A tale of spinors and positive energy theorems

THE UNIVERSITY
of EDINBURGH

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Doctor of Philosophy
April 2026

Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text. This work has not been submitted for any other degree or qualification.

(*Virinchi Rallabhandi*)

Acknowledgements

No dream is ever chased alone.
- Rahul Dravid

A PhD can be a highly rewarding experience, but also one that can be long and challenging - both from an academic and personal point of view. Looking back over these four years, I'd like to take this moment to thank some of those who helped me survive, and occasionally even thrive, in this journey.

To my partner, Sophie, amidst the endless winter of Edinburgh, you have been my greatest source of warmth. It is probably no exaggeration to say that I couldn't have done this without you. I realise this thesis is also a mark of all the time we've had to spend on opposite sides of the planet and all the difficulties that entails. Whatever the future holds, I know how lucky I am to have your support.

To my sister, Monica, we are the closest of siblings. I couldn't always be there for you - not just in these last four years in Edinburgh, but these last five in the UK. I've missed a fair bit - including your incredible growth as a violinist. Our weekly phone calls were always one of the highlights of my week though. You remain one of my closest relations and I hope we never drift apart.

To my parents, you've obviously been there from the start. You taught me the value of education and this thesis represents my highest point in the pursuit of it.

Turning my attention to those here in Edinburgh, I could never have asked for a better supervisor than James Lucietti. To James, you went above and beyond in helping me with all parts of the academic journey - from long meetings in front of the whiteboard to reading notes as long as books and unpolished research statements. I've learnt such an enormous amount in the last four years and that's in large part due to you.

In addition to James, my papers were greatly improved by the discussions I had with numerous others. I'd like to thank Tim Adamo, Piotr Chrusciel, Jelle Hartong, Hari Kunduri, Paul Tod and especially Harvey Reall for this.

Amongst the faculty at the University of Edinburgh, I'd also like to mention Jan Sbierski for the GR reading group - which was where I first studied quasilocal mass - and all the other assistance along the way.

This thesis marks the end of my formal education - a journey that begin far before I arrived in Edinburgh. I've had too many great teachers to thank them all here, but I'd like to specifically mention Greg Gamble and Sergei Kuzenko for the influential roles they played.

The School of Mathematics at the University of Edinburgh is greatly enriched by the many excellent PhD students and postdocs that pass through it. I would like to especially thank Anastasia, Bernardo, Dávid, Gianni, Gerben, Iustin, Lucile, Sean, Sonja, Vijay and Wei for their friendship and all that I've learnt from them.

Outside the academic sphere, I'd like to give a shout-out to Marchmont Cricket Club. I've seen a lot of cricket clubs over the years and Marchmont has certainly been one of the most inclusive and enjoyable to play for - so much so that I'll almost miss the freezing gales and erratic bounce of Cavalry Park.

Finally, no matter how wonderful or kind people may be, at the end of the day, somebody has to pay the bills. For that, I will be eternally grateful to the UK's taxpayers, who ultimately funded the incredibly generous School of Mathematics studentship which supported me financially through this PhD.

Lay summary

Albert Einstein's general theory of relativity is our best known theory of gravity. Having been experimentally verified now for over a century, it stands alongside the standard model of particle physics as the most successful theory of a physical phenomenon that humanity has ever devised.

Within the study of general relativity, this thesis is dedicated to quantifying the energy contained in the gravitational field. In our everyday experience, this is not a difficult task. Using Newton's theory of gravitation, we can very precisely calculate the energy required to lift a brick, or even send one to the Moon.

However, for a variety of technical reasons, it turns out gravitational energy is far more subtle in general relativity. One of the theory's many striking consequences is that it is impossible to measure the gravitational field's energy at any single, given point; instead, energy must be measured over some extended region.

In this thesis, I study energy in two such contexts - "global" and "quasilocal." In the former, one tries to quantify the energy of the entire spacetime, effectively the entire Universe, or at least the energy of an isolated system measured from very far away. In the latter context, one instead tries to ascertain the energy of some smaller region of interest.

Another one of general relativity's striking features is an undetermined free parameter known as the cosmological constant. In fact, a positive cosmological constant is one possible explanation for what is colloquially known as "dark energy." Most of the results I prove in this thesis are about the case when the cosmological constant is negative. As I find in this thesis, gravitational energy has some unique mathematical properties in such Universes.

The main tools I use to prove these properties are known as "spinor" methods. Spinors are peculiar objects in abstract algebra which are difficult to visualise in any simple terms. One of their characteristic features is that a rotation of 360 degrees does not leave them invariant. Instead, a full two rotations are required to return them to their original state. Spinors don't naturally have anything to do with general relativity, but an astonishing piece of work by Edward Witten in 1981 developed a method that proved otherwise. It is primarily Witten's method that I deploy in this thesis.

Abstract

In this thesis, I apply spinorial techniques to develop new positive energy theorems in general relativity, both at the global and quasilocal levels. The new results are mainly in the context of spacetimes with negative cosmological constant. At the global level, I focus on asymptotically, locally AdS spacetimes and give particular attention to spacetimes where conformal infinity has compact, Einstein cross-sections admitting real Killing or parallel spinors. A new positive energy theorem is derived for such spacetimes in terms of geometric data intrinsic to the cross-section. This is followed by the first complete proofs of the BPS inequalities in (the bosonic sectors of) 4D and 5D minimal, gauged supergravity, including with magnetic fields, provided the Maxwell field is exact. The BPS inequalities are proven for asymptotically AdS spacetimes, but also generalised to the aforementioned class of asymptotically, locally AdS spacetimes. Meanwhile, at the quasilocal level, I develop a new notion of quasilocal mass for generic, compact, two dimensional, spacelike surfaces in four dimensional spacetimes with negative cosmological constant. The definition is spinorial and based on work for vanishing cosmological constant by Penrose and Dougan & Mason. Furthermore, this mass is non-negative, equal to the Misner-Sharp mass in spherical symmetry, equal to zero for every generic surface in AdS, has an appropriate form for gravity linearised about AdS and has an appropriate limit for large spheres in asymptotically AdS spacetimes. I finish the thesis with some remarks on natural questions and extensions raised by the results and the methods used to derive them.

List of publications

The following publications were produced through the course of my PhD:

- V. Rallabhandi. Spinorial quasilocal mass for spacetimes with negative cosmological constant, 2025. arXiv[gr-qc/2504.11971]. To appear in *Advances in Theoretical and Mathematical Physics*.

- V. Rallabhandi. On energy bounds in asymptotically locally AdS spacetimes. *Classical and Quantum Gravity*, 43(1):015019, 2026.

Chapter 3 is based on both papers, chapter 4 is based on the second paper and chapter 5 is based on the first paper.

Contents

Chapter 1

Introduction

Madness, as you know, is like gravity; all it takes is a little push.
- The Joker in the *The Dark Knight*

Albert Einstein's general theory of relativity is our best known theory of gravity. Having been experimentally verified now for over a century, it stands alongside the standard model of particle physics as the most successful theory of a physical phenomenon that humanity has ever devised.

General relativity is a famously technically challenging theory which brings together deep geometric ideas to explain gravitation and the response of objects to the gravitational field. In particular, spacetime - the grand arena in which physical phenomenon occurs - is a Lorentzian manifold, (M, g), where the gravitational field is encoded in the metric, g, and is required to satisfy the Einstein equation,

$$R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = 8\pi T_{ab}. \tag{1.1}$$

The left hand side is built out of purely geometric quantities - the Ricci tensor, R_{ab}, the Ricci scalar, R, the metric itself and an undetermined constant, Λ, known as the cosmological constant. In contrast, the right hand side is proportional to the energy-momentum tensor, T_{ab}, a tensor determined by the matter content in the spacetime. As Wheeler famously remarked, "spacetime tells matter how to move; matter tells spacetime how to curve." Hence, understanding gravitational dynamics requires a detailed study of geometry.

This has led to the flourishing research field of mathematical relativity - a program of applying techniques from differential geometry, geometric analysis and theoretical physics to answer questions about gravitation. Partly because of these technical challenges of general relativity, there are several fundamental aspects of the theory which remain under active research despite the theory being well over a hundred years old.

One such aspect is the study of gravitational energy, which is the overriding theme of this thesis. It has long been known that energy is a very subtle concept in general relativity. Firstly, Wheeler's famous remark is only half-true. Since the left hand side of the Einstein equation is highly non-linear, the metric can be highly non-trivial even when $T_{ab} = 0$; spacetime can curve even without matter telling it how. Therefore gravitational energy is not simply quantified by T_{ab}. Besides any implications of the Einstein equation, an additional source of difficulty is the equivalence principle, another foundational tenet of general relativity. In it's simplest expression, the equivalence principle is the assertion that freely falling particles follow causal geodesics of the spacetime [116]. In particular, since all bodies fall precisely the same way in a gravitational field and the gravitational field must be determined by nothing more than the spacetime structure, (M, g), it is impossible to detect the gravitational field through an experiment localised to any one given point in the spacetime. Therefore, energy in general relativity must be measured over some extended region.

Faced with this situation, the typical method to define energy is to work with geometric invariants measured at infinity, thereby capturing some notion of the total energy of the entire spacetime. This approach has lead to commonly accepted definitions such as the ADM [2] and Bondi [12] energies in asymptotically flat spacetimes. These global energies are different from our everyday experience in that under some reasonable assumptions, they are always positive - in contrast, Newton's gravitational potential energy is typically negative. This remarkable positive energy theorem was proven by Schoen & Yau [107, 108] and Witten [125] independently in the early 1980s through very different means. These seminal results have gone on to shape much of the research in the field in the subsequent decades, including this thesis.

But one may be inclined to launch a vociferous protest at this state of affairs. After all, the real Universe has regions of substructure. Can we truly not quantify the energy contained in a smaller region of physical interest, instead of the entire spacetime? Answering this question has lead to the study of "quasilocal mass" - a desire to quantify the total energy bounded by some closed, compact, 2D surface such as a sphere. In this thesis I will study energy at both the global and quasilocal levels.

Another major theme of this thesis will be the cosmological constant, Λ, whose value is left undetermined by the postulates of general relativity. While $\Lambda = 0$ can be assumed for most practical purposes, its sign is critical on the largest length scales. This thesis is an in-depth study of the $\Lambda < 0$ case. My motivation is mainly mathematical interest and a desire for completeness in considering all possible cases of Λ. However, I am also partly motivated by the intense interest being received in physics by the AdS/CFT correspondence, which critically relies on $\Lambda < 0$.

In performing this study, I will follow Witten's original positive energy theorem proof [125]. Witten's proof is astonishing in its simplicity and its surprising use of spinors, which a priori have nothing to do with the problems at hand. But, as with Witten's pioneering work, spinor methods will be the primary tool with which I proceed.

In recounting this tale, I will start afresh in chapter 2 with a survey of the technical background required for the later chapters and a review of the relevant literature to contextualise my work. The crux of the thesis then begins in chapter 3, where I establish various identities and results in analysis which underpin the main methods deployed in this thesis. Then, I'm finally ready in chapter 4 to prove the first tranche of new results, namely a series of positive energy theorems and BPS inequalities for a certain class of asymptotically, locally AdS spacetimes. In chapter 5 I turn my attention to the quasilocal level, where I define a new quasilocal mass for spacetimes with negative cosmological constant and prove that the new definition satisfies a number of physically desirable properties. Finally, pathways for further research are discussed in chapter 6. Additionally, I provide a pair of appendices. Appendix A collates the various conventions I adhere to in presenting the material while appendix B provides a reminder of some frequently used spinor identities.

1.1 Summary of main results

While theses may be as long as novels, unlike the latter, suspense is rarely a desirable feature of the former. Thus, I collect here the two main results in this thesis. The first is a series of positive energy theorems for a certain class of asymptotically, locally AdS spacetimes. The precise definitions of asymptotically, locally AdS spacetime, energy and "conserved quantities" will be given later in definitions 2.11, 2.16, 4.6 and 4.11. The theorem below is collated from theorems 4.2, 4.4, 4.7, 4.8, 4.12 and corollary 4.19.1.

