} - E_\alpha{}^\gamma{}_{\dot\gamma}{}^{\dot\alpha}\xi_{\beta\gamma\dot\alpha\dot\beta} \tag{C.78}$$

$$\implies [\nabla^\gamma{}_{\dot\beta}, \nabla_{(\alpha}{}^{\dot\alpha}]\xi_{\beta)\gamma\dot\alpha\dot\gamma}$$

$$= \overline{C}_{\dot\beta\dot\gamma}{}^{\dot\alpha\dot\mu}\xi_{\alpha\beta\dot\alpha\dot\mu} - C_{\alpha\beta}{}^{\gamma\mu}\xi_{\gamma\mu\dot\beta\dot\gamma} + \frac{1}{2}E_\beta{}^\gamma{}_{\dot\beta}{}^{\dot\alpha}\xi_{\alpha\gamma\dot\alpha\dot\gamma} + \frac{1}{2}E_\alpha{}^\gamma{}_{\dot\beta}{}^{\dot\alpha}\xi_{\beta\gamma\dot\alpha\dot\gamma} - \frac{1}{2}E_\alpha{}^\gamma{}_{\dot\gamma}{}^{\dot\alpha}\xi_{\beta\gamma\dot\alpha\dot\beta}$$

$$- \frac{1}{2}E_\beta{}^\gamma{}_{\dot\gamma}{}^{\dot\alpha}\xi_{\alpha\gamma\dot\alpha\dot\beta}$$

$$= \overline{C}_{\dot\beta\dot\gamma}{}^{\dot\alpha\dot\mu}\xi_{\alpha\beta\dot\alpha\dot\mu} - C_{\alpha\beta}{}^{\gamma\mu}\xi_{\gamma\mu\dot\beta\dot\gamma} + 2E_{(\beta}{}^\gamma{}_{[\dot\beta}{}^{\dot\alpha}\xi_{\alpha)\gamma\dot\gamma]\dot\alpha} \tag{C.79}$$

$$[\nabla^\gamma{}_{\dot\alpha}, \nabla_\alpha{}^{\dot\alpha}]\xi_{\beta\gamma\dot\beta\dot\gamma}$$

$$= (R^\gamma{}_{\dot\alpha\alpha}{}^{\dot\alpha\mu\nu}M_{\mu\nu} + \overline{R}^\gamma{}_{\dot\alpha\alpha}{}^{\dot\alpha\dot\mu\dot\nu}\overline{M}_{\dot\mu\dot\nu})\xi_{\beta\gamma\dot\beta\dot\gamma}$$

$$= R^\gamma{}_{\dot\alpha\alpha}{}^{\dot\alpha}{}_\beta{}^\mu\xi_{\mu\gamma\dot\beta\dot\gamma} + R^\gamma{}_{\dot\alpha\alpha}{}^{\dot\alpha}{}_\gamma{}^\mu\xi_{\beta\mu\dot\beta\dot\gamma} + \overline{R}^\gamma{}_{\dot\alpha\alpha}{}^{\dot\alpha}{}_{\dot\beta}{}^{\dot\mu}\xi_{\beta\gamma\dot\mu\dot\gamma} + \overline{R}^\gamma{}_{\dot\alpha\alpha}{}^{\dot\alpha}{}_{\dot\gamma}{}^{\dot\mu}\xi_{\beta\gamma\dot\beta\dot\mu}$$

$$= (-\delta^{\dot\alpha}{}_{\dot\alpha}C^\gamma{}_{\alpha\beta}{}^\mu + \delta^\gamma{}_\alpha E_\beta{}^\mu{}_{\dot\alpha}{}^{\dot\alpha} - \delta^{\dot\alpha}{}_{\dot\alpha}(-\delta^\gamma{}_\beta\delta^\mu{}_\alpha + \varepsilon_{\alpha\beta}\varepsilon^{\mu\gamma})F)\xi_{\mu\gamma\dot\beta\dot\gamma}$$

$$+ (-\delta^{\dot\alpha}{}_{\dot\alpha}C^\gamma{}_{\alpha\gamma}{}^\mu + \delta^\gamma{}_\alpha E_\gamma{}^\mu{}_{\dot\alpha}{}^{\dot\alpha} - \delta^{\dot\alpha}{}_{\dot\alpha}(-\delta^\gamma{}_\gamma\delta^\mu{}_\alpha + \varepsilon_{\alpha\gamma}\varepsilon^{\mu\gamma})F)\xi_{\beta\mu\dot\beta\dot\gamma}$$

$$+ (\delta^\gamma{}_\alpha\overline{C}_{\dot\alpha}{}^{\dot\alpha}{}_{\dot\beta}{}^{\dot\mu} - \delta^{\dot\alpha}{}_{\dot\alpha}E^\gamma{}_{\alpha\dot\beta}{}^{\dot\mu} + \delta^\gamma{}_\alpha(\varepsilon_{\dot\alpha\dot\beta}\varepsilon^{\dot\mu\dot\alpha} - \delta^{\dot\mu}{}_{\dot\alpha}\delta^{\dot\alpha}{}_{\dot\beta})F)\xi_{\beta\gamma\dot\mu\dot\gamma}$$

$$+ (\delta^\gamma{}_\alpha\overline{C}_{\dot\alpha}{}^{\dot\alpha}{}_{\dot\gamma}{}^{\dot\mu} - \delta^{\dot\alpha}{}_{\dot\alpha}E^\gamma{}_{\alpha\dot\gamma}{}^{\dot\mu} + \delta^\gamma{}_\alpha(\varepsilon_{\dot\alpha\dot\gamma}\varepsilon^{\dot\mu\dot\alpha} - \delta^{\dot\mu}{}_{\dot\alpha}\delta^{\dot\alpha}{}_{\dot\gamma})F)\xi_{\beta\gamma\dot\beta\dot\mu}$$

$$= -2C_{\alpha\beta}{}^{\gamma\mu}\xi_{\gamma\mu\dot\beta\dot\gamma} - 2E_\alpha{}^\gamma{}_{\dot\beta}{}^{\dot\alpha}\xi_{\beta\gamma\dot\alpha\dot\gamma} - 2E_\alpha{}^\gamma{}_{\dot\gamma}{}^{\dot\alpha}\xi_{\beta\gamma\dot\alpha\dot\beta} + 8F\xi_{\alpha\beta\dot\beta\dot\gamma}$$

$$= -2C_{\alpha\beta}{}^{\gamma\mu}\xi_{\gamma\mu\dot\beta\dot\gamma} - 4E_\alpha{}^\gamma{}_{(\dot\beta}{}^{\dot\alpha}\xi_{\beta\gamma\dot\gamma)\dot\alpha} + 8F\xi_{\alpha\beta\dot\beta\dot\gamma} \tag{C.80}$$

$$\implies [\nabla^\gamma{}_{\dot\alpha}, \nabla_{(\alpha}{}^{\dot\alpha}]\xi_{\beta)\gamma\dot\beta\dot\gamma} = -2C_{\alpha\beta}{}^{\gamma\mu}\xi_{\gamma\mu\dot\beta\dot\gamma} - 4E_{(\alpha}{}^\gamma{}_{(\dot\beta}{}^{\dot\alpha}\xi_{\beta)\gamma\dot\gamma)\dot\alpha} + 8F\xi_{\alpha\beta\dot\beta\dot\gamma} \tag{C.81}$$

$$\implies [\nabla^\gamma{}_{(\dot\beta}, \nabla_{(\alpha}{}^{\dot\alpha}]\xi_{\beta)\gamma\dot\alpha\dot\gamma)}$$

$$= \frac{1}{3}(2\overline{C}_{\dot\beta\dot\gamma}{}^{\dot\alpha\dot\mu}\xi_{\alpha\beta\dot\alpha\dot\mu} - 2C_{\alpha\beta}{}^{\gamma\mu}\xi_{\gamma\mu\dot\beta\dot\gamma} + 2E_{(\beta}{}^\gamma{}_{[\dot\beta}{}^{\dot\alpha}\xi_{\alpha)\gamma\dot\gamma]\dot\alpha} + 2E_{(\beta}{}^\gamma{}_{[\dot\gamma}{}^{\dot\alpha}\xi_{\alpha)\gamma\dot\beta]\dot\alpha} - 2C_{\alpha\beta}{}^{\gamma\mu}\xi_{\gamma\mu\dot\beta\dot\gamma}$$

$$- 4E_{(\alpha}{}^\gamma{}_{(\dot\beta}{}^{\dot\alpha}\xi_{\beta)\gamma\dot\gamma)\dot\alpha} + 8F\xi_{\alpha\beta\dot\beta\dot\gamma})$$

$$= \frac{1}{3}(2\overline{C}_{\dot\beta\dot\gamma}{}^{\dot\alpha\dot\mu}\xi_{\alpha\beta\dot\alpha\dot\mu} - 4C_{\alpha\beta}{}^{\gamma\mu}\xi_{\gamma\mu\dot\beta\dot\gamma} - 4E_{(\alpha}{}^\gamma{}_{(\dot\beta}{}^{\dot\alpha}\xi_{\beta)\gamma\dot\gamma)\dot\alpha} + 8F\xi_{\alpha\beta\dot\beta\dot\gamma}) \tag{C.82}$$

Putting the results for $[\nabla^{\gamma}{}_{(\dot\beta}, \nabla_{(\alpha}{}^{\dot\alpha}]\xi_{\beta)\gamma\dot\alpha\dot\gamma)}$ and $[\nabla_{(\alpha(\dot\gamma}, \nabla^{\gamma\dot\alpha}]\xi_{\beta)\gamma\dot\beta)\dot\alpha}$ back into equation C.74,

$$\begin{aligned}
&\nabla_{(\alpha}{}^{\dot\alpha}\xi_{\beta)\dot\alpha\dot\beta\dot\gamma} + \nabla_{(\alpha(\dot\gamma}\xi_{\beta)\dot\beta)}\\
&= \frac{1}{9}(2\bar C_{\dot\beta\dot\gamma}{}^{\dot\alpha\dot\mu}\xi_{\alpha\beta\dot\alpha\dot\mu} - 4C_{\alpha\beta}{}^{\gamma\mu}\xi_{\gamma\mu\dot\beta\dot\gamma} - 4E_{(\alpha}{}^{\gamma}{}_{(\dot\beta}{}^{\dot\alpha}\xi_{\beta)\gamma\dot\gamma)\dot\alpha} + 8F\xi_{\alpha\beta\dot\beta\dot\gamma})\\
&\quad + \frac{8}{9}(-C_{\alpha\beta}{}^{\gamma\mu}\xi_{\gamma\mu\dot\beta\dot\gamma} - \bar C_{\dot\beta}{}^{\dot\alpha}{}_{\dot\gamma}{}^{\dot\mu}\xi_{\alpha\beta\dot\alpha\dot\mu} - 4E_{(\alpha}{}^{\gamma}{}_{(\dot\beta}{}^{\dot\alpha}\xi_{\beta)\gamma\dot\gamma)\dot\alpha} + 8F\xi_{\alpha\beta\dot\beta\dot\gamma})\\
&= -\frac{2}{3}\bar C_{\dot\beta\dot\gamma}{}^{\dot\alpha\dot\mu}\xi_{\alpha\beta\dot\alpha\dot\mu} - \frac{4}{3}C_{\alpha\beta}{}^{\gamma\mu}\xi_{\gamma\mu\dot\beta\dot\gamma} - 4E_{(\alpha}{}^{\gamma}{}_{(\dot\beta}{}^{\dot\alpha}\xi_{\beta)\gamma\dot\gamma)\dot\alpha} + 8F\xi_{\alpha\beta\dot\beta\dot\gamma}\,.
\end{aligned}\tag{C.83}$$

The only other terms in equation C.61 without a curvature factor are $\nabla_{\alpha\dot\alpha}(\xi^{\dot\alpha\dot\beta})\bar\chi_{\dot\beta}$ and $\nabla_{\alpha\dot\alpha}(\xi)\bar\chi^{\dot\alpha}$.

$$\begin{aligned}
&\nabla_{\alpha\dot\alpha}(\xi^{\dot\alpha\dot\beta})\bar\chi_{\dot\beta} + \nabla_{\alpha\dot\alpha}(\xi)\bar\chi^{\dot\alpha}\\
&= \Big(-\frac{2}{9}\nabla_{\alpha}{}^{\dot\beta}\nabla^{\beta}{}_{(\dot\alpha}\nabla^{\gamma\dot\gamma}\xi_{\beta\gamma\dot\beta)\dot\gamma} - \frac{1}{3}\nabla_{\alpha}{}^{\dot\beta}(E^{\beta\gamma\dot\gamma}{}_{(\dot\alpha}\xi_{\beta\gamma\dot\beta)\dot\gamma}) + \frac{2}{15}\nabla_{\alpha\dot\alpha}\nabla_{\beta\dot\beta}\nabla_{\gamma\dot\gamma}(\xi^{\beta\gamma\dot\beta\dot\gamma})\\
&\quad - \frac{7}{10}\nabla_{\alpha\dot\alpha}(E_{\beta\gamma\dot\beta\dot\gamma}\xi^{\beta\gamma\dot\beta\dot\gamma})\Big)\bar\chi^{\dot\alpha}
\end{aligned}\tag{C.84}$$

$$\begin{aligned}
&\nabla_{\alpha\dot\alpha}\nabla_{\beta\dot\beta}\nabla_{\gamma\dot\gamma}\xi^{\beta\gamma\dot\beta\dot\gamma}\\
&= \nabla_{\beta\dot\beta}\nabla_{\gamma\dot\gamma}\nabla_{\alpha\dot\alpha}\xi^{\beta\gamma\dot\beta\dot\gamma} + [\nabla_{\alpha\dot\alpha}, \nabla_{\beta\dot\beta}\nabla_{\gamma\dot\gamma}]\xi^{\beta\gamma\dot\beta\dot\gamma}\\
&= \nabla^{\beta\dot\beta}\nabla^{\gamma\dot\gamma}\nabla_{\alpha\dot\alpha}\xi_{\beta\gamma\dot\beta\dot\gamma} + \nabla_{\beta\dot\beta}[\nabla_{\alpha\dot\alpha}, \nabla_{\gamma\dot\gamma}]\xi^{\beta\gamma\dot\beta\dot\gamma} + [\nabla_{\alpha\dot\alpha}, \nabla_{\beta\dot\beta}]\nabla_{\gamma\dot\gamma}\xi^{\beta\gamma\dot\beta\dot\gamma}
\end{aligned}\tag{C.85}$$

$$\begin{aligned}
&[\nabla_{\alpha\dot\alpha}, \nabla_{\gamma\dot\gamma}]\xi^{\beta\gamma\dot\beta\dot\gamma}\\
&= (R_{\alpha\dot\alpha\gamma\dot\gamma}{}^{\mu\nu}M_{\mu\nu} + \bar R_{\alpha\dot\alpha\gamma\dot\gamma}{}^{\dot\mu\dot\nu}\bar M_{\dot\mu\dot\nu})\xi^{\beta\gamma\dot\beta\dot\gamma}\\
&= -R_{\alpha\dot\alpha\gamma\dot\gamma}{}^{\beta}{}_{\mu}\xi^{\mu\gamma\dot\beta\dot\gamma} - R_{\alpha\dot\alpha\gamma\dot\gamma}{}^{\gamma}{}_{\mu}\xi^{\beta\mu\dot\beta\dot\gamma} - \bar R_{\alpha\dot\alpha\gamma\dot\gamma}{}^{\dot\beta}{}_{\dot\mu}\xi^{\beta\gamma\dot\mu\dot\gamma} - \bar R_{\alpha\dot\alpha\gamma\dot\gamma}{}^{\dot\gamma}{}_{\dot\mu}\xi^{\beta\gamma\dot\beta\dot\mu}\\
&= -(\varepsilon_{\dot\alpha\dot\gamma}C_{\alpha\gamma}{}^{\beta}{}_{\mu} + \varepsilon_{\alpha\gamma}E^{\beta}{}_{\mu\dot\alpha\dot\gamma} + \varepsilon_{\dot\alpha\dot\gamma}(-\delta^{\beta}{}_{\alpha}\varepsilon_{\gamma\mu} - \delta^{\beta}{}_{\gamma}\varepsilon_{\alpha\mu})F)\xi^{\mu\gamma\dot\beta\dot\gamma}\\
&\quad - (\varepsilon_{\dot\alpha\dot\gamma}C_{\alpha\gamma}{}^{\gamma}{}_{\mu} + \varepsilon_{\alpha\gamma}E^{\gamma}{}_{\mu\dot\alpha\dot\gamma} + \varepsilon_{\dot\alpha\dot\gamma}(-\delta^{\gamma}{}_{\alpha}\varepsilon_{\gamma\mu} - \delta^{\gamma}{}_{\gamma}\varepsilon_{\alpha\mu})F)\xi^{\beta\mu\dot\beta\dot\gamma}\\
&\quad - (\varepsilon_{\alpha\gamma}\bar C_{\dot\alpha\dot\gamma}{}^{\dot\beta}{}_{\dot\mu} + \varepsilon_{\dot\alpha\dot\gamma}E_{\alpha\gamma}{}^{\dot\beta}{}_{\dot\mu} + \varepsilon_{\alpha\gamma}(-\delta^{\dot\beta}{}_{\dot\alpha}\varepsilon_{\dot\gamma\dot\mu} - \delta^{\dot\beta}{}_{\dot\gamma}\varepsilon_{\dot\alpha\dot\mu})F)\xi^{\beta\gamma\dot\mu\dot\gamma}\\
&\quad - (\varepsilon_{\alpha\gamma}\bar C_{\dot\alpha\dot\gamma}{}^{\dot\gamma}{}_{\dot\mu} + \varepsilon_{\dot\alpha\dot\gamma}E_{\alpha\gamma}{}^{\dot\gamma}{}_{\dot\mu} + \varepsilon_{\alpha\gamma}(-\delta^{\dot\gamma}{}_{\dot\alpha}\varepsilon_{\dot\gamma\dot\mu} - \delta^{\dot\gamma}{}_{\dot\gamma}\varepsilon_{\dot\alpha\dot\mu})F)\xi^{\beta\gamma\dot\beta\dot\mu}\\
&= -C_{\alpha}{}^{\beta\gamma\mu}\xi_{\gamma\mu\dot\alpha}{}^{\dot\beta} + E^{\beta\gamma}{}_{\dot\alpha\dot\gamma}\xi_{\alpha\gamma}{}^{\dot\beta\dot\gamma} + F\xi_{\alpha}{}^{\beta}{}_{\dot\alpha}{}^{\dot\beta} - E_{\alpha\gamma\dot\alpha\dot\gamma}\xi^{\beta\gamma\dot\beta\dot\gamma} + 3F\xi_{\alpha}{}^{\beta}{}_{\dot\alpha}{}^{\dot\beta} - \bar C_{\dot\alpha}{}^{\dot\beta\dot\gamma\dot\mu}\xi^{\beta}{}_{\alpha\dot\gamma\dot\mu}\\
&\quad + E_{\alpha\gamma}{}^{\dot\beta\dot\gamma}\xi^{\beta\gamma}{}_{\dot\alpha\dot\gamma} + F\xi_{\alpha}{}^{\beta}{}_{\dot\alpha}{}^{\dot\beta} - E_{\alpha\gamma\dot\alpha\dot\gamma}\xi^{\beta\gamma\dot\beta\dot\gamma} + 3F\xi_{\alpha}{}^{\beta}{}_{\dot\alpha}{}^{\dot\beta}
\end{aligned}\tag{C.86}$$

Therefore,

$$\begin{aligned}
&\nabla_{\beta\dot\beta}[\nabla_{\alpha\dot\alpha}, \nabla_{\gamma\dot\gamma}]\xi^{\beta\gamma\dot\beta\dot\gamma}\\
&= \nabla^{\beta\dot\beta}(-C_{\alpha\beta}{}^{\gamma\mu}\xi_{\gamma\mu\dot\alpha\dot\beta} - \bar C_{\dot\alpha\dot\beta}{}^{\dot\gamma\dot\mu}\xi_{\alpha\beta\dot\gamma\dot\mu} - 2E^{\gamma}{}_{(\alpha}{}^{\dot\gamma}{}_{\dot\alpha}\xi_{\beta)\gamma\dot\beta\dot\gamma} - 2E^{\gamma}{}_{\alpha}{}^{\dot\gamma}{}_{(\dot\alpha}\xi_{\beta\gamma\dot\beta)\dot\gamma} + 8F\xi_{\alpha\beta\dot\alpha\dot\beta})
\end{aligned}\tag{C.87}$$

$$\begin{aligned}
&[\nabla_{\alpha\dot\alpha}, \nabla_{\beta\dot\beta}]\nabla_{\gamma\dot\gamma}\xi^{\beta\gamma\dot\beta\dot\gamma}\\
&= (R_{\alpha\dot\alpha\beta\dot\beta}{}^{\mu\nu}M_{\mu\nu} + \bar R_{\alpha\dot\alpha\beta\dot\beta}{}^{\dot\mu\dot\nu}\bar M_{\dot\mu\dot\nu})\nabla_{\gamma\dot\gamma}\xi^{\beta\gamma\dot\beta\dot\gamma}\\
&= -R_{\alpha\dot\alpha\beta\dot\beta}{}^{\beta}{}_{\mu}\nabla_{\gamma\dot\gamma}\xi^{\mu\gamma\dot\beta\dot\gamma} - \bar R_{\alpha\dot\alpha\beta\dot\beta}{}^{\dot\beta}{}_{\dot\mu}\nabla_{\gamma\dot\gamma}\xi^{\beta\gamma\dot\mu\dot\gamma}\\
&= -(\varepsilon_{\dot\alpha\dot\beta}C_{\alpha\beta}{}^{\beta}{}_{\mu} + \varepsilon_{\alpha\beta}E^{\beta}{}_{\mu\dot\alpha\dot\beta} + \varepsilon_{\dot\alpha\dot\beta}(-\delta^{\beta}{}_{\alpha}\varepsilon_{\beta\mu} - \delta^{\beta}{}_{\beta}\varepsilon_{\alpha\mu})F)\nabla_{\gamma\dot\gamma}\xi^{\mu\gamma\dot\beta\dot\gamma}\\
&\quad - (\varepsilon_{\alpha\beta}\bar C_{\dot\alpha\dot\beta}{}^{\dot\beta}{}_{\dot\mu} + \varepsilon_{\dot\alpha\dot\beta}E_{\alpha\beta}{}^{\dot\beta}{}_{\dot\mu} + \varepsilon_{\alpha\beta}(-\delta^{\dot\beta}{}_{\dot\alpha}\varepsilon_{\dot\beta\dot\mu} - \delta^{\dot\beta}{}_{\dot\beta}\varepsilon_{\dot\alpha\dot\mu})F)\nabla_{\gamma\dot\gamma}\xi^{\beta\gamma\dot\mu\dot\gamma}\\
&= -2E^{\gamma}{}_{\alpha}{}^{\dot\gamma}{}_{\dot\alpha}\nabla^{\beta\dot\beta}\xi_{\beta\gamma\dot\beta\dot\gamma} + 6F\nabla^{\beta\dot\beta}\xi_{\alpha\beta\dot\alpha\dot\beta}
\end{aligned}\tag{C.88}$$

$$
\begin{aligned}
&\nabla^{\beta\dot{\beta}}\nabla^{\gamma\dot{\gamma}}\nabla_{\alpha\dot{\alpha}}\xi_{\beta\gamma\dot{\beta}\dot{\gamma}} \\
&= \nabla^{\beta\dot{\beta}}\nabla^{\gamma\dot{\gamma}}\nabla_{(\alpha(\dot{\alpha}}\xi_{\beta\gamma)\dot{\beta}\dot{\gamma})} + \frac{1}{3}\varepsilon_{\alpha\beta}\nabla^{\beta\dot{\beta}}\nabla^{\gamma\dot{\gamma}}\nabla^{\mu}{}_{(\dot{\alpha}}\xi_{\mu\gamma\dot{\beta}\dot{\gamma})} + \frac{1}{3}\varepsilon_{\alpha\gamma}\nabla^{\beta\dot{\beta}}\nabla^{\gamma\dot{\gamma}}\nabla^{\mu}{}_{(\dot{\alpha}}\xi_{\beta\mu\dot{\beta}\dot{\gamma})} \\
&\quad + \frac{1}{3}\varepsilon_{\dot{\alpha}\dot{\beta}}\nabla^{\beta\dot{\beta}}\nabla^{\gamma\dot{\gamma}}\nabla_{(\alpha}{}^{\dot{\mu}}\xi_{\beta\gamma)\dot{\mu}\dot{\gamma}} + \frac{1}{3}\varepsilon_{\dot{\alpha}\dot{\gamma}}\nabla^{\beta\dot{\beta}}\nabla^{\gamma\dot{\gamma}}\nabla_{(\alpha}{}^{\dot{\mu}}\xi_{\beta\gamma)\dot{\beta}\dot{\mu}} + \frac{1}{9}\varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}}\nabla^{\beta\dot{\beta}}\nabla^{\gamma\dot{\gamma}}\nabla^{\mu\dot{\mu}}\xi_{\mu\gamma\dot{\mu}\dot{\gamma}} \\
&\quad + \frac{1}{9}\varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\gamma}}\nabla^{\beta\dot{\beta}}\nabla^{\gamma\dot{\gamma}}\nabla^{\mu\dot{\mu}}\xi_{\mu\gamma\dot{\beta}\dot{\mu}} + \frac{1}{9}\varepsilon_{\alpha\gamma}\varepsilon_{\dot{\alpha}\dot{\beta}}\nabla^{\beta\dot{\beta}}\nabla^{\gamma\dot{\gamma}}\nabla^{\mu\dot{\mu}}\xi_{\beta\mu\dot{\mu}\dot{\gamma}} \\
&\quad + \frac{1}{9}\varepsilon_{\alpha\gamma}\varepsilon_{\dot{\alpha}\dot{\gamma}}\nabla^{\beta\dot{\beta}}\nabla^{\gamma\dot{\gamma}}\nabla^{\mu\dot{\mu}}\xi_{\beta\mu\dot{\beta}\dot{\mu}} \\
&= \frac{1}{3}\nabla_{\alpha}{}^{\dot{\beta}}\nabla^{\gamma\dot{\gamma}}\nabla^{\beta}{}_{(\dot{\alpha}}\xi_{\beta\gamma\dot{\beta}\dot{\gamma})} + \frac{1}{3}\nabla^{\beta\dot{\beta}}\nabla_{\alpha}{}^{\dot{\gamma}}\nabla^{\gamma}{}_{(\dot{\alpha}}\xi_{\beta\gamma\dot{\beta}\dot{\gamma})} + \frac{1}{3}\nabla^{\beta}{}_{\dot{\alpha}}\nabla^{\gamma\dot{\gamma}}\nabla_{(\alpha}{}^{\dot{\beta}}\xi_{\beta\gamma)\dot{\beta}\dot{\gamma}} + \frac{1}{3}\nabla^{\beta\dot{\beta}}\nabla^{\gamma}{}_{\dot{\alpha}}\nabla_{(\alpha}{}^{\dot{\gamma}}\xi_{\beta\gamma)\dot{\beta}\dot{\gamma}} \\
&\quad + \frac{1}{9}\nabla_{\alpha\dot{\alpha}}\nabla_{\beta\dot{\beta}}\nabla_{\gamma\dot{\gamma}}\xi^{\beta\gamma\dot{\beta}\dot{\gamma}} + \frac{1}{9}\nabla_{\alpha}{}^{\dot{\beta}}\nabla^{\gamma}{}_{\dot{\alpha}}\nabla^{\beta\dot{\gamma}}\xi_{\beta\gamma\dot{\beta}\dot{\gamma}} + \frac{1}{9}\nabla^{\beta}{}_{\dot{\alpha}}\nabla_{\alpha}{}^{\dot{\gamma}}\nabla^{\gamma\dot{\beta}}\xi_{\beta\gamma\dot{\beta}\dot{\gamma}} \\
&\quad + \frac{1}{9}\nabla_{\beta\dot{\beta}}\nabla_{\alpha\dot{\alpha}}\nabla_{\gamma\dot{\gamma}}\xi^{\beta\gamma\dot{\beta}\dot{\gamma}} \\
&= \frac{2}{3}\nabla_{\alpha}{}^{\dot{\beta}}\nabla^{\gamma\dot{\gamma}}\nabla^{\beta}{}_{(\dot{\alpha}}\xi_{\beta\gamma\dot{\beta}\dot{\gamma})} + \frac{2}{3}\nabla^{\beta}{}_{\dot{\alpha}}\nabla^{\gamma\dot{\gamma}}\nabla_{(\alpha}{}^{\dot{\beta}}\xi_{\beta\gamma)\dot{\beta}\dot{\gamma}} + \frac{2}{9}\nabla_{\alpha\dot{\alpha}}\nabla_{\beta\dot{\beta}}\nabla_{\gamma\dot{\gamma}}\xi^{\beta\gamma\dot{\beta}\dot{\gamma}} + \frac{1}{9}\nabla_{\alpha}{}^{\dot{\beta}}\nabla^{\gamma}{}_{\dot{\alpha}}\nabla^{\beta\dot{\gamma}}\xi_{\beta\gamma\dot{\beta}\dot{\gamma}} \\
&\quad + \frac{1}{9}\nabla^{\beta}{}_{\dot{\alpha}}\nabla_{\alpha}{}^{\dot{\gamma}}\nabla^{\gamma\dot{\beta}}\xi_{\beta\gamma\dot{\beta}\dot{\gamma}} + \frac{1}{3}[\nabla^{\gamma\dot{\gamma}}, \nabla_{\alpha}{}^{\dot{\beta}}]\nabla^{\beta}{}_{(\dot{\alpha}}\xi_{\beta\gamma\dot{\beta}\dot{\gamma})} + \frac{1}{3}[\nabla^{\gamma\dot{\gamma}}, \nabla^{\beta}{}_{\dot{\alpha}}]\nabla_{(\alpha}{}^{\dot{\beta}}\xi_{\beta\gamma)\dot{\beta}\dot{\gamma}} \\
&\quad + \frac{1}{9}[\nabla_{\beta\dot{\beta}}, \nabla_{\alpha\dot{\alpha}}]\nabla_{\gamma\dot{\gamma}}\xi^{\beta\gamma\dot{\beta}\dot{\gamma}}
\end{aligned}
\tag{C.89}
$$

$$
\begin{aligned}
&[\nabla^{\gamma\dot{\gamma}}, \nabla_{\alpha}{}^{\dot{\beta}}]\nabla^{\beta}{}_{(\dot{\alpha}}\xi_{\beta\gamma\dot{\beta}\dot{\gamma})} \\
&= (R^{\gamma\dot{\gamma}}{}_{\alpha}{}^{\dot{\beta}\mu\nu}M_{\mu\nu} + \overline{R}^{\gamma\dot{\gamma}}{}_{\alpha}{}^{\dot{\beta}\dot{\mu}\dot{\nu}}\overline{M}_{\dot{\mu}\dot{\nu}})\nabla^{\beta}{}_{(\dot{\alpha}}\xi_{\beta\gamma\dot{\beta}\dot{\gamma})} \\
&= R^{\gamma\dot{\gamma}}{}_{\alpha}{}^{\dot{\beta}}{}_{\gamma}{}^{\mu}\nabla^{\beta}{}_{(\dot{\alpha}}\xi_{\beta\mu\dot{\beta}\dot{\gamma})} + \overline{R}^{\gamma\dot{\gamma}}{}_{\alpha}{}^{\dot{\beta}}{}_{\dot{\alpha}}{}^{\dot{\mu}}\nabla^{\beta}{}_{(\dot{\mu}}\xi_{\beta\gamma\dot{\beta}\dot{\gamma})} + \overline{R}^{\gamma\dot{\gamma}}{}_{\alpha}{}^{\dot{\beta}}{}_{\dot{\beta}}{}^{\dot{\mu}}\nabla^{\beta}{}_{(\dot{\alpha}}\xi_{\beta\gamma\dot{\mu}\dot{\gamma})} \\
&\quad + \overline{R}^{\gamma\dot{\gamma}}{}_{\alpha}{}^{\dot{\beta}}{}_{\dot{\gamma}}{}^{\dot{\mu}}\nabla^{\beta}{}_{(\dot{\alpha}}\xi_{\beta\gamma\dot{\beta}\dot{\mu})} \\
&= (\varepsilon^{\dot{\beta}\dot{\gamma}}C^{\gamma}{}_{\alpha\gamma}{}^{\mu} + \delta^{\gamma}{}_{\alpha}E_{\gamma}{}^{\mu\dot{\gamma}\dot{\beta}} + \varepsilon^{\dot{\beta}\dot{\gamma}}(-\delta^{\gamma}{}_{\gamma}\delta^{\mu}{}_{\alpha} + \varepsilon^{\mu\gamma}\varepsilon_{\alpha\gamma})F)\nabla^{\beta}{}_{(\dot{\alpha}}\xi_{\beta\mu\dot{\beta}\dot{\gamma})} \\
&\quad + (\delta^{\gamma}{}_{\alpha}\overline{C}^{\dot{\gamma}\dot{\beta}}{}_{\dot{\alpha}}{}^{\dot{\mu}} + \varepsilon^{\dot{\beta}\dot{\gamma}}E^{\gamma}{}_{\alpha\dot{\alpha}}{}^{\dot{\mu}} + \delta^{\gamma}{}_{\alpha}(\delta^{\dot{\gamma}}{}_{\dot{\alpha}}\varepsilon^{\dot{\mu}\dot{\beta}} + \delta^{\dot{\beta}}{}_{\dot{\alpha}}\varepsilon^{\dot{\mu}\dot{\gamma}})F)\nabla^{\beta}{}_{(\dot{\mu}}\xi_{\beta\gamma\dot{\beta}\dot{\gamma})} \\
&\quad + (\delta^{\gamma}{}_{\alpha}\overline{C}^{\dot{\gamma}\dot{\beta}}{}_{\dot{\beta}}{}^{\dot{\mu}} + \varepsilon^{\dot{\beta}\dot{\gamma}}E^{\gamma}{}_{\alpha\dot{\beta}}{}^{\dot{\mu}} + \delta^{\gamma}{}_{\alpha}(\delta^{\dot{\gamma}}{}_{\dot{\beta}}\varepsilon^{\dot{\mu}\dot{\beta}} + \delta^{\dot{\beta}}{}_{\dot{\beta}}\varepsilon^{\dot{\mu}\dot{\gamma}})F)\nabla^{\beta}{}_{(\dot{\alpha}}\xi_{\beta\gamma\dot{\mu}\dot{\gamma})} \\
&\quad + (\delta^{\gamma}{}_{\alpha}\overline{C}^{\dot{\gamma}\dot{\beta}}{}_{\dot{\gamma}}{}^{\dot{\mu}} + \varepsilon^{\dot{\beta}\dot{\gamma}}E^{\gamma}{}_{\alpha\dot{\gamma}}{}^{\dot{\mu}} + \delta^{\gamma}{}_{\alpha}(\delta^{\dot{\gamma}}{}_{\dot{\gamma}}\varepsilon^{\dot{\mu}\dot{\beta}} + \delta^{\dot{\beta}}{}_{\dot{\gamma}}\varepsilon^{\dot{\mu}\dot{\gamma}})F)\nabla^{\beta}{}_{(\dot{\alpha}}\xi_{\beta\gamma\dot{\beta}\dot{\mu})} \\
&= E_{\alpha}{}^{\gamma\dot{\beta}\dot{\gamma}}\nabla^{\beta}{}_{(\dot{\alpha}}\xi_{\beta\gamma\dot{\beta}\dot{\gamma})} + \overline{C}_{\dot{\alpha}}{}^{\dot{\beta}\dot{\gamma}\dot{\mu}}\nabla^{\beta}{}_{(\dot{\mu}}\xi_{\alpha\beta\dot{\beta}\dot{\gamma})}
\end{aligned}
\tag{C.90}
$$

$$
\begin{aligned}
&[\nabla^{\gamma\dot{\gamma}}, \nabla^{\beta}{}_{\dot{\alpha}}]\nabla_{(\alpha}{}^{\dot{\beta}}\xi_{\beta\gamma)\dot{\beta}\dot{\gamma}} \\
&= (R^{\gamma\dot{\gamma}\beta}{}_{\dot{\alpha}}{}^{\mu\nu}M_{\mu\nu} + \overline{R}^{\gamma\dot{\gamma}\beta}{}_{\dot{\alpha}}{}^{\dot{\mu}\dot{\nu}}\overline{M}_{\dot{\mu}\dot{\nu}})\nabla_{(\alpha}{}^{\dot{\beta}}\xi_{\beta\gamma)\dot{\beta}\dot{\gamma}} \\
&= R^{\gamma\dot{\gamma}\beta}{}_{\dot{\alpha}\alpha}{}^{\mu}\nabla_{(\mu}{}^{\dot{\beta}}\xi_{\beta\gamma)\dot{\beta}\dot{\gamma}} + R^{\gamma\dot{\gamma}\beta}{}_{\dot{\alpha}\beta}{}^{\mu}\nabla_{(\alpha}{}^{\dot{\beta}}\xi_{\mu\gamma)\dot{\beta}\dot{\gamma}} + R^{\gamma\dot{\gamma}\beta}{}_{\dot{\alpha}\gamma}{}^{\mu}\nabla_{(\alpha}{}^{\dot{\beta}}\xi_{\beta\mu)\dot{\beta}\dot{\gamma}} \\
&\quad + \overline{R}^{\gamma\dot{\gamma}\beta}{}_{\dot{\alpha}\dot{\gamma}}{}^{\dot{\mu}}\nabla_{(\alpha}{}^{\dot{\beta}}\xi_{\beta\gamma)\dot{\beta}\dot{\mu}} \\
&= (\delta^{\dot{\gamma}}{}_{\dot{\alpha}}C^{\gamma\beta}{}_{\alpha}{}^{\mu} + \varepsilon^{\beta\gamma}E_{\alpha}{}^{\mu\dot{\gamma}}{}_{\dot{\alpha}} + \delta^{\dot{\gamma}}{}_{\dot{\alpha}}(\delta^{\gamma}{}_{\alpha}\varepsilon^{\mu\beta} + \varepsilon^{\mu\gamma}\delta^{\beta}{}_{\alpha})F)\nabla_{(\mu}{}^{\dot{\beta}}\xi_{\beta\gamma)\dot{\beta}\dot{\gamma}} \\
&\quad + (\delta^{\dot{\gamma}}{}_{\dot{\alpha}}C^{\gamma\beta}{}_{\beta}{}^{\mu} + \varepsilon^{\beta\gamma}E_{\beta}{}^{\mu\dot{\gamma}}{}_{\dot{\alpha}} + \delta^{\dot{\gamma}}{}_{\dot{\alpha}}(\delta^{\gamma}{}_{\beta}\varepsilon^{\mu\beta} + \varepsilon^{\mu\gamma}\delta^{\beta}{}_{\beta})F)\nabla_{(\alpha}{}^{\dot{\beta}}\xi_{\mu\gamma)\dot{\beta}\dot{\gamma}} \\
&\quad + (\delta^{\dot{\gamma}}{}_{\dot{\alpha}}C^{\gamma\beta}{}_{\gamma}{}^{\mu} + \varepsilon^{\beta\gamma}E_{\gamma}{}^{\mu\dot{\gamma}}{}_{\dot{\alpha}} + \delta^{\dot{\gamma}}{}_{\dot{\alpha}}(\delta^{\gamma}{}_{\gamma}\varepsilon^{\mu\beta} + \varepsilon^{\mu\gamma}\delta^{\beta}{}_{\gamma})F)\nabla_{(\alpha}{}^{\dot{\beta}}\xi_{\beta\mu)\dot{\beta}\dot{\gamma}} \\
&\quad + (\varepsilon^{\beta\gamma}\overline{C}^{\dot{\gamma}}{}_{\dot{\alpha}\dot{\gamma}}{}^{\dot{\mu}} + \delta^{\dot{\gamma}}{}_{\dot{\alpha}}E^{\gamma\beta}{}_{\dot{\gamma}}{}^{\dot{\mu}} + \varepsilon^{\beta\gamma}(-\delta^{\dot{\gamma}}{}_{\dot{\gamma}}\delta^{\dot{\mu}}{}_{\dot{\alpha}} + \varepsilon^{\dot{\mu}\dot{\beta}}\varepsilon_{\dot{\alpha}\dot{\gamma}})F)\nabla_{(\alpha}{}^{\dot{\beta}}\xi_{\beta\gamma)\dot{\beta}\dot{\mu}} \\
&= C_{\alpha}{}^{\beta\gamma\mu}\nabla_{(\mu}{}^{\dot{\beta}}\xi_{\beta\gamma)\dot{\alpha}\dot{\beta}} + E^{\gamma\beta\dot{\gamma}}{}_{\dot{\alpha}}\nabla_{(\alpha}{}^{\dot{\beta}}\xi_{\beta\gamma)\dot{\beta}\dot{\gamma}}
\end{aligned}
\tag{C.91}
$$

$$
[\nabla_{\beta\dot{\beta}}, \nabla_{\alpha\dot{\alpha}}]\nabla_{\gamma\dot{\gamma}}\xi^{\beta\gamma\dot{\beta}\dot{\gamma}} = 2E^{\gamma}{}_{\alpha}{}^{\dot{\gamma}}{}_{\dot{\alpha}}\nabla^{\beta\dot{\beta}}\xi_{\beta\gamma\dot{\beta}\dot{\gamma}} - 6F\nabla^{\beta\dot{\beta}}\xi_{\alpha\beta\dot{\alpha}\dot{\beta}} \quad \text{from earlier.} \tag{C.92}
$$