Theorem 1.1. *Let (M, g) be an nD, asymptotically, locally AdS spacetime solving the Einstein equation. Suppose conformal infinity, $\mathcal{I}$, is $\mathbb{R} \times S$ with the metric, $-\mathrm{d}t \otimes \mathrm{d}t + h$, where t is the coordinate along the $\mathbb{R}$ factor in the topology. Assume (S, h) is an Einstein manifold with $Ric(h) = c(n-3)h$ and $c = 0$ or $c = 1$ for each value of t. Let E denote the energy of the spacetime and let $Q_{\hat{k}}$ denote an (appropriately normalised) "conserved quantity" associated to a Killing vector, $\hat{k}$, of (S, h). Let $\hat{D}^{(h)}_A$ denote the Levi-Civita connection of h and $\{\hat{\gamma}^A\}$ the $(n-2)D$ gamma matrices. Suppose $c = 0$ and there exists a non-zero spinor, $\hat{\psi}$, solving $\hat{D}^{(h)}_A \hat{\psi} = 0$ on (S, h). Then, with $\hat{k}^A = -\mathrm{i}\hat{\psi}^\dagger \hat{\gamma}^A \hat{\psi}$ the following properties hold.*

- *If the dominant energy condition holds, then $E + Q_{\hat{k}} \geq 0$.*
- *If $n = 4$, the matter content of the spacetime is an exact electromagnetic field and the Maxwell equations hold, then $E + Q_{\hat{k}} + c(\hat{\psi})q_e \geq 0$, where q_e is the electric charge of the spacetime and $c(\hat{\psi})$ is a constant depending on $c(\hat{\psi})$.*
- *If $n = 5$, the matter content of the spacetime is an exact electromagnetic field and the Maxwell-Chern-Simons equations hold, then $E + Q_{\hat{k}} + c(\hat{\psi})q_e \geq 0$. Note that this $c(\hat{\psi})$ need not equal the $c(\hat{\psi})$ in the previous case.*

Instead, suppose that (S, h) is the round sphere and let $\{\gamma^0, \gamma^I\}$ be the nD, Lorentzian gamma matrices. Then, the following properties hold.

- *If the dominant energy condition holds, then*

$$EI - \mathrm{i}P_I\gamma^I + \frac{\mathrm{i}}{2}J_{IJ}\gamma^0\gamma^{IJ} + K_I\gamma^0\gamma^I \tag{1.2}$$

  is a non-negative definite matrix, where P_I, J_{IJ} and K_I represent the linear momentum, angular momentum and centre of mass position of the spacetime.

- *If $n = 4$, the matter content of the spacetime is an exact electromagnetic field and the Maxwell equations hold, then*

$$EI - \mathrm{i}P_I\gamma^I + \frac{\mathrm{i}}{2}J_{IJ}\gamma^0\gamma^{IJ} + K_I\gamma^0\gamma^I - q_e\gamma^0 \tag{1.3}$$

  is a non-negative definite matrix.

- *If $n = 5$, the matter content of the spacetime is an exact electromagnetic field and the Maxwell-Chern-Simons equations hold, then*

$$EI - \mathrm{i}P_I\gamma^I + \frac{\mathrm{i}}{2}J_{IJ}\gamma^0\gamma^{IJ} + K_I\gamma^0\gamma^I - \frac{\sqrt{3}}{2}q_e\gamma^0 \tag{1.4}$$

  is a non-negative definite matrix.

Finally, suppose $c = 1$ but (S, h) is not the round sphere. Let $\hat{\varepsilon}^{(\pm)}_h$ denote spinors solving $\hat{D}^{(h)}_A \hat{\varepsilon}^{(\pm)}_h = \pm\frac{1}{2}\hat{\gamma}_A\hat{\varepsilon}^{(\pm)}_h$ and define $\hat{k}^{(\pm)A} = -\mathrm{i}\hat{\varepsilon}^{(\pm)\dagger}_h \hat{\gamma}^A \hat{\varepsilon}^{(\pm)}_h$. Without loss of generality, scale $\hat{\varepsilon}^{(\pm)}_h$ so that $\hat{\varepsilon}^{(\pm)\dagger}_h \hat{\varepsilon}^{(\pm)}_h = \hat{\delta}^{(\pm)}$, where $\hat{\delta}^{(\pm)} = 1$ if a non-trivial $\hat{\varepsilon}^{(\pm)}_h$ exists and $\hat{\delta}^{(\pm)} = 0$ otherwise. Suppose at least one of $\hat{\delta}^{(+)}$ or $\hat{\delta}^{(-)}$ is non-zero. Then, the following properties hold.

- *If the dominant energy condition holds, then $E\big(\hat{\delta}^{(+)} + \hat{\delta}^{(-)}\big) + Q_{\hat{k}^{(+)}} + Q_{\hat{k}^{(-)}} \geq 0$.*
- *If $n = 4$, the matter content of the spacetime is an exact electromagnetic field and the Maxwell equations hold, then $E\big(\hat{\delta}^{(+)} + \hat{\delta}^{(-)}\big) + Q_{\hat{k}^{(+)}} + Q_{\hat{k}^{(-)}} + c(\hat{\varepsilon}^{(\pm)}_h)q_e \geq 0$.*

- *If $n = 5$, the matter content of the spacetime is an exact electromagnetic field and the Maxwell-Chern-Simons equations hold, then* $E\big(\hat{\delta}^{(+)}+\hat{\delta}^{(-)}\big)+Q_{\hat{k}^{(+)}}+Q_{\hat{k}^{(-)}}+c(\hat{\varepsilon}_h^{(\pm)})q_e \geq 0$.

The second main result is the development of a new quasilocal mass for 4D spacetimes with negative cosmological constant and an analysis of the new quasilocal mass' properties.

Definition 1.2 (Quasilocal mass)**.** *Given a generic, 2D, spacelike surface, S, within a 4D spacetime, (M, g), satisfying the Einstein equation with negative cosmological constant, Λ, and matter fields satisfying the dominant energy condition, make the following constructions. Let $\{l, n, m, \overline{m}\}$ be a Newman-Penrose tetrad adapted to S. Assume the null expansions of S satisfy $\theta_l > 0$, $\theta_n < 0$ and $\theta_l\theta_n < \frac{2\Lambda}{3}$. Let D_a be the Levi-Civita connection of g and let*

$$\nabla_a\Psi = D_a\Psi + \mathrm{i}\sqrt{-\frac{\Lambda}{12}}\gamma_a\Psi \tag{1.5}$$

for any Dirac spinor, Ψ. Let Φ denote a Dirac spinor satisfying $\overline{m}^a\nabla_a\Phi = 0$ on S and let $\{\Phi^A\}$ be a basis of solutions. Then, define matrices, Q^{AB} and T^{AB}, by

$$Q^{AB} = \int_S l_a n_b \left(\overline{\Phi}^A\gamma^{abc}\nabla_c\Phi^B - \overline{\nabla_c(\Phi^A)}\gamma^{abc}\Phi^B\right) \mathrm{d}A \tag{1.6}$$

$$\text{and } T^{AB} = (\Phi^A)^T C^{-1}\Phi^B, \tag{1.7}$$

$$\text{where } C = \text{charge conjugation matrix.} \tag{1.8}$$

Finally, define the quasilocal mass as

$$m(S) = \frac{1}{16\pi}\sqrt{-\mathrm{tr}(QT^{-1}\overline{Q}\overline{T}^{-1})}. \tag{1.9}$$

The precise meaning of "generic" is given in definitions 5.5, 5.6 and discussed afterwards. The proof that $m(S)$ is well-defined is the main subject of section 5.1 and relies crucially on theorem 5.3.

Theorem 1.3. *The quasilocal mass in definition 1.2 satisfies:*

- $m(S) \geq 0$.
- $m(S) = 0$ *for every generic surface in AdS.*
- *In a spherically symmetric spacetime, the round spheres have $m(S_r^2) = m_{MS}(S_r^2)$, where m_{MS} is the Misner-Sharp mass.*
- $m(S^2_\infty) = \sqrt{E^2 - P_IP^I + J_IJ^I - K_IK^I}$, *where S^2_∞ is a sphere at infinity in an asymptotically AdS spacetime.*
- *For generic S in a spacetime where AdS is perturbed by an infinitesimal energy-momentum tensor, T_{ab}, $m(S^2_\infty) = \sqrt{E^2 - P_IP^I + J_IJ^I - K_IK^I}$ and E, P_I, J_I & K_I are conserved quantities built out of T_{ab}.*

The precise definitions of asymptotically AdS spacetime, conserved quantities and Misner-Sharp mass are given in definitions 2.11, 2.16, 4.11, 5.15 and equation 2.170. The theorem collates the proof that definition 5.9 is well-defined, lemma 5.10, theorem 5.13, corollary 5.14.1 and theorem 5.16.

Chapter 2

Background

First learn stand, then learn fly.
- Mr. Miyagi in *The Karate Kid*

I'll begin the technical chapters of this thesis with a wide-ranging review of the ideas and existing results which motivate everything else to come. In particular, I will discuss spinors, their application in general relativity, asymptotically locally AdS spacetimes and various definitions of energy in general relativity, before bringing all these concepts together in a sketch of Witten's positive energy theorem. I've sought to make the presentation self-contained without being overly excessive. In that vein, certain proofs have been either omitted or given in complete detail on the basis of instructive-ness, future utility and personal indulgence.

2.1 Spinors and the Geroch-Held-Penrose formalism

Few topics in mathematics are as enigmatic as spinors. Even as frequent a practitioner of spin geometry as Michael Atiyah is quoted[1] to have said that "no one fully understands spinors; their algebra is formally understood but their general significance is mysterious." Notwithstanding such cautionary remarks, in this subsection I attempt my own review of spinors, including their application to general relativity in the Geroch-Held-Penrose (GHP) formalism [48]. I will largely adapt the expositions in [43, 17, 35, 116, 93, 48].

2.1.1 Spinors on Minkowski space

The story begins with the Clifford algebra, which I will discuss from a very "physicist" perspective. For now I'll discuss spinors only on Minkowski space, i.e. $\mathbb{R}^n$ with the metric, $\eta = \text{diag}(-1, 1, \cdots, 1)$. More general manifolds and metrics are postponed for now. The nD Clifford algebra is defined to be the unital, associate algebra generated by n elements, $\{\gamma^a\}_{a=0}^{n-1}$, satisfying

$$\gamma^a\gamma^b + \gamma^b\gamma^a = -2\eta^{ab} I, \tag{2.1}$$

where I is the multiplicative identity. If one takes a complex, irreducible, matrix representation of the Clifford algebra, then the generators, γ^a, are known as the "gamma matrices." Indeed, I will always consider the Clifford algebra in such a representation. Furthermore, the perspective I'll take in this thesis is that Dirac spinors are the elements of the representation space, say $\mathbb{S}$. Therefore, the gamma matrices act as endomorphsims on $\mathbb{S}$. It is well known that there exists a unique complex, irreducible representation of the Clifford algebra for even n, there

[1]As far as I can tell, this quote first appeared in Graham Farmelo's comprehensive biography of Paul Dirac.

exist two inequivalent such representations for odd n related by $\gamma^a \to -\gamma^a$ and in both cases $\dim(\mathbb{S}) = 2^{\lfloor n/2 \rfloor}$ [35]. It can also be shown [35] that one can always choose an equivalence class representative in which $(\gamma^0)^\dagger = \gamma^0$ and $(\gamma^I)^\dagger = -\gamma^I$.

Meanwhile, the spin group, $Spin(n-1,1)$, is defined to be the universal covering group[2] of the subgroup of $O(n-1,1)$ connected to the identity matrix, I. At first sight, there appears to be no discernible connection between spin groups and spinors.