Putting the results of the last two pages together,

$$
\begin{aligned}
&\nabla_{\alpha\dot\alpha}\nabla_{\beta\dot\beta}\nabla_{\gamma\dot\gamma}\xi^{\beta\gamma\dot\beta\dot\gamma}\\
&=\frac{2}{3}\nabla_\alpha{}^{\dot\beta}\nabla^{\gamma\dot\gamma}\nabla^\beta{}_{(\dot\alpha}\xi_{\beta\gamma\dot\beta\dot\gamma)}+\frac{2}{3}\nabla^\beta{}_{\dot\alpha}\nabla^{\gamma\dot\gamma}\nabla_{(\alpha}{}^{\dot\beta}\xi_{\beta\gamma)\dot\beta\dot\gamma}+\frac{2}{9}\nabla_{\alpha\dot\alpha}\nabla_{\beta\dot\beta}\nabla_{\gamma\dot\gamma}\xi^{\beta\gamma\dot\beta\dot\gamma}+\frac{1}{9}\nabla_\alpha{}^{\dot\beta}\nabla^\gamma{}_{\dot\alpha}\nabla^{\beta\dot\gamma}\xi_{\beta\gamma\dot\beta\dot\gamma}\\
&\quad+\frac{1}{9}\nabla^\beta{}_{\dot\alpha}\nabla_\alpha{}^{\dot\gamma}\nabla^{\gamma\dot\beta}\xi_{\beta\gamma\dot\beta\dot\gamma}+\frac{1}{3}(E_\alpha{}^{\gamma\dot\beta\dot\gamma}\nabla^\beta{}_{(\dot\alpha}\xi_{\beta\gamma\dot\beta\dot\gamma)}+\overline{C}_{\dot\alpha}{}^{\dot\beta\dot\gamma\dot\mu}\nabla^\beta{}_{(\dot\mu}\xi_{\alpha\beta\dot\beta\dot\gamma)})+\frac{1}{3}(C_\alpha{}^{\beta\gamma\mu}\nabla_{(\mu}{}^{\dot\beta}\xi_{\beta\gamma)\dot\alpha\dot\beta}\\
&\quad+E^{\gamma\beta\dot\gamma}{}_{\dot\alpha}\nabla_{(\alpha}{}^{\dot\beta}\xi_{\beta\gamma)\dot\beta\dot\gamma})+\frac{1}{9}(2E^\gamma{}_\alpha{}^{\dot\gamma}{}_{\dot\alpha}\nabla^{\beta\dot\beta}\xi_{\beta\gamma\dot\beta\dot\gamma}-6F\nabla^{\beta\dot\beta}\xi_{\alpha\beta\dot\alpha\dot\beta})+\nabla^{\beta\dot\beta}(-C_{\alpha\beta}{}^{\gamma\mu}\xi_{\gamma\mu\dot\alpha\dot\beta}\\
&\quad-\overline{C}_{\dot\alpha\dot\beta}{}^{\dot\gamma\dot\mu}\xi_{\alpha\beta\dot\gamma\dot\mu}-2E^\gamma{}_{(\alpha}{}^{\dot\gamma}{}_{\dot\alpha}\xi_{\beta)\gamma\dot\beta\dot\gamma}-2E^\gamma{}_\alpha{}^{\dot\gamma}{}_{(\dot\alpha}\xi_{\beta\gamma\dot\beta)\dot\gamma}+8F\xi_{\alpha\beta\dot\alpha\dot\beta})\\
&\quad-2E^\gamma{}_\alpha{}^{\dot\gamma}{}_{\dot\alpha}\nabla^{\beta\dot\beta}\xi_{\beta\gamma\dot\beta\dot\gamma}+6F\nabla^{\beta\dot\beta}\xi_{\alpha\beta\dot\alpha\dot\beta}
\end{aligned}\tag{C.93}
$$

$$
\begin{aligned}
&\implies\ \nabla_{\alpha\dot\alpha}\nabla_{\beta\dot\beta}\nabla_{\gamma\dot\gamma}\xi^{\beta\gamma\dot\beta\dot\gamma}\\
&=\frac{6}{7}\nabla_\alpha{}^{\dot\beta}\nabla^{\gamma\dot\gamma}\nabla^\beta{}_{(\dot\alpha}\xi_{\beta\gamma\dot\beta\dot\gamma)}+\frac{6}{7}\nabla^\beta{}_{\dot\alpha}\nabla^{\gamma\dot\gamma}\nabla_{(\alpha}{}^{\dot\beta}\xi_{\beta\gamma)\dot\beta\dot\gamma}+\frac{1}{7}\nabla_\alpha{}^{\dot\beta}\nabla^\gamma{}_{\dot\alpha}\nabla^{\beta\dot\gamma}\xi_{\beta\gamma\dot\beta\dot\gamma}+\frac{1}{7}\nabla^\beta{}_{\dot\alpha}\nabla_\alpha{}^{\dot\gamma}\nabla^{\gamma\dot\beta}\xi_{\beta\gamma\dot\beta\dot\gamma}\\
&\quad+\frac{3}{7}(E_\alpha{}^{\gamma\dot\beta\dot\gamma}\nabla^\beta{}_{(\dot\alpha}\xi_{\beta\gamma\dot\beta\dot\gamma)}+\overline{C}_{\dot\alpha}{}^{\dot\beta\dot\gamma\dot\mu}\nabla^\beta{}_{(\dot\mu}\xi_{\alpha\beta\dot\beta\dot\gamma)})+\frac{3}{7}(C_\alpha{}^{\beta\gamma\mu}\nabla_{(\mu}{}^{\dot\beta}\xi_{\beta\gamma)\dot\alpha\dot\beta}+E^{\gamma\beta\dot\gamma}{}_{\dot\alpha}\nabla_{(\alpha}{}^{\dot\beta}\xi_{\beta\gamma)\dot\beta\dot\gamma})\\
&\quad-\frac{9}{7}\nabla^{\beta\dot\beta}(C_{\alpha\beta}{}^{\gamma\mu}\xi_{\gamma\mu\dot\alpha\dot\beta}+\overline{C}_{\dot\alpha\dot\beta}{}^{\dot\gamma\dot\mu}\xi_{\alpha\beta\dot\gamma\dot\mu}+2E^\gamma{}_{(\alpha}{}^{\dot\gamma}{}_{\dot\alpha}\xi_{\beta)\gamma\dot\beta\dot\gamma}+2E^\gamma{}_\alpha{}^{\dot\gamma}{}_{(\dot\alpha}\xi_{\beta\gamma\dot\beta)\dot\gamma}-8F\xi_{\alpha\beta\dot\alpha\dot\beta})\\
&\quad-\frac{16}{7}E^\gamma{}_\alpha{}^{\dot\gamma}{}_{\dot\alpha}\nabla^{\beta\dot\beta}\xi_{\beta\gamma\dot\beta\dot\gamma}+\frac{48}{7}F\nabla^{\beta\dot\beta}\xi_{\alpha\beta\dot\alpha\dot\beta}
\end{aligned}\tag{C.94}
$$

$$
\begin{aligned}
&\nabla_\alpha{}^{\dot\beta}\nabla^{\gamma\dot\gamma}\nabla^\beta{}_{(\dot\alpha}\xi_{\beta\gamma\dot\beta\dot\gamma)}\\
&=\nabla_\alpha{}^{\dot\beta}\nabla^{\gamma\dot\gamma}\nabla^\beta{}_{(\dot\alpha}\xi_{\beta\gamma\dot\beta)\dot\gamma}-\frac{1}{3}\varepsilon_{\dot\gamma\dot\alpha}\nabla_\alpha{}^{\dot\beta}\nabla^{\gamma\dot\gamma}\nabla^\beta{}_{(\dot\mu}\xi_{\beta\gamma\dot\beta)}{}^{\dot\mu}-\frac{1}{3}\varepsilon_{\dot\gamma\dot\beta}\nabla_\alpha{}^{\dot\beta}\nabla^{\gamma\dot\gamma}\nabla^\beta{}_{(\dot\alpha}\xi_{\beta\gamma\dot\mu)}{}^{\dot\mu}\\
&=\nabla_\alpha{}^{\dot\beta}\nabla^{\gamma\dot\gamma}\nabla^\beta{}_{(\dot\alpha}\xi_{\beta\gamma\dot\beta)\dot\gamma}-\frac{1}{6}\nabla_\alpha{}^{\dot\beta}\nabla^\gamma{}_{\dot\alpha}\nabla^{\beta\dot\gamma}\xi_{\beta\gamma\dot\beta\dot\gamma}-\frac{1}{6}\nabla_\alpha{}^{\dot\beta}\nabla^\gamma{}_{\dot\alpha}\nabla^{\beta\dot\gamma}\xi_{\beta\gamma\dot\alpha\dot\gamma}\\
&=\nabla_\alpha{}^{\dot\beta}\nabla^{\gamma\dot\gamma}\nabla^\beta{}_{(\dot\alpha}\xi_{\beta\gamma\dot\beta)\dot\gamma}-\frac{1}{3}\nabla_\alpha{}^{\dot\beta}\nabla^\beta{}_{(\dot\alpha}\nabla^{\gamma\dot\gamma}\xi_{\beta\gamma\dot\beta)\dot\gamma}\\
&=\frac{2}{3}\nabla_\alpha{}^{\dot\beta}\nabla^\beta{}_{(\dot\alpha}\nabla^{\gamma\dot\gamma}\xi_{\beta\gamma\dot\beta)\dot\gamma}-\nabla_\alpha{}^{\dot\beta}[\nabla^\beta{}_{(\dot\alpha},\nabla^{\gamma\dot\gamma}]\xi_{\beta\gamma\dot\beta)\dot\gamma}\\
&=\frac{2}{3}\nabla_\alpha{}^{\dot\beta}\nabla^\beta{}_{(\dot\alpha}\nabla^{\gamma\dot\gamma}\xi_{\beta\gamma\dot\beta)\dot\gamma}-\nabla_\alpha{}^{\dot\beta}((R^\beta{}_{(\dot\alpha}{}^{\gamma\dot\gamma\mu\nu}M_{\mu\nu}+\overline{R}^\beta{}_{(\dot\alpha}{}^{\gamma\dot\gamma\dot\mu\dot\nu}\overline{M}_{\mu\nu})\xi_{\beta\gamma\dot\beta)\dot\gamma})\\
&=\frac{2}{3}\nabla_\alpha{}^{\dot\beta}\nabla^\beta{}_{(\dot\alpha}\nabla^{\gamma\dot\gamma}\xi_{\beta\gamma\dot\beta)\dot\gamma}-\nabla_\alpha{}^{\dot\beta}(R^\beta{}_{(\dot\alpha}{}^{\gamma\dot\gamma}{}_\beta{}^\mu\xi_{\mu\gamma\dot\beta)\dot\gamma}+R^\beta{}_{(\dot\alpha}{}^{\gamma\dot\gamma}{}_\gamma{}^\mu\xi_{\beta\mu\dot\beta)\dot\gamma}+\overline{R}^\beta{}_{(\dot\alpha}{}^{\gamma\dot\gamma}{}_{\dot\beta)}{}^{\dot\mu}\xi_{\beta\gamma\dot\mu\dot\gamma}\\
&\quad+\overline{R}^\beta{}_{(\dot\alpha}{}^{\gamma\dot\gamma}{}_{\dot\gamma}{}^{\dot\mu}\xi_{\beta\gamma\dot\beta)\dot\mu})\\
&=\frac{2}{3}\nabla_\alpha{}^{\dot\beta}\nabla^\beta{}_{(\dot\alpha}\nabla^{\gamma\dot\gamma}\xi_{\beta\gamma\dot\beta)\dot\gamma}-\nabla_\alpha{}^{\dot\beta}((-\delta^{\dot\gamma}{}_{(\dot\alpha}C^{\beta\gamma}{}_\beta{}^\mu+\varepsilon^{\gamma\beta}E_\beta{}^\mu{}_{(\dot\alpha}{}^{\dot\gamma}-\delta^{\dot\gamma}{}_{(\dot\alpha}(\delta^\beta{}_\beta\varepsilon^{\mu\gamma}+\delta^\gamma{}_\beta\varepsilon^{\mu\beta})F)\xi_{\mu\gamma\dot\beta)\dot\gamma}\\
&\quad+(-\delta^{\dot\gamma}{}_{(\dot\alpha}C^{\beta\gamma}{}_\gamma{}^\mu+\varepsilon^{\gamma\beta}E_\gamma{}^\mu{}_{(\dot\alpha}{}^{\dot\gamma}-\delta^{\dot\gamma}{}_{(\dot\alpha}(\delta^\beta{}_\gamma\varepsilon^{\mu\gamma}+\delta^\gamma{}_\gamma\varepsilon^{\mu\beta})F)\xi_{\beta\mu\dot\beta)\dot\gamma}\\
&\quad+(\varepsilon^{\gamma\beta}C_{(\dot\alpha}{}^{\dot\gamma}{}_{\dot\beta)}{}^{\dot\mu}-\delta^{\dot\gamma}{}_{(\dot\alpha}E^{\beta\gamma}{}_{\dot\beta)}{}^{\dot\mu}+\varepsilon^{\gamma\beta}(\varepsilon_{(\dot\alpha\dot\beta)}\varepsilon^{\dot\mu\dot\gamma}-\delta^{\dot\mu}{}_{(\dot\alpha}\delta^{\dot\gamma}{}_{\dot\beta)})F)\xi_{\beta\gamma\dot\mu\dot\gamma}\\
&\quad+(\varepsilon^{\gamma\beta}C_{(\dot\alpha}{}^{\dot\gamma}{}_{\dot\gamma}{}^{\dot\mu}-\delta^{\dot\gamma}{}_{(\dot\alpha}E^{\beta\gamma}{}_{\dot\gamma}{}^{\dot\mu}+\varepsilon^{\gamma\beta}(-\varepsilon_{\dot\gamma(\dot\alpha}\varepsilon^{\dot\mu\dot\gamma}-\delta^{\dot\mu}{}_{(\dot\alpha}\delta^{\dot\gamma}{}_{\dot\gamma})F)\xi_{\beta\gamma\dot\beta)\dot\mu})\\
&=\frac{2}{3}\nabla_\alpha{}^{\dot\beta}\nabla^\beta{}_{(\dot\alpha}\nabla^{\gamma\dot\gamma}\xi_{\beta\gamma\dot\beta)\dot\gamma}+2\nabla_\alpha{}^{\dot\beta}(E^{\beta\gamma\dot\gamma}{}_{(\dot\alpha}\xi_{\beta\gamma\dot\beta)\dot\gamma})
\end{aligned}\tag{C.95}
$$

$$
\begin{aligned}
&\nabla^\beta{}_{\dot\alpha}\nabla^{\gamma\dot\gamma}\nabla_{(\alpha}{}^{\dot\beta}\xi_{\beta\gamma)\dot\beta\dot\gamma}\\
&=\nabla^\beta{}_{\dot\alpha}\nabla^{\gamma\dot\gamma}\nabla_\alpha{}^{\dot\beta}\xi_{\beta\gamma\dot\beta\dot\gamma}-\frac{1}{3}\varepsilon_{\alpha\beta}\nabla^\beta{}_{\dot\alpha}\nabla^{\gamma\dot\gamma}\nabla^{\mu\dot\beta}\xi_{\mu\gamma\dot\beta\dot\gamma}-\frac{1}{3}\varepsilon_{\alpha\gamma}\nabla^\beta{}_{\dot\alpha}\nabla^{\gamma\dot\gamma}\nabla^{\mu\dot\beta}\xi_{\beta\mu\dot\beta\dot\gamma}\\
&=\nabla^\beta{}_{\dot\alpha}\nabla^{\gamma\dot\gamma}\nabla_\alpha{}^{\dot\beta}\xi_{\beta\gamma\dot\beta\dot\gamma}-\frac{1}{3}\nabla_{\alpha\dot\alpha}\nabla^{\gamma\dot\gamma}\nabla^{\beta\dot\beta}\xi_{\beta\gamma\dot\beta\dot\gamma}-\frac{1}{3}\nabla^\beta{}_{\dot\alpha}\nabla_\alpha{}^{\dot\gamma}\nabla^{\gamma\dot\beta}\xi_{\beta\gamma\dot\beta\dot\gamma}\\
&=\frac{2}{3}\nabla^\beta{}_{\dot\alpha}\nabla_\alpha{}^{\dot\beta}\nabla^{\gamma\dot\gamma}\xi_{\beta\gamma\dot\beta\dot\gamma}-\frac{1}{3}\nabla_{\alpha\dot\alpha}\nabla^{\gamma\dot\gamma}\nabla^{\beta\dot\beta}\xi_{\beta\gamma\dot\beta\dot\gamma}+\nabla^\beta{}_{\dot\alpha}[\nabla^{\gamma\dot\gamma},\nabla_\alpha{}^{\dot\beta}]\xi_{\beta\gamma\dot\beta\dot\gamma}
\end{aligned}\tag{C.96}
$$

$$
\begin{aligned}
&\nabla^{\beta}{}_{\dot\alpha}[\nabla^{\gamma\dot\gamma},\nabla_{\alpha}{}^{\dot\beta}]\xi_{\beta\gamma\dot\beta\dot\gamma}\\
&=\nabla^{\beta}{}_{\dot\alpha}((R^{\gamma\dot\gamma}{}_{\alpha}{}^{\dot\beta\mu\nu}M_{\mu\nu}+\overline{R}^{\gamma\dot\gamma}{}_{\alpha}{}^{\dot\beta\dot\mu\dot\nu}\overline{M}_{\mu\nu})\xi_{\beta\gamma\dot\beta\dot\gamma})\\
&=\nabla^{\beta}{}_{\dot\alpha}(R^{\gamma\dot\gamma}{}_{\alpha}{}^{\dot\beta}{}_{\beta}{}^{\mu}\xi_{\mu\gamma\dot\beta\dot\gamma}+R^{\gamma\dot\gamma}{}_{\alpha}{}^{\dot\beta}{}_{\gamma}{}^{\mu}\xi_{\beta\mu\dot\beta\dot\gamma}+\overline{R}^{\gamma\dot\gamma}{}_{\alpha}{}^{\dot\beta}{}_{\dot\beta}{}^{\dot\mu}\xi_{\beta\gamma\dot\mu\dot\gamma}+\overline{R}^{\gamma\dot\gamma}{}_{\alpha}{}^{\dot\beta}{}_{\dot\gamma}{}^{\dot\mu}\xi_{\beta\gamma\dot\beta\dot\mu})\\
&=\nabla^{\beta}{}_{\dot\alpha}((\varepsilon^{\dot\beta\dot\gamma}C^{\gamma}{}_{\alpha\beta}{}^{\mu}+\delta^{\gamma}{}_{\alpha}E_{\beta}{}^{\mu\dot\gamma\dot\beta}+\varepsilon^{\dot\beta\dot\gamma}(-\delta^{\gamma}{}_{\beta}\delta^{\mu}{}_{\alpha}+\varepsilon^{\mu\gamma}\varepsilon_{\alpha\beta})F)\xi_{\mu\gamma\dot\beta\dot\gamma}\\
&\quad+(\varepsilon^{\dot\beta\dot\gamma}C^{\gamma}{}_{\alpha\gamma}{}^{\mu}+\delta^{\gamma}{}_{\alpha}E_{\gamma}{}^{\mu\dot\gamma\dot\beta}+\varepsilon^{\dot\beta\dot\gamma}(-\delta^{\gamma}{}_{\gamma}\delta^{\mu}{}_{\alpha}+\varepsilon^{\mu\gamma}\varepsilon_{\alpha\gamma})F)\xi_{\beta\mu\dot\beta\dot\gamma}\\
&\quad+(\delta^{\gamma}{}_{\alpha}\overline{C}^{\dot\gamma\dot\beta}{}_{\dot\beta}{}^{\dot\mu}+\varepsilon^{\dot\beta\dot\gamma}E^{\gamma}{}_{\alpha\dot\beta}{}^{\dot\mu}+\delta^{\gamma}{}_{\alpha}(\delta^{\dot\gamma}{}_{\dot\beta}\varepsilon^{\dot\mu\dot\beta}+\delta^{\dot\beta}{}_{\dot\beta}\varepsilon^{\dot\mu\dot\gamma})F)\xi_{\beta\gamma\dot\mu\dot\gamma}\\
&\quad+(\delta^{\gamma}{}_{\alpha}\overline{C}^{\dot\gamma\dot\beta}{}_{\dot\gamma}{}^{\dot\mu}+\varepsilon^{\dot\beta\dot\gamma}E^{\gamma}{}_{\alpha\dot\gamma}{}^{\dot\mu}+\delta^{\gamma}{}_{\alpha}(\delta^{\dot\gamma}{}_{\dot\gamma}\varepsilon^{\dot\mu\dot\beta}+\delta^{\dot\beta}{}_{\dot\gamma}\varepsilon^{\dot\mu\dot\gamma})F)\xi_{\beta\gamma\dot\beta\dot\mu})\\
&=\nabla^{\beta}{}_{\dot\alpha}(E_{\beta}{}^{\gamma\dot\beta\dot\gamma}\xi_{\alpha\gamma\dot\beta\dot\gamma}+E_{\alpha}{}^{\gamma\dot\beta\dot\gamma}\xi_{\beta\gamma\dot\beta\dot\gamma})\\
&=2\nabla^{\beta}{}_{\dot\alpha}(E_{(\alpha}{}^{\gamma\dot\beta\dot\gamma}\xi_{\beta)\gamma\dot\beta\dot\gamma}) && \text{(C.97)}\\
&\implies\ \nabla^{\beta}{}_{\dot\alpha}\nabla^{\gamma\dot\gamma}\nabla_{(\alpha}{}^{\dot\beta}\xi_{\beta\gamma)\dot\beta\dot\gamma}\\
&=\frac{2}{3}\nabla^{\beta}{}_{\dot\alpha}\nabla_{\alpha}{}^{\dot\beta}\nabla^{\gamma\dot\gamma}\xi_{\beta\gamma\dot\beta\dot\gamma}-\frac{1}{3}\nabla_{\alpha\dot\alpha}\nabla^{\gamma\dot\gamma}\nabla^{\beta\dot\beta}\xi_{\beta\gamma\dot\beta\dot\gamma}+2\nabla^{\beta}{}_{\dot\alpha}(E_{(\alpha}{}^{\gamma\dot\beta\dot\gamma}\xi_{\beta)\gamma\dot\beta\dot\gamma})\\
&=\frac{2}{3}\nabla^{\beta}{}_{(\dot\alpha}\nabla_{\alpha}{}^{\dot\beta}\nabla^{\gamma\dot\gamma}\xi_{\beta\gamma\dot\beta)\dot\gamma}+\frac{1}{3}\varepsilon_{\dot\alpha\dot\beta}\nabla^{\beta\dot\mu}\nabla_{\alpha}{}^{\dot\beta}\nabla^{\gamma\dot\gamma}\xi_{\beta\gamma\dot\mu\dot\gamma}-\frac{1}{3}\nabla_{\alpha\dot\alpha}\nabla^{\gamma\dot\gamma}\nabla^{\beta\dot\beta}\xi_{\beta\gamma\dot\beta\dot\gamma}\\
&\quad+2\nabla^{\beta}{}_{\dot\alpha}(E_{(\alpha}{}^{\gamma\dot\beta\dot\gamma}\xi_{\beta)\gamma\dot\beta\dot\gamma})\\
&=\frac{2}{3}\nabla^{\beta}{}_{(\dot\alpha}\nabla_{\alpha}{}^{\dot\beta}\nabla^{\gamma\dot\gamma}\xi_{\beta\gamma\dot\beta)\dot\gamma}+\frac{1}{3}\nabla^{\beta\dot\beta}\nabla_{\alpha\dot\alpha}\nabla^{\gamma\dot\gamma}\xi_{\beta\gamma\dot\mu\dot\gamma}-\frac{1}{3}\nabla_{\alpha\dot\alpha}\nabla^{\gamma\dot\gamma}\nabla^{\beta\dot\beta}\xi_{\beta\gamma\dot\beta\dot\gamma}\\
&\quad+2\nabla^{\beta}{}_{\dot\alpha}(E_{(\alpha}{}^{\gamma\dot\beta\dot\gamma}\xi_{\beta)\gamma\dot\beta\dot\gamma})\\
&=\frac{2}{3}\nabla_{\alpha}{}^{\dot\beta}\nabla^{\beta}{}_{(\dot\alpha}\nabla^{\gamma\dot\gamma}\xi_{\beta\gamma\dot\beta)\dot\gamma}+\frac{2}{3}[\nabla^{\beta}{}_{(\dot\alpha},\nabla_{\alpha}{}^{\dot\beta}]\nabla^{\gamma\dot\gamma}\xi_{\beta\gamma\dot\beta)\dot\gamma}+\frac{1}{3}[\nabla_{\beta\dot\beta},\nabla_{\alpha\dot\alpha}]\nabla_{\gamma\dot\gamma}\xi^{\beta\gamma\dot\beta\dot\gamma}\\
&\quad+2\nabla^{\beta}{}_{\dot\alpha}(E_{(\alpha}{}^{\gamma\dot\beta\dot\gamma}\xi_{\beta)\gamma\dot\beta\dot\gamma})\\
&=\frac{2}{3}\nabla_{\alpha}{}^{\dot\beta}\nabla^{\beta}{}_{(\dot\alpha}\nabla^{\gamma\dot\gamma}\xi_{\beta\gamma\dot\beta)\dot\gamma}+\frac{2}{3}[\nabla^{\beta}{}_{(\dot\alpha},\nabla_{\alpha}{}^{\dot\beta}]\nabla^{\gamma\dot\gamma}\xi_{\beta\gamma\dot\beta)\dot\gamma}+\frac{2}{3}E_{\alpha\beta\dot\alpha\dot\beta}\nabla_{\gamma\dot\gamma}\xi^{\beta\gamma\dot\beta\dot\gamma}-2F\nabla^{\beta\dot\beta}\xi_{\alpha\beta\dot\alpha\dot\beta}\\
&\quad+2\nabla^{\beta}{}_{\dot\alpha}(E_{(\alpha}{}^{\gamma\dot\beta\dot\gamma}\xi_{\beta)\gamma\dot\beta\dot\gamma})\ \ \text{via an earlier result} && \text{(C.98)}\\
&[\nabla^{\beta}{}_{\dot\alpha},\nabla_{\alpha}{}^{\dot\beta}]\nabla^{\gamma\dot\gamma}\xi_{\beta\gamma\dot\beta\dot\gamma}\\
&=(R^{\beta}{}_{\dot\alpha\alpha}{}^{\dot\beta\mu\nu}M_{\mu\nu}+\overline{R}^{\beta}{}_{\dot\alpha\alpha}{}^{\dot\beta\dot\mu\dot\nu}\overline{M}_{\dot\mu\dot\nu})\nabla^{\gamma\dot\gamma}\xi_{\beta\gamma\dot\beta\dot\gamma}\\
&=R^{\beta}{}_{\dot\alpha\alpha}{}^{\dot\beta}{}_{\beta}{}^{\mu}\nabla^{\gamma\dot\gamma}\xi_{\mu\gamma\dot\beta\dot\gamma}+\overline{R}^{\beta}{}_{\dot\alpha\alpha}{}^{\dot\beta}{}_{\dot\beta}{}^{\dot\mu}\nabla^{\gamma\dot\gamma}\xi_{\beta\gamma\dot\mu\dot\gamma}\\
&=(-\delta^{\dot\beta}{}_{\dot\alpha}C^{\beta}{}_{\alpha\beta}{}^{\mu}+\delta^{\beta}{}_{\alpha}E_{\beta}{}^{\mu}{}_{\dot\alpha}{}^{\dot\beta}-\delta^{\dot\beta}{}_{\dot\alpha}(-\delta^{\beta}{}_{\beta}\delta^{\mu}{}_{\alpha}+\varepsilon^{\mu\beta}\varepsilon_{\alpha\beta})F)\nabla^{\gamma\dot\gamma}\xi_{\mu\gamma\dot\beta\dot\gamma}\\
&\quad+(\delta^{\beta}{}_{\alpha}\overline{C}_{\dot\alpha}{}^{\dot\beta}{}_{\dot\beta}{}^{\dot\mu}-\delta^{\dot\beta}{}_{\dot\alpha}E^{\beta}{}_{\alpha\dot\beta}{}^{\dot\mu}+\delta^{\beta}{}_{\alpha}(\varepsilon_{\dot\alpha\dot\beta}\varepsilon^{\dot\mu\dot\beta}-\delta^{\dot\mu}{}_{\dot\alpha}\delta^{\dot\beta}{}_{\dot\beta})F)\nabla^{\gamma\dot\gamma}\xi_{\beta\gamma\dot\mu\dot\gamma}\\
&=0 && \text{(C.99)}\\
&[\nabla^{\beta}{}_{\dot\beta},\nabla_{\alpha}{}^{\dot\beta}]\nabla^{\gamma\dot\gamma}\xi_{\beta\gamma\dot\alpha\dot\gamma}\\
&=(R^{\beta}{}_{\dot\beta\alpha}{}^{\dot\beta\mu\nu}M_{\mu\nu}+\overline{R}^{\beta}{}_{\dot\beta\alpha}{}^{\dot\beta\dot\mu\dot\nu}\overline{M}_{\dot\mu\dot\nu})\nabla^{\gamma\dot\gamma}\xi_{\beta\gamma\dot\alpha\dot\gamma}\\
&=R^{\beta}{}_{\dot\beta\alpha}{}^{\dot\beta}{}_{\beta}{}^{\mu}\nabla^{\gamma\dot\gamma}\xi_{\mu\gamma\dot\alpha\dot\gamma}+\overline{R}^{\beta}{}_{\dot\beta\alpha}{}^{\dot\beta}{}_{\dot\alpha}{}^{\dot\mu}\nabla^{\gamma\dot\gamma}\xi_{\beta\gamma\dot\mu\dot\gamma}\\
&=(-\delta^{\dot\beta}{}_{\dot\beta}C^{\beta}{}_{\alpha\beta}{}^{\mu}+\delta^{\beta}{}_{\alpha}E_{\beta}{}^{\mu}{}_{\dot\beta}{}^{\dot\beta}-\delta^{\dot\beta}{}_{\dot\beta}(-\delta^{\beta}{}_{\beta}\delta^{\mu}{}_{\alpha}+\varepsilon^{\mu\beta}\varepsilon_{\alpha\beta})F)\nabla^{\gamma\dot\gamma}\xi_{\mu\gamma\dot\alpha\dot\gamma}\\
&\quad+(\delta^{\beta}{}_{\alpha}\overline{C}_{\dot\beta}{}^{\dot\beta}{}_{\dot\alpha}{}^{\dot\mu}-\delta^{\dot\beta}{}_{\dot\beta}E^{\beta}{}_{\alpha\dot\alpha}{}^{\dot\mu}+\delta^{\beta}{}_{\alpha}(\varepsilon_{\dot\beta\dot\alpha}\varepsilon^{\dot\mu\dot\beta}-\delta^{\dot\mu}{}_{\dot\beta}\delta^{\dot\beta}{}_{\dot\alpha})F)\nabla^{\gamma\dot\gamma}\xi_{\beta\gamma\dot\mu\dot\gamma}\\
&=-2E_{\alpha\beta\dot\alpha\dot\beta}+6F\nabla^{\beta\dot\beta}\xi_{\alpha\beta\dot\alpha\dot\beta}\,. && \text{(C.100)}
\end{aligned}
$$

Putting together the past page of results,

$$
\begin{aligned}
&\nabla^{\beta}{}_{\dot\alpha}\nabla^{\gamma\dot\gamma}\nabla_{(\alpha}{}^{\dot\beta}\xi_{\beta\gamma)\dot\beta\dot\gamma}\\
&=\frac{2}{3}\nabla_{\alpha}{}^{\dot\beta}\nabla^{\beta}{}_{(\dot\alpha}\nabla^{\gamma\dot\gamma}\xi_{\beta\gamma\dot\beta)\dot\gamma}+\frac{2}{3}\times\frac{1}{2}(0-2E_{\alpha\beta\dot\alpha\dot\beta}+6F\nabla^{\beta\dot\beta}\xi_{\alpha\beta\dot\alpha\dot\beta})+\frac{2}{3}E_{\alpha\beta\dot\alpha\dot\beta}\nabla_{\gamma\dot\gamma}\xi^{\beta\gamma\dot\beta\dot\gamma}\\
&\quad-2F\nabla^{\beta\dot\beta}\xi_{\alpha\beta\dot\alpha\dot\beta}+2\nabla^{\beta}{}_{\dot\alpha}(E_{(\alpha}{}^{\gamma\dot\beta\dot\gamma}\xi_{\beta)\gamma\dot\beta\dot\gamma})\\
&=\frac{2}{3}\nabla_{\alpha}{}^{\dot\beta}\nabla^{\beta}{}_{(\dot\alpha}\nabla^{\gamma\dot\gamma}\xi_{\beta\gamma\dot\beta)\dot\gamma}+2\nabla^{\beta}{}_{\dot\alpha}(E_{(\alpha}{}^{\gamma\dot\beta\dot\gamma}\xi_{\beta)\gamma\dot\beta\dot\gamma})\,. \qquad \text{(C.101)}\\
&\nabla_{\alpha}{}^{\dot\beta}\nabla^{\gamma}{}_{\dot\alpha}\nabla^{\beta\dot\gamma}\xi_{\beta\gamma\dot\beta\dot\gamma}\\
&=\nabla_{\alpha}{}^{\dot\beta}\nabla^{\beta}{}_{\dot\alpha}\nabla^{\gamma\dot\gamma}\xi_{\beta\gamma\dot\beta\dot\gamma}\\
&=\nabla_{\alpha}{}^{\dot\beta}\nabla^{\beta}{}_{(\dot\alpha}\nabla^{\gamma\dot\gamma}\xi_{\beta\gamma\dot\beta)\dot\gamma}+\frac{1}{2}\varepsilon_{\dot\alpha\dot\beta}\nabla_{\alpha}{}^{\dot\beta}\nabla^{\beta\dot\mu}\nabla^{\gamma\dot\gamma}\xi_{\beta\gamma\dot\mu\dot\gamma}\\
&=\nabla_{\alpha}{}^{\dot\beta}\nabla^{\beta}{}_{(\dot\alpha}\nabla^{\gamma\dot\gamma}\xi_{\beta\gamma\dot\beta)\dot\gamma}+\frac{1}{2}\nabla_{\alpha\dot\alpha}\nabla_{\beta\dot\beta}\nabla_{\gamma\dot\gamma}\xi^{\beta\gamma\dot\beta\dot\gamma} \qquad \text{(C.102)}\\
&\nabla^{\beta}{}_{\dot\alpha}\nabla_{\alpha}{}^{\dot\gamma}\nabla^{\gamma\dot\beta}\xi_{\beta\gamma\dot\beta\dot\gamma}\\
&=\nabla^{\beta}{}_{\dot\alpha}\nabla_{\alpha}{}^{\dot\beta}\nabla^{\gamma\dot\gamma}\xi_{\beta\gamma\dot\beta\dot\gamma}\\
&=\nabla_{\alpha}{}^{\dot\beta}\nabla^{\beta}{}_{\dot\alpha}\nabla^{\gamma\dot\gamma}\xi_{\beta\gamma\dot\beta\dot\gamma}+[\nabla^{\beta}{}_{\dot\alpha},\nabla_{\alpha}{}^{\dot\beta}]\nabla^{\gamma\dot\gamma}\xi_{\beta\gamma\dot\beta\dot\gamma}\\
&=\nabla_{\alpha}{}^{\dot\beta}\nabla^{\beta}{}_{(\dot\alpha}\nabla^{\gamma\dot\gamma}\xi_{\beta\gamma\dot\beta)\dot\gamma}+\frac{1}{2}\varepsilon_{\dot\alpha\dot\beta}\nabla_{\alpha}{}^{\dot\beta}\nabla^{\beta\dot\mu}\nabla^{\gamma\dot\gamma}\xi_{\beta\gamma\dot\mu\dot\gamma}+0\ \ \text{from above}\\
&=\nabla_{\alpha}{}^{\dot\beta}\nabla^{\beta}{}_{(\dot\alpha}\nabla^{\gamma\dot\gamma}\xi_{\beta\gamma\dot\beta)\dot\gamma}+\frac{1}{2}\nabla_{\alpha\dot\alpha}\nabla_{\beta\dot\beta}\nabla_{\gamma\dot\gamma}\xi^{\beta\gamma\dot\beta\dot\gamma} \qquad \text{(C.103)}
\end{aligned}
$$

Then, plugging equations C.95, C.101, C.102 and C.103 into C.94,

$$
\begin{aligned}
&\nabla_{\alpha\dot\alpha}\nabla_{\beta\dot\beta}\nabla_{\gamma\dot\gamma}\xi^{\beta\gamma\dot\beta\dot\gamma}\\
&=\frac{6}{7}\left(\frac{2}{3}\nabla_{\alpha}{}^{\dot\beta}\nabla^{\beta}{}_{(\dot\alpha}\nabla^{\gamma\dot\gamma}\xi_{\beta\gamma\dot\beta)\dot\gamma}+2\nabla_{\alpha}{}^{\dot\beta}(E^{\beta\gamma\dot\gamma}{}_{(\dot\alpha}\xi_{\beta\gamma\dot\beta)\dot\gamma})\right)+\frac{6}{7}\left(\frac{2}{3}\nabla_{\alpha}{}^{\dot\beta}\nabla^{\beta}{}_{(\dot\alpha}\nabla^{\gamma\dot\gamma}\xi_{\beta\gamma\dot\beta)\dot\gamma}\right.\\
&\quad\left.+2\nabla^{\beta}{}_{\dot\alpha}(E_{(\alpha}{}^{\gamma\dot\beta\dot\gamma}\xi_{\beta)\gamma\dot\beta\dot\gamma})\right)+\frac{1}{7}\left(\nabla_{\alpha}{}^{\dot\beta}\nabla^{\beta}{}_{(\dot\alpha}\nabla^{\gamma\dot\gamma}\xi_{\beta\gamma\dot\beta)\dot\gamma}+\frac{1}{2}\nabla_{\alpha\dot\alpha}\nabla_{\beta\dot\beta}\nabla_{\gamma\dot\gamma}\xi^{\beta\gamma\dot\beta\dot\gamma}\right)\\
&\quad+\frac{1}{7}\left(\nabla_{\alpha}{}^{\dot\beta}\nabla^{\beta}{}_{(\dot\alpha}\nabla^{\gamma\dot\gamma}\xi_{\beta\gamma\dot\beta)\dot\gamma}+\frac{1}{2}\nabla_{\alpha\dot\alpha}\nabla_{\beta\dot\beta}\nabla_{\gamma\dot\gamma}\xi^{\beta\gamma\dot\beta\dot\gamma}\right)\\
&\quad+\frac{3}{7}(E_{\alpha}{}^{\gamma\dot\beta\dot\gamma}\nabla^{\beta}{}_{(\dot\alpha}\xi_{\beta\gamma\dot\beta\dot\gamma)}+\overline{C}_{\dot\alpha}{}^{\dot\beta\dot\gamma\dot\mu}\nabla^{\beta}{}_{(\dot\mu}\xi_{\alpha\beta\dot\beta\dot\gamma)})+\frac{3}{7}(C_{\alpha}{}^{\beta\gamma\mu}\nabla_{(\mu}{}^{\dot\beta}\xi_{\beta\gamma)\dot\alpha\dot\beta}+E^{\gamma\beta\dot\gamma}{}_{\dot\alpha}\nabla_{(\alpha}{}^{\dot\beta}\xi_{\beta\gamma)\dot\beta\dot\gamma})\\
&\quad-\frac{9}{7}\nabla^{\beta\dot\beta}(C_{\alpha\beta}{}^{\gamma\mu}\xi_{\gamma\mu\dot\alpha\dot\beta}+\overline{C}_{\dot\alpha\dot\beta}{}^{\dot\gamma\dot\mu}\xi_{\alpha\beta\dot\gamma\dot\mu}+2E^{\gamma}{}_{(\alpha}{}^{\dot\gamma}{}_{\dot\alpha}\xi_{\beta)\gamma\dot\beta\dot\gamma}+2E^{\gamma}{}_{\alpha}{}^{\dot\gamma}{}_{(\dot\alpha}\xi_{\beta\gamma\dot\beta)\dot\gamma}-8F\xi_{\alpha\beta\dot\alpha\dot\beta})\\
&\quad-\frac{16}{7}E^{\gamma}{}_{\alpha}{}^{\dot\gamma}{}_{\dot\alpha}\nabla^{\beta\dot\beta}\xi_{\beta\gamma\dot\beta\dot\gamma}+\frac{48}{7}F\nabla^{\beta\dot\beta}\xi_{\alpha\beta\dot\alpha\dot\beta}\\
&=\frac{10}{7}\nabla_{\alpha}{}^{\dot\beta}\nabla^{\beta}{}_{(\dot\alpha}\nabla^{\gamma\dot\gamma}\xi_{\beta\gamma\dot\beta)\dot\gamma}+\frac{12}{7}(\nabla_{\alpha}{}^{\dot\beta}(E^{\beta\gamma\dot\gamma}{}_{(\dot\alpha}\xi_{\beta\gamma\dot\beta)\dot\gamma})+\nabla^{\beta}{}_{\dot\alpha}(E_{(\alpha}{}^{\gamma\dot\beta\dot\gamma}\xi_{\beta)\gamma\dot\beta\dot\gamma}))+\frac{1}{7}\nabla_{\alpha\dot\alpha}\nabla_{\beta\dot\beta}\nabla_{\gamma\dot\gamma}\xi^{\beta\gamma\dot\beta\dot\gamma}\\
&\quad+\frac{3}{7}(E_{\alpha}{}^{\gamma\dot\beta\dot\gamma}\nabla^{\beta}{}_{(\dot\alpha}\xi_{\beta\gamma\dot\beta\dot\gamma)}+\overline{C}_{\dot\alpha}{}^{\dot\beta\dot\gamma\dot\mu}\nabla^{\beta}{}_{(\dot\mu}\xi_{\alpha\beta\dot\beta\dot\gamma)}+C_{\alpha}{}^{\beta\gamma\mu}\nabla_{(\mu}{}^{\dot\beta}\xi_{\beta\gamma)\dot\alpha\dot\beta}+E^{\gamma\beta\dot\gamma}{}_{\dot\alpha}\nabla_{(\alpha}{}^{\dot\beta}\xi_{\beta\gamma)\dot\beta\dot\gamma})\\
&\quad-\frac{9}{7}\nabla^{\beta\dot\beta}(C_{\alpha\beta}{}^{\gamma\mu}\xi_{\gamma\mu\dot\alpha\dot\beta}+\overline{C}_{\dot\alpha\dot\beta}{}^{\dot\gamma\dot\mu}\xi_{\alpha\beta\dot\gamma\dot\mu}+2E^{\gamma}{}_{(\alpha}{}^{\dot\gamma}{}_{\dot\alpha}\xi_{\beta)\gamma\dot\beta\dot\gamma}+2E^{\gamma}{}_{\alpha}{}^{\dot\gamma}{}_{(\dot\alpha}\xi_{\beta\gamma\dot\beta)\dot\gamma}-8F\xi_{\alpha\beta\dot\alpha\dot\beta})\\
&\quad-\frac{16}{7}E^{\gamma}{}_{\alpha}{}^{\dot\gamma}{}_{\dot\alpha}\nabla^{\beta\dot\beta}\xi_{\beta\gamma\dot\beta\dot\gamma}+\frac{48}{7}F\nabla^{\beta\dot\beta}\xi_{\alpha\beta\dot\alpha\dot\beta} \qquad \text{(C.104)}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\frac{2}{15}\nabla_{\alpha\dot\alpha}\nabla_{\beta\dot\beta}\nabla_{\gamma\dot\gamma}\xi^{\beta\gamma\dot\beta\dot\gamma}\\
&=\frac{2}{9}\nabla_\alpha{}^{\dot\beta}\nabla^\beta{}_{(\dot\alpha}\nabla^{\gamma\dot\gamma}\xi_{\beta\gamma\dot\beta)\dot\gamma}+\frac{4}{15}(\nabla_\alpha{}^{\dot\beta}(E^{\beta\gamma\dot\gamma}{}_{(\dot\alpha}\xi_{\beta\gamma\dot\beta)\dot\gamma})+\nabla^\beta{}_{\dot\alpha}(E_{(\alpha}{}^{\gamma\dot\beta\dot\gamma}\xi_{\beta)\gamma\dot\beta\dot\gamma}))\\
&\quad+\frac{1}{15}(E_\alpha{}^{\gamma\dot\beta\dot\gamma}\nabla^\beta{}_{(\dot\alpha}\xi_{\beta\gamma\dot\beta\dot\gamma)}+\overline{C}_{\dot\alpha}{}^{\dot\beta\dot\gamma\dot\mu}\nabla^\beta{}_{(\dot\mu}\xi_{\alpha\beta\dot\beta\dot\gamma)}+C_\alpha{}^{\beta\gamma\mu}\nabla_{(\mu}{}^{\dot\beta}\xi_{\beta\gamma)\dot\alpha\dot\beta}+E^{\gamma\beta\dot\gamma}{}_{\dot\alpha}\nabla_{(\alpha}{}^{\dot\beta}\xi_{\beta\gamma)\dot\beta\dot\gamma})\\
&\quad-\frac{1}{5}\nabla^{\beta\dot\beta}(C_{\alpha\beta}{}^{\gamma\mu}\xi_{\gamma\mu\dot\alpha\dot\beta}+\overline{C}_{\dot\alpha\dot\beta}{}^{\dot\gamma\dot\mu}\xi_{\alpha\beta\dot\gamma\dot\mu}+2E^\gamma{}_{(\alpha}{}^{\dot\gamma}{}_{\dot\alpha}\xi_{\beta)\gamma\dot\beta\dot\gamma}+2E^\gamma{}_\alpha{}^{\dot\gamma}{}_{(\dot\alpha}\xi_{\beta\gamma\dot\beta)\dot\gamma}-8F\xi_{\alpha\beta\dot\alpha\dot\beta})\\
&\quad-\frac{16}{45}E^\gamma{}_\alpha{}^{\dot\gamma}{}_{\dot\alpha}\nabla^{\beta\dot\beta}\xi_{\beta\gamma\dot\beta\dot\gamma}+\frac{16}{15}F\nabla^{\beta\dot\beta}\xi_{\alpha\beta\dot\alpha\dot\beta}\,.
\end{aligned}\tag{C.105}
$$

Then, using this expression in equation C.84 results in

$$
\begin{aligned}
&\nabla_{\alpha\dot\alpha}(\xi^{\dot\alpha\dot\beta})\overline{\chi}_{\dot\beta}+\nabla_{\alpha\dot\alpha}(\xi)\overline{\chi}^{\dot\alpha}\\
&=\Bigg(-\frac{1}{15}\nabla_\alpha{}^{\dot\beta}(E^{\beta\gamma\dot\gamma}{}_{(\dot\alpha}\xi_{\beta\gamma\dot\beta)\dot\gamma})+\frac{4}{15}\nabla^\beta{}_{\dot\alpha}(E_{(\alpha}{}^{\gamma\dot\beta\dot\gamma}\xi_{\beta)\gamma\dot\beta\dot\gamma})\\
&\quad+\frac{1}{15}(E_\alpha{}^{\gamma\dot\beta\dot\gamma}\nabla^\beta{}_{(\dot\alpha}\xi_{\beta\gamma\dot\beta\dot\gamma)}+\overline{C}_{\dot\alpha}{}^{\dot\beta\dot\gamma\dot\mu}\nabla^\beta{}_{(\dot\mu}\xi_{\alpha\beta\dot\beta\dot\gamma)}+C_\alpha{}^{\beta\gamma\mu}\nabla_{(\mu}{}^{\dot\beta}\xi_{\beta\gamma)\dot\alpha\dot\beta}+E^{\gamma\beta\dot\gamma}{}_{\dot\alpha}\nabla_{(\alpha}{}^{\dot\beta}\xi_{\beta\gamma)\dot\beta\dot\gamma})\\
&\quad-\frac{1}{5}\nabla^{\beta\dot\beta}(C_{\alpha\beta}{}^{\gamma\mu}\xi_{\gamma\mu\dot\alpha\dot\beta}+\overline{C}_{\dot\alpha\dot\beta}{}^{\dot\gamma\dot\mu}\xi_{\alpha\beta\dot\gamma\dot\mu}+2E^\gamma{}_{(\alpha}{}^{\dot\gamma}{}_{\dot\alpha}\xi_{\beta)\gamma\dot\beta\dot\gamma}+2E^\gamma{}_\alpha{}^{\dot\gamma}{}_{(\dot\alpha}\xi_{\beta\gamma\dot\beta)\dot\gamma}-8F\xi_{\alpha\beta\dot\alpha\dot\beta})\\
&\quad-\frac{16}{45}E^\gamma{}_\alpha{}^{\dot\gamma}{}_{\dot\alpha}\nabla^{\beta\dot\beta}\xi_{\beta\gamma\dot\beta\dot\gamma}+\frac{16}{15}F\nabla^{\beta\dot\beta}\xi_{\alpha\beta\dot\alpha\dot\beta}-\frac{7}{10}\nabla_{\alpha\dot\alpha}(E_{\beta\gamma\dot\beta\dot\gamma}\xi^{\beta\gamma\dot\beta\dot\gamma})\Bigg)\overline{\chi}^{\dot\alpha}\,.
\end{aligned}\tag{C.106}
$$

Finally, putting equations C.106, C.83, C.69 and C.65 into equation C.61 gives

$$
\begin{aligned}
\psi'_\alpha&=\frac{1}{3}\nabla^\beta{}_{(\dot\alpha}\xi_{\alpha\beta\dot\beta\dot\gamma)}\overline{C}^{\dot\alpha\dot\beta\dot\gamma\dot\mu}\overline{\chi}_{\dot\mu}+\frac{1}{3}E^{\beta\gamma\dot\gamma}{}_{\dot\alpha}\nabla_{(\alpha}{}^{\dot\beta}\xi_{\beta\gamma)\dot\beta\dot\gamma}\overline{\chi}^{\dot\alpha}+\frac{10}{9}E_{\alpha\beta\dot\alpha\dot\beta}\nabla_{\gamma\dot\gamma}(\xi^{\beta\gamma\dot\beta\dot\gamma})\overline{\chi}^{\dot\alpha}\\
&\quad+\xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}(E_{\alpha\gamma\dot\alpha\dot\gamma})\overline{\chi}^{\dot\alpha}+2\xi^{\beta\gamma\dot\beta\dot\gamma}E_{\alpha\gamma\dot\alpha\dot\gamma}\nabla_{\beta\dot\beta}\overline{\chi}^{\dot\alpha}-3\xi_{\alpha\beta\dot\alpha\dot\beta}\nabla^{\beta\dot\beta}(F)\overline{\chi}^{\dot\alpha}-8\xi_{\alpha\beta\dot\alpha\dot\beta}F\nabla^{\beta\dot\beta}\overline{\chi}^{\dot\alpha}\\
&\quad+\xi^{\beta\gamma}{}_{\dot\alpha}{}^{\dot\gamma}C_{\alpha\beta\gamma}{}^\mu\nabla_{\mu\dot\gamma}\overline{\chi}^{\dot\alpha}+\xi_\alpha{}^{\beta\dot\beta\dot\gamma}E_\beta{}^\gamma{}_{\dot\alpha\dot\beta}\nabla_{\gamma\dot\gamma}\overline{\chi}^{\dot\alpha}+\xi_\alpha{}^{\gamma\dot\beta\dot\gamma}\overline{C}_{\dot\alpha\dot\beta\dot\gamma}{}^{\dot\mu}\nabla_{\gamma\dot\mu}\overline{\chi}^{\dot\alpha}+\xi^{\beta\gamma}{}_{\dot\alpha}{}^{\dot\gamma}E_{\alpha\beta\dot\gamma}{}^{\dot\mu}\nabla_{\gamma\dot\mu}\overline{\chi}^{\dot\alpha}\\
&\quad-\frac{8}{3}\nabla^{\beta\dot\beta}(\xi_{\alpha\beta\dot\alpha\dot\beta})F\overline{\chi}^{\dot\alpha}+\Bigg(-\frac{2}{3}\overline{C}_{\dot\beta\dot\gamma}{}^{\dot\alpha\dot\mu}\xi_{\alpha\beta\dot\alpha\dot\mu}-\frac{4}{3}C_{\alpha\beta}{}^{\gamma\mu}\xi_{\gamma\mu\dot\beta\dot\gamma}-4E_{(\alpha}{}^\gamma{}_{(\dot\beta}{}^{\dot\alpha}\xi_{\beta)\gamma\dot\gamma)\dot\alpha}+8F\xi_{\alpha\beta\dot\beta\dot\gamma}\\
&\quad-\varepsilon_{\alpha\beta}E^{\gamma\mu\dot\alpha}{}_{(\dot\beta}\xi_{\gamma\mu\dot\gamma)\dot\alpha}\Bigg)\nabla^{\beta\dot\beta}\overline{\chi}^{\dot\gamma}+\Bigg(-\frac{1}{15}\nabla_\alpha{}^{\dot\beta}(E^{\beta\gamma\dot\gamma}{}_{(\dot\alpha}\xi_{\beta\gamma\dot\beta)\dot\gamma})+\frac{4}{15}\nabla^\beta{}_{\dot\alpha}(E_{(\alpha}{}^{\gamma\dot\beta\dot\gamma}\xi_{\beta)\gamma\dot\beta\dot\gamma})\\
&\quad+\frac{1}{15}(E_\alpha{}^{\gamma\dot\beta\dot\gamma}\nabla^\beta{}_{(\dot\alpha}\xi_{\beta\gamma\dot\beta\dot\gamma)}+\overline{C}_{\dot\alpha}{}^{\dot\beta\dot\gamma\dot\mu}\nabla^\beta{}_{(\dot\mu}\xi_{\alpha\beta\dot\beta\dot\gamma)}+C_\alpha{}^{\beta\gamma\mu}\nabla_{(\mu}{}^{\dot\beta}\xi_{\beta\gamma)\dot\alpha\dot\beta}+E^{\gamma\beta\dot\gamma}{}_{\dot\alpha}\nabla_{(\alpha}{}^{\dot\beta}\xi_{\beta\gamma)\dot\beta\dot\gamma})\\
&\quad-\frac{1}{5}\nabla^{\beta\dot\beta}(C_{\alpha\beta}{}^{\gamma\mu}\xi_{\gamma\mu\dot\alpha\dot\beta}+\overline{C}_{\dot\alpha\dot\beta}{}^{\dot\gamma\dot\mu}\xi_{\alpha\beta\dot\gamma\dot\mu}+2E^\gamma{}_{(\alpha}{}^{\dot\gamma}{}_{\dot\alpha}\xi_{\beta)\gamma\dot\beta\dot\gamma}+2E^\gamma{}_\alpha{}^{\dot\gamma}{}_{(\dot\alpha}\xi_{\beta\gamma\dot\beta)\dot\gamma}-8F\xi_{\alpha\beta\dot\alpha\dot\beta})\\
&\quad-\frac{16}{45}E^\gamma{}_\alpha{}^{\dot\gamma}{}_{\dot\alpha}\nabla^{\beta\dot\beta}(\xi_{\beta\gamma\dot\beta\dot\gamma})+\frac{16}{15}F\nabla^{\beta\dot\beta}(\xi_{\alpha\beta\dot\alpha\dot\beta})-\frac{7}{10}\nabla_{\alpha\dot\alpha}(E_{\beta\gamma\dot\beta\dot\gamma}\xi^{\beta\gamma\dot\beta\dot\gamma})\Bigg)\overline{\chi}^{\dot\alpha}\,,
\end{aligned}\tag{C.107}
$$

thereby completing the task of removing curvature-less terms from ψ'_α. This expression still needs to to be simplified a lot to look like the 1st two components of equation C.2. Collecting like terms in the last equation,

$$
\begin{aligned}
\psi'_\alpha = \Big(&2\xi_\beta{}^\gamma{}_{\dot\beta}{}^{\dot\gamma}E_{\alpha\gamma\dot\alpha\dot\gamma} - 8\xi_{\alpha\beta\dot\alpha\dot\beta}F + C_{\alpha\beta}{}^{\gamma\mu}\xi_{\gamma\mu\dot\alpha\dot\beta} + \xi_\alpha{}^\gamma{}_{\dot\beta}{}^{\dot\gamma}E_{\beta\gamma\dot\alpha\dot\gamma} + \overline{C}_{\dot\alpha\dot\beta}{}^{\dot\gamma\dot\mu}\xi_{\alpha\beta\dot\gamma\dot\mu} + \xi_\beta{}^\gamma{}_{\dot\alpha}{}^{\dot\gamma}E_{\alpha\gamma\dot\beta\dot\gamma}\\
&- \frac{2}{3}\overline{C}_{\dot\alpha\dot\beta}{}^{\dot\gamma\dot\mu}\xi_{\alpha\beta\dot\gamma\dot\mu} - \frac{4}{3}C_{\alpha\beta}{}^{\gamma\mu}\xi_{\gamma\mu\dot\alpha\dot\beta} - 4E_{(\alpha}{}^\gamma{}_{(\dot\alpha}{}^{\dot\gamma}\xi_{\beta)\gamma\dot\beta)\dot\gamma} + 8F\xi_{\alpha\beta\dot\alpha\dot\beta} - \varepsilon_{\alpha\beta}E^{\gamma\mu\dot\gamma}{}_{(\dot\alpha}\xi_{\gamma\mu\dot\beta)\dot\gamma}\Big)\nabla^{\beta\dot\beta}\overline{\chi}^{\dot\alpha}\\
&+\Big(-\frac{1}{3}\overline{C}_{\dot\alpha}{}^{\dot\beta\dot\gamma\dot\mu}\nabla^\beta{}_{(\dot\mu}\xi_{\alpha\beta\dot\gamma\dot\beta)} + \frac{1}{3}E^{\beta\gamma\dot\gamma}{}_{\dot\alpha}\nabla_{(\alpha}{}^{\dot\beta}\xi_{\beta\gamma)\dot\beta\dot\gamma} + \frac{10}{9}E_{\alpha\beta\dot\alpha\dot\beta}\nabla_{\gamma\dot\gamma}(\xi^{\beta\gamma\dot\beta\dot\gamma}) + \xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}(E_{\alpha\gamma\dot\alpha\dot\gamma})\\
&- 3\xi_{\alpha\beta\dot\alpha\dot\beta}\nabla^{\beta\dot\beta}(F) - \frac{8}{3}\nabla^{\beta\dot\beta}(\xi_{\alpha\beta\dot\alpha\dot\beta})F - \frac{1}{15}\nabla_\alpha{}^{\dot\beta}(E^{\beta\gamma\dot\gamma}{}_{(\dot\alpha}\xi_{\beta\gamma\dot\beta)\dot\gamma}) + \frac{4}{15}\nabla^\beta{}_{\dot\alpha}(E_{(\alpha}{}^{\gamma\dot\beta\dot\gamma}\xi_{\beta)\gamma\dot\beta\dot\gamma})\\
&+ \frac{1}{15}(E_\alpha{}^{\gamma\dot\beta\dot\gamma}\nabla^\beta{}_{(\dot\alpha}\xi_{\beta\gamma\dot\beta\dot\gamma)} + \overline{C}_{\dot\alpha}{}^{\dot\beta\dot\gamma\dot\mu}\nabla^\beta{}_{(\dot\mu}\xi_{\alpha\beta\dot\beta\dot\gamma)} + C_\alpha{}^{\beta\gamma\mu}\nabla_{(\mu}{}^{\dot\beta}\xi_{\beta\gamma)\dot\alpha\dot\beta} + E^{\gamma\beta\dot\gamma}{}_{\dot\alpha}\nabla_{(\alpha}{}^{\dot\beta}\xi_{\beta\gamma)\dot\beta\dot\gamma})\\
&- \frac{1}{5}\nabla^{\beta\dot\beta}(C_{\alpha\beta}{}^{\gamma\mu}\xi_{\gamma\mu\dot\alpha\dot\beta} + \overline{C}_{\dot\alpha\dot\beta}{}^{\dot\gamma\dot\mu}\xi_{\alpha\beta\dot\gamma\dot\mu} + 2E^\gamma{}_{(\alpha}{}^{\dot\gamma}{}_{\dot\alpha}\xi_{\beta)\gamma\dot\beta\dot\gamma} + 2E^\gamma{}_\alpha{}^{\dot\gamma}{}_{(\dot\alpha}\xi_{\beta\gamma\dot\beta)\dot\gamma} - 8F\xi_{\alpha\beta\dot\alpha\dot\beta})\\
&- \frac{16}{45}E^\gamma{}_\alpha{}^{\dot\gamma}{}_{\dot\alpha}\nabla^{\beta\dot\beta}(\xi_{\beta\gamma\dot\beta\dot\gamma}) + \frac{16}{15}F\nabla^{\beta\dot\beta}(\xi_{\alpha\beta\dot\alpha\dot\beta}) - \frac{7}{10}\nabla_{\alpha\dot\alpha}(E_{\beta\gamma\dot\beta\dot\gamma}\xi^{\beta\gamma\dot\beta\dot\gamma})\Big)\overline{\chi}^{\dot\alpha}\\
= \Big(&\frac{1}{3}\overline{C}_{\dot\alpha\dot\beta}{}^{\dot\gamma\dot\mu}\xi_{\alpha\beta\dot\gamma\dot\mu} - \frac{1}{3}C_{\alpha\beta}{}^{\gamma\mu}\xi_{\gamma\mu\dot\alpha\dot\beta} + 2\xi_\beta{}^\gamma{}_{\dot\beta}{}^{\dot\gamma}E_{\alpha\gamma\dot\alpha\dot\gamma} + \xi_\alpha{}^\gamma{}_{\dot\beta}{}^{\dot\gamma}E_{\beta\gamma\dot\alpha\dot\gamma} + \xi_\beta{}^\gamma{}_{\dot\alpha}{}^{\dot\gamma}E_{\alpha\gamma\dot\beta\dot\gamma}\\
&- 4E_{(\alpha}{}^\gamma{}_{(\dot\alpha}{}^{\dot\gamma}\xi_{\beta)\gamma\dot\beta)\dot\gamma} - \varepsilon_{\alpha\beta}E^{\gamma\mu\dot\gamma}{}_{(\dot\alpha}\xi_{\gamma\mu\dot\beta)\dot\gamma}\Big)\nabla^{\beta\dot\beta}\overline{\chi}^{\dot\alpha}\\
&+\Big(-\frac{4}{15}\overline{C}_{\dot\alpha}{}^{\dot\beta\dot\gamma\dot\mu}\nabla^\beta{}_{(\dot\mu}\xi_{\alpha\beta\dot\gamma\dot\beta)} + \frac{2}{5}E^{\beta\gamma\dot\gamma}{}_{\dot\alpha}\nabla_{(\alpha}{}^{\dot\beta}\xi_{\beta\gamma)\dot\beta\dot\gamma} + \frac{34}{45}E_{\alpha\beta\dot\alpha\dot\beta}\nabla_{\gamma\dot\gamma}(\xi^{\beta\gamma\dot\beta\dot\gamma}) + \xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}(E_{\alpha\gamma\dot\alpha\dot\gamma})\\
&- \frac{7}{5}\xi_{\alpha\beta\dot\alpha\dot\beta}\nabla^{\beta\dot\beta}(F) - \frac{1}{15}\nabla_\alpha{}^{\dot\beta}(E^{\beta\gamma\dot\gamma}{}_{(\dot\alpha}\xi_{\beta\gamma\dot\beta)\dot\gamma}) + \frac{4}{15}\nabla^\beta{}_{\dot\alpha}(E_{(\alpha}{}^{\gamma\dot\beta\dot\gamma}\xi_{\beta)\gamma\dot\beta\dot\gamma}) - \frac{7}{10}\nabla_{\alpha\dot\alpha}(E_{\beta\gamma\dot\beta\dot\gamma}\xi^{\beta\gamma\dot\beta\dot\gamma})\\
&+ \frac{1}{15}(E_\alpha{}^{\gamma\dot\beta\dot\gamma}\nabla^\beta{}_{(\dot\alpha}\xi_{\beta\gamma\dot\beta\dot\gamma)} + C_\alpha{}^{\beta\gamma\mu}\nabla_{(\mu}{}^{\dot\beta}\xi_{\beta\gamma)\dot\alpha\dot\beta})\\
&- \frac{1}{5}\nabla^{\beta\dot\beta}(C_{\alpha\beta}{}^{\gamma\mu}\xi_{\gamma\mu\dot\alpha\dot\beta} + \overline{C}_{\dot\alpha\dot\beta}{}^{\dot\gamma\dot\mu}\xi_{\alpha\beta\dot\gamma\dot\mu} + 2E^\gamma{}_{(\alpha}{}^{\dot\gamma}{}_{\dot\alpha}\xi_{\beta)\gamma\dot\beta\dot\gamma} + 2E^\gamma{}_\alpha{}^{\dot\gamma}{}_{(\dot\alpha}\xi_{\beta\gamma\dot\beta)\dot\gamma})\Big)\overline{\chi}^{\dot\alpha}\\
= \Big(&\frac{1}{3}\overline{C}_{\dot\alpha\dot\beta}{}^{\dot\gamma\dot\mu}\xi_{\alpha\beta\dot\gamma\dot\mu} - \frac{1}{3}C_{\alpha\beta}{}^{\gamma\mu}\xi_{\gamma\mu\dot\alpha\dot\beta} + \{1\}\Big)\nabla^{\beta\dot\beta}\overline{\chi}^{\dot\alpha}\\
&+\Big(-\frac{4}{15}\overline{C}_{\dot\alpha}{}^{\dot\beta\dot\gamma\dot\mu}\nabla^\beta{}_{(\dot\mu}\xi_{\alpha\beta\dot\gamma\dot\beta)} + \frac{1}{15}C_\alpha{}^{\beta\gamma\mu}\nabla_{(\mu}{}^{\dot\beta}\xi_{\beta\gamma)\dot\alpha\dot\beta}\\
&- \frac{1}{5}\nabla^{\beta\dot\beta}(C_{\alpha\beta}{}^{\gamma\mu}\xi_{\gamma\mu\dot\alpha\dot\beta} + \overline{C}_{\dot\alpha\dot\beta}{}^{\dot\gamma\dot\mu}\xi_{\alpha\beta\dot\gamma\dot\mu}) + \{0\}\Big)\overline{\chi}^{\dot\alpha} \quad \text{say,}
\end{aligned}
\tag{C.108}
$$

i.e. $\{1\}$ and $\{0\}$ are a collection coefficients not involving the Weyl tensor. Since $\nabla^{\beta\dot\beta}\overline{\chi}_{\dot\beta} = 0$, $\{1\}$ can be symmetrised between and $\dot\alpha$ and $\dot\beta$. Thus,

$$
\begin{aligned}
\{1\} &= 2\xi_\beta{}^\gamma{}_{(\dot\beta}{}^{\dot\gamma}E_{\alpha\gamma\dot\alpha)\dot\gamma} + \xi_\alpha{}^\gamma{}_{(\dot\beta}{}^{\dot\gamma}E_{\beta\gamma\dot\alpha)\dot\gamma} + \xi_\beta{}^\gamma{}_{(\dot\alpha}{}^{\dot\gamma}E_{\alpha\gamma\dot\beta)\dot\gamma} - 4E_{(\alpha}{}^\gamma{}_{(\dot\alpha}{}^{\dot\gamma}\xi_{\beta)\gamma\dot\beta)\dot\gamma}\\
&\quad - \varepsilon_{\alpha\beta}E^{\gamma\mu\dot\gamma}{}_{(\dot\alpha}\xi_{\gamma\mu\dot\beta)\dot\gamma}\\
&= 2E_\alpha{}^\gamma{}_{(\dot\alpha}{}^{\dot\gamma}\xi_{\beta\gamma\dot\beta)\dot\gamma} - 2E_{(\alpha}{}^\gamma{}_{(\dot\alpha}{}^{\dot\gamma}\xi_{\beta)\gamma\dot\beta)\dot\gamma} - \varepsilon_{\alpha\beta}E^{\gamma\mu\dot\gamma}{}_{(\dot\alpha}\xi_{\gamma\mu\dot\beta)\dot\gamma}\\
&= 2E_{(\alpha}{}^\gamma{}_{(\dot\alpha}{}^{\dot\gamma}\xi_{\beta)\gamma\dot\beta)\dot\gamma} + \varepsilon_{\alpha\beta}E^{\mu\gamma}{}_{(\dot\alpha}{}^{\dot\gamma}\xi_{\mu\gamma\dot\beta)\dot\gamma} - 2E_{(\alpha}{}^\gamma{}_{(\dot\alpha}{}^{\dot\gamma}\xi_{\beta)\gamma\dot\beta)\dot\gamma} - \varepsilon_{\alpha\beta}E^{\gamma\mu\dot\gamma}{}_{(\dot\alpha}\xi_{\gamma\mu\dot\beta)\dot\gamma}\\
&= 0\,.
\end{aligned}
\tag{C.109}
$$

That leaves only {0} to simplify.

$$
\begin{aligned}
\{0\} = {}& \frac{2}{5}E^{\beta\gamma\dot\gamma}{}_{\dot\alpha}\nabla_{(\alpha}{}^{\dot\beta}\xi_{\beta\gamma)\dot\beta\dot\gamma} + \frac{34}{45}E_{\alpha\beta\dot\alpha\dot\beta}\nabla_{\gamma\dot\gamma}(\xi^{\beta\gamma\dot\beta\dot\gamma}) + \xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}(E_{\alpha\gamma\dot\alpha\dot\gamma}) - \frac{7}{5}\xi_{\alpha\beta\dot\alpha\dot\beta}\nabla^{\beta\dot\beta}(F) \\
& - \frac{1}{15}\nabla_\alpha{}^{\dot\beta}(E^{\beta\gamma\dot\gamma}{}_{(\dot\alpha}\xi_{\beta\gamma\dot\beta)\dot\gamma}) + \frac{4}{15}\nabla^\beta{}_{\dot\alpha}(E_{(\alpha}{}^{\gamma\dot\beta\dot\gamma}\xi_{\beta)\gamma\dot\beta\dot\gamma}) - \frac{7}{10}\nabla_{\alpha\dot\alpha}(E_{\beta\gamma\dot\beta\dot\gamma}\xi^{\beta\gamma\dot\beta\dot\gamma}) \\
& + \frac{1}{15}E_\alpha{}^{\gamma\dot\beta\dot\gamma}\nabla^\beta{}_{(\dot\alpha}\xi_{\beta\gamma\dot\beta\dot\gamma)} - \frac{2}{5}\nabla^{\beta\dot\beta}(E^\gamma{}_{(\alpha}{}^{\dot\gamma}{}_{\dot\alpha}\xi_{\beta)\gamma\dot\beta\dot\gamma} + E^\gamma{}_\alpha{}^{\dot\gamma}{}_{(\dot\alpha}\xi_{\beta\gamma\dot\beta)\dot\gamma}) \\
= {}& \frac{2}{5}E^{\beta\gamma\dot\gamma}{}_{\dot\alpha}\nabla_{(\alpha}{}^{\dot\beta}\xi_{\beta\gamma)\dot\beta\dot\gamma} + \frac{34}{45}E_{\alpha\beta\dot\alpha\dot\beta}\nabla_{\gamma\dot\gamma}(\xi^{\beta\gamma\dot\beta\dot\gamma}) + \xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}(E_{\alpha\gamma\dot\alpha\dot\gamma}) - \frac{7}{5}\xi_{\alpha\beta\dot\alpha\dot\beta}\nabla^{\beta\dot\beta}(F) \\
& - \frac{1}{15}\nabla_\alpha{}^{\dot\beta}(E^{\beta\gamma\dot\gamma}{}_{(\dot\alpha})\xi_{\beta\gamma\dot\beta)\dot\gamma} - \frac{1}{15}E^{\beta\gamma\dot\gamma}{}_{(\dot\alpha}\nabla_\alpha{}^{\dot\beta}\xi_{\beta\gamma\dot\beta)\dot\gamma} + \frac{4}{15}\nabla^\beta{}_{\dot\alpha}(E_{(\alpha}{}^{\gamma\dot\beta\dot\gamma})\xi_{\beta)\gamma\dot\beta\dot\gamma} \\
& + \frac{4}{15}E_{(\alpha}{}^{\gamma\dot\beta\dot\gamma}\nabla^\beta{}_{\dot\alpha}\xi_{\beta)\gamma\dot\beta\dot\gamma} - \frac{7}{10}\nabla_{\alpha\dot\alpha}(E_{\beta\gamma\dot\beta\dot\gamma})\xi^{\beta\gamma\dot\beta\dot\gamma} - \frac{7}{10}E^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\alpha\dot\alpha}\xi_{\beta\gamma\dot\beta\dot\gamma} + \frac{1}{15}E_\alpha{}^{\gamma\dot\beta\dot\gamma}\nabla^\beta{}_{(\dot\alpha}\xi_{\beta\gamma\dot\beta\dot\gamma)} \\
& - \frac{2}{5}\nabla^{\beta\dot\beta}(E^\gamma{}_{(\alpha}{}^{\dot\gamma}{}_{\dot\alpha})\xi_{\beta)\gamma\dot\beta\dot\gamma} - \frac{2}{5}E^\gamma{}_{(\alpha}{}^{\dot\gamma}{}_{\dot\alpha}\nabla^{\beta\dot\beta}\xi_{\beta)\gamma\dot\beta\dot\gamma} - \frac{2}{5}\nabla^{\beta\dot\beta}(E^\gamma{}_\alpha{}^{\dot\gamma}{}_{(\dot\alpha})\xi_{\beta\gamma\dot\beta)\dot\gamma} - \frac{2}{5}E^\gamma{}_\alpha{}^{\dot\gamma}{}_{(\dot\alpha}\nabla^{\beta\dot\beta}\xi_{\beta\gamma\dot\beta)\dot\gamma}
\end{aligned}
\tag{C.110}
$$

$$
\begin{aligned}
& E^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\alpha\dot\alpha}\xi_{\beta\gamma\dot\beta\dot\gamma} \\
& = E^{\beta\gamma\dot\beta\dot\gamma}\Big(\nabla_{(\alpha(\dot\alpha}\xi_{\beta\gamma)\dot\beta\dot\gamma)} + \frac{1}{3}\varepsilon_{\alpha\beta}\nabla^\mu{}_{(\dot\alpha}\xi_{\mu\gamma\dot\beta\dot\gamma)} + \frac{1}{3}\varepsilon_{\alpha\gamma}\nabla^\mu{}_{(\dot\alpha}\xi_{\beta\mu\dot\beta\dot\gamma)} + \frac{1}{3}\varepsilon_{\dot\alpha\dot\beta}\nabla_{(\alpha}{}^{\dot\mu}\xi_{\beta\gamma)\dot\mu\dot\gamma} \\
& \quad + \frac{1}{3}\varepsilon_{\dot\alpha\dot\gamma}\nabla_{(\alpha}{}^{\dot\mu}\xi_{\beta\gamma)\dot\beta\dot\mu} + \frac{1}{9}\varepsilon_{\alpha\beta}\varepsilon_{\dot\alpha\dot\beta}\nabla^{\mu\dot\mu}\xi_{\mu\gamma\dot\mu\dot\gamma} + \frac{1}{9}\varepsilon_{\alpha\gamma}\varepsilon_{\dot\alpha\dot\beta}\nabla^{\mu\dot\mu}\xi_{\beta\mu\dot\mu\dot\gamma} + \frac{1}{9}\varepsilon_{\alpha\beta}\varepsilon_{\dot\alpha\dot\gamma}\nabla^{\mu\dot\mu}\xi_{\mu\gamma\dot\beta\dot\mu} \\
& \quad + \frac{1}{9}\varepsilon_{\alpha\gamma}\varepsilon_{\dot\alpha\dot\gamma}\nabla^{\mu\dot\mu}\xi_{\beta\mu\dot\beta\dot\mu}\Big) \\
& = \frac{2}{3}E_\alpha{}^{\gamma\dot\beta\dot\gamma}\nabla^\beta{}_{(\dot\alpha}\xi_{\beta\gamma\dot\beta\dot\gamma)} + \frac{2}{3}E^{\beta\gamma\dot\gamma}{}_{\dot\alpha}\nabla_{(\alpha}{}^{\dot\beta}\xi_{\beta\gamma)\dot\beta\dot\gamma} + \frac{4}{9}E_{\alpha\beta\dot\alpha\dot\beta}\nabla_{\gamma\dot\gamma}\xi^{\beta\gamma\dot\beta\dot\gamma}
\end{aligned}
\tag{C.111}
$$