However, their connection can be unraveled by first descending to the Lie algebra. The Lorentz algebra, $\mathfrak{o}(\mathfrak{n}-\mathfrak{1},\mathfrak{1})$, is defined by generators, $\{M_{ab} = -M_{ba}\}$, satisfying

$$[M_{ab}, M_{bc}] = -(\eta_{ac}M_{bd} - \eta_{ad}M_{bc} - \eta_{bc}M_{ad} + \eta_{bd}M_{ac}). \tag{2.2}$$

In particular, the fundamental representation has the matrices,

$$(M_{ab})_{cd} = \eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc}. \tag{2.3}$$

It can be checked - using the Clifford algebra - that this same Lie algebra is satisfied by

$$S_{ab} = -\frac{1}{4}[\gamma_a, \gamma_b] = -\frac{1}{2}\gamma_{ab}. \tag{2.4}$$

Since the spin group and Lorentz group only differ by universal cover, they share a Lie algebra. This fact allows a representation of the spin group to be generated from representations of the Lorentz group as follows.

Suppose $\gamma'^a = \Lambda^a{}_b\gamma^b$ for $\Lambda \in O(n-1,1)$. By the definition of $O(n-1,1)$, it follows that γ'^a also satisfies the Clifford algebra. However, as I've explained, the algebra's representation is unique up to equivalence. Therefore, there exists a matrix, $S[\Lambda]$, such that[3] $\gamma'^a = S[\Lambda]^{-1}\gamma^a S[\Lambda]$. Suppose that $\Lambda = \mathrm{e}^{\omega^{ab}M_{ab}/2}$ for some constants, ω_{ab}. Then, $S[\Lambda] = \mathrm{e}^{\omega^{ab}S_{ab}/2}$, which is proven as follows. In general, the study of the adjoint representation shows $\mathrm{e}^{-A}B\mathrm{e}^{A} = \mathrm{e}^{C_A}(B)$, where C_A is the operator, $C_A(B) = -[A,B]$. In this case,

$$C_{-\omega^{bc}\gamma_{bc}/4}(\gamma^a) = \frac{1}{4}\omega^{bc}[\gamma_{bc}, \gamma^a] = \omega^{ab}\gamma_b = \frac{1}{2}(\omega^{cd}M_{cd})^a{}_b\gamma^b, \tag{2.5}$$

which proves that upon exponentiation,

$$S[\Lambda]^{-1}\gamma^a S[\Lambda] = \mathrm{e}^{\omega^{bc}M_{bc}/2}(\gamma^a) = \Lambda^a{}_b\gamma^b. \tag{2.6}$$

A Dirac spinor, $\Psi \in \mathbb{S}$, is then defined to transform as $\Psi \to S[\Lambda]\Psi$. As a consequence, an object such as the Dirac operator, $\gamma^a\partial_a\Psi$, is covariant.

From equation 2.6 and Schur's lemma,

$$\begin{aligned} S[\Lambda_1\Lambda_2]^{-1}\gamma^a S[\Lambda_1\Lambda_2] &= (\Lambda_1)^a{}_c(\Lambda_2)^c{}_b\gamma^b = (S[\Lambda_1]S[\Lambda_2])^{-1}\gamma^a S[\Lambda_1]S[\Lambda_2] \\ \implies S[\Lambda_1\Lambda_2] &= f(\Lambda_1,\Lambda_2)S[\Lambda_1]S[\Lambda_2] \end{aligned} \tag{2.7}$$

for some function, f. Therefore $S[\Lambda]$ only provides a projective representation of $SO^\uparrow(n-1,1)$. This is because S is in fact providing a true representation of the universal cover, $Spin(n-1,1)$. The classic example which demonstrates this is the rotation,

$$\Lambda = \mathrm{e}^{\theta M} \text{ for } M = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{2.8}$$

[2] This is not a valid definition for $Spin(p,q)$ when both $p,q \geq 2$ [43].

[3] In odd dimensions, there are two inequivalent representations related by sign. However, it can be checked that $\gamma'^a = S[\Lambda]^{-1}(-\gamma^a)S[\Lambda]$ would lead to a contradiction when considering $\gamma^0\cdots\gamma^{n-1}$ (which must be proportional to I by Schur's lemma), provided that $\det(\Lambda) = 1$.

which (in the 4D Weyl representation of the gamma matrices) leads to

$$S[\Lambda] = \begin{bmatrix} e^{i\theta\sigma_3/2} & 0 \\ 0 & e^{i\theta\sigma_3/2} \end{bmatrix}. \tag{2.9}$$

Λ being a 2π rotation therefore does not lead to $S[\Lambda]$ being the identity matrix.

Since $\dim(\mathbb{S}) = 2^{\lfloor n/2 \rfloor}$, a Dirac spinor is naturally a $2^{\lfloor n/2 \rfloor}$-component object. However, while $\mathbb{S}$ is the representation space for an irreducible representation of the Clifford algebra, $S[\Lambda]$ acting on $\mathbb{S}$ is not an irreducible representation of $Spin(n-1,1)$ when n is even. One example of a non-trivial, invariant subspace is the space of Weyl spinors, which are defined to be eigenspinors of $\gamma^0 \cdots \gamma^{n-1}$. Since $(\gamma^0 \cdots \gamma^{n-1})^2 = (-1)^{(n+2)(n-1)/2} I$, $(\gamma^0 \cdots \gamma^{n-1})^\dagger \gamma^0 \cdots \gamma^{n-1} = I$ and $\{\gamma^a, \gamma^0 \cdots \gamma^{n-1}\} = 0$, $\mathbb{S}$ splits into two eigenspaces of equal dimension, both of which are invariant under $S[\Lambda]$ $\forall \Lambda$. When $n = 4$, it is actually particularly convenient to work with these two-component Weyl spinors instead of the four-component Dirac spinors.

To introduce the formalism of two-component spinors, I'll start by studying the spin group in four dimensions. From its outset, this discussion will need the extended Pauli matrices, $\sigma_a \equiv (I, \sigma_i)$ and $\tilde{\sigma}_a \equiv (I, -\sigma_i)$ where σ_i are the usual Pauli matrices.

Theorem 2.1. *Define* $\Lambda : SL(2,\mathbb{C})/\mathbb{Z}_2 \to SO^\uparrow(3,1)$ *by*

$$\Lambda(A)^a{}_b = -\frac{1}{2}\mathrm{tr}(\tilde{\sigma}^a A \sigma_b A^\dagger). \tag{2.10}$$

Then, Λ *is a group isomorphism.*

Note this theorem proves $Spin(3,1) \cong SL(2,\mathbb{C})$.

Proof. Let V^a be a Lorentz vector and let $\mathbb{H}$ be the space of hermitian 2×2 matrices. Observe that vectors can be put into bijection with $\mathbb{H}$ by the map,

$$\mathbb{V} = V^a \sigma_a = \begin{bmatrix} V^0 + V^3 & V^1 - iV^2 \\ V^1 + iV^2 & V^0 - V^3 \end{bmatrix}. \tag{2.11}$$

Since $\mathrm{tr}(\tilde{\sigma}^a \sigma_b) = -2\delta^a{}_b$, it follows that

$$V^a = -\frac{1}{2}\mathrm{tr}(\tilde{\sigma}^a \mathbb{V}). \tag{2.12}$$

Furthermore, this map has the property that

$$\det(\mathbb{V}) = (V^0)^2 - (V^1)^2 - (V^2)^2 - (V^3)^2 = -V^a V_a. \tag{2.13}$$

Observe that for any matrix, $A \in SL(2,\mathbb{C})$, $\det(A\mathbb{V}A^\dagger) = \det(\mathbb{V}) = -V^a V_a$ and $A\mathbb{V}A^\dagger$ is still hermitian. Therefore, $\exists$ a map, $\Lambda : SL(2,\mathbb{C}) \to O(3,1)$ such that

$$A\mathbb{V}A^\dagger = \Lambda(A)^a{}_b V^b \sigma_a. \tag{2.14}$$

From equation 2.12, it must be that

$$\Lambda(A)^a{}_b = -\frac{1}{2}\mathrm{tr}(\tilde{\sigma}^a A \sigma_b A^\dagger). \tag{2.15}$$

It remains to better analyse the properties of Λ. Firstly, it's a group homomorphism since

$$\Lambda(A_1)^a{}_c \Lambda(A_2)^c{}_b V^b \sigma_a = A_1(A_2 \mathbb{V} A_2^\dagger)A_1^\dagger = (A_1 A_2)\mathbb{V}(A_1 A_2)^\dagger = \Lambda(A_1 A_2)^a{}_b V^b \sigma_a. \tag{2.16}$$

Next, if $A \in \ker(\Lambda)$, then $A\mathbb{V}A^\dagger = \mathbb{V}$ for every hermitian matrix $\mathbb{V}$. By choosing different options for $\mathbb{V}$ (e.g. even $\mathbb{V} = I$, which would imply A commutes with every hermitian matrix), it follows that $A = \pm I$.

It remains to find the range of Λ. It can be checked that choosing $A = \mathrm{e}^{\mathrm{i}\theta\sigma_i/2}$ leads to Λ being a rotation about the x^i axis while choosing $A = \mathrm{e}^{\Theta\sigma_i/2}$ leads to a boost in the $t - x^i$ plane. Since rotations and boosts generate $SO^\uparrow(3,1)$, the range is at least that big. However, $\{\sigma_i, \mathrm{i}\sigma_i\}$ is a basis for $\mathfrak{sl}(2,\mathbb{C})$, meaning every element of $SL(2,\mathbb{C})$ can be written as a product of $\mathrm{e}^{\mathrm{i}\theta\sigma_i/2}$s and $\mathrm{e}^{\Theta\sigma_i/2}$s.

Therefore the range is exactly $SO^\uparrow(3,1)$. Using the first isomorphism theorem, one concludes that Λ is a group isomorphism between $SL(2,\mathbb{C})/\mathbb{Z}_2$ and $SO^\uparrow(3,1)$. □

As I foreshadowed earlier, Weyl spinors will be related to the irreducible representations of the spin group. Four irreducible representations of $SL(2,\mathbb{C})$ are particularly central to the study of two-component spinors. Given ${A_\alpha}^\beta \in SL(2,\mathbb{C})$, these representations are defined by the following transformations.

- Fundamental/left-handed Weyl/(1/2, 0): $\psi'_\alpha = {A_\alpha}^\beta \psi_\beta$
- Contragradient: $\psi'^\alpha = \psi'^\beta {(A^{-1})_\beta}^\alpha$
- Complex conjugate/right-handed Weyl/(0, 1/2): $\overline{\psi}_{\dot\alpha} = {\overline{A}_{\dot\alpha}}^{\dot\beta} \overline{\psi}_{\dot\beta}$
- Contragradient complex conjugate: $\overline{\psi}^{\dot\alpha} = \overline{\psi}^{\dot\beta} {(\overline{A}^{-1})_{\dot\beta}}^{\dot\alpha}$

Whether a two-component index is dotted or undotted and whether it's upstairs or downstairs therefore uniquely identifies the representation that spinor is in. One can now construct spinors with many indices by tensor product.

It's manifest that complex conjugation turns a dotted index into an undotted index and vice-versa. Furthermore, since $\varepsilon_{\gamma\delta} {A_\alpha}^\gamma {A_\beta}^\delta = \det(A)\varepsilon_{\alpha\beta} = \varepsilon_{\alpha\beta}$, $\varepsilon_{\alpha\beta}$ and $\varepsilon_{\dot\alpha\dot\beta}$ are invariant tensors and it can be checked that they can be used to raise and lower indices.