Hence,

$$
\begin{aligned}
& \{0\} \\
& = -\frac{1}{15}E^{\gamma\beta\dot\gamma}{}_{\dot\alpha}\nabla_{(\alpha}{}^{\dot\beta}\xi_{\beta\gamma)\dot\beta\dot\gamma} + \frac{4}{9}E_{\alpha\beta\dot\alpha\dot\beta}\nabla_{\gamma\dot\gamma}(\xi^{\beta\gamma\dot\beta\dot\gamma}) + \xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}(E_{\alpha\gamma\dot\alpha\dot\gamma}) - \frac{7}{5}\xi_{\alpha\beta\dot\alpha\dot\beta}\nabla^{\beta\dot\beta}(F) \\
& - \frac{1}{15}\nabla_\alpha{}^{\dot\beta}(E^{\beta\gamma\dot\gamma}{}_{(\dot\alpha})\xi_{\beta\gamma\dot\beta)\dot\gamma} - \frac{1}{15}E^{\beta\gamma\dot\gamma}{}_{(\dot\alpha}\nabla_\alpha{}^{\dot\beta}\xi_{\beta\gamma\dot\beta)\dot\gamma} + \frac{4}{15}\nabla^\beta{}_{\dot\alpha}(E_{(\alpha}{}^{\gamma\dot\beta\dot\gamma})\xi_{\beta)\gamma\dot\beta\dot\gamma} \\
& + \frac{4}{15}E_{(\alpha}{}^{\gamma\dot\beta\dot\gamma}\nabla^\beta{}_{\dot\alpha}\xi_{\beta)\gamma\dot\beta\dot\gamma} - \frac{7}{10}\nabla_{\alpha\dot\alpha}(E_{\beta\gamma\dot\beta\dot\gamma})\xi^{\beta\gamma\dot\beta\dot\gamma} - \frac{2}{5}E_\alpha{}^{\gamma\dot\beta\dot\gamma}\nabla^\beta{}_{(\dot\alpha}\xi_{\beta\gamma\dot\beta\dot\gamma)} - \frac{2}{5}\nabla^{\beta\dot\beta}(E^\gamma{}_{(\alpha}{}^{\dot\gamma}{}_{\dot\alpha})\xi_{\beta)\gamma\dot\beta\dot\gamma} \\
& - \frac{2}{5}E^\gamma{}_{(\alpha}{}^{\dot\gamma}{}_{\dot\alpha}\nabla^{\beta\dot\beta}\xi_{\beta)\gamma\dot\beta\dot\gamma} - \frac{2}{5}\nabla^{\beta\dot\beta}(E^\gamma{}_\alpha{}^{\dot\gamma}{}_{(\dot\alpha})\xi_{\beta\gamma\dot\beta)\dot\gamma} - \frac{2}{5}E^\gamma{}_\alpha{}^{\dot\gamma}{}_{(\dot\alpha}\nabla^{\beta\dot\beta}\xi_{\beta\gamma\dot\beta)\dot\gamma}\,.
\end{aligned}
\tag{C.112}
$$

$$
\begin{aligned}
& E^{\beta\gamma\dot\gamma}{}_{(\dot\alpha}\nabla_\alpha{}^{\dot\beta}\xi_{\beta\gamma\dot\beta)\dot\gamma} \\
& = E^{\beta\gamma\dot\gamma}{}_{\dot\alpha}\nabla_\alpha{}^{\dot\beta}\xi_{\beta\gamma\dot\beta\dot\gamma} - \frac{1}{2}\varepsilon_{\dot\alpha\dot\beta}E^{\beta\gamma\dot\beta\dot\mu}\nabla_\alpha{}^{\dot\beta}\xi_{\beta\gamma\dot\mu\dot\gamma} \\
& = E^{\beta\gamma\dot\gamma}{}_{\dot\alpha}\nabla_\alpha{}^{\dot\beta}\xi_{\beta\gamma\dot\beta\dot\gamma} - \frac{1}{2}E^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\alpha\dot\alpha}\xi_{\beta\gamma\dot\beta\dot\gamma} \\
& = E^{\gamma\beta\dot\gamma}{}_{\dot\alpha}\nabla_{(\alpha}{}^{\dot\beta}\xi_{\beta\gamma)\dot\beta\dot\gamma} + \frac{1}{3}\varepsilon_{\alpha\beta}E^{\gamma\beta\dot\gamma}{}_{\dot\alpha}\nabla^{\mu\dot\beta}\xi_{\mu\gamma\dot\beta\dot\gamma} + \frac{1}{3}\varepsilon_{\alpha\gamma}E^{\gamma\beta\dot\gamma}{}_{\dot\alpha}\nabla^{\mu\dot\beta}\xi_{\beta\mu\dot\beta\dot\gamma} - \frac{1}{3}E_\alpha{}^{\gamma\dot\beta\dot\gamma}\nabla^\beta{}_{(\dot\alpha}\xi_{\beta\gamma\dot\beta\dot\gamma)} \\
& \quad - \frac{1}{3}E^{\gamma\beta\dot\gamma}{}_{\dot\alpha}\nabla_{(\alpha}{}^{\dot\beta}\xi_{\beta\gamma)\dot\beta\dot\gamma} - \frac{2}{9}E_{\alpha\beta\dot\alpha\dot\beta}\nabla_{\gamma\dot\gamma}\xi^{\beta\gamma\dot\beta\dot\gamma} \quad \text{using the result above} \\
& = \frac{2}{3}E^{\gamma\beta\dot\gamma}{}_{\dot\alpha}\nabla_{(\alpha}{}^{\dot\beta}\xi_{\beta\gamma)\dot\beta\dot\gamma} - \frac{1}{3}E_\alpha{}^{\gamma\dot\beta\dot\gamma}\nabla^\beta{}_{(\dot\alpha}\xi_{\beta\gamma\dot\beta\dot\gamma)} + \frac{4}{9}E_{\alpha\beta\dot\alpha\dot\beta}\nabla_{\gamma\dot\gamma}\xi^{\beta\gamma\dot\beta\dot\gamma}
\end{aligned}
\tag{C.113}
$$

$$
\begin{aligned}
\{0\} = &-\frac{1}{9}E^{\gamma\beta\dot\gamma}{}_{\dot\alpha}\nabla_{(\alpha}{}^{\dot\beta}\xi_{\beta\gamma)\dot\beta\dot\gamma} + \frac{56}{135}E_{\alpha\beta\dot\alpha\dot\beta}\nabla_{\gamma\dot\gamma}(\xi^{\beta\gamma\dot\beta\dot\gamma}) + \xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}(E_{\alpha\gamma\dot\alpha\dot\gamma}) - \frac{7}{5}\xi_{\alpha\beta\dot\alpha\dot\beta}\nabla^{\beta\dot\beta}(F) \\
&- \frac{1}{15}\nabla_{\alpha}{}^{\dot\beta}(E^{\beta\gamma\dot\gamma}{}_{(\dot\alpha})\xi_{\beta\gamma\dot\beta)\dot\gamma} + \frac{4}{15}\nabla^{\beta}{}_{\dot\alpha}(E_{(\alpha}{}^{\gamma\dot\beta\dot\gamma})\xi_{\beta)\gamma\dot\beta\dot\gamma} + \frac{4}{15}E_{(\alpha}{}^{\gamma\dot\beta\dot\gamma}\nabla^{\beta}{}_{\dot\alpha}\xi_{\beta)\gamma\dot\beta\dot\gamma} \\
&- \frac{7}{10}\nabla_{\alpha\dot\alpha}(E_{\beta\gamma\dot\beta\dot\gamma})\xi^{\beta\gamma\dot\beta\dot\gamma} - \frac{17}{45}E_{\alpha}{}^{\gamma\dot\beta\dot\gamma}\nabla^{\beta}{}_{(\dot\alpha}\xi_{\beta\gamma\dot\beta\dot\gamma)} - \frac{2}{5}\nabla^{\beta\dot\beta}(E^{\gamma}{}_{(\alpha}{}^{\dot\gamma}{}_{\dot\alpha})\xi_{\beta)\gamma\dot\beta\dot\gamma} \\
&- \frac{2}{5}E^{\gamma}{}_{(\alpha}{}^{\dot\gamma}{}_{\dot\alpha}\nabla^{\beta\dot\beta}\xi_{\beta)\gamma\dot\beta\dot\gamma} - \frac{2}{5}\nabla^{\beta\dot\beta}(E^{\gamma}{}_{\alpha}{}^{\dot\gamma}{}_{(\dot\alpha})\xi_{\beta\gamma\dot\beta)\dot\gamma} - \frac{2}{5}E^{\gamma}{}_{\alpha}{}^{\dot\gamma}{}_{(\dot\alpha}\nabla^{\beta\dot\beta}\xi_{\beta\gamma\dot\beta)\dot\gamma} \qquad \text{(C.114)}
\end{aligned}
$$

$$
\begin{aligned}
&E_{(\alpha}{}^{\gamma\dot\beta\dot\gamma}\nabla^{\beta}{}_{\dot\alpha}\xi_{\beta)\gamma\dot\beta\dot\gamma} \\
&= E_{\alpha}{}^{\gamma\dot\beta\dot\gamma}\nabla^{\beta}{}_{\dot\alpha}\xi_{\beta\gamma\dot\beta\dot\gamma} - \frac{1}{2}\varepsilon_{\alpha\beta}E^{\mu\gamma\dot\beta\dot\gamma}\nabla^{\beta}{}_{\dot\alpha}\xi_{\mu\gamma\dot\beta\dot\gamma} \\
&= E_{\alpha}{}^{\gamma\dot\beta\dot\gamma}\nabla^{\beta}{}_{\dot\alpha}\xi_{\beta\gamma\dot\beta\dot\gamma} - \frac{1}{2}E^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\alpha\dot\alpha}\xi_{\beta\gamma\dot\beta\dot\gamma} \\
&= E_{\alpha}{}^{\gamma\dot\beta\dot\gamma}\nabla^{\beta}{}_{(\dot\alpha}\xi_{\beta\gamma\dot\beta\dot\gamma)} + \frac{1}{3}\varepsilon_{\dot\alpha\dot\beta}E_{\alpha}{}^{\gamma\dot\beta\dot\gamma}\nabla^{\beta\dot\mu}\xi_{\beta\gamma\dot\mu\dot\gamma} + \frac{1}{3}\varepsilon_{\dot\alpha\dot\gamma}E_{\alpha}{}^{\gamma\dot\beta\dot\gamma}\nabla^{\beta\dot\mu}\xi_{\beta\gamma\dot\beta\dot\mu} - \frac{1}{3}E_{\alpha}{}^{\gamma\dot\beta\dot\gamma}\nabla^{\beta}{}_{(\dot\alpha}\xi_{\beta\gamma\dot\beta\dot\gamma)} \\
&\quad - \frac{1}{3}E^{\gamma\beta\dot\gamma}{}_{\dot\alpha}\nabla_{(\alpha}{}^{\dot\beta}\xi_{\beta\gamma)\dot\beta\dot\gamma} - \frac{2}{9}E_{\alpha\beta\dot\alpha\dot\beta}\nabla_{\gamma\dot\gamma}\xi^{\beta\gamma\dot\beta\dot\gamma} \\
&= \frac{2}{3}E_{\alpha}{}^{\gamma\dot\beta\dot\gamma}\nabla^{\beta}{}_{(\dot\alpha}\xi_{\beta\gamma\dot\beta\dot\gamma)} - \frac{1}{3}E^{\gamma\beta\dot\gamma}{}_{\dot\alpha}\nabla_{(\alpha}{}^{\dot\beta}\xi_{\beta\gamma)\dot\beta\dot\gamma} + \frac{4}{9}E_{\alpha\beta\dot\alpha\dot\beta}\nabla_{\gamma\dot\gamma}\xi^{\beta\gamma\dot\beta\dot\gamma} \qquad \text{(C.115)}
\end{aligned}
$$

$$
\begin{aligned}
\implies \{0\} = &-\frac{1}{5}E^{\gamma\beta\dot\gamma}{}_{\dot\alpha}\nabla_{(\alpha}{}^{\dot\beta}\xi_{\beta\gamma)\dot\beta\dot\gamma} + \frac{8}{15}E_{\alpha\beta\dot\alpha\dot\beta}\nabla_{\gamma\dot\gamma}(\xi^{\beta\gamma\dot\beta\dot\gamma}) + \xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}(E_{\alpha\gamma\dot\alpha\dot\gamma}) - \frac{7}{5}\xi_{\alpha\beta\dot\alpha\dot\beta}\nabla^{\beta\dot\beta}(F) \\
&- \frac{1}{15}\nabla_{\alpha}{}^{\dot\beta}(E^{\beta\gamma\dot\gamma}{}_{(\dot\alpha})\xi_{\beta\gamma\dot\beta)\dot\gamma} + \frac{4}{15}\nabla^{\beta}{}_{\dot\alpha}(E_{(\alpha}{}^{\gamma\dot\beta\dot\gamma})\xi_{\beta)\gamma\dot\beta\dot\gamma} - \frac{7}{10}\nabla_{\alpha\dot\alpha}(E_{\beta\gamma\dot\beta\dot\gamma})\xi^{\beta\gamma\dot\beta\dot\gamma} \\
&- \frac{1}{5}E_{\alpha}{}^{\gamma\dot\beta\dot\gamma}\nabla^{\beta}{}_{(\dot\alpha}\xi_{\beta\gamma\dot\beta\dot\gamma)} - \frac{2}{5}\nabla^{\beta\dot\beta}(E^{\gamma}{}_{(\alpha}{}^{\dot\gamma}{}_{\dot\alpha})\xi_{\beta)\gamma\dot\beta\dot\gamma} - \frac{2}{5}E^{\gamma}{}_{(\alpha}{}^{\dot\gamma}{}_{\dot\alpha}\nabla^{\beta\dot\beta}\xi_{\beta)\gamma\dot\beta\dot\gamma} \\
&- \frac{2}{5}\nabla^{\beta\dot\beta}(E^{\gamma}{}_{\alpha}{}^{\dot\gamma}{}_{(\dot\alpha})\xi_{\beta\gamma\dot\beta)\dot\gamma} - \frac{2}{5}E^{\gamma}{}_{\alpha}{}^{\dot\gamma}{}_{(\dot\alpha}\nabla^{\beta\dot\beta}\xi_{\beta\gamma\dot\beta)\dot\gamma} \qquad \text{(C.116)}
\end{aligned}
$$

$$
\begin{aligned}
&E^{\gamma}{}_{(\alpha}{}^{\dot\gamma}{}_{\dot\alpha}\nabla^{\beta\dot\beta}\xi_{\beta)\gamma\dot\beta\dot\gamma} + E^{\gamma}{}_{\alpha}{}^{\dot\gamma}{}_{(\dot\alpha}\nabla^{\beta\dot\beta}\xi_{\beta\gamma\dot\beta)\dot\gamma} \\
&= E^{\gamma}{}_{\alpha}{}^{\dot\gamma}{}_{\dot\alpha}\nabla^{\beta\dot\beta}\xi_{\beta\gamma\dot\beta\dot\gamma} + E^{\gamma}{}_{\alpha}{}^{\dot\gamma}{}_{\dot\alpha}\nabla^{\beta\dot\beta}\xi_{\beta\gamma\dot\beta\dot\gamma} - \frac{1}{2}\varepsilon_{\alpha\beta}E^{\gamma\mu\dot\gamma}{}_{\dot\alpha}\nabla^{\beta\dot\beta}\xi_{\mu\gamma\dot\beta\dot\gamma} - \frac{1}{2}\varepsilon_{\dot\alpha\dot\beta}E^{\gamma}{}_{\alpha}{}^{\dot\gamma\dot\mu}\nabla^{\beta\dot\beta}\xi_{\beta\gamma\dot\mu\dot\gamma} \\
&= 2E_{\alpha\beta\dot\alpha\dot\beta}\nabla_{\gamma\dot\gamma}\xi^{\beta\gamma\dot\beta\dot\gamma} - \frac{1}{2}E^{\gamma\beta\dot\gamma}{}_{\dot\alpha}\nabla_{\alpha}{}^{\dot\beta}\xi_{\beta\gamma\dot\beta\dot\gamma} - \frac{1}{2}E_{\alpha}{}^{\gamma\dot\beta\dot\gamma}\nabla^{\beta}{}_{\dot\alpha}\xi_{\beta\gamma\dot\beta\dot\gamma} \\
&= 2E_{\alpha\beta\dot\alpha\dot\beta}\nabla_{\gamma\dot\gamma}\xi^{\beta\gamma\dot\beta\dot\gamma} - \frac{1}{2}E^{\gamma\beta\dot\gamma}{}_{\dot\alpha}\nabla_{(\alpha}{}^{\dot\beta}\xi_{\beta\gamma)\dot\beta\dot\gamma} - \frac{1}{6}\varepsilon_{\alpha\beta}E^{\gamma\beta\dot\gamma}{}_{\dot\alpha}\nabla^{\mu\dot\beta}\xi_{\mu\gamma\dot\beta\dot\gamma} - \frac{1}{6}\varepsilon_{\alpha\gamma}E^{\gamma\beta\dot\gamma}{}_{\dot\alpha}\nabla^{\mu\dot\beta}\xi_{\beta\mu\dot\beta\dot\gamma} \\
&\quad - \frac{1}{2}E_{\alpha}{}^{\gamma\dot\beta\dot\gamma}\nabla^{\beta}{}_{(\dot\alpha}\xi_{\beta\gamma\dot\beta\dot\gamma)} - \frac{1}{6}\varepsilon_{\dot\alpha\dot\beta}E_{\alpha}{}^{\gamma\dot\beta\dot\gamma}\nabla^{\beta\dot\mu}\xi_{\beta\gamma\dot\mu\dot\gamma} - \frac{1}{6}\varepsilon_{\dot\alpha\dot\gamma}E_{\alpha}{}^{\gamma\dot\beta\dot\gamma}\nabla^{\beta\dot\mu}\xi_{\beta\gamma\dot\beta\dot\mu} \\
&= \frac{4}{3}E_{\alpha\beta\dot\alpha\dot\beta}\nabla_{\gamma\dot\gamma}\xi^{\beta\gamma\dot\beta\dot\gamma} - \frac{1}{2}E^{\gamma\beta\dot\gamma}{}_{\dot\alpha}\nabla_{(\alpha}{}^{\dot\beta}\xi_{\beta\gamma)\dot\beta\dot\gamma} - \frac{1}{2}E_{\alpha}{}^{\gamma\dot\beta\dot\gamma}\nabla^{\beta}{}_{(\dot\alpha}\xi_{\beta\gamma\dot\beta\dot\gamma)} \qquad \text{(C.117)}
\end{aligned}
$$

$$
\begin{aligned}
\implies \{0\} &= \xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}(E_{\alpha\gamma\dot\alpha\dot\gamma}) - \frac{7}{5}\xi_{\alpha\beta\dot\alpha\dot\beta}\nabla^{\beta\dot\beta}(F) - \frac{1}{15}\nabla_{\alpha}{}^{\dot\beta}(E^{\beta\gamma\dot\gamma}{}_{(\dot\alpha})\xi_{\beta\gamma\dot\beta)\dot\gamma} + \frac{4}{15}\nabla^{\beta}{}_{\dot\alpha}(E_{(\alpha}{}^{\gamma\dot\beta\dot\gamma})\xi_{\beta)\gamma\dot\beta\dot\gamma} \\
&\quad - \frac{7}{10}\nabla_{\alpha\dot\alpha}(E_{\beta\gamma\dot\beta\dot\gamma})\xi^{\beta\gamma\dot\beta\dot\gamma} - \frac{2}{5}\nabla^{\beta\dot\beta}(E^{\gamma}{}_{(\alpha}{}^{\dot\gamma}{}_{\dot\alpha})\xi_{\beta)\gamma\dot\beta\dot\gamma} - \frac{2}{5}\nabla^{\beta\dot\beta}(E^{\gamma}{}_{\alpha}{}^{\dot\gamma}{}_{(\dot\alpha})\xi_{\beta\gamma\dot\beta)\dot\gamma} \\
&= \xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}(E_{\alpha\gamma\dot\alpha\dot\gamma}) - \frac{7}{5}\xi_{\alpha\beta\dot\alpha\dot\beta}\nabla^{\beta\dot\beta}(F) - \frac{7}{10}\nabla_{\alpha\dot\alpha}(E_{\beta\gamma\dot\beta\dot\gamma})\xi^{\beta\gamma\dot\beta\dot\gamma} - \frac{1}{15}\nabla_{\alpha}{}^{\dot\beta}(E^{\beta\gamma\dot\gamma}{}_{\dot\alpha})\xi_{\beta\gamma\dot\beta\dot\gamma} \\
&\quad + \frac{4}{15}\nabla^{\beta}{}_{\dot\alpha}(E_{\alpha}{}^{\gamma\dot\beta\dot\gamma})\xi_{\beta\gamma\dot\beta\dot\gamma} - \frac{2}{5}\nabla^{\beta\dot\beta}(E^{\gamma}{}_{\alpha}{}^{\dot\gamma}{}_{\dot\alpha})\xi_{\beta\gamma\dot\beta\dot\gamma} - \frac{2}{5}\nabla^{\beta\dot\beta}(E^{\gamma}{}_{\alpha}{}^{\dot\gamma}{}_{\dot\alpha})\xi_{\beta\gamma\dot\beta\dot\gamma} \\
&\quad + \frac{1}{30}\varepsilon_{\dot\alpha\dot\beta}\nabla_{\alpha}{}^{\dot\beta}(E^{\beta\gamma\dot\gamma\dot\mu})\xi_{\beta\gamma\dot\mu\dot\gamma} - \frac{2}{15}\varepsilon_{\alpha\beta}\nabla^{\beta}{}_{\dot\alpha}(E^{\mu\gamma\dot\beta\dot\gamma})\xi_{\mu\gamma\dot\beta\dot\gamma} + \frac{1}{5}\varepsilon_{\alpha\beta}\nabla^{\beta\dot\beta}(E^{\gamma\mu\dot\gamma}{}_{\dot\alpha})\xi_{\mu\gamma\dot\beta\dot\gamma} \\
&\quad + \frac{1}{5}\varepsilon_{\dot\alpha\dot\beta}\nabla^{\beta\dot\beta}(E^{\gamma}{}_{\alpha}{}^{\dot\gamma\dot\mu})\xi_{\beta\gamma\dot\mu\dot\gamma} \qquad \text{(C.118)}
\end{aligned}
$$

$$
\begin{aligned}
\{0\} &= \xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}(E_{\alpha\gamma\dot\alpha\dot\gamma}) - \frac{7}{5}\xi_{\alpha\beta\dot\alpha\dot\beta}\nabla^{\beta\dot\beta}(F) - \frac{7}{10}\nabla_{\alpha\dot\alpha}(E_{\beta\gamma\dot\beta\dot\gamma})\xi^{\beta\gamma\dot\beta\dot\gamma} - \frac{1}{15}\nabla_{\alpha\dot\beta}(E_{\beta\gamma\dot\alpha\dot\gamma})\xi^{\beta\gamma\dot\beta\dot\gamma} \\
&\quad + \frac{4}{15}\nabla_{\beta\dot\alpha}(E_{\alpha\gamma\dot\beta\dot\gamma})\xi^{\beta\gamma\dot\beta\dot\gamma} - \frac{4}{5}\nabla_{\beta\dot\beta}(E_{\alpha\gamma\dot\alpha\dot\gamma})\xi^{\beta\gamma\dot\beta\dot\gamma} + \frac{1}{30}\nabla_{\alpha\dot\alpha}(E^{\beta\gamma\dot\beta\dot\gamma})\xi_{\beta\gamma\dot\beta\dot\gamma} \\
&\quad - \frac{2}{15}\nabla_{\alpha\dot\alpha}(E^{\beta\gamma\dot\beta\dot\gamma})\xi_{\beta\gamma\dot\beta\dot\gamma} + \frac{1}{5}\nabla_{\alpha\dot\beta}(E_{\beta\gamma\dot\alpha\dot\gamma})\xi^{\beta\gamma\dot\beta\dot\gamma} + \frac{1}{5}\nabla_{\beta\dot\alpha}(E_{\alpha\gamma\dot\beta\dot\gamma})\xi_{\beta\gamma\dot\beta\dot\gamma} \\
&= \xi^{\beta\gamma\dot\beta\dot\gamma}\nabla_{\beta\dot\beta}(E_{\alpha\gamma\dot\alpha\dot\gamma}) - \frac{7}{5}\xi_{\alpha\beta\dot\alpha\dot\beta}\nabla^{\beta\dot\beta}(F) - \frac{7}{10}\nabla_{\alpha\dot\alpha}(E_{\beta\gamma\dot\beta\dot\gamma})\xi^{\beta\gamma\dot\beta\dot\gamma} - \frac{1}{15}\nabla_{\alpha\dot\beta}(E_{\beta\gamma\dot\alpha\dot\gamma})\xi^{\beta\gamma\dot\beta\dot\gamma} \\
&\quad + \frac{4}{15}\nabla_{\beta\dot\alpha}(E_{\alpha\gamma\dot\beta\dot\gamma})\xi^{\beta\gamma\dot\beta\dot\gamma} - \frac{4}{5}\nabla_{\beta\dot\beta}(E_{\alpha\gamma\dot\alpha\dot\gamma})\xi^{\beta\gamma\dot\beta\dot\gamma} + \frac{1}{30}\nabla_{\alpha\dot\alpha}(E^{\beta\gamma\dot\beta\dot\gamma})\xi_{\beta\gamma\dot\beta\dot\gamma} \\
&\quad - \frac{2}{15}\nabla_{\alpha\dot\alpha}(E^{\beta\gamma\dot\beta\dot\gamma})\xi_{\beta\gamma\dot\beta\dot\gamma} + \frac{1}{5}\nabla_{\alpha\dot\beta}(E_{\beta\gamma\dot\alpha\dot\gamma})\xi^{\beta\gamma\dot\beta\dot\gamma} + \frac{1}{5}\nabla_{\beta\dot\alpha}(E_{\alpha\gamma\dot\beta\dot\gamma})\xi_{\beta\gamma\dot\beta\dot\gamma} \\
&= \xi^{\beta\gamma\dot\beta\dot\gamma}\Bigg(\frac{1}{5}\nabla_{\beta\dot\beta}E_{\alpha\gamma\dot\alpha\dot\gamma} - \frac{4}{5}\nabla_{\alpha\dot\alpha}E_{\beta\gamma\dot\beta\dot\gamma} + \frac{2}{15}\nabla_{\alpha\dot\beta}E_{\beta\gamma\dot\alpha\dot\gamma} + \frac{7}{15}\nabla_{\beta\dot\alpha}E_{\alpha\gamma\dot\beta\dot\gamma}\Bigg) \\
&\quad - \frac{7}{5}\xi_{\alpha\beta\dot\alpha\dot\beta}\nabla^{\beta\dot\beta}F \\
&= \xi^{\beta\gamma\dot\beta\dot\gamma}\Bigg(\frac{1}{5}\nabla_{(\beta(\dot\beta}E_{\alpha\gamma)\dot\alpha\dot\gamma)} + \frac{1}{15}\varepsilon_{\beta\alpha}\nabla^{\mu}{}_{(\dot\beta}E_{\mu\gamma\dot\alpha\dot\gamma)} + \frac{1}{15}\varepsilon_{\beta\gamma}\nabla^{\mu}{}_{(\dot\beta}E_{\alpha\mu\dot\alpha\dot\gamma)} + \frac{1}{15}\varepsilon_{\dot\beta\dot\alpha}\nabla_{(\beta}{}^{\dot\mu}E_{\alpha\gamma)\dot\mu\dot\gamma} \\
&\quad + \frac{1}{15}\varepsilon_{\dot\beta\dot\gamma}\nabla_{(\beta}{}^{\dot\mu}E_{\alpha\gamma)\dot\alpha\dot\mu} + \frac{1}{45}\varepsilon_{\beta\alpha}\varepsilon_{\dot\beta\dot\alpha}\nabla^{\mu\dot\mu}E_{\mu\gamma\dot\mu\dot\gamma} + \frac{1}{45}\varepsilon_{\beta\alpha}\varepsilon_{\dot\beta\dot\gamma}\nabla^{\mu\dot\mu}E_{\mu\gamma\dot\alpha\dot\mu} + \frac{1}{45}\varepsilon_{\beta\gamma}\varepsilon_{\dot\beta\dot\alpha}\nabla^{\mu\dot\mu}E_{\alpha\mu\dot\mu\dot\gamma} \\
&\quad + \frac{1}{45}\varepsilon_{\beta\gamma}\varepsilon_{\dot\beta\dot\gamma}\nabla^{\mu\dot\mu}E_{\alpha\mu\dot\alpha\dot\mu} - \frac{4}{5}\nabla_{(\alpha(\dot\alpha}E_{\beta\gamma)\dot\beta\dot\gamma)} - \frac{4}{15}\varepsilon_{\alpha\beta}\nabla^{\mu}{}_{(\dot\alpha}E_{\mu\gamma\dot\beta\dot\gamma)} - \frac{4}{15}\varepsilon_{\alpha\gamma}\nabla^{\mu}{}_{(\dot\alpha}E_{\beta\mu\dot\beta\dot\gamma)} \\
&\quad - \frac{4}{15}\varepsilon_{\dot\alpha\dot\beta}\nabla_{(\alpha}{}^{\dot\mu}E_{\beta\gamma)\dot\mu\dot\gamma} - \frac{4}{15}\varepsilon_{\dot\alpha\dot\gamma}\nabla_{(\alpha}{}^{\dot\mu}E_{\beta\gamma)\dot\beta\dot\mu} - \frac{4}{45}\varepsilon_{\alpha\beta}\varepsilon_{\dot\alpha\dot\beta}\nabla^{\mu\dot\mu}E_{\mu\gamma\dot\mu\dot\gamma} - \frac{4}{45}\varepsilon_{\alpha\beta}\varepsilon_{\dot\alpha\dot\gamma}\nabla^{\mu\dot\mu}E_{\mu\gamma\dot\beta\dot\mu} \\
&\quad - \frac{4}{45}\varepsilon_{\alpha\gamma}\varepsilon_{\dot\alpha\dot\beta}\nabla^{\mu\dot\mu}E_{\beta\mu\dot\mu\dot\gamma} - \frac{4}{45}\varepsilon_{\alpha\gamma}\varepsilon_{\dot\alpha\dot\gamma}\nabla^{\mu\dot\mu}E_{\beta\mu\dot\beta\dot\mu} + \frac{2}{15}\nabla_{(\alpha(\dot\beta}E_{\beta\gamma)\dot\alpha\dot\gamma)} + \frac{2}{45}\varepsilon_{\alpha\beta}\nabla^{\mu}{}_{(\dot\beta}E_{\mu\gamma\dot\alpha\dot\gamma)} \\
&\quad + \frac{2}{45}\varepsilon_{\alpha\gamma}\nabla^{\mu}{}_{(\dot\beta}E_{\beta\mu\dot\alpha\dot\gamma)} + \frac{2}{45}\varepsilon_{\dot\beta\dot\alpha}\nabla_{(\alpha}{}^{\dot\mu}E_{\beta\gamma)\dot\mu\dot\gamma} + \frac{2}{45}\varepsilon_{\dot\beta\dot\gamma}\nabla_{(\alpha}{}^{\dot\mu}E_{\beta\gamma)\dot\alpha\dot\mu} + \frac{2}{135}\varepsilon_{\alpha\beta}\varepsilon_{\dot\beta\dot\alpha}\nabla^{\mu\dot\mu}E_{\mu\gamma\dot\mu\dot\gamma} \\
&\quad + \frac{2}{135}\varepsilon_{\alpha\beta}\varepsilon_{\dot\beta\dot\gamma}\nabla^{\mu\dot\mu}E_{\mu\gamma\dot\alpha\dot\mu} + \frac{2}{135}\varepsilon_{\alpha\gamma}\varepsilon_{\dot\beta\dot\alpha}\nabla^{\mu\dot\mu}E_{\beta\mu\dot\mu\dot\gamma} + \frac{2}{135}\varepsilon_{\alpha\gamma}\varepsilon_{\dot\beta\dot\gamma}\nabla^{\mu\dot\mu}E_{\beta\mu\dot\alpha\dot\mu} \\
&\quad + \frac{7}{15}\nabla_{(\beta(\dot\alpha}E_{\alpha\gamma)\dot\beta\dot\gamma)} + \frac{7}{45}\varepsilon_{\beta\alpha}\nabla^{\mu}{}_{(\dot\alpha}E_{\mu\gamma\dot\beta\dot\gamma)} + \frac{7}{45}\varepsilon_{\beta\gamma}\nabla^{\mu}{}_{(\dot\alpha}E_{\alpha\mu\dot\beta\dot\gamma)} + \frac{7}{45}\varepsilon_{\dot\alpha\dot\beta}\nabla_{(\beta}{}^{\dot\mu}E_{\alpha\gamma)\dot\mu\dot\gamma} \\
&\quad + \frac{7}{45}\varepsilon_{\dot\alpha\dot\gamma}\nabla_{(\beta}{}^{\dot\mu}E_{\alpha\gamma)\dot\beta\dot\mu} + \frac{7}{135}\varepsilon_{\beta\alpha}\varepsilon_{\dot\alpha\dot\beta}\nabla^{\mu\dot\mu}E_{\mu\gamma\dot\mu\dot\gamma} + \frac{7}{135}\varepsilon_{\beta\alpha}\varepsilon_{\dot\alpha\dot\gamma}\nabla^{\mu\dot\mu}E_{\mu\gamma\dot\beta\dot\mu} \\
&\quad + \frac{7}{135}\varepsilon_{\beta\gamma}\varepsilon_{\dot\alpha\dot\beta}\nabla^{\mu\dot\mu}E_{\alpha\mu\dot\mu\dot\gamma} + \frac{7}{135}\varepsilon_{\beta\gamma}\varepsilon_{\dot\alpha\dot\gamma}\nabla^{\mu\dot\mu}E_{\alpha\mu\dot\beta\dot\mu}\Bigg) - \frac{7}{5}\xi_{\alpha\beta\dot\alpha\dot\beta}\nabla^{\beta\dot\beta}F \\
&= -\frac{2}{3}\xi_{\alpha}{}^{\gamma\dot\beta\dot\gamma}\nabla^{\beta}{}_{(\dot\alpha}E_{\beta\gamma\dot\beta\dot\gamma)} - \frac{1}{3}\xi^{\gamma\beta\dot\gamma}{}_{\dot\alpha}\nabla_{(\alpha}{}^{\dot\beta}E_{\beta\gamma)\dot\beta\dot\gamma} - \frac{7}{15}\xi_{\alpha\beta\dot\alpha\dot\beta}\nabla_{\gamma\dot\gamma}E^{\beta\gamma\dot\beta\dot\gamma} - \frac{7}{5}\xi_{\alpha\beta\dot\alpha\dot\beta}\nabla^{\beta\dot\beta}F \\
&= -\frac{2}{3}\xi_{\alpha}{}^{\gamma\dot\beta\dot\gamma}\nabla^{\beta}{}_{(\dot\alpha}E_{\beta\gamma\dot\beta\dot\gamma)} - \frac{1}{3}\xi^{\gamma\beta\dot\gamma}{}_{\dot\alpha}\nabla_{(\alpha}{}^{\dot\beta}E_{\beta\gamma)\dot\beta\dot\gamma} \\
&= -\frac{2}{3}\xi_{\alpha}{}^{\gamma\dot\beta\dot\gamma}\nabla_{\gamma}{}^{\dot\mu}\overline{C}_{\dot\alpha\dot\beta\dot\gamma\dot\mu} - \frac{1}{3}\xi^{\gamma\beta\dot\gamma}{}_{\dot\alpha}\nabla^{\mu}{}_{\dot\gamma}C_{\alpha\beta\gamma\mu} \qquad \text{(C.119)}
\end{aligned}
$$