Likewise, equation 2.14 implies

$$\sigma_a = A\sigma_b A^\dagger {(\Lambda(A)^{-1})^b}_a, \tag{2.17}$$

which means σ_a is effectively an invariant tensor carrying one Lorentz index, one undotted spinor index and one dotted spinor index, i.e. $\sigma_a \equiv (\sigma_a)_{\alpha\dot\alpha}$. It can then be checked that $\tilde\sigma_a \equiv (\tilde\sigma_a)^{\dot\alpha\alpha} = \varepsilon^{\alpha\beta}\varepsilon^{\dot\alpha\dot\beta}(\sigma_a)_{\beta\dot\beta}$. Hence, equations 2.11 and 2.12 mean any Lorentz index can be swapped for a pair of dotted & undotted spinor indices and vice versa, i.e. $V_{\alpha\dot\alpha} = V^a(\sigma_a)_{\alpha\dot\alpha}$ and $V^a = -\frac{1}{2}(\tilde\sigma^a)^{\dot\alpha\alpha}V_{\alpha\dot\alpha}$.

I'm now in a position to complete the loop and return to Dirac spinors. It can be checked that if

$$\gamma^a = \begin{bmatrix} 0 & \sigma^a \\ \tilde\sigma^a & 0 \end{bmatrix}, \tag{2.18}$$

then the Clifford algebra holds, $(\gamma^0)^\dagger = \gamma^0$, $(\gamma^I)^\dagger = -\gamma^I$ and the collection, $\{\gamma^a\}_{a=0}^3$, has no non-trivial, invariant subspaces. This is called the Weyl representation of the gamma matrices. Furthermore,

$$\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{bmatrix} \mathrm{i}I & 0 \\ 0 & -\mathrm{i}I \end{bmatrix}. \tag{2.19}$$

Therefore, the Weyl spinors of each eigenspace are simply the top two and bottom two components in this four-component decomposition of the gamma matrices. This allows Dirac spinors to be written in terms of the $SL(2,\mathbb{C})$ indices as $\Psi = (\psi_\alpha, \bar{\chi}^{\dot{\alpha}})^T$ as follows. With the chosen gamma matrices, the spinor representation of $\mathfrak{o}(3,1)$ is

$$S_{ab} = -\frac{1}{2}\gamma_{ab} = \begin{bmatrix} \sigma_{ab} & 0 \\ 0 & \tilde{\sigma}_{ab} \end{bmatrix}, \tag{2.20}$$

$$\text{where } (\sigma_{ab})_\alpha{}^\beta = -\frac{1}{4}((\sigma_a)_{\alpha\dot{\alpha}}(\tilde{\sigma}_b)^{\dot{\alpha}\beta} - (\sigma_b)_{\alpha\dot{\alpha}}(\tilde{\sigma}_a)^{\dot{\alpha}\beta}) \equiv \frac{1}{2}\begin{bmatrix} 0 & \sigma_1 & \sigma_2 & \sigma_3 \\ -\sigma_1 & 0 & \mathrm{i}\sigma_3 & -\mathrm{i}\sigma_2 \\ -\sigma_2 & -\mathrm{i}\sigma_3 & 0 & \mathrm{i}\sigma_1 \\ -\sigma_3 & \mathrm{i}\sigma_2 & -\mathrm{i}\sigma_1 & 0 \end{bmatrix} \tag{2.21}$$

$$\text{and } (\tilde{\sigma}_{ab})^{\dot{\alpha}}{}_{\dot{\beta}} = -\frac{1}{4}((\tilde{\sigma}_a)^{\dot{\alpha}\alpha}(\sigma_b)_{\alpha\dot{\beta}} - (\tilde{\sigma}_b)^{\dot{\alpha}\alpha}(\sigma_a)_{\alpha\dot{\beta}}) \equiv \frac{1}{2}\begin{bmatrix} 0 & -\sigma_1 & -\sigma_2 & -\sigma_3 \\ \sigma_1 & 0 & \mathrm{i}\sigma_3 & -\mathrm{i}\sigma_2 \\ \sigma_2 & -\mathrm{i}\sigma_3 & 0 & \mathrm{i}\sigma_1 \\ \sigma_3 & \mathrm{i}\sigma_2 & -\mathrm{i}\sigma_1 & 0 \end{bmatrix}. \tag{2.22}$$

Observe that $\{\sigma_{ab}\}$ forms a basis for $\mathfrak{sl}(2,\mathbb{C})$ and $\tilde{\sigma}_{ab} = -\sigma_{ab}^\dagger$. Hence, if $\mathrm{e}^{\omega_{ab}\sigma^{ab}/2} = A \in SL(2,\mathbb{C})$, then the spinor representation has

$$S[\Lambda] = \begin{bmatrix} \mathrm{e}^{\omega_{ab}\sigma^{ab}/2} & \\ & \mathrm{e}^{\omega_{ab}\tilde{\sigma}^{ab}/2} \end{bmatrix} = \begin{bmatrix} A & \\ & A^{-\dagger} \end{bmatrix}. \tag{2.23}$$

Hence, the lower two components transform under a complex conjugated contragradient representation relative to the top two components, which in turn can be taken to transform in the defining representation of $SL(2,\mathbb{C})$. Therefore, it is indeed valid to decompose a Dirac spinor in four dimensions into Weyl spinors as $\Psi = (\psi_\alpha, \bar{\chi}^{\dot{\alpha}})^T$.

I'll finally also mention the Dirac conjugate, $\overline{\Psi} = \Psi^\dagger\gamma^0$, which is the appropriate analogue of complex conjugation for Dirac spinors to maintain covariance of different equations and compatibility with $(\gamma^a)^\dagger = \gamma^0\gamma^a\gamma^0$. In four dimensions, this leads to $\overline{\Psi} = -(\chi^\alpha, \overline{\psi}_{\dot{\alpha}})$.

2.1.2 Spinors on curved space

So far, these spinors have only been defined on nD Minkowski space. Additional complications arise for curved manifolds, (M,g). For a general manifold, the tangent bundle is a vector bundle with structure group, $GL(n,\mathbb{R})$. The presence of a non-generate, Lorentzian metric, g, means the structure group can be refined to $O(n-1,1)$. This can be viewed as the associated vector bundle for the principal bundle of orthonormal frames and effectively translates to being able to consistently choose vielbeins across (M,g). In general relativity it's always assumed (M,g) is orientable and time-orientable. These topological assumptions amount to being able to reduce the structure group further to $SO^\uparrow(n-1,1)$. A spin structure on a manifold is then the ability to improve the structure group to $Spin(n-1,1)$, the universal cover of $SO^\uparrow(n-1,1)$, which leads to the spin bundle, where the model fibre is $\mathbb{S}$.

This improvement is not always possible though. Dirac spinors were defined to transform as $\Psi \to S[\Lambda]\Psi$ upon a Lorentz transformation. But, a local Lorentz transformation is exactly the effect of the transition function when moving from one trivialising patch of the frame bundle to another, e.g. $V_B^a = (\Lambda_{BA}(x))^a{}_b V_A^b$ for a vector when going from a trivialising patch, U_A to a trivialising patch, U_B. Therefore, Dirac spinors must also adjust accordingly, e.g. $\Psi_B = S[\Lambda_{BA}(x)]\Psi_A$. However, $S[\Lambda]$ is really a representation of $Spin(n-1,1)$, not $SO^\uparrow(n-1,1)$, and as I've shown with equation 2.9, a closed loop in $SO^\uparrow(n-1,1)$ need not be a closed loop in $Spin(n-1,1)$; it's possible to start on one sheet of the cover and finish on a different one. Hence, there is no guarantee that Λ_{BA} solving the cocycle condition, $\Lambda_{AC}\Lambda_{CB}\Lambda_{BA} = I$, leads to $S[\Lambda_{BA}]$ solving its cocycle condition.

Like orientability, the existence of a spin structure is a topological restriction saying there exists a trivialisation, $\{U_A\}$, such that the cocycle condition holds for the $Spin(n-1,1)$-valued transition functions whilst also having $\mathrm{pr}_{frame} \circ \mathbf{\Lambda} = \mathrm{pr}_{spin}$, where pr is the base projection map for a bundle and $\mathbf{\Lambda} : Spin(n-1,1) \to SO^{\uparrow}(n-1,1)$ is the covering map. In this thesis I will always assume a spin structure exists. Although not every pseudo-Riemannian manifold admits a spin structure, when it does, it can be shown [43] there are $\mathrm{Hom}(\pi_1(M), \mathbb{Z}_2)$ different inequivalent spin structures[4].

There is also the notion of a Clifford bundle for a manifold. However, I will never write a gamma matrix or a Pauli matrix with a coordinate index[5] in this thesis, e.g. γ^μ. Hence, for me γ^a are just matrices of constants even in curved space and I don't need to worry about uplifting the Clifford algebra.

Having now defined spinors in curved space, I need to also extend the definition of the covariant derivative to spinors. Given a connection on a principal bundle, the connection on an associated vector bundle is simply the pushforward of the connection under the representation of the structure group which is used to define the vector bundle. Since I already know $\mathfrak{o}(n-1,1)$'s action on spinors, I immediately get the following expressions. Given a vielbein, ${e_a}^\mu$, and spin connection coefficients, ω_{abc}, the covariant derivative acts on different types of spinors as below.

- Dirac spinors: $D_a\Psi = {e_a}^\mu \partial_\mu \Psi - \frac{1}{4}\omega_{bca}\gamma^{bc}\Psi$
- Dirac conjugate: $D_a\overline{\Psi} = {e_a}^\mu \partial_\mu \overline{\Psi} + \frac{1}{4}\omega_{bca}\overline{\Psi}\gamma^{bc} = (D_a\Psi)^\dagger\gamma^0$
- Left-handed Weyl: $D_a\psi_\alpha = {e_a}^\mu \partial_\mu \psi_\alpha + \frac{1}{2}\omega_{bca}{(\sigma^{bc})_\alpha}^\beta\psi_\beta$
- Right-handed Weyl: $D_a\overline{\psi}^{\dot\alpha} = {e_a}^\mu \partial_\mu \overline{\psi}^{\dot\alpha} + \frac{1}{2}\omega_{bca}{(\tilde\sigma^{bc})^{\dot\alpha}}_{\dot\beta}\overline{\psi}^{\dot\beta}$
- Gamma matrix: $D_a\gamma^b = 0$
- Pauli matrix: $D_a(\sigma_b)_{\alpha\dot\alpha} = 0$
- Levi-Civita symbol: $D_a\varepsilon_{\alpha\beta} = 0$

The last three items in the list are not definitions, but rather derived from the other three, the spin connection coefficients and various identities of the gamma & Pauli matrices.

It's now finally time to start applying some of these spinor constructions to general relativity. The main application I'll discuss is the GHP formalism [48]. However, to do so, I'll first have to introduce the more fundamental Newman-Penrose formalism [93].

Let Σ be a 3D, spacelike, compact submanifold with boundary, S, within the spacetime, (M, g). Let $\{P, Q, X, Y\}$ be a vierbein with X^a & Y^a tangent to S, Q^a an outward-pointing normal to S in Σ and P^a a timelike, future-directed normal to Σ. See figure 2.1 for a visual depiction of this set-up.

Having chosen $\{P, Q, X, Y\}$ as described, an NP tetrad is defined by

$$l^a = \frac{1}{\sqrt{2}}\left(P^a + Q^a\right), \; n^a = \frac{1}{\sqrt{2}}\left(P^a - Q^a\right) \text{ and } m^a = \frac{1}{\sqrt{2}}\left(X^a + \mathrm{i}Y^a\right). \tag{2.24}$$

Equivalently, an NP tetrad can be defined as a collection of four vectors, $\{l, n, m, \overline{m}\}$, such that they locally form a basis for the tangent space, all four vectors are null, l & n are real, l

[4] A fact that will repeatedly haunt me through this thesis.

[5] In this thesis, $a, b, \cdots$ will denote vielbein indices running $0, 1, \cdots, n-1$, although in most cases they could equally well represent abstract indices. In contrast, coordinate indices running $0, 1, \cdots, n-1$ will be $\mu, \nu, \cdots$.

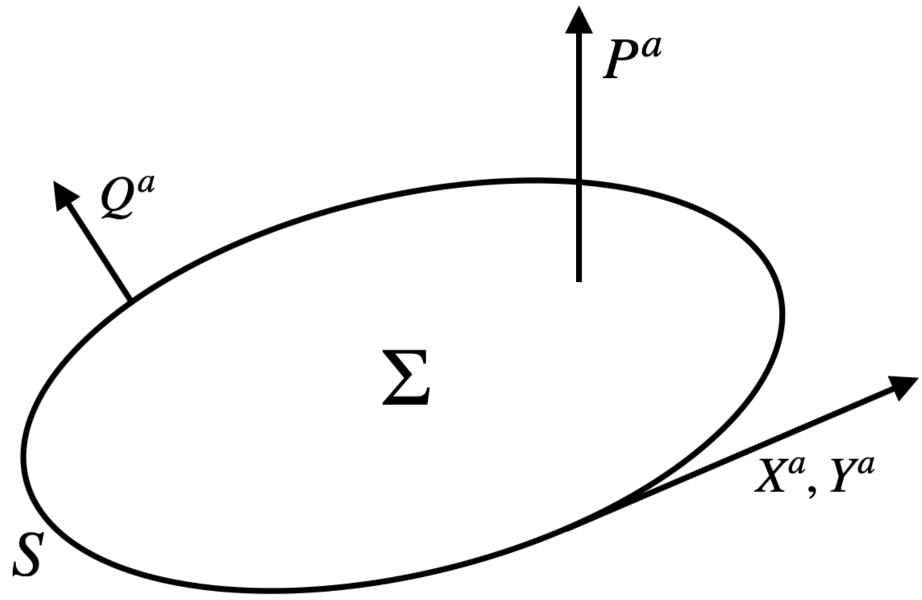

Figure 2.1: Σ is a 3D, spacelike, compact hypersurface with boundary, S. X^a & Y^a are orthonormal tangents to S, P^a is a timelike, future-directed, unit normal to Σ and Q^a is the unit, outward-pointing normal to S that is tangent to Σ. This set-up will be used for defining quasilocal mass and for applying the GHP formalism.

& n are normal to S, l is future-directed and outgoing, n is future-directed and ingoing, m is complex, m is tangent to S, $l^a n_a = -1$ and $m^a \overline{m}_a = 1$. Then, one could define

$$P^a = \frac{1}{\sqrt{2}}\left(l^a + n^a\right), \; Q^a = \frac{1}{\sqrt{2}}\left(l^a - n^a\right), \; X^a = \frac{1}{\sqrt{2}}\left(m^a + \overline{m}^a\right)$$
$$\text{and } Y^a = \frac{1}{\mathrm{i}\sqrt{2}}\left(m^a - \overline{m}^a\right). \tag{2.25}$$

The idea of the NP formalism is to decompose everything into its components in this well-chosen tetrad. For example, the covariant derivative, D_a, is replaced by the four derivative operators,

$$D = l^a D_a, \; \Delta = n^a D_a, \; \delta = m^a D_a \text{ and } \bar{\delta} = \overline{m}^a D_a. \tag{2.26}$$

Similarly the connection coefficients are also decomposed into scalars, e.g. $\rho = -m^a \bar{\delta} l_a$ and $\mu = \overline{m}^a \delta n_a$. The full list of NP coefficients, κ, τ, σ, ρ, π, ν, μ, λ, ε, γ, β and α is given in appendix A. The spacetime metric in this tetrad is

$$g_{ab} = -l_a n_b - n_a l_b + m_a \overline{m}_b + \overline{m}_a m_b, \tag{2.27}$$

while the metric induced on S is

$$\beta_{ab} = m_a \overline{m}_b + \overline{m}_a m_b. \tag{2.28}$$

The various components of the Weyl and Ricci tensors are also decomposed into scalars using the tetrad, allowing the Cartan structure equations and the Einstein equation to be written as a series of scalar equations. However, I will not need those equations in the NP formalism for this thesis.

The NP formalism turns out to be a particularly convenient way to study the extrinsic geometry of S. This is defined in general in much the same way as textbook derivations for codimension-1 surfaces [104, 116].

Definition 2.2 (Extrinsic curvature)**.** *The extrinsic curvature tensor of a surface (of arbitrary co-dimension) is defined by*

$$\mathbb{K}(X, Y) = -(D_{X_{||}} Y_{||})^{\perp}, \tag{2.29}$$

where X & Y are arbitrary vectors and $||$ & $\perp$ denote the components parallel & orthogonal to the surface respectively.

Lemma 2.3. *Let S be a spacelike, 2D surface. Choose a NP tetrad adapted to S. Then, the extrinsic curvature can be written as*

$$\mathbb{K}^a{}_{bc} = -\beta^d{}_b \beta^e{}_c (D_d(l_e) n^a + D_d(n_e) l^a). \tag{2.30}$$

Furthermore, $\mathbb{K}^a{}_{bc} = \mathbb{K}^a{}_{cb}$.

Proof. $\{l, n\}$ forms a basis for the normal bundle of S. Hence, for any vector, X,

$$X^a = X^a_{||} - X^b n_b l^a - X^b l_b n^a. \tag{2.31}$$

Therefore, using $\beta^a{}_b l^b = \beta^a{}_b n^b = 0$, the extrinsic curvature is

$$\begin{aligned}
\mathbb{K}(X, Y)^a &= (D_{X_{||}} Y_{||})^b l_b n^a + (D_{X_{||}} Y_{||})^b n_b l^a && (2.32)\\
&= \beta^c{}_d X^d D_c(\beta^b{}_e Y^e) l_b n^a + \beta^c{}_d X^d D_c(\beta^b{}_e Y^e) n_b l^a && (2.33)\\
&= X^b Y^c (\beta^d{}_b D_d(\beta^e{}_c) l_e n^a + \beta^d{}_b D_d(\beta^e{}_c) n_e l^a) && (2.34)\\
&= -X^b Y^c (\beta^d{}_b \beta^e{}_c D_d(l_e) n^a + \beta^d{}_b \beta^e{}_c D_d(n_e) l^a), && (2.35)
\end{aligned}$$

which proves the index expression since X and Y are arbitrary.

Since the extrinsic curvature is invariant under arbitrary choices of how the NP tetrad is extended off S, l and n can be taken to be normal to auxiliary null hypersurfaces, Σ_L and Σ_N. These hypersurfaces can be taken to be the zero sets of some functions, L and N. Thus,

$$l_a = D_a L + L l'_a \text{ and } n_a = D_a N + N n'_a \tag{2.36}$$

for some l' and n'. Hence,

$$\mathbb{K}^a{}_{bc} = -\beta^d{}_b \beta^e{}_c ((D_d D_e L + L D_d l'_e + l'_e D_d L) n^a + (D_d D_e N + N D_d n'_e + n'_e D_d N) l^a). \tag{2.37}$$

On S, where $L = N = 0$, this reduces to

$$\begin{aligned}
\mathbb{K}^a{}_{bc} &= -\beta^d{}_b \beta^e{}_c ((D_d D_e L + 0 + l'_e l_d) n^a + (D_d D_e N + 0 + n'_e n_d) l^a) && (2.38)\\
&= \beta^d{}_b \beta^e{}_c ((D_d D_e L + 0) n^a + (D_d D_e N + 0) l^a), && (2.39)
\end{aligned}$$

which is manifestly symmetric in the lower indices. □

Definition 2.4 (Mean curvature)**.** *The mean curvature vector is defined to be*

$$H^a = \beta^{bc} \mathbb{K}^a{}_{bc} = -\beta^{bc} (D_b(l_c) n^a + D_b(n_c) l^a) = -\theta_l n^a - \theta_n l^a, \tag{2.40}$$

where $\theta_l = \beta^{ab} D_a l_b$ and $\theta_n = \beta^{ab} D_a n_b$ are the expansions along l^a and n^a respectively.

Lemma 2.5. *For a tensor, $T^{a_1 \cdots a_A}_{b_1 \cdots b_B}$, tangent to S, the induced Levi-Civita connection is*

$$\mathbb{D}_c T^{a_1 \cdots a_A}_{b_1 \cdots b_B} = \beta^d{}_c \beta^{a_1}{}_{e_1} \cdots \beta^{a_A}{}_{e_A} \beta^{f_1}{}_{b_1} \cdots \beta^{f_B}{}_{b_B} D_d T^{e_1 \cdots e_A}_{f_1 \cdots f_B}. \tag{2.41}$$

Proof. When acting on a function, f, on S, lemma 2.3 implies

$$\begin{aligned}
\mathbb{D}_a\mathbb{D}_b f &= \beta^c{}_a\beta^d{}_b D_c(\beta^e{}_d D_e f) && (2.42)\\
&= \beta^c{}_a\beta^d{}_b D_c D_d f + \beta^c{}_a\beta^d{}_b D_c(\beta^e{}_d) D_e f && (2.43)\\
&= \beta^c{}_a\beta^d{}_b D_c D_d f + \beta^c{}_a\beta^d{}_b (0 + n^e D_c l_d + 0 + l^e D_c n_d + 0) D_e f && (2.44)\\
&= \beta^c{}_a\beta^d{}_b D_c D_d f - \mathbb{K}^c{}_{ab} D_c f && (2.45)
\end{aligned}$$

is symmetric in a and b. Thus, $\mathbb{D}_a$ is torsion-free. Likewise, when acting on the induced metric,

$$\mathbb{D}_a\beta_{bc} = \beta^d{}_a\beta^e{}_b\beta^f{}_c D_d\beta_{ef} = 0 \tag{2.46}$$

since every term from expanding β_{ef} leads to factors like $\beta^a{}_b l^b = \beta^a{}_b n^b = 0$. Therefore, $\mathbb{D}_a$ is also metric-compatible. □

Lemma 2.6. *The NP coefficients*[6]*, μ and ρ, are real. Furthermore, they are related to the expansions along the null normals by $\theta_l = -2\rho$ and $\theta_n = 2\mu$.*

Proof. Since $\mathbb{K}^a{}_{bc}$ is symmetric in the lower two indices,

$$\begin{aligned}
l_a\mathbb{K}^a{}_{bc}m^b\overline{m}^c = l_a\mathbb{K}^a{}_{bc}\overline{m}^b m^c &\iff \beta^d{}_b\beta^e{}_c D_d(l_e)m^b\overline{m}^c = \beta^d{}_b\beta^e{}_c D_d(l_e)\overline{m}^b m^c && (2.47)\\
&\iff m^d\overline{m}^e D_d(l_e) = \overline{m}^d m^e D_d(l_e) && (2.48)\\
&\iff -\bar{\rho} = -\rho. && (2.49)
\end{aligned}$$

Similarly, $n_a\mathbb{K}^a{}_{bc}m^b\overline{m}^c = n_a\mathbb{K}^a{}_{bc}\overline{m}^b m^c \iff m^d\overline{m}^e D_d(n_e) = \overline{m}^d m^e D_d(n_e) \iff \mu = \bar{\mu}$ for μ's reality. For the expansions, if β_{ab} is the induced metric on S, then

$$\theta_l = \beta^{ab}D_a l_b \text{ and } \theta_n = \beta^{ab}D_a n_b. \tag{2.50}$$

Hence, by equation 2.28, $\theta_l = -\bar{\rho} - \rho = -2\rho$ and $\theta_n = \mu + \bar{\mu} = 2\mu$. □

A NP tetrad could be defined without reference to any surface S, i.e. one could merely impose conditions such as $l^a n_a = -1$, l and n are future-directed etc. In this case, μ and ρ need not be real. Instead, their reality is a diagnostic for whether the tetrad could be adapted to some surface, S.