Substituting this result and the earlier result, $\{1\} = 0$, into equation C.108,

$$
\begin{aligned}
\psi'_{\alpha} &= \Bigg(\frac{1}{3}\overline{C}_{\dot\alpha\dot\beta}{}^{\dot\gamma\dot\mu}\xi_{\alpha\beta\dot\gamma\dot\mu} - \frac{1}{3}C_{\alpha\beta}{}^{\gamma\mu}\xi_{\gamma\mu\dot\alpha\dot\beta}\Bigg)\nabla^{\beta\dot\beta}\overline{\chi}^{\dot\alpha} + \Bigg(- \frac{4}{15}\overline{C}_{\dot\alpha}{}^{\dot\beta\dot\gamma\dot\mu}\nabla^{\beta}{}_{(\dot\mu}\xi_{\alpha\beta\dot\gamma\dot\beta)} + \frac{1}{15}C_{\alpha}{}^{\beta\gamma\mu}\nabla_{(\mu}{}^{\dot\beta}\xi_{\beta\gamma)\dot\alpha\dot\beta} \\
&\quad - \frac{1}{5}\nabla^{\beta\dot\beta}(C_{\alpha\beta}{}^{\gamma\mu}\xi_{\gamma\mu\dot\alpha\dot\beta} + \overline{C}_{\dot\alpha\dot\beta}{}^{\dot\gamma\dot\mu}\xi_{\alpha\beta\dot\gamma\dot\mu}) - \frac{2}{3}\xi_{\alpha}{}^{\gamma\dot\beta\dot\gamma}\nabla_{\gamma}{}^{\dot\mu}(\overline{C}_{\dot\alpha\dot\beta\dot\gamma\dot\mu}) - \frac{1}{3}\xi^{\gamma\beta\dot\gamma}{}_{\dot\alpha}\nabla^{\mu}{}_{\dot\gamma}(C_{\alpha\beta\gamma\mu})\Bigg)\overline{\chi}^{\dot\alpha}\,. \\
&\qquad \text{(C.120)}
\end{aligned}
$$

This expression can be re-written as

$$
\begin{aligned}
\psi'_\alpha = &\left(\frac{1}{3}\overline{C}_{\dot\alpha\dot\beta}{}^{\dot\gamma\dot\mu}\xi_{\alpha\beta\dot\gamma\dot\mu} - \frac{1}{3}C_{\alpha\beta}{}^{\gamma\mu}\xi_{\gamma\mu\dot\alpha\dot\beta}\right)\nabla^{\beta\dot\beta}\overline{\chi}^{\dot\alpha} + \left(\frac{1}{15}C_\alpha{}^{\beta\gamma\mu}\nabla_{(\mu}{}^{\dot\beta}\xi_{\beta\gamma)\dot\alpha\dot\beta} - \frac{4}{15}\overline{C}_{\dot\alpha}{}^{\dot\beta\dot\gamma\dot\mu}\nabla^\beta{}_{(\dot\mu}\xi_{\alpha\beta\dot\gamma\dot\beta)}\right. \\
&- \frac{1}{5}\nabla^{\beta\dot\beta}(C_{\alpha\beta}{}^{\gamma\mu})\xi_{\gamma\mu\dot\alpha\dot\beta} - \frac{1}{5}\nabla^{\beta\dot\beta}(\overline{C}_{\dot\alpha\dot\beta}{}^{\dot\gamma\dot\mu})\xi_{\alpha\beta\dot\gamma\dot\mu} - \frac{1}{5}C_{\alpha\beta}{}^{\gamma\mu}\nabla^{\beta\dot\beta}(\xi_{\gamma\mu\dot\alpha\dot\beta}) - \frac{1}{5}\overline{C}_{\dot\alpha\dot\beta}{}^{\dot\gamma\dot\mu}\nabla^{\beta\dot\beta}(\xi_{\alpha\beta\dot\gamma\dot\mu}) \\
&\left.- \frac{2}{3}\xi_\alpha{}^{\gamma\dot\beta\dot\gamma}\nabla_\gamma{}^{\dot\mu}(\overline{C}_{\dot\alpha\dot\beta\dot\gamma\dot\mu}) - \frac{1}{3}\xi^{\gamma\beta\dot\gamma}{}_{\dot\alpha}\nabla^\mu{}_{\dot\gamma}(C_{\alpha\beta\gamma\mu})\right)\overline{\chi}^{\dot\alpha} \\
= &\left(\frac{1}{3}\overline{C}_{\dot\alpha\dot\beta}{}^{\dot\gamma\dot\mu}\xi_{\alpha\beta\dot\gamma\dot\mu} - \frac{1}{3}C_{\alpha\beta}{}^{\gamma\mu}\xi_{\gamma\mu\dot\alpha\dot\beta}\right)\nabla^{\beta\dot\beta}\overline{\chi}^{\dot\alpha} + \left(\frac{1}{15}C_\alpha{}^{\beta\gamma\mu}\nabla_{(\mu}{}^{\dot\beta}\xi_{\beta\gamma)\dot\alpha\dot\beta} - \frac{4}{15}\overline{C}_{\dot\alpha}{}^{\dot\beta\dot\gamma\dot\mu}\nabla^\beta{}_{(\dot\mu}\xi_{\alpha\beta\dot\gamma\dot\beta)}\right. \\
&+ \frac{1}{5}\nabla^\beta{}_{\dot\beta}(C_{\alpha\beta\gamma\mu})\xi^{\gamma\mu}{}_{\dot\alpha}{}^{\dot\beta} + \frac{1}{5}\nabla_\beta{}^{\dot\beta}(\overline{C}_{\dot\alpha\dot\beta\dot\gamma\dot\mu})\xi_\alpha{}^{\beta\dot\gamma\dot\mu} + \frac{1}{5}C_\alpha{}^{\beta\gamma\mu}\nabla_\beta{}^{\dot\beta}(\xi_{\gamma\mu\dot\alpha\dot\beta}) + \frac{1}{5}\overline{C}_{\dot\alpha}{}^{\dot\beta\dot\gamma\dot\mu}\nabla^\beta{}_{\dot\beta}(\xi_{\alpha\beta\dot\gamma\dot\mu}) \\
&\left.- \frac{2}{3}\xi_\alpha{}^{\gamma\dot\beta\dot\gamma}\nabla_\gamma{}^{\dot\mu}(\overline{C}_{\dot\alpha\dot\beta\dot\gamma\dot\mu}) - \frac{1}{3}\xi^{\gamma\beta\dot\gamma}{}_{\dot\alpha}\nabla^\mu{}_{\dot\gamma}(C_{\alpha\beta\gamma\mu})\right)\overline{\chi}^{\dot\alpha} \\
= &\left(\frac{1}{3}\overline{C}_{\dot\alpha\dot\beta}{}^{\dot\gamma\dot\mu}\xi_{\alpha\beta\dot\gamma\dot\mu} - \frac{1}{3}C_{\alpha\beta}{}^{\gamma\mu}\xi_{\gamma\mu\dot\alpha\dot\beta}\right)\nabla^{\beta\dot\beta}\overline{\chi}^{\dot\alpha} + \left(\frac{1}{15}C_\alpha{}^{\beta\gamma\mu}\nabla_{(\mu}{}^{\dot\beta}\xi_{\beta\gamma)\dot\alpha\dot\beta} - \frac{4}{15}\overline{C}_{\dot\alpha}{}^{\dot\beta\dot\gamma\dot\mu}\nabla^\beta{}_{(\dot\mu}\xi_{\alpha\beta\dot\gamma\dot\beta)}\right. \\
&+ \frac{1}{5}\nabla^\beta{}_{\dot\beta}(C_{\alpha\beta\gamma\mu})\xi^{\gamma\mu}{}_{\dot\alpha}{}^{\dot\beta} + \frac{1}{5}\nabla_\beta{}^{\dot\beta}(\overline{C}_{\dot\alpha\dot\beta\dot\gamma\dot\mu})\xi_\alpha{}^{\beta\dot\gamma\dot\mu} + \frac{1}{5}C_\alpha{}^{\beta\gamma\mu}\nabla_{(\beta}{}^{\dot\beta}\xi_{\gamma\mu)\dot\alpha\dot\beta} + \frac{1}{5}\overline{C}_{\dot\alpha}{}^{\dot\beta\dot\gamma\dot\mu}\nabla^\beta{}_{(\dot\beta}\xi_{\alpha\beta\dot\gamma\dot\mu)} \\
&\left.- \frac{2}{3}\xi_\alpha{}^{\gamma\dot\beta\dot\gamma}\nabla_\gamma{}^{\dot\mu}(\overline{C}_{\dot\alpha\dot\beta\dot\gamma\dot\mu}) - \frac{1}{3}\xi^{\gamma\beta\dot\gamma}{}_{\dot\alpha}\nabla^\mu{}_{\dot\gamma}(C_{\alpha\beta\gamma\mu})\right)\overline{\chi}^{\dot\alpha} \\
= &\left(\frac{1}{3}\overline{C}_{\dot\alpha\dot\beta}{}^{\dot\gamma\dot\mu}\xi_{\alpha\beta\dot\gamma\dot\mu} - \frac{1}{3}C_{\alpha\beta}{}^{\gamma\mu}\xi_{\gamma\mu\dot\alpha\dot\beta}\right)\nabla^{\beta\dot\beta}\overline{\chi}^{\dot\alpha} + \left(\frac{4}{15}C_\alpha{}^{\beta\gamma\mu}\nabla_{(\mu}{}^{\dot\beta}\xi_{\beta\gamma)\dot\alpha\dot\beta} - \frac{1}{15}\overline{C}_{\dot\alpha}{}^{\dot\beta\dot\gamma\dot\mu}\nabla^\beta{}_{(\dot\mu}\xi_{\alpha\beta\dot\gamma\dot\beta)}\right. \\
&\left.- \frac{7}{15}\xi_\alpha{}^{\gamma\dot\beta\dot\gamma}\nabla_\gamma{}^{\dot\mu}(\overline{C}_{\dot\alpha\dot\beta\dot\gamma\dot\mu}) - \frac{2}{15}\xi^{\gamma\beta\dot\gamma}{}_{\dot\alpha}\nabla^\mu{}_{\dot\gamma}(C_{\alpha\beta\gamma\mu})\right)\overline{\chi}^{\dot\alpha}\,, \qquad \text{(C.121)}
\end{aligned}
$$

which are the 1st two components of equation C.2. Hence, by the aforementioned symmetry between dotted and undotted indices, the equation relating the 2nd two components of equation C.2 also holds true - thereby completing the proof.

Appendix D

A primer on spinors

Given the emphasis on spinors in my thesis, I thought it best to include a general mathematical overview of them "from first principles." My presentation here is a collation of results in [24], [25], [26] and [27][1]. My only contributions are some details of proofs omitted in [26] and [27]. For this appendix alone, rather than the specific case of 3 space and 1 time dimension, I will work in D-dimensional spacetime with s space and t time dimensions. I will also break the convention of presenting four-component spinors in boldface.

D.1 Arbitrary spacetimes

The study of spinors is intimately connected with the representation theory of "Clifford algebras." A Clifford algebra is a set of D objects (which can be thought of as matrices as only their representations in finite dimensional vector spaces are relevant[2]), $\{\gamma_a\}_{a=0}^{D-1}$, such that

$$\{\gamma_a, \gamma_b\} = \gamma_a\gamma_b + \gamma_b\gamma_a = -2\eta_{ab}I\,, \tag{D.1}$$

where $\eta_{ab} = \mathrm{diag}(-1, \cdots -1, 1, \cdots 1)$ with t minus ones, s plus ones and $s + t = D$.

The first task is to study finite dimensional, complex, irreducible representations of this algebra. As I will show, for questions such as the existence, uniqueness and dimension of the irreducible representations, it suffices to study the algebra, $\{\gamma_a, \gamma_b\} = 2\delta_{ab}I$.
Let $\{\tilde{\gamma}_a, \tilde{\gamma}_b\} = -2\eta_{ab}I$, $\gamma_a = \mathrm{i}\tilde{\gamma}_a$ for $t \le a \le D-1$ and $\gamma_a = \tilde{\gamma}_a$ for $0 \le a \le t-1$.
Then, for $a, b \ge t$,

$$\gamma_a\gamma_b + \gamma_b\gamma_a = \mathrm{i}\tilde{\gamma}_a\mathrm{i}\tilde{\gamma}_b + \mathrm{i}\tilde{\gamma}_b\mathrm{i}\tilde{\gamma}_a = -(\tilde{\gamma}_a\tilde{\gamma}_b + \tilde{\gamma}_b\tilde{\gamma}_a) = 2\eta_{ab}I = 2\delta_{ab}I\,. \tag{D.2}$$

Likewise, for $a, b < t$,

$$\gamma_a\gamma_b + \gamma_b\gamma_a = \tilde{\gamma}_a\tilde{\gamma}_b + \tilde{\gamma}_b\tilde{\gamma}_a = -2\eta_{ab}I = 2\delta_{ab}I\,. \tag{D.3}$$

Finally, when one of a and b is less than t and the other is greater than or equal to t,

$$\gamma_a\gamma_b + \gamma_b\gamma_a = \mathrm{i}(\tilde{\gamma}_a\tilde{\gamma}_b + \tilde{\gamma}_b\tilde{\gamma}_a) = -2\mathrm{i}\eta_{ab}I = 0 = 2\delta_{ab}I\,. \tag{D.4}$$

Therefore, the original Clifford algebra can be transformed to one where $-\eta_{ab} \to \delta_{ab}$. Conversely, if $\{\gamma_a, \gamma_b\} = 2\delta_{ab}I$, then letting $\tilde{\gamma}_a = \gamma_a$ for $0 \le a \le t-1$ and $\tilde{\gamma}_a = -\mathrm{i}\gamma_a$ for $t \le a \le D-1$ yields $\{\tilde{\gamma}_a, \tilde{\gamma}_b\} = -2\eta_{ab}I$.

[1] I have taken some proofs almost exactly as presented in these references.
[2] The Clifford algebra must be assumed to be associative for a matrix representation to be well defined.

Since the two Clifford algebras are equivalent, for now consider $\{\gamma_a, \gamma_b\} = 2\delta_{ab} I$.

Let $\{\gamma_a\}_{a=0}^{D-1}$ be a finite dimensional, complex, irreducible representation of the Clifford algebra, $\{\gamma_a, \gamma_b\} = 2\delta_{ab} I$. Denote the dimension of the representation space by N.
Let $\{\Gamma_A\}_{A=0}^{2^D-1} = \{I, \gamma_a, \gamma_a\gamma_b \text{ with } a < b, \gamma_a\gamma_b\gamma_c \text{ with } a < b < c, \cdots, \gamma_0 \cdots \gamma_{D-1}\}$. By definition, all the Γ_A are $N \times N$ matrices.
$\gamma_a\gamma_b + \gamma_b\gamma_a = 2\delta_{ab} I \implies (\gamma_a)^2 = I$ and $\gamma_a\gamma_b = -\gamma_b\gamma_a$ for $a \neq b$. Thus,

$$\begin{aligned}(\Gamma_A)^2 &= \gamma_{a_1} \cdots \gamma_{a_n}\gamma_{a_1} \cdots \gamma_{a_n} \text{ for some } 0 \leq n \leq D-1 \text{ and } a_1 < \cdots < a_n \\ &= \gamma_{a_1}\gamma_{a_1}(-1)^{n-1}\gamma_{a_2} \cdots \gamma_{a_n}\gamma_{a_2} \cdots \gamma_{a_n} \\ &= (-1)^{n-1}\gamma_{a_2} \cdots \gamma_{a_n}\gamma_{a_2} \cdots \gamma_{a_n} \\ &= (-1)^{n-1+n-2+\cdots+1} I \\ &= (-1)^{n(n-1)/2} I \end{aligned} \tag{D.5}$$

Hence, all the Γ_A are invertible and $(\Gamma_A)^{-1} = (-1)^{n(n-1)/2}\Gamma_A$.

Lemma D.1. *$G = \{\pm\Gamma_A\}_{A=0}^{2^D-1}$ is a finite group of order 2^{D+1} under multiplication.*

Proof. That G has 2^{D+1} elements follows directly from the definition.
Matrix multiplication is already associative.
The identity matrix, I, is Γ_0 by definition and hence in G.
$(\pm\Gamma_A)^{-1} = \pm(-1)^{n(n-1)/2}\Gamma_A \in G$.
All that is left to show is that multiplication is a well defined binary operation on G.
Let $\Gamma_A = \gamma_{a_1} \cdots \gamma_{a_m}$ and $\Gamma_B = \gamma_{b_1} \cdots \gamma_{b_n} \implies \Gamma_A\Gamma_B = \gamma_{a_1} \cdots \gamma_{a_m}\gamma_{b_1} \cdots \gamma_{b_n}$.
If $a_i \neq b_j \, \forall i, j$, then changing the order of the γ_{a_i} and γ_{b_j} (at the expense of some -1 factors) to make the sequence in ascending order of indices means $\Gamma_A\Gamma_B \in G$. If $a_i = b_j$ for some i and j, then changing the order to make them adjacent means $\gamma_{a_i}\gamma_{b_j} = I$ and those two γs are removed. This can be done until no a_i and b_j are equal.
Therefore, $\Gamma_A\Gamma_B \in G$ again $\implies$ The binary operation is well defined. □

$\{\gamma_a\}_{a=0}^{D-1}$ is irreducible $\iff$ there is no subspace of $\mathbb{C}^N$ invariant under all γ_a.
As $\{\gamma_a\}_{a=0}^{D-1} \subset G$, the elements of G also have no common invariant subspace.
Hence the irreducible representation of the Clifford algebra automatically leads to an irreducible representation of G in the same representation space.

Theorem D.2. *The dimension of an irreducible representation's representation space, N, can only be $2^{\lfloor D/2 \rfloor}$.*

Proof. Let Y be an arbitrary $N \times N$ matrix and let

$$S = \sum_{A=0}^{2^D-1} (\Gamma_A)^{-1} Y \Gamma_A \, . \tag{D.6}$$

where I have adopted the convention of explicitly showing all summations on the $A, B, \ldots$ indices. Then,

$$(\Gamma_B)^{-1} S \Gamma_B = \sum_{A=0}^{2^D-1} (\Gamma_B)^{-1}(\Gamma_A)^{-1} Y \Gamma_A \Gamma_B = \sum_{A=0}^{2^D-1} (\Gamma_A\Gamma_B)^{-1} Y \Gamma_A \Gamma_B \, . \tag{D.7}$$

$\Gamma_B\Gamma_A \in G$ and as Γ_B is invertible, $\Gamma_{A_1}\Gamma_B = \pm\Gamma_{A_2}\Gamma_B \implies \Gamma_{A_1} = \pm\Gamma_{A_2}$.
Thus, $\{\Gamma_A\Gamma_B\}_{A=0}^{2^D-1} = \{\pm\Gamma_C\}_{C=0}^{2^D-1}$ where on the RHS, a $+$ or $-$ is chosen for each C depending on whether $\Gamma_A\Gamma_B = \Gamma_C$ or $\Gamma_A\Gamma_B = -\Gamma_C$ (hence $\{\pm\Gamma_C\}_{C=0}^{2^D-1}$ has only half as many elements as the group, G). That means the equation above can be simplified to

$$(\Gamma_B)^{-1}S\Gamma_B = \sum_{C=0}^{2^D-1}(\pm\Gamma_C)^{-1}Y(\pm\Gamma_C) = \sum_{C=0}^{2^D-1}(\Gamma_C)^{-1}Y\Gamma_C = S\,. \tag{D.8}$$

Therefore, $S\Gamma_B = \Gamma_B S \;\; \forall B$.
Hence, $S = \lambda I$ for some $\lambda \in \mathbb{C}$ by Schur's lemma. That means

$$\begin{aligned}
\lambda I &= \sum_{A=0}^{2^D-1}(\Gamma_A)^{-1}Y\Gamma_A \\
\implies \mathrm{tr}(\lambda I) &= \mathrm{tr}\left(\sum_{A=0}^{2^D-1}(\Gamma_A)^{-1}Y\Gamma_A\right) \\
\implies \lambda N &= \sum_{A=0}^{2^D-1}\mathrm{tr}((\Gamma_A)^{-1}Y\Gamma_A) = \sum_{A=0}^{2^D-1}\mathrm{tr}(\Gamma_A(\Gamma_A)^{-1}Y) = 2^D\mathrm{tr}(Y) \\
\implies \lambda &= \frac{2^D\mathrm{tr}(Y)}{N} \implies \frac{2^D\mathrm{tr}(Y)}{N}I = \sum_{A=0}^{2^D-1}(\Gamma_A)^{-1}Y\Gamma_A\,.
\end{aligned} \tag{D.9}$$

In the last equation,

$$\mathrm{LHS} = \frac{2^D Y_{kk}}{N}\delta_{ij} = \frac{2^D}{N}\delta_{kl}\delta_{ij}Y_{kl} \text{ and} \tag{D.10}$$

$$\mathrm{RHS} = \sum_{A=0}^{2^D-1}(\Gamma_A^{-1})_{ik}Y_{kl}(\Gamma_A)_{lj}\,. \tag{D.11}$$

Then, since Y_{kl} is arbitrary,

$$\begin{aligned}
\mathrm{LHS} = \mathrm{RHS} \implies \frac{2^D}{N}\delta_{kl}\delta_{ij} &= \sum_{A=0}^{2^D-1}(\Gamma_A^{-1})_{ik}(\Gamma_A)_{lj} \\
\implies \frac{2^D}{N}\delta_{ij}\delta_{ij} &= \sum_{A=0}^{2^D-1}(\Gamma_A^{-1})_{ii}(\Gamma_A)_{jj} \\
\iff 2^D &= \sum_{A=0}^{2^D-1}\mathrm{tr}(\Gamma_A)\mathrm{tr}((\Gamma_A)^{-1})\,.
\end{aligned} \tag{D.12}$$

Let $\Gamma_A = \gamma_{a_1}\cdots\gamma_{a_n}$ for some $1 \le n \le D-1$ (any Γ_A other than $\Gamma_0 = I$ and $\Gamma_{2^D-1} = \gamma_0\cdots\gamma_{D-1}$ can be written in such a form by definition).
Therefore, $\exists b \in \{0, 1, \cdots, D-1\}$ such that $b \neq a_i \;\; \forall i$. Then, if n is odd,

$$\begin{aligned}
(\gamma_b)^{-1}\Gamma_A\gamma_b &= \gamma_b\gamma_{a_1}\cdots\gamma_{a_n}\gamma_b \\
&= (\gamma_b)^2(-1)^n\gamma_{a_1}\cdots\gamma_{a_n} \\
&= (-1)^n\Gamma_A \\
&= -\Gamma_A \quad \text{as } n \text{ is odd.}
\end{aligned} \tag{D.13}$$

Therefore,

$$\mathrm{tr}((\gamma_b)^{-1}\Gamma_A\gamma_b) = \mathrm{tr}(-\Gamma_A) \iff \mathrm{tr}(\Gamma_A) = \mathrm{tr}(-\Gamma_A) \implies \mathrm{tr}(\Gamma_A) = 0\,. \tag{D.14}$$

On the other hand, if n is even,

$$\begin{aligned}(\gamma_{a_1})^{-1}\Gamma_A\gamma_{a_1} &= \gamma_{a_1}\gamma_{a_1}\cdots\gamma_{a_n}\gamma_{a_1}\\ &= \gamma_{a_1}\gamma_{a_1}\gamma_{a_1}(-1)^{n-1}\gamma_{a_2}\cdots\gamma_{a_n}\\ &= (-1)^{n-1}\gamma_{a_1}\cdots\gamma_{a_n}\\ &= -\Gamma_A \quad \text{as } n \text{ is even.}\end{aligned} \tag{D.15}$$

Hence, $\mathrm{tr}(\Gamma_A) = 0$ by the same logic as before.
Then, in equation D.12, the only non-traceless matrices in the sum are when $A = 0$ and when $A = 2^D - 1$. Thus,

$$\begin{aligned}2^D &= \mathrm{tr}(I)\mathrm{tr}(I^{-1}) + \mathrm{tr}(\gamma_0\cdots\gamma_{D-1})\mathrm{tr}((\gamma_0\cdots\gamma_{D-1})^{-1})\\ &= N^2 + \mathrm{tr}(\gamma_0\cdots\gamma_{D-1})\mathrm{tr}((\gamma_0\cdots\gamma_{D-1})^{-1})\,.\end{aligned} \tag{D.16}$$

It will now be necessary to consider D even and odd separately; I will start with the former.

$$\begin{aligned}\mathrm{tr}(\gamma_0\cdots\gamma_{D-1}) &= \mathrm{tr}(\gamma_{D-1}\gamma_0\cdots\gamma_{D-2})\\ &= \mathrm{tr}(\gamma_0\cdots\gamma_{D-1}(-1)^{D-1})\\ &= \mathrm{tr}(-\gamma_0\cdots\gamma_{D-1}) \quad \text{as } D \text{ is even}\end{aligned} \tag{D.17}$$

$$\begin{aligned}\implies \mathrm{tr}(\gamma_0\cdots\gamma_{D-1}) &= 0\\ \implies 2^D &= N^2\\ \implies N &= 2^{D/2} = 2^{\lfloor D/2\rfloor}\end{aligned} \tag{D.18}$$

However, when D is odd,

$$\begin{aligned}\gamma_a\gamma_0\cdots\gamma_{D-1} &= \gamma_a\gamma_0\cdots\gamma_{a-1}\gamma_a\gamma_{a+1}\cdots\gamma_{D-1}\\ &= \gamma_0\cdots\gamma_{a-1}\gamma_a(-1)^a\gamma_a\gamma_{a+1}\cdots\gamma_{D-1}\\ &= \gamma_0\cdots\gamma_{a-1}\gamma_a(-1)^a\gamma_{a+1}\cdots\gamma_{D-1}\gamma_a(-1)^{D-a-1}\\ &= (-1)^{D-1}\gamma_0\cdots\gamma_{D-1}\gamma_a\\ &= \gamma_0\cdots\gamma_{D-1}\gamma_a \quad \text{as } D \text{ is odd.}\end{aligned} \tag{D.19}$$

Then, since all elements of G are products of the γs and possibly a factor of -1,
$g\gamma_0\cdots\gamma_{D-1} = \gamma_0\cdots\gamma_{D-1}g \ \ \forall g \in G$.
Therefore, $\gamma_0\cdots\gamma_{D-1} = \lambda I$ for some $\lambda \in \mathbb{C}\setminus\{0\}$ by Schur's lemma (not the same λ as before).
Thus,

$$\begin{aligned}2^D &= N^2 + \mathrm{tr}(\lambda I)\mathrm{tr}((\lambda I)^{-1})\\ &= N^2 + (N\lambda)\left(\frac{N}{\lambda}\right)\\ &= 2N^2\\ \implies N &= 2^{(D-1)/2} = 2^{\lfloor D/2\rfloor}\,.\end{aligned} \tag{D.20}$$

Hence, for any dimension, D, N is uniquely determined to be $2^{\lfloor D/2\rfloor}$. □

The previous theorem uniquely determines the representation space's dimension, but as yet I have said nothing about the number of inequivalent representations in $\mathbb{C}^N$.

Theorem D.3. *For even dimensions, a finite dimensional, complex, irreducible representation of the Clifford algebra is unique up to equivalence, where as in odd dimensions, there are two inequivalent representations related by a factor of* -1.

Proof. Let $\{\gamma_a\}_{a=0}^{D-1}$ and $\{\tilde{\gamma}_a\}_{a=0}^{D-1}$ be two inequivalent, finite dimensional, complex irreducible representations of the Clifford algebra. Let G and $\tilde{G}$ be the two corresponding finite groups generated as before. For an arbitrary $N \times N$ matrix, Y, this time let

$$S = \sum_{A=0}^{2^D-1} (\Gamma_A)^{-1} Y \tilde{\Gamma}_A \tag{D.21}$$

$$\begin{aligned}
\implies (\Gamma_B)^{-1} S \tilde{\Gamma}_B &= \sum_{A=0}^{2^D-1} (\Gamma_B)^{-1} (\Gamma_A)^{-1} Y \tilde{\Gamma}_A \tilde{\Gamma}_B \\
&= \sum_{A=0}^{2^D-1} (\Gamma_A \Gamma_B)^{-1} Y \tilde{\Gamma}_A \tilde{\Gamma}_B \\
&= \sum_{C=0}^{2^D-1} (\Gamma_C)^{-1} Y \tilde{\Gamma}_C \\
&= S
\end{aligned}$$

$$\iff S\tilde{\Gamma}_B = \Gamma_B S \quad \forall B \tag{D.22}$$

with the 3rd last line following by the same reasoning as equation D.8. Now, since the representations of G & $\tilde{G}$ are inequivalent, $S\tilde{\Gamma}_B = \Gamma_B S \implies S = 0$ by Schur's 2nd lemma, i.e.

$$\sum_{A=0}^{2^D-1} (\Gamma_A^{-1})_{ik} Y_{kl} (\tilde{\Gamma}_A)_{lj} = 0\,. \tag{D.23}$$

However, since Y_{kl} is arbitrary, it must be that

$$0 = \sum_{A=0}^{2^D-1} (\Gamma_A^{-1})_{ik} (\tilde{\Gamma}_A)_{lj} \tag{D.24}$$

$$\begin{aligned}
\implies 0 &= \sum_{A=0}^{2^D-1} (\Gamma_A^{-1})_{ii} (\tilde{\Gamma}_A)_{jj} \\
&= \sum_{A=0}^{2^D-1} \mathrm{tr}((\Gamma_A)^{-1}) \mathrm{tr}(\tilde{\Gamma}_A)\,.
\end{aligned} \tag{D.25}$$

For even D, it was shown in the proof of theorem D.2 that $\tilde{\Gamma}_0 = I$ is the only one of the $\tilde{\Gamma}_A$s that is not traceless.
Hence, $0 = \mathrm{tr}(I^{-1})\mathrm{tr}(I) = N^2 \implies N = 0$, contradicting theorem D.2.
Therefore, for even dimensions, there could not have been two inequivalent representations to begin with, thereby proving the 1st half of the theorem.

Meanwhile for odd D, it was shown in the proof of theorem D.2 that $\tilde{\Gamma}_0 = I$ and $\tilde{\Gamma}_{D-1} = \tilde{\lambda} I$ are the only non-traceless Γ_As. Hence,

$$\begin{aligned}
0 &= \mathrm{tr}(I^{-1})\mathrm{tr}(I) + \mathrm{tr}((\lambda I)^{-1})\mathrm{tr}(\tilde{\lambda} I) \\
&= N^2 + \frac{\tilde{\lambda}}{\lambda} N^2
\end{aligned}$$

$$\iff \tilde{\lambda} = -\lambda\,. \tag{D.26}$$

Because of this result, there cannot be a 3rd inequivalent representation as follows.
Let $\{\gamma'_a\}_{a=0}^{D-1}$ be a 3rd inequivalent representation. Then, considering the three representations pairwise, $\lambda = -\tilde{\lambda}$, $\lambda' = -\tilde{\lambda}$ and $\lambda' = -\lambda$. The 1st and 3rd of these equations together imply $\lambda' = \tilde{\lambda}$, which contradicts the 2nd equation.

There could yet be two inequivalent representations though. Let $\tilde{\gamma}_a = -\gamma_a$. Then,

$$\tilde{\gamma}_a\tilde{\gamma}_b + \tilde{\gamma}_b\tilde{\gamma}_a = (-\gamma_a)(-\gamma_b) + (-\gamma_b)(-\gamma_a) = \gamma_a\gamma_b + \gamma_b\gamma_a = 2\delta_{ab}I\,. \tag{D.27}$$

Therefore, $\{\tilde{\gamma}_a\}_{a=0}^{D-1} = \{-\gamma_a\}_{a=0}^{D-1}$ also satisfies the Clifford algebra.
Assume $\exists$ an $N \times N$ matrix, C, such that $\tilde{\gamma}_a = C^{-1}\gamma_a C$ for a contradiction.

$$\begin{aligned}\tilde{\gamma}_0 \cdots \tilde{\gamma}_{D-1} &= C^{-1}\gamma_0 C \cdots C^{-1}\gamma_{D-1}C \\ &= C^{-1}\gamma_0 \cdots \gamma_{D-1}C \\ &= C^{-1}\lambda I C \\ &= \lambda I\end{aligned} \tag{D.28}$$

However, $\tilde{\gamma}_0 \cdots \tilde{\gamma}_{D-1} = (-1)^D\gamma_0 \cdots \gamma_{D-1} = -\lambda I$.
That means $\lambda I = -\lambda I$, which contradicts $\lambda \neq 0$.
Hence, in odd dimensions, $\{\gamma_a\}_{a=0}^{D-1}$ and $\{-\gamma_a\}_{a=0}^{D-1}$ are indeed inequivalent representations. □

Having established these properties, it is time to return to the general Clifford algebra, $\{\gamma_a, \gamma_b\} = -2\eta_{ab}I$, where the previous two theorems will continue to hold via the reasons outlined earlier. Spinors can now be defined as the N-component objects of $\mathbb{C}^N$, the representation space of the Clifford algebra. As I will outline, these spinors will allow representations of the spin groups (the universal covering groups of $\mathrm{SO}^\uparrow(s,t)$).
From hereon, let $\gamma_0 \cdots \gamma_{D-1}$ be denoted by γ_{D+1}.

Let $\Lambda^a{}_b \in \mathrm{SO}^\uparrow(s,t)$ and let $\gamma'_a = (\Lambda^{-1})^b{}_a\gamma_b$, i.e. as if γ^a was a Lorentz vector. Then,

$$\begin{aligned}\gamma'_a\gamma'_b + \gamma'_b\gamma'_a &= (\Lambda^{-1})^c{}_a(\Lambda^{-1})^d{}_b(\gamma_c\gamma_d + \gamma_d\gamma_c) \\ &= -2\eta_{cd}(\Lambda^{-1})^c{}_a(\Lambda^{-1})^d{}_b I \\ &= -2\eta_{ab}I \quad \text{by the defining properties of } \mathrm{SO}^\uparrow(s,t)\,.\end{aligned} \tag{D.29}$$

Therefore $\{\gamma'_a\}_{a=0}^{D-1}$ also satisfy the Clifford algebra.
In even dimensions, since the irreducible representation is unique, $\exists S(\Lambda)$ such that $\gamma'_a = S(\Lambda)^{-1}\gamma_a S(\Lambda)$. However, in odd dimensions, both $\gamma'_a = S(\Lambda)^{-1}\gamma_a S(\Lambda)$ and $\gamma'_a = S(\Lambda)^{-1}(-\gamma_a)S(\Lambda)$ could be possible by the previous theorem. Consider the latter case.
$\gamma_{D+1} = \gamma_0 \cdots \gamma_{D-1} = \frac{1}{N!}\varepsilon^{a_1\cdots a_D}\gamma_{a_1}\cdots\gamma_{a_D}$ by anticommutativity. Hence,

$$\begin{aligned}S(\Lambda)^{-1}\gamma_{D+1}S(\Lambda) &= \frac{1}{N!}\varepsilon^{a_1\cdots a_D}S(\Lambda)^{-1}\gamma_{a_1}S(\Lambda)\cdots S(\Lambda)^{-1}\gamma_{a_D}S(\Lambda) \\ &= \frac{(-1)^D}{N!}\varepsilon^{a_1\cdots a_D}\gamma'_{a_1}\cdots\gamma'_{a_D} \\ &= \frac{(-1)^D}{N!}\varepsilon^{a_1\cdots a_D}(\Lambda^{-1})^{b_1}{}_{a_1}\cdots(\Lambda^{-1})^{b_D}{}_{a_D}\gamma_{b_1}\cdots\gamma_{b_D} \\ &= \frac{(-1)^D}{N!}\det(\Lambda^{-1})\varepsilon^{b_1\cdots b_D}\gamma'_{b_1}\cdots\gamma'_{b_D} \\ &= -\gamma_{D+1} \quad \text{as } D \text{ is odd and } \det(\Lambda^{-1}) = 1\,.\end{aligned} \tag{D.30}$$

However, I showed earlier that in odd dimensions, $\gamma_{D+1} = \lambda I$ for some complex $\lambda \neq 0$. Thus, the last equation says $S(\Lambda)^{-1}\lambda I S(\Lambda) = -\lambda I \iff \lambda I = -\lambda I \iff \lambda = 0$, which contradicts

$\lambda \neq 0$.
Therefore, even in odd dimensions, $\gamma'_a = S(\Lambda)^{-1}\gamma_a S(\Lambda)$. Hence, in any dimension,

$$\begin{aligned} S(\Lambda_1)^{-1}S(\Lambda_2)^{-1}\gamma_a S(\Lambda_2)S(\Lambda_1) &= S(\Lambda_1)^{-1}(\Lambda_2^{-1})^b{}_a\gamma_b S(\Lambda_1) \\ &= (\Lambda_1^{-1})^c{}_b(\Lambda_2^{-1})^b{}_a\gamma_c \\ &= ((\Lambda_2\Lambda_1)^{-1})^b{}_a\gamma_b \\ &= S(\Lambda_2\Lambda_1)^{-1}\gamma_a S(\Lambda_2\Lambda_1) \\ \iff \gamma_a S(\Lambda_2)S(\Lambda_1)S(\Lambda_2\Lambda_1)^{-1} &= S(\Lambda_2)S(\Lambda_1)S(\Lambda_2\Lambda_1)^{-1}\gamma_a\,. \end{aligned} \tag{D.31}$$

Since the last equation holds $\forall a$, $gS(\Lambda_2)S(\Lambda_1)S(\Lambda_2\Lambda_1)^{-1} = S(\Lambda_2)S(\Lambda_1)S(\Lambda_2\Lambda_1)^{-1}g \;\; \forall g \in G$.
By Schur's lemma, $S(\Lambda_2)S(\Lambda_1)S(\Lambda_2\Lambda_1)^{-1} = f(\Lambda_2, \Lambda_1)I \iff S(\Lambda_1)S(\Lambda_2) = f(\Lambda_1, \Lambda_2)S(\Lambda_1\Lambda_2)$ for some $f(\Lambda_1, \Lambda_2) \in \mathbb{C}$.
Therefore, S is a projective representation of $\mathrm{SO}^{\uparrow}(s,t)$.
In general, this is the best that can be done for $\mathrm{SO}^{\uparrow}(s,t)$. However, since $S(\Lambda)^{-1}\gamma_a S(\Lambda) = (\Lambda^{-1})^b{}_a\gamma_b$ is invariant under $S(\Lambda) \to \beta S(\Lambda)$ for any $\beta \in \mathbb{C}\setminus\{0\}$, S can be extended to a representation of $\mathrm{Spin}(s,t)$, the universal covering group of $\mathrm{SO}^{\uparrow}(s,t)$. In this case, it can be shown[3] S can be made into a linear representation, rather than only a projective representation. This property distinguishes the spinor representation from other tensor representations; spinors facilitate a representation of $\mathrm{Spin}(s,t)$, not $\mathrm{SO}^{\uparrow}(s,t)$.
From henceforth, let $S(\Lambda) \equiv S(N)$ where N is a pre-image of Λ under the covering map.