Lemma 2.7. *$\{l, n\}$ is hypersurface orthogonal if and only if both μ and ρ are real.*

Proof. By Frobenius' theorem, l_a and n_a are hypersurface orthogonal if and only if $l \wedge n \wedge \mathrm{d}l$ and $l \wedge n \wedge \mathrm{d}n$ are both zero [116]. Since $\{l, n, m, \overline{m}\}$ forms a pointwise basis and $l \wedge n \wedge \mathrm{d}l$ & $l \wedge n \wedge \mathrm{d}n$ are both 4-forms, they must be proportional to $l \wedge n \wedge m \wedge \overline{m}$ (with potentially zero proportionality function). Hence, it suffices to find the $m \wedge \overline{m}$ component of $\mathrm{d}l$ and $\mathrm{d}n$.

$$\begin{aligned}
\beta^c{}_a\beta^d{}_b(\mathrm{d}l)_{cd} &= (m^c\overline{m}_a + \overline{m}^c m_a)(m^d\overline{m}_b + \overline{m}^d m_b)(D_c l_d - D_d l_c) && (2.51)\\
&= (\bar{\rho} - \rho)(m_a\overline{m}_b - \overline{m}_a m_b) && (2.52)\\
\implies l \wedge n \wedge \mathrm{d}l &= (\bar{\rho} - \rho) l \wedge n \wedge m \wedge \overline{m}. && (2.53)\\
\text{Similarly, } l \wedge n \wedge \mathrm{d}n &= (\bar{\mu} - \mu) l \wedge n \wedge m \wedge \overline{m}. && (2.54)
\end{aligned}$$

Therefore, $l \wedge n \wedge \mathrm{d}l = l \wedge n \wedge \mathrm{d}n = 0 \iff$ both μ and ρ are real. □

[6]See appendix A for the definitions of all the NP coefficients.

The main idea of the Geroch-Held-Penrose (GHP) formalism [48] is to further decompose all the NP quantities into spinorial quantities in a canonical way. Observe that for a null vector, V^a, equation 2.11 implies $\det(V_{\alpha\dot{\alpha}}) = 0$.

Therefore $V_{\alpha\dot{\alpha}}$ is a 2×2, non-zero, rank-1 matrix, meaning the columns of $V_{\alpha\dot{\alpha}}$ must be proportional to each other. Hence, $\exists\, u_\alpha$ and v_α such that

$$V_{\alpha\dot{\alpha}} = u_\alpha \bar{v}_{\dot{\alpha}} \equiv \begin{bmatrix} u_1\bar{v}_{\dot{1}} & u_1\bar{v}_{\dot{2}} \\ u_2\bar{v}_{\dot{1}} & u_2\bar{v}_{\dot{2}} \end{bmatrix}. \tag{2.55}$$

Then, $V_{\alpha\dot{\alpha}}$ is hermitian $\implies u_1\bar{v}_{\dot{1}},\, u_2\bar{v}_{\dot{2}} \in \mathbb{R}$ and $u_1\bar{v}_{\dot{2}} = \bar{u}_{\dot{2}}v_1$. Combining these two properties, $u_1\bar{v}_{\dot{1}} = \bar{u}_{\dot{2}}|v_1|^2/\bar{v}_{\dot{2}}$ and thus $\bar{u}_{\dot{2}}/\bar{v}_{\dot{2}} \in \mathbb{R}$, say c_2 (if v_1 or v_2 is zero then I could immediately write $V_{\alpha\dot{\alpha}} = v_\alpha\bar{v}_{\dot{\alpha}}$ for some v_α with v_1 or v_2 being zero).

Similarly, $u_2\bar{v}_{\dot{2}} = \bar{u}_{\dot{1}}|v_2|^2/\bar{v}_{\dot{1}} \implies \bar{u}_{\dot{1}}/\bar{v}_{\dot{1}} = c_1 \in \mathbb{R}$. So far, that means

$$V_{\alpha\dot{\alpha}} \equiv \begin{bmatrix} c_1|v_1|^2 & c_1v_1\bar{v}_{\dot{2}} \\ c_2v_2\bar{v}_{\dot{1}} & c_2|v_2|^2 \end{bmatrix}. \tag{2.56}$$

Now, $c_2v_2\bar{v}_{\dot{1}} = \overline{(c_1v_1\bar{v}_{\dot{2}})} \implies c_1 = c_2$ and hence $V_{\alpha\dot{\alpha}} = cv_\alpha\bar{v}_{\dot{\alpha}}$ for some two-component spinor, v_α, and some real number, c. Assuming $V^a \neq 0$, $c \neq 0$. Furthermore, if V^a is future-directed and causal, then $V_{1\dot{1}} = V^0 + V^3 \geq 0 \implies c > 0$. Finally, re-defining $\sqrt{c}v_\alpha \to v_\alpha$ gives $V_{\alpha\dot{\alpha}} = v_\alpha\bar{v}_{\dot{\alpha}}$.

Applying all this to the NP tetrad means $\exists o_\alpha, \iota_\alpha$ such that $l_{\alpha\dot{\alpha}} = o_\alpha\bar{o}_{\dot{\alpha}}$ and $n_{\alpha\dot{\alpha}} = \iota_\alpha\bar{\iota}_{\dot{\alpha}}$[7]. Furthermore, $-1 = l^a n_a = -\frac{1}{2}l^{\alpha\dot{\alpha}}n_{\alpha\dot{\alpha}} = -\frac{1}{2}o^\alpha\bar{o}^{\dot{\alpha}}\iota_\alpha\bar{\iota}_{\dot{\alpha}} = -\frac{1}{2}|o^\alpha\iota_\alpha|^2 \implies |o^\alpha\iota_\alpha| = \sqrt{2}$.

In these definitions I still have the freedom to change o_α or ι_α by a phase without changing $l_{\alpha\dot{\alpha}}$ or $n_{\alpha\dot{\alpha}}$. I'll use that to fix $\iota^\alpha o_\alpha = \sqrt{2}$.

As a consequence, o_α and ι_α are pointwise linearly independent and thus form a pointwise basis for two-component spinors. Therefore, any two component spinor, ψ_α, can be decomposed as $\psi_\alpha = \psi_o o_\alpha + \psi_\iota\iota_\alpha$ for some functions, ψ_o and ψ_ι. These functions are determined by

$$o^\alpha\psi_\alpha = o^\alpha\left(\psi_o o_\alpha + \psi_\iota\iota_\alpha\right) = 0 - \sqrt{2}\psi_\iota \text{ and} \tag{2.57}$$

$$\iota^\alpha\psi_\alpha = \iota^\alpha\left(\psi_o o_\alpha + \psi_\iota\iota_\alpha\right) = \sqrt{2}\psi_o + 0. \tag{2.58}$$

Once l^a and n^a are chosen, the choice of m^a is fixed uniquely up to an $SO(2)$ rotation. This freedom matches with the remaining phase freedom left after choosing $\iota^\alpha o_\alpha = \sqrt{2}$. Indeed, any choice/guess that works for $m_{\alpha\dot{\alpha}}$ in terms of o_α and ι_α is good enough.

For example, choosing $m_{\alpha\dot{\alpha}} = \iota_\alpha\bar{o}_{\dot{\alpha}}$ implies $m^a m_a = -\frac{1}{2}m^{\alpha\dot{\alpha}}m_{\alpha\dot{\alpha}} = -\frac{1}{2}\iota^\alpha\bar{o}^{\dot{\alpha}}\iota_\alpha\bar{o}_{\dot{\alpha}} = 0$ and $m^a\bar{m}_a = -\frac{1}{2}m^{\alpha\dot{\alpha}}\bar{m}_{\alpha\dot{\alpha}} = -\frac{1}{2}\iota^\alpha\bar{o}^{\dot{\alpha}}\bar{\iota}_{\dot{\alpha}}o_\alpha = 1$ as required.

In summary, I can make the following definitions.

Definition 2.8 (Spinor dyad, o_α, ι_α, ψ_o, ψ_ι)**.** *When converted to two-component spinors, write the NP tetrad in terms of a spinor dyad, $\{o, \iota\}$, as*

$$l_{\alpha\dot{\alpha}} = o_\alpha\bar{o}_{\dot{\alpha}} \text{ and } n_{\alpha\dot{\alpha}} = \iota_\alpha\bar{\iota}_{\dot{\alpha}} \tag{2.59}$$

with $\iota^\alpha o_\alpha = \sqrt{2}$. Subsequently, decompose any two-component spinor, ψ_α, as

$$\psi_\alpha = \psi_o o_\alpha + \psi_\iota\iota_\alpha \tag{2.60}$$

$$\iff \psi_o = \frac{1}{\sqrt{2}}\iota^\alpha\psi_\alpha \text{ and } \psi_\iota = -\frac{1}{\sqrt{2}}o^\alpha\psi_\alpha. \tag{2.61}$$

[7]Omicron and iota are supposed to be the closest Greek letters to O and I, which in turn stand for outgoing and ingoing. Unfortunately, o and ι are awful letters to use when writing mathematics. Nonetheless, they have become the standard choice in the literature. Given this is typed text and it's possible to distinguish o from 0 and ι from i say, I've chosen to succumb to conformity and make the standard letter choice.

Finally, in terms of the spinor dyad, choose

$$m_{\alpha\dot{\alpha}} = \iota_\alpha \bar{o}_{\dot{\alpha}} \text{ and } \overline{m}_{\alpha\dot{\alpha}} = o_\alpha \bar{\iota}_{\dot{\alpha}}. \tag{2.62}$$

Despite the discussion above, there is still some freedom in this definition. In particular, the scaling freedom, $(l, n) \to (fl, n/f)$ for any non-zero, real function, f, in the NP tetrad still survives the spinor decomposition and corresponds to the scaling freedom $(o, \iota) \to (zo, \iota/z)$ for any non-zero, complex, function, z. Rather than a mere curiosity or gauge nuisance, GHP leverage this to make the following constructions.

Definition 2.9 (GHP type and spin-weighted derivatives)**.** *Let z be an arbitrary, non-zero complex function. Then, a scalar, spinor or tensor, $f_{p,q}$, is called GHP type-(p,q) if and only if $f_{p,q} \to z^p \bar{z}^q f_{p,q}$ when $\bar{o}^{\dot{\alpha}} \to z\bar{o}^{\dot{\alpha}}$ and $\bar{\iota}^{\dot{\alpha}} \to \bar{\iota}^{\dot{\alpha}}/z$. Given such an object*[8]*, define the spin weighted derivatives as*

$$\mathcal{P} f_{p,q} = D f_{p,q} - p\varepsilon f_{p,q} - q\bar{\varepsilon} f_{p,q}, \tag{2.63}$$
$$\mathcal{P}' f_{p,q} = \Delta f_{p,q} - p\gamma f_{p,q} - q\bar{\gamma} f_{p,q}, \tag{2.64}$$
$$\eth f_{p,q} = \delta f_{p,q} - p\beta f_{p,q} - q\bar{\alpha} f_{p,q} \tag{2.65}$$
$$\text{and } \bar{\eth} f_{p,q} = \bar{\delta} f_{p,q} - p\alpha f_{p,q} - q\bar{\beta} f_{p,q}, \tag{2.66}$$

where ϵ, γ, α and β are NP coefficients.

It immediately follows that the spinor dyad itself has definite GHP type, namely o^α is type-$(0,1)$ and ι^α is type-$(0,-1)$. Furthermore, since a spinor, ψ_α is defined independently of any choice of spinor dyad, writing $\psi_\alpha = \psi_o o_\alpha + \psi_\iota \iota_\alpha$ implies ψ_o is type-$(0,-1)$ and ψ_ι is type-$(0,1)$.

Lemma 2.10. *If $\mathbb{D}_a$ is the covariant derivative intrinsic to S, then for any functions, f_1 & f_2,*

$$\mathbb{D}_a(f_1 f_2 m^a) = \delta(f_1) f_2 + f_1 \delta(f_2) + (\beta - \bar{\alpha}) f_1 f_2. \tag{2.67}$$

Proof. By equation 2.41,

$$\mathbb{D}_a(f_1 f_2 m^a) = \beta^b{}_a \beta^a{}_c D_b(f_1 f_2 m^c) \tag{2.68}$$
$$= \beta^{ab} D_a(f_1 f_2 m_b) \tag{2.69}$$
$$= (m^a \overline{m}^b + \overline{m}^a m^b)(m_b f_2 D_a f_1 + m_b f_1 D_a f_2 + f_1 f_2 D_a m_b) \tag{2.70}$$
$$= f_2 \delta f_1 + 0 + f_1 \delta f_2 + 0 + (\beta - \bar{\alpha}) f_1 f_2 + 0, \tag{2.71}$$

which is the claimed result. □

Corollary 2.10.1. *Integration by parts holds for $\eth$ on S in the sense that*

$$\int_S \tilde{f}_{-p-1,-q+1} \eth(f_{p,q}) \mathrm{d}A = -\int_S \eth(\tilde{f}_{-p-1,-q+1}) f_{p,q} \, \mathrm{d}A. \tag{2.72}$$

Note that $\eth$ effectively changes the GHP type of a quantity by $(1,-1)$, so if the GHP types of f and $\tilde{f}$ were not related as in the corollary, then the integrand would be basis dependent and hence not a truly geometric quantity.

[8]Note that not everything has a definite GHP type. For example, the NP coefficients, α, β, γ and ε are not type-(p,q) for any p and q.

Proof. Using the lemma and Stokes' theorem,

$$\int_S \tilde{f}_{-p-1,-q+1}\eth(f_{p,q})\mathrm{d}A = \int_S \tilde{f}_{-p-1,-q+1}(\delta f_{p,q} - p\beta f_{p,q} - q\bar{\alpha} f_{p,q})\mathrm{d}A \tag{2.73}$$

$$= \int_S \big(\mathbb{D}_a(\tilde{f}_{-p-1,-q+1}f_{p,q}m^a) - \delta(\tilde{f}_{-p-1,-q+1})f_{p,q} - (\beta - \bar{\alpha})\tilde{f}_{-p-1,-q+1}f_{p,q} - (p\beta + q\bar{\alpha})\tilde{f}_{-p-1,-q+1}f_{p,q}\big)\mathrm{d}A \tag{2.74}$$

$$= 0 - \int_S \eth(\tilde{f}_{-p-1,-q+1})f_{p,q}\,\mathrm{d}A \tag{2.75}$$

as claimed. □

The corollary means that, in a sense, the natural derivatives to use on a surface, S, are not δ and $\bar{\delta}$, but $\eth$ and $\bar{\eth}$ instead.

2.2 Asymptotically locally AdS spacetimes

Definitions of asymptotics form an essential part of the practical study of general relativity. Without asymptotics, it would be difficult, if not impossible, to define global notions of energy, define black holes or solve the Einstein equation with appropriate boundary conditions. For me, one of the main themes of this thesis will be the study of "asymptotically, locally AdS" spacetimes, which I review now. For the purposes of this section, it will be convenient to choose units/length scales such that $\Lambda = -\frac{1}{2}(n-1)(n-2)$.

Definition 2.11 (Asymptotically, locally AdS)**.** *An n-dimensional spacetime, (M, g), is said to be asymptotically, locally AdS if and only if the following conditions hold. (M, g) must admit a conformal compactification such that conformal infinity, $\mathcal{I}$, is a timelike hypersurface. Then, in an open neighbourhood of $\mathcal{I}$, there must exist coordinates, $(r, x^m) = (r, t, x^\alpha)$, such that $\{r = \infty\}$ is $\mathcal{I}$ itself and g admits a Fefferman-Graham expansion [42],*

$$g = \mathrm{e}^{2r}\left(f_{(0)mn} + \mathrm{e}^{-r}f_{(1)mn} + \cdots + \mathrm{e}^{-(n-1)r}f_{(n-1)mn} + r\mathrm{e}^{-(n-1)r}\tilde{f}_{(n-1)} + \cdots\right)\mathrm{d}x^m \otimes \mathrm{d}x^n + \mathrm{d}r \otimes \mathrm{d}r, \tag{2.76}$$

with $f_{(k)mn}$ and $\tilde{f}_{(k)mn}$ independent of r for any k. The series, $f_{(0)mn} + \mathrm{e}^{-r}f_{(1)mn} + \cdots$, will be denoted f_{mn} (when summed). It will be assumed $\mathcal{I}$ is diffeomeorphic to $\mathbb{R} \times S$ for some $(n-2)D$, spacelike, compact manifold, S. t will be chosen as the coordinate along $\mathbb{R}$ and x^α are coordinates along S. Without loss of generality, the coordinates are chosen[9] so that $f_{(0)0\alpha} = 0$.

The lowest order n terms of f_{mn}, i.e $f_{(0)}$, $\cdots$, $f_{(n-2)}$ and $\tilde{f}_{(n-1)}$, are uniquely determined by $f_{(0)mn}$ and its curvature [42, 36, 110], i.e. specifying $f_{(0)}$ specifies g up to $O\big(\mathrm{e}^{-(n-3)r}\big)$. This is because of a recursive relation that appears when solving the (vacuum, $\Lambda < 0$) Einstein equation order by order in e^{-r}. $f_{(n-1)mn}$ is the first distinguishing term and is often thought of as a "boundary stress tensor" [110], although the procedure I will follow will not require holographic renormalisation like [110].

Typically, $f_{(2k+1)mn} = 0$ and there are further conditions on the trace and divergence of $f_{(n-1)mn}$, but these won't matter for the present discussion.

[9]The coordinates will always be ordered so that t is the 0th coordinate, r is the 1st coordinate and x^α are coordinates 2 to $n-1$. (t, x^α) together wll be denoted x^m with $m, n, \cdots$ running $0, 2, 3, \cdots, n-1$. Meanwhile, the corresponding vielbein indices will be $A, B, \cdots$ and $M, N, \cdots$, which run $2, 3, \cdots n-1$ and $0, 2, 3, \cdots n-1$ respectively.

A striking feature of this expansion is that $f_{(0)}$ - the metric on $\mathcal{I}$ itself - can be freely chosen. A major focus of this work will be exploring the effects of this boundary geometry freedom on positive energy theorems. A formal definition of energy itself in these spacetimes is postponed until section 2.3.

Definition 2.12 (Background metric, $\bar{g}$)**.** *Any asymptotically, locally AdS metric, $\bar{g}$, is said to be a background metric for g when $\bar{f}_{(0)mn} = f_{(0)mn}$.*

Hence, g and $\bar{g}$ can only differ from $O\big(e^{-(n-3)r}\big)$ onwards. The most commonly considered background metric will be AdS itself,

$$\bar{g} = \mathrm{d}r \otimes \mathrm{d}r + e^{2r}\left(-\left(1+\frac{1}{4}e^{-2r}\right)^2 \mathrm{d}t \otimes \mathrm{d}t + \left(1-\frac{1}{4}e^{-2r}\right)^2 g_{S^{n-2}}\right). \tag{2.77}$$

For global coordinates on AdS, both t and r range over all of $\mathbb{R}$ and any coordinates can be chosen on the spheres of constant t and r. In terms of Fefferman-Graham expansions, this has $S = S^{n-2}$ cross-section, $f_{(n-1)} = 0$ and $f_{(0)} = -\mathrm{d}t \otimes \mathrm{d}t + g_{S^{n-2}}$ (unless $n = 5$ in which case $f_{(4)} = \frac{1}{16}(-\mathrm{d}t \otimes \mathrm{d}t + g_{S^3})$).

Note that $f_{(0)}$ in an asymptotically, locally AdS spacetime need not be locally isometric to the $f_{(0)} = -\mathrm{d}t \otimes \mathrm{d}t + g_{S^{n-2}}$ of AdS; it is truly arbitrary. This is unlike the naming of ALE - asymptotically, locally Euclidean - in the study of gravitational instantons, where the boundary geometry is some quotient of Euclidean space. Instead, asymptotically, locally AdS spacetimes get their name from the property [97] that to leading order, in this case $O(e^{4r})$,

$$R_{abcd} = -(g_{ac}g_{bd} - g_{ad}g_{bc}). \tag{2.78}$$

Therefore, to leading order the Riemann tensor appears maximally symmetric with negative curvature constant.

For the purpose of defining energy later, I will also need the following constructions. Let Σ_t be any $(n-1)$D, spacelike hypersurface intersecting $\mathcal{I}$ such that t is a constant on Σ_t in an open neighbourhood of $\mathcal{I}$. Then, let $\Sigma_{t,r}$ be a constant r cross-section of Σ_t and let $\Sigma_{t,\infty} = \Sigma_t \cap \mathcal{I} = S$. Coordinate and vielbein indices in this context will be denoted $i, j, \cdots$ and $I, J, \cdots$ respectively, both running $1, 2, \cdots, n-1$. Furthermore, throughout this thesis, $\int_{\Sigma_{t,\infty}}$ should be interpreted as $\lim_{r\to\infty}\int_{\Sigma_{t,r}}$.

I've illustrated the overall set-up in figure 2.2.

2.3 Energy in general relativity

Quantifying the energy of the gravitational field is a famously subtle problem in general relativity. The toughest obstacles are the equivalence principle and the principle of general covariance themselves - the most fundamental tenets of the theory. In particular, since all bodies fall precisely the same way in a gravitational field and the gravitational field must be determined by nothing more than the spacetime structure, (M, g), it is impossible to detect the gravitational field through an experiment performed at any one given point, $p \in M$. Rather, the effects of gravity manifest themselves via data throughout a local neighbourhood of a point, most famously the (intrinsic) curvature of (M, g). Therefore, it becomes impossible to define any notion of gravitational energy measured at a given point, p. Any notion of energy in general relativity must instead ascribe a number to an extended region within M. In this section, I will consider two contexts in which this program may be attempted. The first is a global definition of energy, i.e. the aforementioned extended region of M is the entirety of M itself. In this case, it must be assumed that (M, g) is non-compact and possesses some notion of a surface at infinity. The second context is a quasilocal definition of energy, where the task is to quantify the energy within some compact region instead.

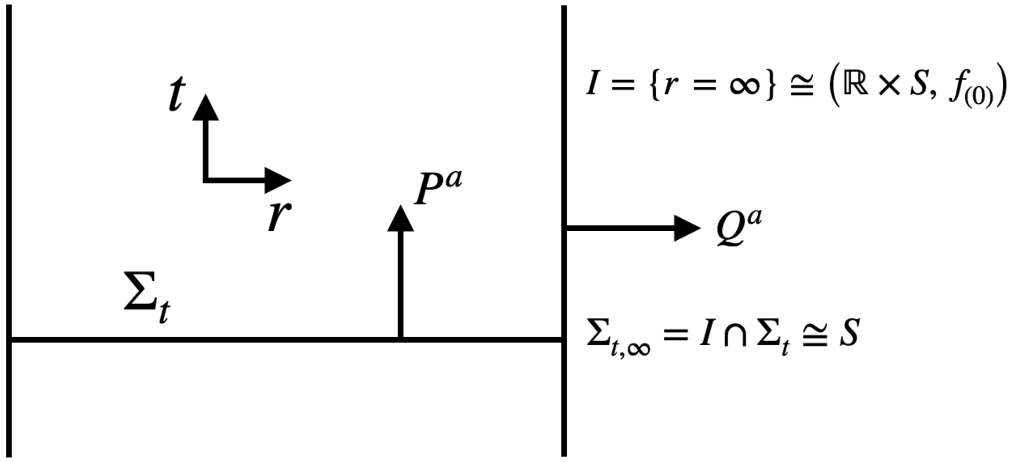

Figure 2.2: This is a Penrose diagram for defining asymptotically, locally AdS spacetimes and energy within them. Conformal infinity, $\mathcal{I}$, is $\{r = \infty\}$, where r is the Fefferman-Graham coordinate. $\mathcal{I}$ is topologically $\mathbb{R} \times S$ with metric $f_{(0)mn}$. t is a coordinate along the $\mathbb{R}$ direction, Σ_t is a spacelike hypersurface which has constant t near $\mathcal{I}$, P^a is the future-directed, timelike, unit normal to Σ_t and Q^a is the outward, unit normal to constant r surfaces near $\mathcal{I}$. Energy is measured at $\Sigma_{t,\infty} = \mathcal{I} \cap \Sigma_t$.