A natural way to generate a representation of $\mathrm{Spin}(s,t)$, is to exponentiate[4] elements of $\mathfrak{spin}(s,t)$. Since a group and its universal cover are locally isomorphic, $\mathfrak{spin}(s,t) \cong \mathfrak{o}(\mathfrak{s},\mathfrak{t})$. Hence, one must study the the connection between Lorentz groups and Clifford algebras at the level of Lie algebras. To do so, let $M_{ab} = -\frac{1}{4}[\gamma_a, \gamma_b]$. Then,

$$\begin{aligned} [M_{ab}, M_{cd}] &= \frac{1}{16}[[\gamma_a,\gamma_b],[\gamma_c,\gamma_d]] \\ &= \frac{1}{16}[\gamma_a\gamma_b - \gamma_b\gamma_a, \gamma_c\gamma_d - \gamma_d\gamma_c] \\ &= \frac{1}{16}(\gamma_a\gamma_b - \gamma_b\gamma_a)(\gamma_c\gamma_d - \gamma_d\gamma_c) - \frac{1}{16}(\gamma_c\gamma_d - \gamma_d\gamma_c)(\gamma_a\gamma_b - \gamma_b\gamma_a)v \\ &= \frac{1}{16}(\gamma_a\gamma_b\gamma_c\gamma_d - \gamma_a\gamma_b\gamma_d\gamma_c - \gamma_b\gamma_a\gamma_c\gamma_d + \gamma_b\gamma_a\gamma_d\gamma_c - \gamma_c\gamma_d\gamma_a\gamma_b + \gamma_c\gamma_d\gamma_b\gamma_a \\ &\qquad + \gamma_d\gamma_c\gamma_a\gamma_b - \gamma_d\gamma_c\gamma_b\gamma_a)\,. \end{aligned} \tag{D.32}$$

Using the Clifford algebra,

$$\begin{aligned} \gamma_c\gamma_d\gamma_a\gamma_b &= -\gamma_c\gamma_a\gamma_d\gamma_b - 2\eta_{ad}\gamma_c\gamma_b \\ &= \gamma_a\gamma_c\gamma_d\gamma_b + 2\eta_{ac}\gamma_d\gamma_b - 2\eta_{ad}\gamma_c\gamma_b \\ &= -\gamma_a\gamma_c\gamma_b\gamma_d - 2\eta_{bd}\gamma_a\gamma_c + 2\eta_{ac}\gamma_d\gamma_b - 2\eta_{ad}\gamma_c\gamma_b \\ &= \gamma_a\gamma_b\gamma_c\gamma_d + 2\eta_{bc}\gamma_a\gamma_d - 2\eta_{bd}\gamma_a\gamma_c + 2\eta_{ac}\gamma_d\gamma_b - 2\eta_{ad}\gamma_c\gamma_b \\ \iff \gamma_a\gamma_b\gamma_c\gamma_d - \gamma_c\gamma_d\gamma_a\gamma_b &= 2(\eta_{ad}\gamma_c\gamma_b - \eta_{ac}\gamma_d\gamma_b + \eta_{bd}\gamma_a\gamma_c - \eta_{bc}\gamma_a\gamma_d)\,. \end{aligned} \tag{D.33}$$

$\gamma_c\gamma_d\gamma_b\gamma_a - \gamma_b\gamma_a\gamma_c\gamma_d$, $\gamma_d\gamma_c\gamma_a\gamma_b - \gamma_a\gamma_b\gamma_d\gamma_c$ and $\gamma_b\gamma_a\gamma_d\gamma_c - \gamma_d\gamma_c\gamma_b\gamma_a$ follow by relabelling indices.

[3] I will sketch how this can be done below and in the next subsection of this appendix.
[4] I will have an example later in the appendix.

Substituting these expressions,

$$
\begin{aligned}
[M_{ab}, M_{cd}] = \frac{1}{8}(&\eta_{ad}\gamma_c\gamma_b - \eta_{ac}\gamma_d\gamma_b + \eta_{bd}\gamma_a\gamma_c - \eta_{bc}\gamma_a\gamma_d \\
&+ \eta_{ca}\gamma_b\gamma_d - \eta_{cb}\gamma_a\gamma_d + \eta_{da}\gamma_c\gamma_b - \eta_{db}\gamma_c\gamma_a \\
&+ \eta_{db}\gamma_a\gamma_c - \eta_{da}\gamma_b\gamma_c + \eta_{cb}\gamma_d\gamma_a - \eta_{ca}\gamma_d\gamma_b \\
&+ \eta_{bc}\gamma_d\gamma_a - \eta_{bd}\gamma_c\gamma_a + \eta_{ac}\gamma_b\gamma_d - \eta_{ad}\gamma_b\gamma_c) \\
= &\frac{1}{4}(\eta_{ad}[\gamma_c, \gamma_b] + \eta_{ac}[\gamma_b, \gamma_d] + \eta_{bd}[\gamma_a, \gamma_c] + \eta_{bc}[\gamma_d, \gamma_a]) \\
= &\eta_{ad}M_{bc} - \eta_{ac}M_{bd} + \eta_{bc}M_{ad} - \eta_{bd}M_{ac} \,. \qquad \text{(D.34)}
\end{aligned}
$$

In summary, $M_{ab} = -\frac{1}{4}[\gamma_a, \gamma_b]$ satisfy the Lie algebra of $\mathfrak{o}(3, 1)$, i.e. M_{ab} are Lorentz generators.

It is now time to study the effects of these transformation properties of the Clifford algebra on the properties of spinors themselves. Spinors were originally used most prominently in physics in the context of the Dirac equation,

$$
(\mathrm{i}\gamma^a\nabla_a - q\gamma^a A_a(x) - m)\Psi(x) = 0 \,, \qquad \text{(D.35)}
$$

where Ψ is a $2^{\lfloor D/2 \rfloor}$-component spinor. To be a well defined equation of motion, the Dirac equation must transform covariantly.
That means under a local Lorentz transformation, ${e'}_a{}^m(x) = (\Lambda^{-1})^b{}_a e_b{}^m(x)$, the Dirac equation must be $0 = (\mathrm{i}\gamma^a\nabla'_a - q\gamma^a A'_a(x) - m)\Psi'(x)$. This equation still has γ^a, not γ'^a, because despite appearances, γ^a are supposed to be a set of constant matrices; they cannot be different for different observers.
Since $\nabla_a = \Lambda^b{}_a\nabla'_b$ and $A_a = \Lambda^b{}_a A'_b$, the original Dirac equation can be re-written as

$$
\begin{aligned}
0 &= (\mathrm{i}\gamma^a\nabla_a - q\gamma^a A_a(x) - m)\Psi(x) \\
&= (\Lambda^b{}_a\gamma^a(\mathrm{i}\nabla'_b - qA'_b(x)) - m)\Psi(x) \,. \qquad \text{(D.36)}
\end{aligned}
$$

Earlier, I showed that $\gamma'_a = (\Lambda^{-1})^b{}_a\gamma_a \implies \gamma'_a = S(\Lambda)^{-1}\gamma_a S(\Lambda)$ for some group representation, $S(\Lambda)$. Let $T(\Lambda)$ be the corresponding representation for contravariant indices,
i.e. $\gamma'^a = \Lambda^a{}_b\gamma^b \implies \gamma'^a = T(\Lambda)^{-1}\gamma^a T(\Lambda)$. Hence, the Dirac equation becomes

$$
\begin{aligned}
0 &= (T(\Lambda)^{-1}\gamma^b T(\Lambda)(\mathrm{i}\nabla'_b - qA'_b(x)) - m)\Psi(x) \\
&= T(\Lambda)^{-1}(\gamma^a(\mathrm{i}\nabla'_a - qA'_a(x)) - m)T(\Lambda)\Psi(x) \\
\iff 0 &= (\mathrm{i}\gamma^a\nabla'_a - q\gamma^a A'_a(x) - m)T(\Lambda)\Psi(x) \,. \qquad \text{(D.37)}
\end{aligned}
$$

Therefore, it must be that $\Psi'(x) = T(\Lambda)\Psi(x)$. This defines the transformation property of spinors[5].
If one restricts attention to special relativity, then the transformation of interest is $x'^a = \Lambda^a{}_b x^b$. Then, the Dirac equation is $0 = (\mathrm{i}\gamma^a\partial_a - q\gamma^a A_a(x) - m)\Psi(x)$ and the transformation property required of spinors is $\Psi'(x') = T(\Lambda)\Psi(x)$, or equivalently $\Psi'(x) = T(\Lambda)\Psi(\Lambda^{-1}x)$.

There are still many properties of spinors left to consider. For "calculation" purposes, it will be useful to choose a basis in the spinor/representation space of the Clifford algebra. As G is a finite group, $\exists$ an inner product (that is unique up to scaling) invariant under the action of the representation. Since scaling is arbitrary, any scaling of this unique inner product can

[5]Rather than take the Dirac equation as fundamental and derive spinors' transformation properties from there, a more mathematical perspective would be to define spinors to transform as $\Psi'(x) = T(\Lambda)\Psi(x)$ and use that to prove the Dirac equation transforms covariantly.

be chosen. Then, choose a basis that is orthonormal with respect to this inner product.
In this basis, all γ_a are unitary, i.e. $\gamma_a^\dagger = (\gamma_a)^{-1}$.
However, $\gamma_a\gamma_b + \gamma_b\gamma_a = -2\eta_{ab}I \implies (\gamma_a)^2 = -\eta_{aa}I$ (no sum).
Hence, $(\gamma_a)^{-1} = \gamma_a$ for $0 \le a \le t-1$ and $(\gamma_a)^{-1} = -\gamma_a$ for $t \le a \le s+t-1$.
Or equivalently, $\gamma_a^\dagger = \gamma_a$ for $0 \le a \le t-1$ and $\gamma_a^\dagger = -\gamma_a$ for $t \le a \le s+t-1$.

Theorem D.4. *Let $A = \gamma_0\gamma_1\cdots\gamma_{t-1}$. Then, A is unitary and $\gamma_a^\dagger = (-1)^{t+1}A\gamma_a A^{-1}$.*

Proof. For $0 \le a \le t-1$, $\gamma_a^\dagger = (\gamma_a)^{-1} = \gamma_a$. Then,

$$\begin{aligned} A^\dagger A &= (\gamma_0\cdots\gamma_{t-1})^\dagger(\gamma_0\cdots\gamma_{t-1}) \\ &= \gamma_{t-1}^\dagger\cdots\gamma_0^\dagger\gamma_0\cdots\gamma_{t-1} \\ &= (\gamma_{t-1})^{-1}\cdots(\gamma_0)^{-1}\gamma_0\cdots\gamma_{t-1} \\ &= I \quad \implies A \text{ is unitary.} \end{aligned} \tag{D.38}$$

For $0 \le b \le t-1$, $(\gamma_b)^{-1} = \gamma_b$ and hence $A^{-1} = \gamma_{t-1}\cdots\gamma_0$.
For $t \le a \le s+t-1$,

$$\begin{aligned} (-1)^{t+1}A\gamma_a A^{-1} &= (-1)^{t+1}\gamma_0\cdots\gamma_{t-1}\gamma_a\gamma_{t-1}\cdots\gamma_0 \\ &= (-1)^{t-1}\gamma_a(-1)^t\gamma_0\cdots\gamma_{t-1}\gamma_{t-1}\cdots\gamma_0 \\ &= (-1)^{2t+1}\gamma_a \\ &= -\gamma_a \\ &= \gamma_a^\dagger\,. \end{aligned} \tag{D.39}$$

For $0 \le a \le t-1$,

$$\begin{aligned} (-1)^{t+1}A\gamma_a A^{-1} &= (-1)^{t+1}\gamma_0\cdots\gamma_{t-1}\gamma_a\gamma_{t-1}\cdots\gamma_0 \\ &= (-1)^{t-1}\gamma_0\cdots\gamma_{t-1}\gamma_a\gamma_{t-1}\cdots\gamma_a\cdots\gamma_0 \;\; \text{(no sum)} \\ &= (-1)^{t+1}\gamma_0\cdots\gamma_{t-1}\gamma_{t-1}\cdots\gamma_a(-1)^{t-a-1}\gamma_a\cdots\gamma_0 \\ &= (-1)^{2t-a}\gamma_0\cdots\gamma_a\gamma_a\gamma_a\gamma_{a-1}\cdots\gamma_0 \\ &= (-1)^a\gamma_0\cdots\gamma_{a-1}\gamma_a\gamma_{a-1}\cdots\gamma_0 \\ &= (-1)^a\gamma_0\cdots\gamma_{a-1}\gamma_{a-1}\cdots\gamma_0\gamma_a(-1)^a \\ &= (-1)^{2a}\gamma_a \\ &= \gamma_a \\ &= \gamma_a^\dagger\,. \end{aligned} \tag{D.40}$$

Putting together all the cases, $\gamma_a^\dagger = (-1)^{t+1}A\gamma_a A^{-1}$ in general. □

To derive the next few results, restrict attention to the case of D being even.

$$\begin{aligned} (\pm\gamma_a)^*(\pm\gamma_b)^* + (\pm\gamma_b)^*(\pm\gamma_a)^* &= (\gamma_a\gamma_b + \gamma_b\gamma_a)^* \\ &= (-2\eta_{ab}I)^* \\ &= -2\eta_{ab}I \end{aligned} \tag{D.41}$$

Therefore, $\{\pm\gamma_a^*\}_{a=0}^{D-1}$ also satisfy the Clifford algebra.
Since the irreducible representation of the Clifford algebra is unique in even dimensions, $\exists$ matrices, B_1 and B_2, such that $\gamma_a^* = B_1\gamma_a(B_1)^{-1}$ and $-\gamma_a^* = B_2\gamma_a(B_2)^{-1}$. These two

equations can be wrapped together by saying $\gamma_a^* = \mu B\gamma_a B^{-1}$ where $\mu = \pm 1$. Here μ and B are taken to be interdependent, e.g. if $\mu = 1$, then $B = B_1$ while if $\mu = -1$, then $B = B_2$.

$$\begin{aligned}
\gamma_a^* = \mu B\gamma_a B^{-1} \implies \gamma_a &= \mu B^*\gamma_a^* B^{-*} \\
&= \mu B^* \mu B\gamma_a B^{-1}B^{-*} \\
&= B^*B\gamma_a(B^*B)^{-1} \\
\iff \gamma_a B^*B &= B^*B\gamma_a \quad \forall a
\end{aligned} \tag{D.42}$$

Then, by Schur's lemma, $B^*B = \nu I$ for some $\nu \in \mathbb{C} \setminus \{0\}$.
But, $BB^* = \nu I$ as well since a matrix and its inverse commute.
Therefore, $(BB^*)^* = \nu^* I \implies B^*B = \nu^* I \implies \nu I = \nu^* I \implies \nu \in \mathbb{R} \setminus \{0\}$. Then,

$$\begin{aligned}
BB^* = \nu I &\implies \det(BB^*) = \det(\nu I) \\
&\implies \det(B)\det(B^*) = \nu^{2^{D/2}}\det(I) \\
&\implies \nu^{2^{D/2}} = |\det(B)|^2 \,.
\end{aligned} \tag{D.43}$$

For any $k \in \mathbb{C} \setminus \{0\}$, $(kB)\gamma_a(kB)^{-1} = B\gamma_a B^{-1} = \gamma_a^*$, i.e. B can be scaled without loss of generality as its definition only relies on $\mu B\gamma_a B^{-1} = \gamma_a^*$.
I will scale B so that $|\det(B)| = 1$.
Therefore, $\nu^{2^{D/2}} = 1$ and hence $\nu = \pm 1$.
Since γ_a are unitary,

$$\begin{aligned}
I = \gamma_a\gamma_a^\dagger &= \gamma_a(\gamma_a^*)^T = \gamma_a\mu B^{-T}\gamma_a^T B^T \\
\implies I^* &= (\mu\gamma_a B^{-T}\gamma_a^T B^T)^* \\
\iff I &= \mu(\gamma_a B^{-T}\gamma_a^T B^T)^* \\
&= \mu\gamma_a^* B^{-\dagger}\gamma_a^\dagger B^\dagger \\
&= \mu^2 B\gamma_a B^{-1}B^{-\dagger}\gamma_a^\dagger B^\dagger \\
&= B\gamma_a B^{-1}B^{-\dagger}\gamma_a^\dagger B^\dagger \\
\iff B^{-\dagger}\gamma_a &= B\gamma_a B^{-1}B^{-\dagger} \\
\iff \gamma_a B^\dagger B &= B^\dagger B\gamma_a \quad \forall, .
\end{aligned} \tag{D.44}$$

Then, by Schur's lemma, $B^\dagger B = \rho I$ for some $\rho \in \mathbb{C} \setminus \{0\}$. Hence, $\rho = \pm 1$ by the exact same reasoning by which ν was constrained to be ± 1.
For any vector, $v \in \mathbb{C}^{2^{D/2}}$, $v^\dagger B^\dagger B v = v^\dagger \rho I v \implies ||Bv||^2 = \rho||v||^2$. Then, as $||Bv||^2 \geq 0$ and $||v||^2 \geq 0$, it must be that $\rho \geq 0$.
Thus, ρ can only equal 1, thereby making B unitary.

Theorem D.5. *Let $C = B^T A$. Then, C is unitary and $\gamma_a^T = (-1)^{t+1}\mu C\gamma_a C^{-1}$.*

Proof. $C^\dagger C = (B^T A)^\dagger B^T A = A^\dagger B^* B^T A = A^\dagger (BB^\dagger)^* A = A^\dagger A = I \implies C$ is unitary.
For the other part of the proof, applying theorem D.4 along the way,

$$\begin{aligned}
\gamma_a^T &= (\gamma_a^\dagger)^* \\
&= ((-1)^{t+1}A\gamma_a A^{-1})^* \\
&= (-1)^{t+1}A^*\gamma_a^* A^{-*} \,.
\end{aligned} \tag{D.45}$$

$$\begin{aligned}
A^* &= (\gamma_0 \cdots \gamma_{t-1})^* \\
&= \gamma_0^* \cdots \gamma_{t-1}^* \\
&= \mu B\gamma_0 B^{-1} \cdots \mu B\gamma_{t-1}B^{-1} \\
&= \mu^t BAB^{-1} \quad \implies A^{-*} = \frac{1}{\mu^t}BA^{-1}B^{-1}
\end{aligned} \tag{D.46}$$

Therefore,

$$\begin{aligned}\gamma_a^T &= (-1)^{t+1}(\mu^t BAB^{-1})(\mu B\gamma_a B^{-1})\left(\frac{1}{\mu^t}BA^{-1}B^{-1}\right)\\ &= (-1)^{t+1}\mu BA\gamma_a A^{-1}B^{-1}\,.\end{aligned} \tag{D.47}$$

$B^*B = \nu I$ and $B^\dagger B = I \implies B^* = \nu B^\dagger \implies B = \nu B^T$. Thus,

$$\begin{aligned}\gamma_a^T &= (-1)^{t+1}\mu\nu B^T A\gamma_a A^{-1}B^{-T}/\nu\\ &= (-1)^{t+1}C\gamma_a C^{-1}\,.\end{aligned} \tag{D.48}$$

□

Consider the effect of B and C on spinors in the context of the Dirac equation.

$$\begin{aligned} & 0 = (\mathrm{i}\gamma^a\nabla_a - q\gamma^a A_a - m)\Psi\\ \iff\ & 0 = (-\mathrm{i}\gamma^{a*}\nabla_a - q\gamma^{a*}A_a - m)\Psi^*\\ & \quad = (-\mathrm{i}\mu B\gamma^a B^{-1}\nabla_a - q\mu B\gamma^a B^{-1}A_a - m)\Psi^*\\ & \quad = B(-\mathrm{i}\mu\gamma^a\nabla_a - q\mu\gamma^a A_a - m)B^{-1}\Psi^*\\ \iff\ & 0 = (-\mathrm{i}\mu\gamma^a\nabla_a - q\mu\gamma^a A_a - m)B^{-1}\Psi^* \end{aligned} \tag{D.49}$$

If $\mu = -1$, then $B^{-1}\Psi^*$ satisfies the same Dirac equation as Ψ but with $q \to -q$.
If $\mu = -1$, then $B^{-1}\Psi^*$ describes the antiparticle of the particle described by Ψ.
On the other hand, if $\mu = 1$, then $B^{-1}\Psi^*$ satisfies the same Dirac equation as Ψ but with both $q \to -q$ and $m \to -m$.
When $\mu = -1$, a particle is its own antiparticle if and only if $B^{-1}\Psi^* = \Psi \iff \Psi^* = B\Psi$.
Therefore, $\Psi = (B\Psi)^* = B^*\Psi^* = B^*B\Psi = \nu\Psi \implies \nu = 1$.

Definition D.6. *If $\mu = -1$, $\nu = 1$ and $\Psi^* = B\Psi$, then Ψ is called a Majorana spinor. If $\mu = 1$, $\nu = 1$ and $\Psi^* = B\Psi$, then Ψ is called a pseudo-Majorana spinor.*
If $\nu = -1$ and one has two spinors, $\Psi_i (i = 1, 2)$, then one can impose an “SU(2) *reality condition,” $\overline{\Psi}^i = (\Psi_i)^* = \varepsilon^{ij}B\Psi_j$. In this case, the $\mu = -1$ and $\mu = 1$ cases are called* SU(2) *Majorana and* SU(2) *pseudo-Majorana spinors respectively.*

The matrix, C, can also be related to antiparticles as follows. From the Dirac equation,

$$\begin{aligned} 0 &= ((\mathrm{i}\gamma^a\nabla_a - q\gamma^a A_a - m)\Psi)^\dagger\\ &= -\mathrm{i}\nabla_a(\Psi^\dagger)(\gamma^a)^\dagger - q\Psi^\dagger(\gamma^a)^\dagger A_a - m\Psi^\dagger\\ &= -\mathrm{i}\nabla_a(\Psi^\dagger)(-1)^{t+1}A\gamma^a A^{-1} - (-1)^{t+1}q\Psi^\dagger A\gamma^a A^{-1}A_a - m\Psi^\dagger\\ &= (-\mathrm{i}\nabla_a(\Psi^\dagger A)(-1)^{t+1}\gamma^a - (-1)^{t+1}q\Psi^\dagger A\gamma^a A_a - m\Psi^\dagger)A^{-1}\,. \end{aligned} \tag{D.50}$$

Let $\Psi^\dagger A = \overline{\Psi}$; $\overline{\Psi}$ is called the adjoint spinor. With this notation,

$$\begin{aligned} & 0 = \overline{\Psi}((-1)^{t+1}\mathrm{i}\gamma^a\overleftarrow{\nabla}_a + (-1)^{t+1}q\gamma^a A_a + m)\\ \implies\ & 0 = ((-1)^{t+1}\mathrm{i}(\gamma^a)^T\nabla_a + (-1)^{t+1}q(\gamma^a)^T A_a + m)\overline{\Psi}^T\\ & \quad = ((-1)^{t+1}\mathrm{i}(-1)^{t+1}\mu C\gamma^a C^{-1}\nabla_a + (-1)^{t+1}q(-1)^{t+1}\mu C\gamma^a C^{-1}A_a + m)\overline{\Psi}^T\\ \implies\ & 0 = (\mathrm{i}\mu\gamma^a\nabla_a + q\mu\gamma^a A_a + m)C^{-1}\overline{\Psi}^T \end{aligned} \tag{D.51}$$

Again, if $\mu = -1$, then $C^{-1}\overline{\Psi}^T$ describes the antiparticle of the particle described by Ψ. For this reason, $C^{-1}\overline{\Psi}^T$ is denoted $\overline{\Psi}_C$ and C is called the charge conjugation matrix. For reasons unknown, B does not have a special name despite the similarity. It is however no coincidence that $B^{-1}\Psi^*$ and $C^{-1}\overline{\Psi}^T$ serve the same purpose.

Theorem D.7. *$B^{-1}\Psi^*$ and $C^{-1}\overline{\Psi}^T$ are proportional to each other.*

Proof. $C^{-1}\overline{\Psi}^T = (B^TA)^{-1}(\Psi^\dagger A)^T = A^{-1}B^{-T}A^T\Psi^* = A^{-1}(AB^{-1})^T\Psi^*$
$A^* = \gamma_0^* \cdots \gamma_{t-1}^* = \mu B\gamma_0 B^{-1}\cdots \mu B\gamma_{t-1}B^{-1} = \mu^t BAB^{-1} \implies AB^{-1} = \mu^t B^{-1}A^*$
Therefore, $C^{-1}\overline{\Psi}^T = \mu^t A^{-1}A^\dagger B^{-T}\Psi^*$.
However, I showed earlier that $B = \nu B^T$. Thus, $B^{-1} = \nu B^{-T} \iff B^{-T} = \nu B^{-1}$ since $\nu^2 = 1$.
Meanwhile, for the other two matrices,

$$\begin{aligned} A^{-1}A^\dagger &= (\gamma_0\cdots\gamma_{t-1})^{-1}(\gamma_0\cdots\gamma_{t-1})^\dagger \\ &= \gamma_{t-1}^{-1}\cdots\gamma_0^{-1}\gamma_{t-1}^\dagger\cdots\gamma_0^\dagger \\ &= \gamma_{t-1}\cdots\gamma_0\gamma_{t-1}\cdots\gamma_0 \\ &= (-1)^{t-1+t-2+\cdots+1}I \\ &= (-1)^{t(t-1)/2}I \qquad \text{(D.52)} \\ \implies C^{-1}\overline{\Psi}^T &= \nu\mu^t(-1)^{t(t-1)/2}B^{-1}\Psi^* \,. \qquad \text{(D.53)} \end{aligned}$$

□

As it happens, ν and μ are not independent.

Theorem D.8. *ν is a function of μ, t and s by*

$$\nu = \cos\left(\frac{\pi}{4}(s-t)\right) - \mu\sin\left(\frac{\pi}{4}(s-t)\right) . \qquad \text{(D.54)}$$

Proof. I have already shown $B^T = \nu B$. Then, using theorems D.4 and D.5,

$$\begin{aligned} C^T &= (B^TA)^T \\ &= \gamma_{t-1}^T\cdots\gamma_0^T B \\ &= (-1)^{t+1}\mu C\gamma_{t-1}C^{-1}\cdots(-1)^{t+1}\mu C\gamma_0 C^{-1}B \\ &= (-1)^{t(t-1)}\mu^t C\gamma_{t-1}\cdots\gamma_0 C^{-1}B \\ &= (-1)^{t(t-1)}\mu^t(-1)^{t-1+t-2+\cdots+1}C\gamma_0\cdots\gamma_{t-1}C^{-1}B \\ &= (-1)^{t(3t+1)/2}\mu^t CAC^{-1}B \\ &= (-1)^{t(3t+1)/2}\mu^t CAA^{-1}B^{-T}B \\ &= (-1)^{t(3t+1)/2}\mu^t C\nu B^{-1}B \\ &= (-1)^{t(t-1)/2}\mu^t\nu C\,. \qquad \text{(D.55)} \end{aligned}$$

Thus, B and C may be symmetric or antisymmetric (independently). To see how this is relevant, consider the group, G, introduced earlier. In particular, consider the subset, $\{\Gamma_A\}_{A=0}^{2^D-1}$. Let $\sum_{A=0}^{2^D-1} C_A\Gamma_A = 0$ for some constants, $C_A \in \mathbb{C}$. Then,

$$0 = \sum_{A=0}^{2^D-1} C_A\Gamma_A\Gamma_B \qquad \text{(D.56)}$$

$$\implies 0 = \sum_{A=0}^{2^D-1} C_A\mathrm{tr}(\Gamma_A\Gamma_B)\,. \qquad \text{(D.57)}$$

However, I showed earlier that $\Gamma_A\Gamma_B = \pm\Gamma_C$ for some C and $\mathrm{tr}(\Gamma_C) = 0$ unless $\Gamma_C = I$ (in even dimensions).

Hence, $\mathrm{tr}(\Gamma_C) \neq 0 \implies \Gamma_B = (\Gamma_A)^{-1} = \pm\Gamma_A \implies A = B$.
Therefore, the sum in D.57 collapses to $C_B = 0$.
As B is arbitrary, $\{\Gamma_A\}_{A=0}^{2^D-1}$ is a linearly independent set. The size of the set is $2^D = 2^{D/2}\times 2^{D/2}$, which is the dimension of the vector space of $2^{D/2} \times 2^{D/2}$ matrices.
Thus, $\{\Gamma_A\}_{A=0}^{2^D-1}$ is a basis for the set of $2^{D/2} \times 2^{D/2}$ matrices. This basis can be "antisymmetrised" to $\{\tilde{\Gamma}^{(n)}\}$, where $\tilde{\Gamma}^{(n)} = \gamma_{[a_1}\cdots\gamma_{a_n]}$, i.e. rather than $\gamma_{a_1}\cdots\gamma_{a_n}$ with $a_1 < a_2 < \cdots < a_n$, the indices are antisymmetrised. There are ${}^DC_n$ matrices of type, $\tilde{\Gamma}^{(n)}$. Furthermore, as C is invertible, $\{\Gamma_A\}_{A=0}^{2^D-1}$ is a basis $\implies \{C\Gamma_A\}_{A=0}^{2^D-1}$ is a basis $\implies \{C\tilde{\Gamma}^{(n)}\}$ is a basis.

$$\begin{aligned}
(C\tilde{\Gamma}^{(n)})^T &= (\tilde{\Gamma}^{(n)})^T C^T \\
&= (\gamma_{[a_1}\cdots\gamma_{a_n]})^T(-1)^{t(t-1)/2}\mu^t\nu C \\
&= \gamma^T_{[a_n}\cdots\gamma^T_{a_1]}(-1)^{t(t-1)/2}\mu^t\nu C \\
&= (-1)^{t+1}\mu C\gamma_{[a_n}C^{-1}\cdots(-1)^{t+1}\mu C\gamma_{a_1]}C^{-1}(-1)^{t(t-1)/2}\mu^t\nu C \\
&= (-1)^{n(t+1)}\mu^{n+t}C(-1)^{t(t-1)/2}\nu(-1)^{n-1+n-2+\cdots+1}\gamma_{[a_1}\cdots\gamma_{a_n]} \\
&= (-1)^{(n^2+n+2nt-t+t^2)/2}\mu^{n+t}\nu C\tilde{\Gamma}^{(n)}
\end{aligned} \tag{D.58}$$

The last equation means each of the $C\tilde{\Gamma}^{(n)}$ is either symmetric or antisymmetric.
Since every matrix can be decomposed into symmetric and antisymmetric parts, the antisymmetric $C\tilde{\Gamma}^{(n)}$ must form a basis for the antisymmetric $2^{D/2} \times 2^{D/2}$ matrices.
However, the set of antisymmetric matrices is known to have dimension,
${}^{2^{D/2}}C_2 = \frac{1}{2}2^{D/2}(2^{D/2} - 1)$.
That means there are $\frac{1}{2}2^{D/2}(2^{D/2} - 1)$ antisymmetric $C\tilde{\Gamma}^{(n)}$. To count the number of antisymmetric $C\tilde{\Gamma}^{(n)}$, note that there are ${}^DC_n$ matrices of type, $C\tilde{\Gamma}^{(n)}$, and $\frac{1}{2}(1-(-1)^{(n^2+n+2nt-t+t^2)/2}\mu^{n+t}\nu) = 0$ for a symmetric $C\tilde{\Gamma}^{(n)}$ and 1 for an antisymmetric $C\tilde{\Gamma}^{(n)}$. Therefore,

$$\begin{aligned}
\frac{1}{2}2^{D/2}(2^{D/2} - 1) &= \sum_{n=0}^{D}\frac{1}{2}(1 - (-1)^{(n^2+n+2nt-t+t^2)/2}\mu^{n+t}\nu)\,{}^DC_n \\
\implies 2^D - 2^D/2 &= \sum_{n=0}^{D}(1 - (-1)^{(n^2+n+2nt-t+t^2)/2}\mu^{n+t}\nu)\,{}^DC_n \\
&= \sum_{n=0}^{D}{}^DC_n - \nu\mu^t(-1)^{t(t-1)/2}\sum_{n=0}^{D}\mu^n(-1)^{n(n+2t+1)/2}\ {}^DC_n \\
\implies 2^{D/2}\mu^t(-1)^{t(t-1)/2} &= \nu\sum_{n=0}^{D}{}^DC_n\mu^n(-1)^{n(n+2t+1)/2}\,.
\end{aligned} \tag{D.59}$$

At this point one might guess that

$$(-1)^{n(n+2t+1)/2} = \frac{(-1)^{nt}}{2}((1+\mathrm{i})\mathrm{i}^n + (1-\mathrm{i})(-\mathrm{i})^n)\,. \tag{D.60}$$

Because of the periodicity in powers of 1 and i, this expression only needs to hold for n, t mod 4, to hold in general. I have checked the equation really does hold for those 16 combinations on *Mathematica*. Hence,

$$
\begin{aligned}
2^{D/2}\mu^t(-1)^{t(t-1)/2} &= \frac{\nu}{2}\sum_{n=0}^{D}(\mu(-1)^t)^n((1+\mathrm{i})\mathrm{i}^n+(1-\mathrm{i})(-\mathrm{i})^n)^D C_n \\
&= \frac{\nu(1+\mathrm{i})}{2}\sum_{n=0}^{D}(\mu(-1)^t)^n(\mathrm{i}^n-\mathrm{i}(-\mathrm{i})^n)^D C_n \\
&= \frac{\nu(1+\mathrm{i})}{2}\left(\sum_{n=0}^{D}(\mathrm{i}\mu(-1)^t)^n\,{}^D C_n - \mathrm{i}\sum_{n=0}^{D}(-\mathrm{i}\mu(-1)^t)^n\,{}^D C_n\right) \\
&= \frac{1}{2}\nu(1+\mathrm{i})((1+\mathrm{i}\mu(-1)^t)^D-\mathrm{i}(1-\mathrm{i}\mu(-1)^t)^D)\,. \qquad \text{(D.61)}
\end{aligned}
$$

Since $1+\mathrm{i}=\sqrt{2}\mathrm{e}^{\mathrm{i}\pi/4}$ and $1-\mathrm{i}=\sqrt{2}\mathrm{e}^{-\mathrm{i}\pi/4}$, the last line can be re-written as

$$
2^{D/2}\mu^t(-1)^{t(t-1)/2} = \frac{1}{2}\nu\sqrt{2}\mathrm{e}^{\mathrm{i}\pi/4}2^{D/2}(\mathrm{e}^{\mathrm{i}\mu(-1)^t D\pi/4}-\mathrm{e}^{\mathrm{i}\pi/2}\mathrm{e}^{-\mathrm{i}\mu(-1)^t D\pi/4})\,, \qquad \text{(D.62)}
$$

which re-arranges to

$$
\nu = \frac{\sqrt{2}\mu^t(-1)^{t(t-1)/2}}{\mathrm{e}^{\mathrm{i}\pi/4}(\mathrm{e}^{\mathrm{i}\mu(-1)^t D\pi/4}-\mathrm{e}^{\mathrm{i}\pi/2}\mathrm{e}^{-\mathrm{i}\mu(-1)^t D\pi/4})}\,. \qquad \text{(D.63)}
$$

Because of the periodicity of $\mathrm{e}^{\mathrm{i}x\pi/4}$ and $(-1)^x$, it only matters whether $\mu=1$ or -1 and what s and t are modulo 8.
Therefore, there are only $2\times 8\times 8 = 128$ different cases. Again, one may guess that

$$
\frac{\mathrm{e}^{\mathrm{i}\pi/4}(\mathrm{e}^{\mathrm{i}\mu(-1)^t D\pi/4}-\mathrm{e}^{\mathrm{i}\pi/2}\mathrm{e}^{-\mathrm{i}\mu(-1)^t D\pi/4})}{\sqrt{2}\mu^t(-1)^{t(t-1)/2}} = \cos\left(\frac{\pi}{4}(s-t)\right)-\mu\sin\left(\frac{\pi}{4}(s-t)\right). \qquad \text{(D.64)}
$$

To check that this equation really holds, one only needs to check the 128 different cases - a task I have completed with the aid of *Mathematica*. Finally, $\nu=\pm 1 \implies \nu=\frac{1}{\nu}$ and thus $\nu=\cos(\frac{\pi}{4}(s-t))-\mu\sin(\frac{\pi}{4}(s-t))$. □

Since equation D.41, the discussion has been limited to even dimensions. It is now time to extend the results to odd dimensions. Let D be even and let the odd dimension of interest be $D+1$. If $D=s+t$, assume without loss of generality that $D+1=(s+1)+t$, i.e. a space dimension is added. Let $\gamma_{D+1}=\gamma_0\cdots\gamma_{D-1}$ as before. Then,

$$
\begin{aligned}
\gamma_{D+1}\gamma_a &= \gamma_0\cdots\gamma_{D-1}\gamma_a \\
&= \gamma_0\cdots\gamma_a\cdots\gamma_{D-1}\gamma_a \quad \text{(no sum)} \\
&= \gamma_0\cdots\gamma_a\gamma_a\cdots\gamma_{D-1}(-1)^{D-a-1} \\
&= (-1)^a\gamma_a\gamma_0\cdots\gamma_a\cdots\gamma_{D-1}(-1)^{D-a-1} \\
&= (-1)^{D-1}\gamma_a\gamma_{D+1} \\
&= -\gamma_a\gamma_{D+1} \quad \text{as } D \text{ is even,} \qquad \text{(D.65)} \\
\iff \gamma_{D+1}\gamma_a+\gamma_a\gamma_{D+1} &= 0 = -2\eta_{a,D}I\,. \qquad \text{(D.66)} \\
\text{Meanwhile, } (\gamma_{D+1})^2 &= \gamma_0\cdots\gamma_{D-1}\gamma_0\cdots\gamma_{D-1} \\
&= (-1)^{D-1+D-2+\cdots+1}(\gamma_0)^2\cdots(\gamma_{D-1})^2 \\
&= (-1)^{D(D-1)/2}(-1)^s I \\
&= (-1)^{D^2/2+(s-t)/2} I \\
&= (-1)^{(s-t)/2} I\,, \qquad \text{(D.67)}
\end{aligned}
$$

as $D^2/2$ is even, $(\gamma_a)^2 = I$ for timelike indices and $(\gamma_a)^2 = -I$ for spacelike indices.
Hence, $\{\gamma_a, \gamma_{D+1}\}_{a=0}^{D-1}$ satisfies the Clifford algebra for $s - t \equiv 2 \pmod 4$ and $\{\gamma_a, \mathrm{i}\gamma_{D+1}\}_{a=0}^{D-1}$ satisfies the Clifford algebra for $s - t \equiv 0 \pmod 4$ ($s - t \equiv 1, 3 \pmod 4$ are not possible for even D).
By theorems D.2 and D.3, in odd dimensions, there are two inequivalent representations, $\{\gamma_a, \gamma_{D+1}\}_{a=0}^{D-1}$ & $\{-\gamma_a, -\gamma_{D+1}\}_{a=0}^{D-1}$ and $\{\gamma_a, \mathrm{i}\gamma_{D+1}\}_{a=0}^{D-1}$ & $\{-\gamma_a, -\mathrm{i}\gamma_{D+1}\}_{a=0}^{D-1}$ respectively.
Unlike the even case, $\{\gamma_a^*, \gamma_{D+1}^*\}_{a=0}^{D-1}$ & $\{-\gamma_a^*, -\gamma_{D+1}^*\}_{a=0}^{D-1}$ and $\{\gamma_a^*, -\mathrm{i}\gamma_{D+1}^*\}_{a=0}^{D-1}$ & $\{-\gamma_a^*, \mathrm{i}\gamma_{D+1}^*\}_{a=0}^{D-1}$ respectively are no longer equivalent.
Thus, in $\gamma_a^* = \mu B\gamma_a B^{-1}$, μ can on longer be freely chosen as 1 or -1. Instead, μ will be fixed by forcing $\gamma_{D+1}^* = \mu B\gamma_{D+1}B^{-1}$ or $-\mathrm{i}\gamma_{D+1}^* = \mu B\mathrm{i}\gamma_{D+1}B^{-1}$.
First, consider $\gamma_{D+1}^* = \mu B\gamma_{D+1}B^{-1}$. For that,

$$\begin{aligned}
\mu B\gamma_{D+1}B^{-1} &= \gamma_{D+1}^* \\
&= \gamma_0^* \cdots \gamma_{D-1}^* \\
&- \mu B\gamma_0 B^{-1} \cdots \mu B\gamma_{D-1}B^{-1} \\
&= \mu^D B\gamma_{D+1}B^{-1} \\
&= B\gamma_{D+1}B^{-1} \quad \text{as } D \text{ is even,} \\
\implies \mu &= 1\,.
\end{aligned} \tag{D.68}$$

Hence, when $s - t \equiv 2 \pmod 4$, $\mu = 1$. Similarly, $-\mathrm{i}\gamma_{D+1}^* = \mu B\mathrm{i}\gamma_{D+1}B^{-1} \implies \mu = -1$ when $s-t \equiv 0 \pmod 4$. These two equations can be summarised in one equation, $\mu = (-1)^{(s-t+2)/2}$.
To proceed, note that $D+1$ odd, the irreducible representations still have dimension, $2^{D/2}$.
Therefore, $\{\gamma_a\}_{a=0}^{D-1}$ can still be used to generate $\{\Gamma_A\}_{A=0}^{2^D-1}$, which will still be a basis for $2^{D/2} \times 2^{D/2}$ matrices. Furthermore, A's properties only depend on t, not s. Likewise, in finding $\nu = \pm 1$ and and the other results, I only needed $2^{D/2}$ is even, not D is even. In fact, looking back over the proofs, all the properties continue to hold. The only difference is $\mu = (-1)^{(s-t+2)/2}$ is fixed rather than free.
Thus far, I have written odd dimensions as $D+1 = (s+1)+t$. To write odd D as $s+t$, I will have to let $s \to s-1$ in the theorems for odd dimensions. Overall, one gets the following.

Theorem D.9 (Summary of results)**.** *For $D = s + t$ (D may be odd or even) and $D > 1$,*

- $\mu = (-1)^{(s-t+1)/2}$ *in odd dimensions.*
- *μ can be freely chosen as 1 or -1 in even dimensions.*
- $\gamma_a^\dagger = (-1)^{t+1} A\gamma_a A^{-1}$ *where* $A = \gamma_0 \cdots \gamma_{t-1}$.
- *$\exists$ a matrix, B, such that* $\gamma_a^* = \mu B\gamma_a B^{-1}$.
- $\gamma_a^T = (-1)^{t+1}\mu C\gamma_a C^{-1}$ *where* $C = B^T A$.
- *A, B and C are all unitary,* $B^*B = \nu I$ *for* $\nu = \pm 1$, $B^T = \nu B$ *and* $C^T = \nu\mu^t(-1)^{t(t-1)/2}C$.
- $\nu = \cos\big(\frac{\pi}{4}(s-t)\big) - \mu\sin\big(\frac{\pi}{4}(s-t)\big)$ *in even dimensions.*
- $\nu = \cos\big(\frac{\pi}{4}(s-t-1)\big) - \mu\sin\big(\frac{\pi}{4}(s-t-1)\big)$ *in odd dimensions.*

Proof. See above. □

I am now in a position to evaluate all possible combinations of ν, μ and $s-t$ (ν and μ only depend on $s-t$).
For $s - t \equiv 1, 3, 5, 7 \pmod 8$, $s - t - 1 \equiv 0, 2, 4, 6 \pmod 8$ and hence $\mu = -1, 1, -1, 1$ and

$\nu = \cos 0 + \sin 0 = 1$, $\cos\pi/2 + \sin\pi/2 = -1$, $\cos\pi + \sin\pi = -1$, $\cos 3\pi/2 + \sin 3\pi/2 = 1$.
In the even cases, $\mu = \pm 1$ and $s - t \equiv 0, 2, 4, 6 \pmod 8$ imply $\nu = \cos 0 \mp \sin 0 = 1$, $\cos\pi/2 \mp \sin\pi/2 = \mp 1$, $\cos\pi \mp \sin\pi = -1$, $\cos 3\pi/2 \mp \sin 3\pi/2 = \pm 1$. These results are summarised in table D.1.

ν	μ	Possible $s-t$ mod 8	Antiparticle related spinor
1	1	0, 6, 7	pseudo-Majorana
1	-1	0, 1, 2	Majorana
-1	1	2, 3, 4	SU(2) pseudo-Majorana
-1	-1	4, 5, 6	SU(2) Majorana

Table D.1: The antiparticle related spinors possible in different spacetimes

Besides the suite of Majorana like spinors, another special type of spinor relevant to physics is the so-called Weyl spinor. Weyl spinors are defined as eigenvectors of γ_{D+1}. However, I already showed in equation D.19 that in odd dimensions $\gamma_{D+1}\gamma_a = \gamma_a\gamma_{D+1}\ \forall a$
$\implies \gamma_{D+1} g = g\gamma_{D+1}\ \forall g \in G \implies \gamma_{D+1} \propto I$ by Schur's lemma.
Hence, in odd dimensions, every spinor is an eigenvector of γ_{D+1} and so the concept of a Weyl spinor would be fruitless.
To accommodate for that, define Weyl spinors to exist only for even dimensional spacetimes. Rather than $\gamma_{D+1}\Psi = \lambda\Psi$ however, it is more customary[6] to consider $(-1)^{(s-t)/4}\gamma_{D+1}\Psi = \lambda\Psi$ with $(-1)^{1/2}$ defined to be $-\mathrm{i}$ without loss of generality[7].

$$\begin{aligned}
\lambda^2\Psi &= (-1)^{(s-t)/4}\gamma_{D+1}(-1)^{(s-t)/4}\gamma_{D+1}\Psi \\
&= (-1)^{(s-t)/2}\gamma_0\cdots\gamma_{D-1}\gamma_0\cdots\gamma_{D-1}\Psi \\
&= (-1)^{(s-t)/2}(-1)^{D-1+D-2+\cdots+1}(\gamma_0)^2\cdots(\gamma_{D-1})^2\Psi \\
&= (-1)^{(s-t)/2}(-1)^{D(D-1)/2}(-1)^s I\Psi \\
&= (-1)^{(s+t)^2/2+s-t}\Psi \\
\implies \lambda &= \pm(-1)^{(s+t)^2/4+(s-t)/2}
\end{aligned} \tag{D.69}$$

In even dimensions, $s-t$ is also even and thus $(s+t)^2/4 + (s-t)/2$ is an integer $\implies \lambda = \pm 1$. Eigenvectors with eigenvalues, $+1$ and -1, are called left handed Weyl spinors and right handed Weyl spinors respectively.

Theorem D.10. *The eigenspaces of left handed and right handed Weyl spinors both have dimension, $2^{D/2-1}$, and hence their direct sum is the entire representation space.*

Proof. In proving theorem D.4, I showed that $\gamma_a^\dagger = \gamma_a$ for $0 \le a \le t-1$ and $\gamma_a^\dagger = -\gamma_a$ for $t \le a \le s+t-1$. Therefore,

$$\begin{aligned}
\gamma_{D+1}^\dagger\gamma_{D+1} &= \gamma_{D-1}^\dagger\cdots\gamma_0^\dagger\gamma_0\cdots\gamma_{D-1} \\
&= (-1)^s\gamma_{D-1}\cdots\gamma_0\gamma_0\cdots\gamma_{D-1} \\
&= (-1)^s(-1)^s I \\
&= I\,,
\end{aligned} \tag{D.70}$$

meaning $\gamma_{D+1}^\dagger$ commutes with γ_{D+1}, i.e. γ_{D+1} is a "normal" operator and thus diagonalisable. Hence, the sum of the dimensions of eigenspaces of $\lambda = 1$ and $\lambda = -1$ equals the dimension of

[6] With the benefit of hindsight, the eigenvalues are nicer with this convention.
[7] There is always a choice to be made between $(-1)^{1/2} = \mathrm{i}$ and $(-1)^{1/2} = -\mathrm{i}$.

the full space, namely $2^{D/2}$.
Next, let $(-1)^{(s-t)/4}\gamma_{D+1}\Psi = \pm\Psi$. As D is even, by equation D.65, $\{\gamma_a, \gamma_{D+1}\} = 0$. Hence,

$$(-1)^{(s-t)/4}\gamma_{D+1}\gamma_a\Psi = -(-1)^{(s-t)/4}\gamma_a\gamma_{D+1}\Psi = \mp\gamma_a\Psi \tag{D.71}$$

If Ψ is in the $\pm$ eigenspace, then $\gamma_a\Psi$ is in the $\mp$ eigenspace. However, all the γ_a are invertible. γ_a induces a bijection between the $\pm$ eigenspace to the $\mp$ eigenspace.
That finally proves that the $\pm$ eigenspaces must have the same dimension, namely $\frac{1}{2}2^{D/2} = 2^{D/2-1}$. □

The component of an arbitrary spinor, Ψ, in each of these eigenspaces can be found by the projection operators, $P_\pm = \frac{1}{2}(I \pm (-1)^{(s-t)/4}\gamma_{D+1})$, since $P_+ + P_- = I$ and (using equation D.67 and $s-t$ being even)

$$\begin{aligned}
(-1)^{(s-t)/4}\gamma_{D+1}P_\pm\Psi &= \frac{1}{2}(-1)^{(s-t)/4}\gamma_{D+1}(I \pm (-1)^{(s-t)/4}\gamma_{D+1})\Psi \\
&= \frac{1}{2}(-1)^{(s-t)/4}\gamma_{D+1}\Psi \pm \frac{1}{2}(-1)^{(s-t)/2}(\gamma_{D+1})^2\Psi \\
&= \frac{1}{2}(-1)^{(s-t)/4}\gamma_{D+1}\Psi \pm \frac{1}{2}(-1)^{(s-t)/2}(-1)^{(s-t)/2}\Psi \\
&= \frac{1}{2}(-1)^{(s-t)/4}\gamma_{D+1}\Psi \pm \frac{1}{2}\Psi \\
&= \pm\frac{1}{2}(\Psi \pm (-1)^{(s-t)/4}\gamma_{D+1}\Psi) \\
&= \pm P_\pm\Psi\,.
\end{aligned} \tag{D.72}$$

Since Weyl spinors can be constructed in any even dimension and (by table D.1) Majorana spinors can be constructed when $s-t \equiv 0,1,2 \pmod 8$, the double of a Majorana-Weyl spinor is possible when $s-t \equiv 0,2 \pmod 8$.

D.2 Three space and one time dimension

Up to now, I have considered spinors very generally. For a specific example, consider the case most relevant to physics, namely $s = 3$ and $t = 1$.
Then, $D = 4$, $2^{D/2} = 4$ and there is a unique irreducible representation[8] of the Clifford algebra (up to equivalence).
It suffices to guess this representation (and thereby prove its existence too). I will use the so-called "Weyl representation,"

$$\gamma_a \equiv \begin{bmatrix} 0 & \sigma_a \\ \tilde{\sigma}_a & 0 \end{bmatrix} \text{ where } \sigma_a \equiv (I, \sigma_1, \sigma_2, \sigma_3),\ \tilde{\sigma}_a \equiv (I, -\sigma_1, -\sigma_2, -\sigma_3) \tag{D.73}$$

and σ_1, σ_2 & σ_3 are the Pauli matrices. I have to check this representation is well defined.

$$\begin{aligned}
\gamma_a\gamma_b + \gamma_b\gamma_a &= \begin{bmatrix} 0 & \sigma_a \\ \tilde{\sigma}_a & 0 \end{bmatrix}\begin{bmatrix} 0 & \sigma_b \\ \tilde{\sigma}_b & 0 \end{bmatrix} + \begin{bmatrix} 0 & \sigma_b \\ \tilde{\sigma}_b & 0 \end{bmatrix}\begin{bmatrix} 0 & \sigma_a \\ \tilde{\sigma}_a & 0 \end{bmatrix} \\
&= \begin{bmatrix} \sigma_a\tilde{\sigma}_b + \sigma_b\tilde{\sigma}_a & 0 \\ 0 & \tilde{\sigma}_a\sigma_b + \tilde{\sigma}_b\sigma_a \end{bmatrix} \\
&= \begin{bmatrix} -2\eta_{ab}I & 0 \\ 0 & -2\eta_{ab}I \end{bmatrix} \\
&= -2\eta_{ab}I \implies \text{the Clifford algebra is satisfied.}
\end{aligned} \tag{D.74}$$

[8]Thus far, I have only proven theorems about the uniqueness of representations, not existence.

Next, it must be shown that the chosen representation is irreducible. Let S be a non-empty subspace of $\mathbb{C}^4$ invariant under all γ_a.
Therefore, $\forall v \in \mathbb{C}^4$ and $\forall a \in \{0, 1, 2, 3\}$, $\gamma_a v \in S$.
But then, $\gamma_a \gamma_b v \in S$ as $\gamma_b v = v'$ for some $v' \in S$ and thus $\gamma_a v' \in S$.
Likewise, $\forall \lambda_1, \lambda_2 \in \mathbb{C}$, $(\lambda_1 \gamma_a + \lambda_2 \gamma_b) v \in S$ as $\gamma_a v, \gamma_b v \in S$ and S is closed under linear combinations by virtue of being a subspace.
Overall, S is invariant under all products and linear combinations of γ_a and thus invariant under all linear combinations of elements in $G = \{\pm \Gamma_A\}_{A=0}^{15}$. By direct evaluation (on *Mathematica*),

$$\{\Gamma_A\}_{A=0}^{15} = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ -\mathrm{i} & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \right.$$
$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \mathrm{i} & 0 & 0 \\ -\mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -\mathrm{i} & 0 & 0 & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ 0 & 0 & -\mathrm{i} & 0 \\ 0 & 0 & 0 & \mathrm{i} \end{bmatrix},$$
$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -\mathrm{i} & 0 & 0 \\ -\mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & -\mathrm{i} & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -\mathrm{i} & 0 \\ 0 & 0 & 0 & \mathrm{i} \\ -\mathrm{i} & 0 & 0 & 0 \\ 0 & \mathrm{i} & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix},$$
$$\left. \begin{bmatrix} 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & -\mathrm{i} & 0 \\ 0 & -\mathrm{i} & 0 & 0 \\ -\mathrm{i} & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -\mathrm{i} & 0 \\ 0 & 0 & 0 & -\mathrm{i} \\ \mathrm{i} & 0 & 0 & 0 \\ 0 & \mathrm{i} & 0 & 0 \end{bmatrix}, \begin{bmatrix} \mathrm{i} & 0 & 0 & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ 0 & 0 & -\mathrm{i} & 0 \\ 0 & 0 & 0 & -\mathrm{i} \end{bmatrix} \right\}. \tag{D.75}$$

However, by inspection, complex linear combinations of these matrices can produce any 4×4 complex matrix (e.g. look at the 4 matrix subsets {0, 7, 8, 15}, {1, 4, 11, 14}, {2, 3, 12, 13} and {5, 6, 9, 10} with matrices labelled as per the order in which they are listed above).
Thus, S is invariant under all 4×4 matrices $\implies S = \mathbb{C}^4$.
Therefore, the Weyl representation of the Clifford algebra is indeed irreducible.
The Weyl representation is also unitary under the standard inner product of $\mathbb{C}^4$ since $\gamma_0^\dagger = \gamma_0$ and $\gamma_i^\dagger = -\gamma_i$. As for Weyl spinors,

$$\begin{aligned} (-1)^{(s-t)/4} \gamma_5 &= (-1)^{1/2} \gamma_0 \cdots \gamma_3 \\ &= -\mathrm{i} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{bmatrix} \begin{bmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{bmatrix} \begin{bmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{bmatrix} \\ &= -\mathrm{i} \begin{bmatrix} -\sigma_1 & 0 \\ 0 & \sigma_1 \end{bmatrix} \begin{bmatrix} -\mathrm{i}\sigma_1 & 0 \\ 0 & -\mathrm{i}\sigma_1 \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \end{aligned} \tag{D.76}$$

$$\implies (-1)^{(s-t)/4} \gamma_5 \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} w \\ x \\ -y \\ -z \end{bmatrix}. \tag{D.77}$$

span($\{(1,0,0,0), (0,1,0,0)\}$) and span($\{(0,0,1,0), (0,0,0,1)\}$) are the eigenspaces of left handed and right handed Weyl spinors respectively. To reflect this, the four-component spinor, Ψ, can be written as $\Psi = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}$, where ψ_α and $\bar{\chi}^{\dot{\alpha}}$ are two-component Weyl spinors. Undotted and dotted indices are left handed and right handed respectively.

As shown by equation D.34, $M_{ab} = -\frac{1}{4}[\gamma_a, \gamma_b]$ are Lorentz generators in spinor space. Let $\sigma_{ab} = -\frac{1}{4}(\sigma_a\tilde{\sigma}_b - \sigma_b\tilde{\sigma}_a)$ and $\tilde{\sigma}_{ab} = -\frac{1}{4}(\tilde{\sigma}_a\sigma_b - \tilde{\sigma}_b\sigma_a)$. σ_{ab} and $\tilde{\sigma}_{ab}$ are called left handed and right handed Lorentz generators respectively because

$$\begin{aligned}
M_{ab} &= -\frac{1}{4}\left(\begin{bmatrix}0 & \sigma_a\\ \tilde{\sigma}_a & 0\end{bmatrix}\begin{bmatrix}0 & \sigma_b\\ \tilde{\sigma}_b & 0\end{bmatrix} - \begin{bmatrix}0 & \sigma_b\\ \tilde{\sigma}_b & 0\end{bmatrix}\begin{bmatrix}0 & \sigma_a\\ \tilde{\sigma}_a & 0\end{bmatrix}\right)\\
&= -\frac{1}{4}\begin{bmatrix}\sigma_a\tilde{\sigma}_b - \sigma_b\tilde{\sigma}_a & 0\\ 0 & \tilde{\sigma}_a\sigma_b - \tilde{\sigma}_b\sigma_a\end{bmatrix}\\
&= \begin{bmatrix}\sigma_{ab} & 0\\ 0 & \tilde{\sigma}_{ab}\end{bmatrix} \qquad \text{(D.78)}\\
\implies M_{ab}\Psi &= \begin{bmatrix}\sigma_{ab} & 0\\ 0 & \tilde{\sigma}_{ab}\end{bmatrix}\begin{bmatrix}\psi_\alpha\\ \bar{\chi}^{\dot{\alpha}}\end{bmatrix} = \begin{bmatrix}\sigma_{ab}\psi_\alpha\\ \tilde{\sigma}_{ab}\bar{\chi}^{\dot{\alpha}}\end{bmatrix}. \qquad \text{(D.79)}
\end{aligned}$$

$M_{ab}\Psi$ must still be a spinor of the same type as Ψ.
Therefore, $\sigma_{ab}\psi_\alpha$ must be a left handed Weyl spinor and $\tilde{\sigma}_{ab}\bar{\chi}^{\dot{\alpha}}$ must be a right handed Weyl spinor.
Since M_{ab} only induces a linear transformation, the spinor indices of σ_{ab} and $\tilde{\sigma}_{ab}$ must be ${(\sigma_{ab})_\alpha}^\beta$ and ${(\tilde{\sigma}_{ab})^{\dot{\alpha}}}_{\dot{\beta}}$ respectively $\implies M_{ab}\Psi = \begin{pmatrix}{(\sigma_{ab})_\alpha}^\beta\psi_\beta\\ {(\tilde{\sigma}_{ab})^{\dot{\alpha}}}_{\dot{\beta}}\bar{\chi}^{\dot{\beta}}\end{pmatrix}$.
This gives the so-called $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representations of the Lie algebra, $\mathfrak{o}(3,1)$, namely $M_{ab}(\psi_\alpha) = {(\sigma_{ab})_\alpha}^\beta\psi_\beta$ and $M_{ab}(\bar{\chi}^{\dot{\alpha}}) = {(\tilde{\sigma}_{ab})^{\dot{\alpha}}}_{\dot{\beta}}\bar{\chi}^{\dot{\beta}}$ respectively. Furthermore, for σ_{ab} and $\tilde{\sigma}_{ab}$ to have the indices they do (in type and position), the spinor indices of the extended Pauli matrices must be $(\sigma_a)_{\alpha\dot{\alpha}}$ and $(\tilde{\sigma}^a)^{\dot{\alpha}\alpha}$. Finally, by direct evaluation, one finds

$${(\sigma_{ab})_\alpha}^\beta \equiv \frac{1}{2}\begin{bmatrix}0 & \sigma_1 & \sigma_2 & \sigma_3\\ -\sigma_1 & 0 & \mathrm{i}\sigma_3 & -\mathrm{i}\sigma_2\\ -\sigma_2 & -\mathrm{i}\sigma_3 & 0 & \mathrm{i}\sigma_1\\ -\sigma_3 & \mathrm{i}\sigma_2 & -\mathrm{i}\sigma_1 & 0\end{bmatrix} \text{ and } {(\tilde{\sigma}_{ab})^{\dot{\alpha}}}_{\dot{\beta}} \equiv \frac{1}{2}\begin{bmatrix}0 & -\sigma_1 & -\sigma_2 & -\sigma_3\\ \sigma_1 & 0 & \mathrm{i}\sigma_3 & -\mathrm{i}\sigma_2\\ \sigma_2 & -\mathrm{i}\sigma_3 & 0 & \mathrm{i}\sigma_1\\ \sigma_3 & \mathrm{i}\sigma_2 & -\mathrm{i}\sigma_1 & 0\end{bmatrix}. \qquad \text{(D.80)}$$

This was all at the level of the Lie algebra. To get to the Lie group, one must use the exponential map. The universal covering group of $\mathrm{SO}^\uparrow(3,1)$ is $\mathrm{SL}(2,\mathbb{C})$ and thus the exponential map will generate representations of $\mathrm{SL}(2,\mathbb{C})$, not $\mathrm{SO}^\uparrow(3,1)$.
Let $I + M \in \mathrm{SL}(2,\mathbb{C})$ for infinitesimal M. Thus, $1 = \det(I + M) = 1 + \mathrm{tr}(M) \implies \mathrm{tr}(M) = 0$. Since the Pauli matrices are a basis for traceless 2×2 matrices, $\mathfrak{sl}(2,\mathbb{C}) = \{z_i\sigma_i | z_i \in \mathbb{C}^3\}$. However, that is the complex Lie algebra. To get the real Lie algebra, let

$$\begin{aligned}
&z_1 = \frac{1}{2}(K^{01} + \mathrm{i}K^{23}),\ z_2 = \frac{1}{2}(K^{02} + \mathrm{i}K^{31}) \text{ and } z_1 = \frac{1}{2}(K^{03} + \mathrm{i}K^{12})\\
\implies &z_i\sigma_i = \frac{1}{2}((K^{01} + \mathrm{i}K^{23})\sigma_1 + (K^{02} + \mathrm{i}K^{31})\sigma_2 + (K^{03} + \mathrm{i}K^{12})\sigma_3) \qquad \text{(D.81)}
\end{aligned}$$

for $K^{ab} \in \mathbb{R}$. Not all the K^{ab} have been defined yet; that is most conveniently accomplished (to make connection with the Lie algebra, $\mathfrak{o}(3,1)$) by letting $K^{ab} = -K^{ba}$. Then,

$$\begin{aligned}
\frac{1}{2}K^{ab}\sigma_{ab} &= K^{01}\sigma_{01} + K^{02}\sigma_{02} + K^{03}\sigma_{03} + K^{12}\sigma_{12} + K^{13}\sigma_{13} + K^{23}\sigma_{23}\\
&= \frac{1}{2}(K^{01}\sigma_1 + K^{02}\sigma_2 + K^{03}\sigma_3 + K^{12}\mathrm{i}\sigma_3 - K^{13}\mathrm{i}\sigma_2 + K^{23}\mathrm{i}\sigma_1)\\
&= \frac{1}{2}((K^{01} + \mathrm{i}K^{23})\sigma_1 + (K^{02} + \mathrm{i}K^{31})\sigma_2 + (K^{03} + \mathrm{i}K^{12})\sigma_3)\\
&= z_i\sigma_i\,. \qquad \text{(D.82)}
\end{aligned}$$

Therefore, $\mathfrak{sl}(2,\mathbb{C}) = \left\{\frac{1}{2}K^{ab}(\sigma_{ab})_\alpha{}^\beta \middle| \, K^{ab} = -K^{ba} \in \mathbb{R}\right\}$.
Finally, as $\mathrm{SL}(2,\mathbb{C})$ is simply connected, $\{N_\alpha{}^\beta = \mathrm{e}^{K^{ab}(\sigma_{ab})_\alpha{}^\beta/2} | \, K^{ab} = -K^{ba} \in \mathbb{R}\}$ is a dense subset of $\mathrm{SL}(2,\mathbb{C})$.

Via equation D.37, I showed that under $\gamma'^a = \Lambda^a{}_b\gamma^b = T(\Lambda)^{-1}\gamma^a T(\Lambda)$, $\Psi'(x) = T(\Lambda)\Psi$. I commented that representation of the Lorentz group, $T(\Lambda)$, could be extended to a representation of the universal covering group. This is exactly what I will do now using the exponential map. As the Lorentz generators when acting on four-component spinors are M_{ab}, $T(N) = \mathrm{e}^{K^{ab}M_{ab}/2}$. The factor of a half is necessary in the exponential because $\mathfrak{o}(3,1)$ is only six-dimensional, where as $K^{ab}M_{ab}$ double counts the 6 independent M_{ab} via $K^{ba}M_{ba} = (-K^{ab})(-M_{ab})$. Thus,

$$\begin{aligned}
T(N) &= \mathrm{e}^{K^{ab}M_{ab}/2} \\
&= \mathrm{e}^{\frac{1}{2}K^{ab}\begin{bmatrix}(\sigma_{ab})_\alpha{}^\beta & 0 \\ 0 & (\tilde{\sigma}_{ab})^{\dot\alpha}{}_{\dot\beta}\end{bmatrix}} \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{K^{ab}}{2}\right)^n \begin{bmatrix}(\sigma_{ab})_\alpha{}^\beta & 0 \\ 0 & (\tilde{\sigma}_{ab})^{\dot\alpha}{}_{\dot\beta}\end{bmatrix}^n \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{K^{ab}}{2}\right)^n \begin{bmatrix}((\sigma_{ab})_\alpha{}^\beta)^n & 0 \\ 0 & ((\tilde{\sigma}_{ab})^{\dot\alpha}{}_{\dot\beta})^n\end{bmatrix} \\
&= \begin{bmatrix}\mathrm{e}^{K^{ab}(\sigma_{ab})_\alpha{}^\beta/2} & 0 \\ 0 & \mathrm{e}^{K^{ab}(\tilde{\sigma}_{ab})^{\dot\alpha}{}_{\dot\beta}/2}\end{bmatrix}.
\end{aligned} \tag{D.83}$$

I have already shown $\mathrm{e}^{K^{ab}(\sigma_{ab})_\alpha{}^\beta/2} = N_\alpha{}^\beta$. Let $M = \mathrm{e}^{K^{ab}(\tilde{\sigma}_{ab})^{\dot\alpha}{}_{\dot\beta}/2}$.

$$\begin{aligned}
\frac{1}{2}K^{ab}\tilde{\sigma}_{ab} &= K^{01}\tilde{\sigma}_{01} + K^{02}\tilde{\sigma}_{02} + K^{03}\tilde{\sigma}_{03} + K^{12}\tilde{\sigma}_{12} + K^{13}\tilde{\sigma}_{13} + K^{23}\tilde{\sigma}_{23} \\
&= \frac{1}{2}(-K^{01}\sigma_1 - K^{02}\sigma_2 - K^{03}\sigma_3 + \mathrm{i}K^{12}\sigma_3 - \mathrm{i}K^{13}\sigma_2 + \mathrm{i}K^{23}\sigma_1) \\
&= \frac{1}{2}((-K^{01} + \mathrm{i}K^{23})\sigma_1 + (-K^{02} + \mathrm{i}K^{31})\sigma_2 + (-K^{03} + \mathrm{i}K^{12})\sigma_3) \\
&= -z_i^*\sigma_i
\end{aligned} \tag{D.84}$$

Then, from $M = \mathrm{e}^{K^{ab}(\tilde{\sigma}_{ab})^{\dot\alpha}{}_{\dot\beta}/2}$,

$$\begin{aligned}
M &= \mathrm{e}^{-z_i^*\sigma_i} \\
\implies M^\dagger &= \mathrm{e}^{-z_i\sigma_i^\dagger} = \mathrm{e}^{-z_i\sigma_i} = N^{-1} \iff M = N^{-\dagger} \\
\implies T(N)\Psi &= \mathrm{e}^{K^{ab}M_{ab}/2}\Psi \\
&= \begin{bmatrix}N_\alpha{}^\beta & 0 \\ 0 & (N^{-\dagger})^{\dot\alpha}{}_{\dot\beta}\end{bmatrix}\begin{bmatrix}\psi_\beta \\ \bar{\chi}^{\dot\beta}\end{bmatrix} \\
&= \begin{bmatrix}N_\alpha{}^\beta\psi_\beta \\ (N^{-\dagger})^{\dot\alpha}{}_{\dot\beta}\bar{\chi}^{\dot\beta}\end{bmatrix}.
\end{aligned} \tag{D.85}$$

Hence, it must be that under the $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ representations of $\mathrm{SL}(2,\mathbb{C})$, left and right handed Weyl spinors respectively transform as $\psi'_\alpha = N_\alpha{}^\beta\psi_\beta$ and $\bar{\chi}'^{\dot\alpha} = (N^{-\dagger})^{\dot\alpha}{}_{\dot\beta}\bar{\chi}^{\dot\beta} = \bar{\chi}^{\dot\beta}(N^{-*})_{\dot\beta}{}^{\dot\alpha}$. One subtlety of this result (in particular the block diagonal form of $\mathrm{e}^{K^{ab}M_{ab}/2}$) is that although the representation of the Clifford algebra is irreducible, the induced $\mathrm{SL}(2,\mathbb{C})$ representation is not. The latter's irreducible components are the spaces of left handed and right handed spinors.

Since $N \in \mathrm{SL}(2,\mathbb{C}) \implies \det(N) = 1$, $N_\alpha{}^\mu N_\beta{}^\nu \varepsilon_{\mu\nu} = \varepsilon_{\alpha\beta}$ and $\varepsilon^{\mu\nu}(N^{-1})_\mu{}^\alpha (N^{-1})_\nu{}^\beta = \varepsilon^{\alpha\beta}$ where $\varepsilon_{\alpha\beta}$ and $\varepsilon^{\alpha\beta}$ are antisymmetric tensors with $\varepsilon_{12} = -1$ and $\varepsilon^{12} = 1$.
As $\varepsilon_{\alpha\beta}$ and $\varepsilon^{\alpha\beta}$ are invariant tensors of $\mathrm{SL}(2,\mathbb{C})$ and $\varepsilon^{\alpha\gamma}\varepsilon_{\gamma\beta} = \delta^\alpha{}_\beta$, they can be used to raise and lower indices.
Then, $\bar{\chi}'_{\dot{\alpha}} = \varepsilon_{\dot{\alpha}\dot{\beta}}\bar{\chi}'^{\dot{\beta}} = \varepsilon_{\dot{\alpha}\dot{\beta}}\bar{\chi}^{\dot{\gamma}}(N^{-*})_{\dot{\gamma}}{}^{\dot{\beta}}$.
$N_\alpha{}^\mu N_\beta{}^\nu \varepsilon_{\mu\nu} = \varepsilon_{\alpha\beta} \iff \varepsilon = N\varepsilon N^T$ in matrix notation. That means
$N^{-1}\varepsilon = \varepsilon N^T \implies -N^{-1}\varepsilon = -\varepsilon N^T \implies N^{-1}\varepsilon^T = \varepsilon^T N^T \implies \varepsilon_{\dot{\alpha}\dot{\beta}}(N^{-*})_{\dot{\gamma}}{}^{\dot{\beta}} = \varepsilon_{\dot{\beta}\dot{\gamma}}(N^*)_{\dot{\alpha}}{}^{\dot{\beta}}$.
Therefore, $\bar{\chi}'_{\dot{\alpha}} = \varepsilon_{\dot{\alpha}\dot{\beta}}\bar{\chi}^{\dot{\gamma}}(N^{-*})_{\dot{\gamma}}{}^{\dot{\beta}} = \varepsilon_{\dot{\beta}\dot{\gamma}}(N^*)_{\dot{\alpha}}{}^{\dot{\beta}}\bar{\chi}^{\dot{\gamma}} = (N^*)_{\dot{\alpha}}{}^{\dot{\beta}}\bar{\chi}_{\dot{\beta}}$
Similarly, raising the index on the left handed spinor, $\psi'^\alpha = \varepsilon^{\alpha\beta}\psi'_\beta = \varepsilon^{\alpha\beta}N_\beta{}^\gamma\psi_\gamma$.
$\varepsilon^{\mu\nu}(N^{-1})_\mu{}^\alpha(N^{-1})_\nu{}^\beta = \varepsilon^{\alpha\beta} \implies \varepsilon = N^{-T}\varepsilon N^{-1} \implies \varepsilon N = N^{-T}\varepsilon \implies \varepsilon^{\alpha\beta}N_\beta{}^\gamma = (N^{-1})_\beta{}^\alpha\varepsilon^{\beta\gamma}$.
Therefore, $\psi'^\alpha = \varepsilon^{\alpha\beta}N_\beta{}^\gamma\psi_\gamma = (N^{-1})_\beta{}^\alpha\varepsilon^{\beta\gamma}\psi_\gamma = \psi^\beta(N^{-1})_\beta{}^\alpha$.
Having established these transformation properties, one can now develop the two-component spinor formalism via tensor products, index raising/lowering etc. like for other tensor types.

The two-component spinor formalism was based on writing the full spinor space as a direct sum of left handed and right handed Weyl spinors. However, I also spent many pages earlier considering Majorana spinors and it would be incomplete of me not not consider them in the special case of $s - t = 3 - 1 = 2$ where (by table D.1) they do exist.
By definition D.6 and theorem D.7, a four-component spinor is Majorana if and only if $\Psi = \nu\mu^t(-1)^{t(t-1)/2}C^{-1}\overline{\Psi}^T = 1 \times (-1)^1(-1)^{1\times 0/2}C^{-1}\overline{\Psi}^T = -C^{-1}\overline{\Psi}^T \implies \overline{\Psi}^T = -C\Psi$.
It suffices to guess C by forcing it to satisfy theorem D.5 and equation D.55. With $\nu = 1$, $\mu = -1$ and $t = 1$, they say $C^\dagger C = I$, $\gamma_a^T = -C\gamma_a C^{-1}$ and $C^T = -C$. Guided by the antisymmetry and the block diagonal nature of the Weyl representation, try

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{D.86}$$

$$\implies C^\dagger C = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I \tag{D.87}$$

$C^{-1} = -C$ by the previous line and thus $-C\gamma_a C^{-1} = C\gamma_a C$. Also, C can also be written slightly more compactly as

$$C = \begin{bmatrix} \varepsilon & 0 \\ 0 & -\varepsilon \end{bmatrix} \text{ where } \varepsilon = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{D.88}$$

This notation allows easier checking of the remaining property, $-C\gamma_a C^{-1} = \gamma_a^T$. Explicitly,

$$\begin{aligned}
-C\gamma_a C^{-1} &= \begin{bmatrix} \varepsilon & 0 \\ 0 & -\varepsilon \end{bmatrix} \begin{bmatrix} 0 & \sigma_a \\ \tilde{\sigma}_a & 0 \end{bmatrix} \begin{bmatrix} \varepsilon & 0 \\ 0 & -\varepsilon \end{bmatrix} = \begin{bmatrix} 0 & -\varepsilon\sigma_a\varepsilon \\ -\varepsilon\tilde{\sigma}_a\varepsilon & 0 \end{bmatrix} \\
\varepsilon\sigma_0\varepsilon = \varepsilon I \varepsilon &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -\sigma_0^T \\
\varepsilon\sigma_1\varepsilon &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \sigma_1^T \\
\varepsilon\sigma_2\varepsilon &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{i} & 0 \\ 0 & \mathrm{i} \end{bmatrix} = \begin{bmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{bmatrix} = \sigma_2^T \\
\varepsilon\sigma_3\varepsilon &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \sigma_3^T
\end{aligned}$$

In summary,

$$-C\gamma_a C^{-1} = \begin{bmatrix} 0 & \tilde{\sigma}_a^T \\ \sigma_a^T & 0 \end{bmatrix} = \gamma_a^T \text{ since } \tilde{\sigma}_a = (I, -\sigma_i)\,. \tag{D.89}$$

Therefore, the chosen matrix for C can indeed be used as the charge conjugation matrix. Then,

$$-C\Psi = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -x \\ w \\ z \\ -y \end{bmatrix}. \tag{D.90}$$

Meanwhile,

$$\begin{aligned}
\overline{\Psi}^T &= (\Psi^\dagger A)^T \\
&= A^T \Psi^* \\
&= \gamma_0^T \Psi^* \\
&= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} w^* \\ x^* \\ y^* \\ z^* \end{bmatrix} \\
&= \begin{bmatrix} y^* \\ z^* \\ w^* \\ x^* \end{bmatrix}
\end{aligned} \tag{D.91}$$

Therefore,

$$-C\Psi = \overline{\Psi}^T \implies \Psi = \begin{bmatrix} w \\ x \\ -x^* \\ w^* \end{bmatrix}. \tag{D.92}$$

In the two-component spinor notation, $\begin{pmatrix} w \\ x \end{pmatrix}$ would be denoted as ψ_α. Then,

$$\psi^\alpha = \varepsilon^{\alpha\beta}\psi_\beta \equiv \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} w \\ x \end{bmatrix} = \begin{bmatrix} -x \\ w \end{bmatrix} \implies \begin{bmatrix} -x^* \\ w^* \end{bmatrix} = (\psi^\alpha)^* \tag{D.93}$$

Conjugation swaps dotted and undotted spinor indices since $\psi'_\alpha = N_\alpha{}^\beta \psi_\beta$
$\implies (\psi'_\alpha)^* = (N^*)_\alpha{}^\beta (\psi_\beta)^*$ (and likewise for conjugating an initially dotted spinor) which is the transformation of right handed Weyl spinor as shown earlier. For this reason, $(\psi_\alpha)^*$ can be denoted as $\bar{\psi}_{\dot{\alpha}}$.
In summary, the most general Majorana spinor for $s = 3$ and $t = 1$ is $\Psi = \begin{pmatrix} \psi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix}$.

Finally, it is worth checking that despite appearances, spinor representations are not the same as vector representations. It is often remarked (e.g. by quoting Michael Atiyah) that spinors are like the square root of a vector. That is because of arguments like the one below.
Let $\delta^a{}_b + X^a{}_b \in \mathrm{SO}^\uparrow(3,1)$ for infinitesimal $X^a{}_b$. Then, by the defining properties of $\mathrm{SO}^\uparrow(3,1)$, $1 = \det(\delta^a{}_b + X^a{}_b) = 1 + \mathrm{tr}(X) \implies X^a{}_a = 0$.
Also, $\eta_{ab} = \eta_{cd}(\delta^c{}_a + X^c{}_a)(\delta^d{}_b + X^d{}_b) = \eta_{ab} + X_{ba} + X_{ab} \implies X_{ba} = -X_{ab}$. Antisymmetry automatically implies tracelessness; thus $\mathfrak{o}(3,1)$ consists of all 4×4 antisymmetric matrices.
Therefore, $\Lambda = \mathrm{e}^{K^{ab} S_{ab}/2} \in \mathrm{SO}^\uparrow(3,1)$ where S_{ab} is a basis (with 6 independent elements) for 4×4 antisymmetric matrices. The corresponding group action on four-component spinors is $T(\Lambda) \equiv T(N) = \mathrm{e}^{K^{ab} M_{ab}/2}$. The standard basis for 4×4 antisymmetric matrices is

$$S_{ab} \equiv \left\{ \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \right\}. \tag{D.94}$$

It can be checked that S_{ab} satisfies the Lie algebra generator commutation relations for $\mathfrak{o}(3,1)$. By Rodrigues' formula and other related identities, if (n_x, n_y, n_z) is a unit vector of $\mathbb{R}^3$, then $\mathrm{e}^{\theta A}$, where

$$A = \begin{bmatrix} 0 & n_z & -n_y \\ -n_z & 0 & n_x \\ n_y & -n_x & 0 \end{bmatrix}, \tag{D.95}$$

is a rotation of θ about $\vec{n}$. A can be represented in term of 4×4 matrices via

$$\begin{aligned} A &= n_z \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - n_y \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} + n_x \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \\ &= n_z S_{12} - n_y S_{13} + n_x S_{23} \end{aligned} \tag{D.96}$$

and thus $\mathrm{e}^{\theta A} = \mathrm{e}^{\theta(n_z S_{12} - n_y S_{13} + n_x S_{23})} \in \mathrm{SO}^{\uparrow}(3,1)$. The corresponding representation on spinor space is

$$\begin{aligned}
T(N) &= \mathrm{e}^{\theta(n_z M_{12} - n_y M_{13} + n_x M_{23})} \\
&= \mathrm{e}^{\theta\left(n_z \begin{bmatrix}\sigma_{12} & 0 \\ 0 & \tilde{\sigma}_{12}\end{bmatrix} - n_y \begin{bmatrix}\sigma_{13} & 0 \\ 0 & \tilde{\sigma}_{13}\end{bmatrix} + n_x \begin{bmatrix}\sigma_{23} & 0 \\ 0 & \tilde{\sigma}_{23}\end{bmatrix}\right)} \\
&= \mathrm{e}^{\frac{\mathrm{i}\theta}{2}\begin{bmatrix} n_x\sigma_1 + n_y\sigma_2 + n_z\sigma_3 & 0 \\ 0 & n_x\sigma_1 + n_y\sigma_2 + n_z\sigma_3\end{bmatrix}} \\
&= \begin{bmatrix}\mathrm{e}^{\mathrm{i}\theta\vec{n}\cdot\vec{\sigma}/2} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\theta\vec{n}\cdot\vec{\sigma}/2}\end{bmatrix} \\
(\vec{n}\cdot\vec{\sigma})^2 &= \begin{bmatrix} n^z & n_x - \mathrm{i}n_y \\ n_x + \mathrm{i}n_y & -n_z\end{bmatrix}\begin{bmatrix} n^z & n_x - \mathrm{i}n_y \\ n_x + \mathrm{i}n_y & -n_z\end{bmatrix} \\
&= \begin{bmatrix} n_z^2 + n_x^2 + n_y^2 & 0 \\ 0 & n_x^2 + n_y^2 + n_z^2\end{bmatrix} \\
&= I \quad \text{as } ||\vec{n}|| = 1\,.
\end{aligned} \tag{D.97}$$

Then, the exponential can be evaluated to

$$\begin{aligned}
\mathrm{e}^{\mathrm{i}\theta\vec{n}\cdot\vec{\sigma}/2} &= \sum_{m=0}^{\infty}\frac{1}{m!}\left(\frac{\mathrm{i}\theta}{2}\right)^m(\vec{n}\cdot\vec{\sigma})^m \\
&= I\sum_{m=0}^{\infty}\frac{1}{(2m)!}\left(\frac{\mathrm{i}\theta}{2}\right)^{2m} + (\vec{n}\cdot\vec{\sigma})\sum_{m=0}^{\infty}\frac{1}{(2m+1)!}\left(\frac{\mathrm{i}\theta}{2}\right)^{2m+1} \\
&= \cos(\theta/2)I + \mathrm{i}\sin(\theta/2)\vec{n}\cdot\vec{\sigma}\,.
\end{aligned} \tag{D.98}$$

Notice that a rotation of θ has lead to a rotation of only $\theta/2$ in the cos and sin terms acting on spinor space.
e.g. Let $\theta = 2\pi \implies \Lambda = \mathrm{e}^{\theta A} = I$ as a 2π rotation does nothing. However,

$$\begin{aligned}
T(N) &= \begin{bmatrix}\cos(\pi)I + \mathrm{i}\sin(\pi)\vec{n}\cdot\vec{\sigma} & 0 \\ 0 & \cos(\pi)I + \mathrm{i}\sin(\pi)\vec{n}\cdot\vec{\sigma}\end{bmatrix} \\
&= -I
\end{aligned} \tag{D.99}$$

$T(N)\Psi = -\Psi$ under a 2π rotation.
Hence, one needs to do a full 2π rotation twice to return the spinor, Ψ, to its original state. Therefore, the spinor representation really is different to the vector representation. This essentially reflects the fact that the spinor is transforming under $\mathrm{SL}(2,\mathbb{C})$, not $\mathrm{SO}^{\uparrow}(3,1)$. The $\mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2 \cong \mathrm{SO}^{\uparrow}(3,1)$ isomorphism means N and $-N$ both correspond to the same Lorentz transformation, Λ. That is why $\Lambda = I$ can still lead to $T(N) = -I$; the two are related by the $\mathbb{Z}_2$ quotienting.

Appendix E

Notational conventions

Most, if not all, of these conventions follow those of [25]. To be well defined, some of these conventions require quite non-trivial concepts and theorems. I will simply be taking them as assumed knowledge. In such cases, it may help to read appendix D if the issue concerns spinors. There are also times when a result could be placed in the present appendix or in appendix F. In such cases, I have chosen not to duplicate results, but instead choose whichever appendix I think is better suited for that result.

The Einstein summation convention will be in effect at all times.

At all times I will be working in units where $c = 1$ and $\hbar = 1$.

* denotes complex conjugate.

Given a matrix, M, the inverse, inverse transpose, inverse conjugate transpose and inverse conjugate are denoted by $M^{-1}, M^{-T}, M^{-\dagger}$ and M^{-*} respectively.

While many results in my thesis generalise to arbitrary manifolds, at all times I have restricted attention to four-dimensional, orientable, path connected, Lorentzian manifolds with a $(-1, 1, 1, 1)$ metric signature.

$[\cdot, \cdot]$ denotes a commutator and $\{\cdot, \cdot\}$ denotes an anticommutator.

$\mathrm{SO}^{\uparrow}(3, 1)$ denotes the proper orthochronous Lorentz group.

All Lie algebras are denoted in fraktur, e.g. $\mathfrak{sl}(2, \mathbb{C})$ is the Lie algebra of $\mathrm{SL}(2, \mathbb{C})$.

Three types of indices - curved space, local Lorentz and spinor - are frequently encountered in this work. They are represented by Latin letters from the middle of the alphabet, Latin letters from the start of the alphabet and Greek letters[1] respectively. When working in flat space, I will use Latin letters from the start of the alphabet.
e.g. The metric would be denoted $g_{mn}(x)$ and would transform as

$$g'_{mn}(x') = \frac{\partial x^p}{\partial x'^m} \frac{\partial x^q}{\partial x'^n} g_{pq}(x) \tag{E.1}$$

under transformations, $x \to x'$, of the general coordinate group.
Then, one can introduce a vierbein[2] by $\{e_a{}^m(x)\partial_m\}_{a=0}^3$ such that $\eta_{ab} = e_a{}^m(x) e_b{}^n(x) g_{mn}(x)$.

[1]There are two types of spinor indices - dotted and undotted - as illustrated below.
[2]A vierbein is a new tangent space basis.

The $\{e_a{}^m\partial_m\}_{a=0}^3$ are only unique up to a Lorentz transformation within the tangent space[3]. e.g. A curved space object such as the Ricci tensor, $R_{mn}(x)$ can be converted into

$$R_{ab}(x) = e_a{}^m(x)e_b{}^n(x)R_{mn}(x) \tag{E.2}$$

and then under a local Lorentz transformation, $e'_a{}^m = (\Lambda^{-1})^b{}_a e_a{}^m$ for some $\Lambda \in \mathrm{SO}^\uparrow(3,1)$ (the proper orthochronous Lorentz group), R_{ab} would transform as

$$R'_{ab}(x) = (\Lambda^{-1})^c{}_a(\Lambda^{-1})^d{}_b R_{cd}(x)\,. \tag{E.3}$$

For the collection, $e_a{}^m$, $e_m{}^a$ denotes $(e_a{}^m)^{-1}$. Then, $\{e_m{}^a(x)\mathrm{d}x^m\}_{a=0}^3$ is a basis for the cotangent space and is called the inverse vierbein.
Finally, two-component spinors are required for objects transforming under representations of $\mathrm{SL}(2,\mathbb{C})$ - the universal covering group of $\mathrm{SO}^\uparrow(3,1)$. Such type-$(m,n)$ spin tensors transform under the "$T^{(m/2,n/2)}$" representation of $\mathrm{SL}(2,\mathbb{C})$ by

$$\psi'_{\alpha_1\cdots\alpha_m\dot\alpha_1\cdots\dot\alpha_n}(x) = N_{\alpha_1}{}^{\beta_1}\cdots N_{\alpha_m}{}^{\beta_m}N^*{}_{\dot\alpha_1}{}^{\dot\beta_1}\cdots N^*{}_{\dot\alpha_m}{}^{\dot\beta_m}\psi_{\beta_1\cdots\beta_m\dot\beta_1\cdots\dot\beta_n}(x) \tag{E.4}$$

for some $N \in \mathrm{SL}(2,\mathbb{C})$. This representation is irreducible when ψ is symmetric in its dotted and undotted indices independently. By equation E.4, $(\psi_{\alpha_1\cdots\alpha_m\dot\alpha_1\cdots\dot\alpha_n}(x))^*$ transforms as a type-(n,m) spin tensor. Motivated by that, let $\bar\psi_{\alpha_1\cdots\alpha_n\dot\alpha_1\cdots\dot\alpha_m}(x)$ denote $(\psi_{\alpha_1\cdots\alpha_m\dot\alpha_1\cdots\dot\alpha_n}(x))^*$.

The three types of indices are lowered an raised by the general metric, Minkowski metric and Levi-Civita symbol and their inverses respectively.
e.g. $V_m = g_{mn}V^n$, $V_a = \eta_{ab}V^b$, $\psi_\alpha = \varepsilon_{\alpha\beta}\psi^\beta$ and $\psi_{\dot\alpha} = \varepsilon_{\dot\alpha\dot\beta}\psi^{\dot\beta}$ to lower indices and analogously with the inverses to raise indices.

Levi-Civita symbols are normalised by $\varepsilon_{12} = -1$, $\varepsilon^{12} = 1$, $\varepsilon_{\dot1\dot2} = -1$, $\varepsilon^{\dot1\dot2} = 1$, $\varepsilon_{0123} = -1$ and $\varepsilon^{0123} = 1$.

$$(\sigma_a)_{\alpha\dot\alpha} \equiv (I,\sigma_1,\sigma_2,\sigma_3) \tag{E.5}$$

$$(\tilde\sigma_a)^{\dot\alpha\alpha} = \varepsilon^{\alpha\beta}\varepsilon^{\dot\alpha\dot\beta}(\sigma_a)_{\beta\dot\beta} \equiv (I,-\sigma_1,-\sigma_2,-\sigma_3) \tag{E.6}$$

$$\sigma_{1,2,3} = \text{Pauli matrices} \tag{E.7}$$

$$(\sigma_{ab})_\alpha{}^\beta = -\frac{1}{4}((\sigma_a)_{\alpha\dot\alpha}(\tilde\sigma_b)^{\dot\alpha\beta} - (\sigma_b)_{\alpha\dot\alpha}(\tilde\sigma_a)^{\dot\alpha\beta}) \equiv \frac{1}{2}\begin{bmatrix} 0 & \sigma_1 & \sigma_2 & \sigma_3 \\ -\sigma_1 & 0 & \mathrm{i}\sigma_3 & -\mathrm{i}\sigma_2 \\ -\sigma_2 & -\mathrm{i}\sigma_3 & 0 & \mathrm{i}\sigma_1 \\ -\sigma_3 & \mathrm{i}\sigma_2 & -\mathrm{i}\sigma_1 & 0 \end{bmatrix} \tag{E.8}$$

$$(\tilde\sigma_{ab})^{\dot\alpha}{}_{\dot\beta} = -\frac{1}{4}((\tilde\sigma_a)^{\dot\alpha\alpha}(\sigma_b)_{\alpha\dot\beta} - (\tilde\sigma_b)^{\dot\alpha\alpha}(\sigma_a)_{\alpha\dot\beta}) \equiv \frac{1}{2}\begin{bmatrix} 0 & -\sigma_1 & -\sigma_2 & -\sigma_3 \\ \sigma_1 & 0 & \mathrm{i}\sigma_3 & -\mathrm{i}\sigma_2 \\ \sigma_2 & -\mathrm{i}\sigma_3 & 0 & \mathrm{i}\sigma_1 \\ \sigma_3 & \mathrm{i}\sigma_2 & -\mathrm{i}\sigma_1 & 0 \end{bmatrix} \tag{E.9}$$

$(\sigma_{ab})_\alpha{}^\beta$ are the Lorentz generators, i.e. it can be shown

$$\left\{N_\alpha{}^\beta = \mathrm{e}^{\frac{1}{2}K^{ab}(\sigma_{ab})_\alpha{}^\beta} \mid K^{ab} \text{ is a constant, real, antisymmetric matrix}\right\} \tag{E.10}$$

[3]That is why the term, "local Lorentz," is used to describe the corresponding indices.

is a dense subset of $\mathrm{SL}(2,\mathbb{C})$. Upon a representation of $\mathrm{SL}(2,\mathbb{C})$ (e.g. in the space of type-(m,n) spin tensors), let M_{ab} denote the pushforward of σ_{ab}. I will (somewhat lazily) call M_{ab} Lorentz generators as well. The "left handed" and "right handed" parts to the Lorentz generator are denoted by $M_{\alpha\beta}$ and $\overline{M}_{\dot\alpha\dot\beta}$ respectively and are connected to the (full) Lorentz generator by $M_{\alpha\beta} = \frac{1}{2}(\sigma^{ab})_{\alpha\beta}M_{ab}$ and $\overline{M}_{\dot\alpha\dot\beta} = -\frac{1}{2}(\tilde\sigma^{ab})_{\dot\alpha\dot\beta}M_{ab}$.

With $(\sigma_a)_{\alpha\dot\alpha}$ and $(\tilde\sigma_a)^{\dot\alpha\alpha}$, one can convert a local Lorentz vector index into a dotted and undotted index pair and vice versa by

$$V_{\alpha\dot\alpha} = (\sigma_a)_{\alpha\dot\alpha}V^a \text{ and } V_a = -\frac{1}{2}(\tilde\sigma_a)^{\dot\alpha\alpha}V_{\alpha\dot\alpha}\,. \tag{E.11}$$

The vierbein is also used to transform the covariant derivative from having a general coordinate to local Lorentz index - just like for normal tensors - by

$$\nabla_a = e_a{}^m\nabla_m\,. \tag{E.12}$$

For this equation to be consistent with the normal action of ∇_m (under a metric compatible, torsion-free connection), one needs to define

$$\nabla_a = e_a{}^m\partial_m + \frac{1}{2}\omega_{abc}M^{bc} \tag{E.13}$$

$$\text{where } \omega_{abc} = \frac{1}{2}(C_{bca} + C_{acb} - C_{abc}) \text{ and} \tag{E.14}$$

$$C_{ab}{}^c = (e_a{}^n\partial_n(e_b{}^m) - e_b{}^n\partial_n(e_a{}^m))e_m{}^c\,. \tag{E.15}$$

These $C_{ab}{}^c$ are called "anholonomy coefficients" and satisfy $[e_a{}^m\partial_m, e_b{}^n\partial_n] = C_{ab}{}^c e_c{}^m\partial_m$. Similarly, I will frequently be using $\nabla_{\alpha\dot\alpha} = (\sigma_a)_{\alpha\dot\alpha}\nabla^a$ and $\nabla^{\alpha\dot\alpha} = \varepsilon^{\alpha\beta}\varepsilon^{\dot\alpha\dot\beta}\nabla_{\beta\dot\beta} = (\tilde\sigma_a)^{\dot\alpha\alpha}\nabla^a$as well. Without vierbeins, Christoffel symbols are denoted by $\Gamma^m{}_{np} = \frac{1}{2}g^{mq}(\partial_n g_{pq} + \partial_p g_{qn} - \partial_q g_{np})$.

A series of derivatives acts on all terms enclosed in brackets, e.g. $\nabla_{a_1}\cdots\nabla_{a_n}(R\varphi)$ means there are n derivatives, ∇_{a_1}, ..., ∇_{a_n}, acting on the product, $R\varphi$, with ∇_{a_n} acting first and ∇_{a_1} acting last. I will try not to write any ambiguous expressions such as $\nabla^a\varphi\nabla_a\varphi$ which in principle could mean $\nabla^a(\varphi)\nabla_a(\varphi)$ or $\nabla^a(\varphi\nabla_a\varphi)$. The only exception to this rule is when there is a symmetrisation or antisymmetrisation which would make brackets around differentiated terms ungainly. In such cases the derivatives are taken to act to the extent of the symmetrisation or antisymmetrisation brackets. For example, in a term like $\nabla^{(\alpha}{}_{\dot\beta}\xi^{\beta\gamma)\dot\alpha\dot\beta}\psi_\alpha$, the derivative acts on ξ only, not ψ; explicitly writing something like $\nabla^{(\alpha}{}_{\dot\beta}(\xi^{\beta\gamma)\dot\alpha\dot\beta})\psi_\alpha$ seems too cumbersome - although in a term like $\nabla_{(\alpha}{}^{\dot\gamma}(\sigma)\xi_{\beta\gamma)\dot\beta\dot\gamma}\nabla^{\beta\dot\beta}\psi^\gamma$, it is unavoidable because the derivative only acts on σ, but the symmetrisation extends beyond that.

I will be working with the Weyl representation of the Dirac matrices,

$$\gamma_a = \begin{bmatrix} 0 & \sigma_a \\ \tilde\sigma_a & 0 \end{bmatrix}\,. \tag{E.16}$$

Four-component spinors are denoted in bold, e.g. $\mathbf{\Psi} = \begin{pmatrix}\psi_\alpha \\ \bar\chi^{\dot\alpha}\end{pmatrix}$. Four-component spinors will always be denoted as column vectors. When the components are too long to fit in a line, I will write $\mathbf{\Psi} = [\psi_\alpha, \bar\chi^{\dot\alpha}]^T$ so that $\mathbf{\Psi}$ is still a column.

If $\mathbf{\Psi} = \begin{pmatrix}\psi_\alpha \\ \bar\chi^{\dot\alpha}\end{pmatrix}$, then $\overline{\mathbf{\Psi}}$ denotes $(\chi^\alpha, \bar\psi_{\dot\alpha}) = \mathbf{\Psi}^\dagger\gamma_0$.

Any integral stated without bounds implicitly means integrate over all possible values of the variables comprising the volume element.

Appendix F

Frequently used identities

Most of the following identities are listed in [25]. I will use them liberally without proof or explicit mention.

Upon an infinitesimal Weyl transformation, ${e'}_a{}^m = (1+\sigma){e}_a{}^m$,

$$
\begin{aligned}
g'_{mn} &= (1-2\sigma)g_{mn} && \text{(F.1)}\\
\nabla'_a &= \nabla_a + \sigma\nabla_a - \nabla^b(\sigma)M_{ab} && \text{(F.2)}\\
R'_{abcd} &= (1+2\sigma)R_{abcd} + \eta_{bd}\nabla_a\nabla_c(\sigma) - \eta_{bc}\nabla_a\nabla_d(\sigma) + \eta_{ac}\nabla_b\nabla_d(\sigma) - \eta_{ad}\nabla_b\nabla_c(\sigma) && \text{(F.3)}\\
R'_{ab} &= (1+2\sigma)R_{ab} + \eta_{ab}\Box(\sigma) + 2\nabla_a\nabla_b(\sigma) && \text{(F.4)}\\
R' &= (1+2\sigma)R + 6\Box(\sigma) && \text{(F.5)}\\
C'_{abcd} &= (1+2\sigma)C_{abcd} && \text{(F.6)}\\
C'^m{}_{npq} &= C^m{}_{npq}\,. && \text{(F.7)}
\end{aligned}
$$

I will also require some of the finite case. Upon ${e'}_a{}^m = \mathrm{e}^\sigma {e}_a{}^m \iff g'_{mn} = \mathrm{e}^{-2\sigma}g_{mn}$,

$$
\begin{aligned}
\nabla'_a &= \mathrm{e}^\sigma(\nabla_a - \nabla^b(\sigma)M_{ab}) && \text{(F.8)}\\
R' &= \mathrm{e}^{2\sigma}(R + 6\Box(\sigma) - 6\nabla^a(\sigma)\nabla_a(\sigma))\,. && \text{(F.9)}
\end{aligned}
$$

When faced with spinor gymnastics, the following identities are invaluable.

$$
\begin{aligned}
\psi_\alpha\chi^\alpha &= -\psi^\alpha\chi_\alpha && \text{(F.10)}\\
\psi_{\dot\alpha}\chi^{\dot\alpha} &= -\psi^{\dot\alpha}\chi_{\dot\alpha} && \text{(F.11)}\\
(\sigma_a)_{\alpha\dot\alpha}(\tilde\sigma_b)^{\dot\alpha\beta} + (\sigma_b)_{\alpha\dot\alpha}(\tilde\sigma_a)^{\dot\alpha\beta} &= -2\eta_{ab}\delta_\alpha{}^\beta && \text{(F.12)}\\
(\tilde\sigma_a)^{\dot\alpha\alpha}(\sigma_b)_{\alpha\dot\beta} + (\tilde\sigma_b)^{\dot\alpha\alpha}(\sigma_a)_{\alpha\dot\beta} &= -2\eta_{ab}\delta^{\dot\alpha}{}_{\dot\beta} && \text{(F.13)}\\
(\sigma_a)_{\alpha\dot\alpha}(\tilde\sigma_b)^{\dot\alpha\alpha} &= -2\eta_{ab} && \text{(F.14)}\\
(\sigma^a)_{\alpha\dot\alpha}(\tilde\sigma_a)^{\dot\beta\beta} &= -2\delta^\beta{}_\alpha\delta^{\dot\beta}{}_{\dot\alpha} && \text{(F.15)}\\
(\sigma_a)_{\alpha\dot\beta}(\tilde\sigma_b)^{\dot\beta\beta}(\sigma_c)_{\beta\dot\alpha} &= \eta_{ca}(\sigma_b)_{\alpha\dot\alpha} - \eta_{bc}(\sigma_a)_{\alpha\dot\alpha} - \eta_{ab}(\sigma_c)_{\alpha\dot\alpha} + \mathrm{i}\varepsilon_{abcd}(\sigma^d)_{\alpha\dot\alpha} && \text{(F.16)}\\
(\tilde\sigma_a)^{\dot\alpha\beta}(\sigma_b)_{\beta\dot\beta}(\tilde\sigma_c)^{\dot\beta\alpha} &= \eta_{ca}(\tilde\sigma_b)^{\dot\alpha\alpha} - \eta_{bc}(\tilde\sigma_a)^{\dot\alpha\alpha} - \eta_{ab}(\tilde\sigma_c)^{\dot\alpha\alpha} - \mathrm{i}\varepsilon_{abcd}(\tilde\sigma^d)^{\dot\alpha\alpha} && \text{(F.17)}\\
(\sigma_a)_{\alpha\dot\beta}(\tilde\sigma_{bc})^{\dot\beta}{}_{\dot\alpha} &= \frac{1}{2}(\eta_{ab}(\sigma_c)_{\alpha\dot\alpha} - \eta_{ac}(\sigma_b)_{\alpha\dot\alpha} - \mathrm{i}\varepsilon_{abcd}(\sigma^d)_{\alpha\dot\alpha}) && \text{(F.18)}\\
(\tilde\sigma_{ab})^{\dot\alpha}{}_{\dot\beta}(\tilde\sigma_c)^{\dot\beta\alpha} &= \frac{1}{2}(\eta_{bc}(\tilde\sigma_a)^{\dot\alpha\alpha} - \eta_{ac}(\tilde\sigma_b)^{\dot\alpha\alpha} + \mathrm{i}\varepsilon_{abcd}(\tilde\sigma^d)^{\dot\alpha\alpha}) && \text{(F.19)}\\
(\sigma_{ab})_\alpha{}^\beta(\sigma_c)_{\beta\dot\alpha} &= \frac{1}{2}(\eta_{bc}(\sigma_a)_{\alpha\dot\alpha} - \eta_{ac}(\sigma_b)_{\alpha\dot\alpha} - \mathrm{i}\varepsilon_{abcd}(\sigma^d)_{\alpha\dot\alpha}) && \text{(F.20)}\\
(\tilde\sigma_a)^{\dot\alpha\beta}(\sigma_{bc})_\beta{}^\alpha &= \frac{1}{2}(\eta_{ab}(\tilde\sigma_c)^{\dot\alpha\alpha} - \eta_{ac}(\tilde\sigma_b)^{\dot\alpha\alpha} + \mathrm{i}\varepsilon_{abcd}(\tilde\sigma^d)^{\dot\alpha\alpha})\,. && \text{(F.21)}
\end{aligned}
$$

I will regularly require the action of Lorentz generators on different tensors. They are

$$M_{ab}(\psi_\alpha) = (\sigma_{ab})_\alpha{}^\beta \psi_\beta \tag{F.22}$$
$$M_{ab}(\psi^{\dot\alpha}) = (\tilde\sigma_{ab})^{\dot\alpha}{}_{\dot\beta}\psi^{\dot\beta} \tag{F.23}$$
$$M_{ab}(V^c) = \delta^c{}_a V_b - \delta^c{}_b V_a \tag{F.24}$$
$$M_{\alpha\beta}(\psi_\gamma) = \frac{1}{2}(\varepsilon_{\gamma\alpha}\psi_\beta + \varepsilon_{\gamma\beta}\psi_\alpha) \tag{F.25}$$
$$\overline{M}_{\dot\alpha\dot\beta}(\psi_\alpha) = 0 \tag{F.26}$$
$$M_{\alpha\beta}(\psi_{\dot\alpha}) = 0 \tag{F.27}$$
$$\overline{M}_{\dot\alpha\dot\beta}(\psi_{\dot\gamma}) = \frac{1}{2}(\varepsilon_{\dot\gamma\dot\alpha}\psi_{\dot\beta} + \varepsilon_{\dot\gamma\dot\beta}\psi_{\dot\alpha})\,. \tag{F.28}$$

For tensors with more than one index, a Leibniz style rule applies index to index, e.g.

$$M_{ab}(T^{cd}) = \delta^c{}_a T_b{}^d - \delta^c{}_b T_a{}^d + \delta^d{}_a T^c{}_b - \delta^d{}_b T^c{}_a\,. \tag{F.29}$$

The Riemann tensor definition, $[\nabla_a, \nabla_b] = \frac{1}{2}R_{ab}{}^{cd}M_{cd}$, written in spinor notation is

$$[\nabla_{\alpha\dot\alpha}, \nabla_{\beta\dot\beta}] = \frac{1}{2}R_{\alpha\dot\alpha\beta\dot\beta}{}^{cd}M_{cd} \tag{F.30}$$
$$= R_{\alpha\dot\alpha\beta\dot\beta\mu\nu}M^{\mu\nu} + \overline{R}_{\alpha\dot\alpha\beta\dot\beta\dot\mu\dot\nu}\overline{M}^{\dot\mu\dot\nu} \tag{F.31}$$
$$\text{where } R_{\alpha\dot\alpha\beta\dot\beta\mu\nu} = \frac{1}{2}(\sigma^a)_{\alpha\dot\alpha}(\sigma^b)_{\beta\dot\beta}(\sigma^{cd})_{\mu\nu}R_{abcd} \tag{F.32}$$
$$\text{and } \overline{R}_{\alpha\dot\alpha\beta\dot\beta\dot\mu\dot\nu} = -\frac{1}{2}(\sigma^a)_{\alpha\dot\alpha}(\sigma^b)_{\beta\dot\beta}(\tilde\sigma^{cd})_{\dot\mu\dot\nu}R_{abcd}\,. \tag{F.33}$$

The following are well known identities of the Riemann tensor and its descendants.

$$R_{abcd} = -R_{abdc} \tag{F.34}$$
$$R_{abcd} = -R_{bacd} \tag{F.35}$$
$$R_{abcd} = R_{cdab} \tag{F.36}$$
$$0 = R_{abcd} + R_{acdb} + R_{adbc} \tag{F.37}$$
$$0 = \nabla_a R_{debc} + \nabla_b R_{deca} + \nabla_c R_{deab} \tag{F.38}$$
$$R_{ab} = R^c{}_{acb} \tag{F.39}$$
$$R_{ab} = R_{ba} \tag{F.40}$$
$$R = R^a{}_a \tag{F.41}$$
$$\nabla^b R_{ab} = \frac{1}{2}\nabla_a R \tag{F.42}$$
$$C_{abcd} = R_{abcd} - \frac{1}{2}\eta_{ac}R_{bd} - \frac{1}{2}\eta_{bd}R_{ac} + \frac{1}{2}\eta_{ad}R_{bc} + \frac{1}{2}\eta_{bc}R_{ad} - \frac{1}{6}R\eta_{bc}\eta_{ad} + \frac{1}{6}R\eta_{ac}\eta_{bd} \tag{F.43}$$

In spinor notation, these properties can be used to define

$$E_{\alpha\beta\dot\alpha\dot\beta} = \frac{1}{2}(\sigma^a)_{\alpha\dot\alpha}(\sigma^b)_{\beta\dot\beta}\left(R_{ab} - \frac{1}{4}R\right) \tag{F.44}$$
$$C_{\alpha\beta\mu\nu} = \frac{1}{12}((\sigma^{ab})_{\alpha\beta}(\sigma^{cd})_{\mu\nu} + (\sigma^{ab})_{\alpha\mu}(\sigma^{cd})_{\nu\beta} + (\sigma^{ab})_{\alpha\nu}(\sigma^{cd})_{\beta\mu})\left(C_{abcd} - \frac{\mathrm{i}}{2}\varepsilon_{ab}{}^{ef}C_{efcd}\right) \tag{F.45}$$
$$\overline{C}_{\dot\alpha\dot\beta\dot\mu\dot\nu} = \frac{1}{12}((\tilde\sigma^{ab})_{\dot\alpha\dot\beta}(\tilde\sigma^{cd})_{\dot\mu\dot\nu} + (\tilde\sigma^{ab})_{\dot\alpha\dot\mu}(\tilde\sigma^{cd})_{\dot\nu\dot\beta} + (\tilde\sigma^{ab})_{\dot\alpha\dot\nu}(\tilde\sigma^{cd})_{\dot\beta\dot\mu})\left(C_{abcd} + \frac{\mathrm{i}}{2}\varepsilon_{ab}{}^{ef}C_{efcd}\right) \tag{F.46}$$
$$F = \frac{1}{12}R \tag{F.47}$$

and thereby derive the following identities,

$$E_{\alpha\beta\dot{\alpha}\dot{\beta}} = E_{(\alpha\beta)(\dot{\alpha}\dot{\beta})} \tag{F.48}$$

$$C_{\alpha\beta\mu\nu} = C_{(\alpha\beta\mu\nu)} \tag{F.49}$$

$$\overline{C}_{\dot{\alpha}\dot{\beta}\dot{\mu}\dot{\nu}} = \overline{C}_{(\dot{\alpha}\dot{\beta}\dot{\mu}\dot{\nu})} \tag{F.50}$$

$$R_{\alpha\dot{\alpha}\beta\dot{\beta}\mu\nu} = \varepsilon_{\dot{\alpha}\dot{\beta}} C_{\alpha\beta\mu\nu} + \varepsilon_{\alpha\beta} E_{\mu\nu\dot{\alpha}\dot{\beta}} + \varepsilon_{\dot{\alpha}\dot{\beta}}(\varepsilon_{\alpha\mu}\varepsilon_{\beta\nu} + \varepsilon_{\alpha\nu}\varepsilon_{\beta\mu})F \tag{F.51}$$

$$R_{\alpha\dot{\alpha}\beta\dot{\beta}\dot{\mu}\dot{\nu}} = \varepsilon_{\alpha\beta} \overline{C}_{\dot{\alpha}\dot{\beta}\dot{\mu}\dot{\nu}} + \varepsilon_{\dot{\alpha}\dot{\beta}} E_{\alpha\beta\dot{\mu}\dot{\nu}} + \varepsilon_{\alpha\beta}(\varepsilon_{\dot{\alpha}\dot{\mu}}\varepsilon_{\dot{\beta}\dot{\nu}} + \varepsilon_{\dot{\alpha}\dot{\nu}}\varepsilon_{\dot{\beta}\dot{\mu}})F \tag{F.52}$$

$$\nabla^{\beta\dot{\beta}} E_{\alpha\beta\dot{\alpha}\dot{\beta}} = -3\nabla_{\alpha\dot{\alpha}} F \tag{F.53}$$

$$\nabla^{\mu}{}_{\dot{\alpha}} C_{\alpha\beta\gamma\mu} = \nabla_{(\alpha}{}^{\dot{\beta}} E_{\beta\gamma)\dot{\beta}\dot{\alpha}} \tag{F.54}$$

$$\nabla_{\alpha}{}^{\dot{\mu}} \overline{C}_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\mu}} = \nabla^{\beta}{}_{(\dot{\alpha}} E_{\beta\alpha\dot{\beta}\dot{\gamma})} \,. \tag{F.55}$$

Appendix G

Student achievements

As the end of 3rd year beckoned, a looming choice approached. What to study next? Like my peers, I subjected myself to the seemingly endless series of advertorial presentations about different physics and mathematics research groups available at UWA. Some may find it curious that after all the Powerpoints and aesthetically pleasing imagery I chose to try join the group that promised nothing more than one line - "for Field Theory and Quantum Gravity [FTQG] projects, please contact Prof. Sergei Kuzenko." But there was little counterintuition about my choice. I did not care about publishing papers or conducting groundbreaking research in an honours or master's. I wanted to learn as much advanced, but fundamental, physics and mathematics as I could. And not niche research areas of interest to few; I wanted to gain skill and knowledge valuable across the mathematical sciences. For that, the FTQG group seemed the perfect fit. I think my biggest achievement (and a great credit to FTQG group) in the Master of Physics - more so than any lemma or theorem that I proved in this thesis - was the progress I made on that path. Producing this document would have been impossible without learning a highly non-trivial amount of differential geometry & general relativity and developing significant fluency with spinor gymnastics. Indeed, of the three semesters I spent in this course, the entirety of the first was dedicated to up-skilling myself in preparation for a technical project in the FTQG group. As for the project itself, my achievements were more in presenting existing knowledge in a coherent and self-contained fashion, rather than developing new knowledge. This is not unusual for master's projects in the theoretical physics community. As I hinted in chapter 1, [20] contains essentially the same results as mine for 2nd order operators. However, I was not aware of this paper until the last month of my master's and I derived my results independently. Also, as I stated in chapter 3, some extensions to the problem I considered (or related problems) about the conformal d'Alembertian have already been solved in [17] and [21]. In practice, I treated the conformal d'Alembertian as more like a training exercise in becoming comfortable with spinors and higher symmetries before tackling the massless Dirac operator - a more technically challenging task. My main achievements in the project were doing and presenting long calculations not readily available in the literature. During my master's, I was perplexed by the culture of scientific publication - in particular the almost complete lack of detailed proofs/calculations and the sheer volume of detail brushed under the carpet. I wanted to do better. In this spirit, the accursed words, "obviously," "clearly," "easily" and "it can be shown" were all banished as far as possible in my thesis. It is my hope that any student who completed the same foundational study as I did in the 1st semester of my course would be able to understand almost everything I have written in this thesis. To assist in that, I also included appendix D, a comprehensive overview of spinors. It was not all a solo effort, of course. Other than Sergei, I had help from Emmanouil Raptakis in particular with my calculations. Although most of the work is my own, he guided me in how to approach the problems and also walked me through some easier cases (e.g. flat space).

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