2.3.1 Global definitions

General relativity can be viewed as a field theory for the metric, g, defined by the Einstein-Hilbert action,

$$S = \frac{1}{16\pi} \int_M R \,\mathrm{d}\mu(g). \tag{2.79}$$

As such, the most natural global definition of energy one might attempt is to find a Hamiltonian formulation for the theory and then define energy to be the on-shell value of the Hamiltonian. I will review this construction, following [2, 106, 58, 104].

For a Hamiltonian formulation, one needs a spacelike hypersurface, Σ_t, with is loosely interpreted as a "constant time" slice. Given a Σ_t, it's natural to introduce an $(n-1)+1$ split,

$$g = -N^2 \mathrm{d}t \otimes \mathrm{d}t + h_{ij}(\mathrm{d}x^i + N^i \mathrm{d}t) \otimes (\mathrm{d}x^j + N^j \mathrm{d}t). \tag{2.80}$$

Now, S can be viewed as a functional of h_{ij}, N^i and N. Choose the orthogonal, but not normalised, basis[10],

$$e^0 = \mathrm{d}t \;\text{ and }\; e^i = \mathrm{d}x^i + N^i \mathrm{d}t. \tag{2.81}$$

The change of basis matrix is therefore

$$e^\mu = \mathrm{d}x^\nu B_\nu{}^\mu \;\text{ with }\; B_\nu{}^\mu = \begin{bmatrix} 1 & N^i \\ 0 & \delta_j{}^i \end{bmatrix} \;\text{ and} \tag{2.82}$$

$$g = -N^2 e^0 \otimes e^0 + h_{ij} e^i \otimes e^j. \tag{2.83}$$

Since $\det(B) = 1$, $\mathrm{d}\mu(g) = N\sqrt{h}\,\mathrm{d}^{n-1}x\,\mathrm{d}t$. Furthermore, the unit normal to Σ_t is

$$P_\mu \equiv -N \mathrm{d}t \iff P^\mu \equiv \frac{1}{N}\partial_t - \frac{1}{N} N^i \partial_i \tag{2.84}$$

[10]Unfortunately, there is some index duplication here, but the meaning of each equation should be apparent from the context.

and the induced metric is

$$h_{\mu\nu} = g_{\mu\nu} + P_\mu P_\nu = h_{ij} e^i \otimes e^j. \tag{2.85}$$

Then, it can be checked by direct calculation that Σ_t's extrinsic curvature is

$$K_{\mu\nu} e^\mu \otimes e^\nu = \frac{1}{2} \left(\mathcal{L}_P h\right)_{\mu\nu} e^\mu \otimes e^\nu = K_{ij} e^i \otimes e^j \tag{2.86}$$

$$\text{where } K_{ij} = \frac{1}{2N} \left(\partial_t h_{ij} - D_i^{(h)} N_j - D_j^{(h)} N_i\right) \tag{2.87}$$

and the $i, j, \cdots$ indices are raised and lower by h. By the Gauss-Codazzi equations, the spacetime and hypersurface Ricci scalars are related by

$$R = R^{(h)} - 2R_{\mu\nu} P^\mu P^\nu + K^2 - K^{\mu\nu} K_{\mu\nu} = R^{(h)} - 2R_{\mu\nu} P^\mu P^\nu + K^2 - K^{ij} K_{ij}. \tag{2.88}$$

The Ricci tensor can be eliminated as follows.

$$R_{\mu\nu} P^\mu P^\nu = R^\rho{}_{\mu\rho\nu} P^\mu P^\nu \tag{2.89}$$

$$= P^\nu [D_\rho, D_\nu] P^\rho \tag{2.90}$$

$$= D_\rho(P^\nu D_\nu P^\rho) - D_\nu(P^\nu D_\rho P^\rho) - D_\rho(P^\nu) D_\nu(P^\rho) + D_\nu(P^\nu) D_\rho(P^\rho), \tag{2.91}$$

$$K^2 = h^{\mu\nu} K_{\mu\nu} = h^{\mu\nu} h_\mu{}^\rho D_\rho P_\nu = h^{\mu\nu} D_\mu P_\nu = D_\mu P^\mu \text{ and} \tag{2.92}$$

$$K^{\mu\nu} K_{\mu\nu} = h^{\mu\rho} D_\rho(P^\nu) h_{\nu\sigma} D^\sigma(P_\mu) = D^\mu(P^\nu) D_\nu(P_\mu) + 0 \text{ imply} \tag{2.93}$$

$$R = R^{(h)} + K^{ij} K_{ij} - K^2 - 2D_\rho(P^\nu D_\nu P^\rho) + 2D_\nu(P^\nu D_\rho P^\rho). \tag{2.94}$$

Therefore, ignoring total derivatives, the Einstein-Hilbert action in these coordinates is

$$S = \frac{1}{16\pi} \int \int_{\Sigma_t} \left(R^{(h)} + K^{ij} K_{ij} - K^2\right) N\sqrt{h}\, \mathrm{d}^{n-1}x\, \mathrm{d}t. \tag{2.95}$$

Since $\partial_t N$ and $\partial_t N^i$ do not appear in the action, they are auxiliary variables and the conjugate momentum to h_{ij} is

$$p_{ij} = \sqrt{h}(K_{ij} - K h_{ij}). \tag{2.96}$$

Hence, swapping K_{ij} for p_{ij} everywhere, the Hamiltonian is

$$H = \frac{1}{16\pi} \int_{\Sigma_t} \left(p^{ij} \partial_t(h_{ij}) - \mathcal{L}\right) \mathrm{d}^{n-1}x \tag{2.97}$$

$$= \frac{1}{16\pi} \int_{\Sigma_t} \left(N \left(\frac{1}{h} p^{ij} p_{ij} - \frac{1}{(n-2)h} p^2 - R^{(h)} + 2\Lambda \right) - 2N^i D^{(h)j} \left(\frac{1}{\sqrt{h}} p_{ij} \right) + D_i^{(h)} \left(2p^{ij} N_j \frac{1}{\sqrt{h}} \right) \right) \mathrm{d}V. \tag{2.98}$$

Again, ignoring the boundary term finally yields

$$H = \frac{1}{16\pi} \int_{\Sigma_t} \left(N \left(\frac{1}{h} p^{ij} p_{ij} - \frac{1}{(n-2)h} p^2 - R^{(h)} + 2\Lambda \right) - 2N^i D^{(h)j} \left(\frac{1}{\sqrt{h}} p_{ij} \right) \right) \mathrm{d}V. \tag{2.99}$$

Due to the constraint equations,

$$R^{(h)} - 2\Lambda - K^{ij} K_{ij} + K^2 = 0 \text{ and } D_j^{(h)} K^j{}_i - D_i^{(h)} K = 0 \tag{2.100}$$

in vacuum, the bulk contribution to the on-shell value of the Hamiltonian is thus simply zero. Therefore, the actual value of the on-shell Hamiltonian comes completely from the boundary term. This boundary term is fixed by ensuring the action and Hamiltonian have a well-defined variational principle.

Suppose Σ_t has an asymptotic end with boundary at infinity called $\Sigma_{t,\infty}$. First consider $\delta_p H$. The first two terms in equation 2.99 have no derivatives on p_{ij} and therefore don't contribute any boundary terms. The only boundary terms comes from the last term, leading to

$$\delta_p H = \text{bulk terms} - \frac{1}{8\pi}\int_{\Sigma_{t,\infty}} Q^j N^i \frac{1}{\sqrt{h}}\delta(p_{ij})\mathrm{d}A, \tag{2.101}$$

where Q^i is the unit normal to $\Sigma_{t,\infty}$ within Σ_t.

Next consider $\delta_h H$. It's well known that the variation of the Ricci scalar is [116]

$$\delta R^{(h)} = -R^{(h)ij}\delta h_{ij} + D^{(h)i}D^{(h)j}\delta h_{ij} - D^{(h)k}D^{(h)}_k(h^{ij}\delta h_{ij}). \tag{2.102}$$

Boundary terms will arise from each of the derivatives in the second and third terms. Thus,

$$\begin{aligned}\int_{\Sigma_t} N\delta(R^{(h)})\mathrm{d}V = \int_{\Sigma_{t,\infty}} Q^i \Big(& N(D^{(h)j}\delta h_{ij} - D^{(h)}_i(h^{jk}\delta h_{jk})) \\ & - D^{(h)j}(N)\delta h_{ij} + D^{(h)}_i(N)h^{jk}\delta h_{jk}\Big)\mathrm{d}A + \text{bulk terms}.\end{aligned} \tag{2.103}$$

The only other way to get boundary terms in $\delta_h H$ is through the last term in equation 2.99. There is a $\delta(1/\sqrt{h})$ acted on by $D^{(h)j}$, but boundary terms also arise from varying the Christoffel symbols in $D^{(h)j}$ itself. In particular,

$$D^{(h)}_j\left(\frac{1}{\sqrt{h}}p_i{}^j\right) = \partial_j\left(\frac{1}{\sqrt{h}}p_i{}^j\right) + \frac{1}{\sqrt{h}}\left(\Gamma^{(h)j}{}_{kj}p_i{}^k - \Gamma^{(h)k}{}_{ij}p_k{}^j\right), \tag{2.104}$$

$$\delta\Gamma^{(h)i}{}_{jk} = \frac{1}{2}h^{il}\left(D^{(h)}_j\delta h_{kl} + D^{(h)}_k\delta h_{lj} - D^{(h)}_l\delta h_{jk}\right) \quad \text{and} \tag{2.105}$$

$$\delta\Gamma^{(h)j}{}_{ij} = \frac{1}{2}h^{jk}D^{(h)}_i\delta h_{jk} \tag{2.106}$$

leads to

$$\begin{aligned}\int_{\Sigma_t} N^i\delta_h\left(D^{(h)j}\left(\frac{1}{\sqrt{h}}p_{ij}\right)\right)\mathrm{d}V = \text{bulk terms} + \int_{\Sigma_{t,\infty}}\Bigg(& Q^j\delta\left(\frac{1}{\sqrt{h}}\right)p_{ij} + \frac{N^i}{2\sqrt{h}}(h^{jl}\delta(h_{jl})Q_k p_i{}^k \\ & - h^{kl}(Q_i\delta h_{jl} + Q_j\delta h_{li} - Q_l\delta h_{ij})p_k{}^j)\Bigg)\mathrm{d}A.\end{aligned} \tag{2.107}$$

Putting it all together,

$$16\pi\delta H = 16\pi\delta_p H + 16\pi\delta_h H \tag{2.108}$$

$$\begin{aligned} &= -\int_{\partial_\infty\Sigma_t} NQ^i\left(D^{(h)j}\delta h_{ij} - D^{(h)}_i(h^{jk}\delta h_{jk})\right)\mathrm{d}A - 2\int_{\partial_\infty\Sigma_t} Q^i N^j\delta\left(K_{ij} - Kh_{ij}\right)\mathrm{d}A \\ &\quad + \int_{\partial_\infty\Sigma_t} Q^i\left(D^{(h)j}(N)\delta h_{ij} - h^{jk}\delta h_{jk}D^{(h)}_i N\right)\mathrm{d}A \\ &\quad + \int_{\partial_\infty\Sigma_t} N^i\frac{1}{\sqrt{h}}\delta(h_{jk})\left(Q_i p^{jk} - p_{il}Q^l h^{jk}\right)\mathrm{d}A + \text{bulk terms}.\end{aligned} \tag{2.109}$$

To make further progress, one needs to choose asymptotics so the integrals in δH can be evaluated more precisely. I'll begin with the simplest - and perhaps physically most typical - case, namely when Σ_t is part of asymptotically flat initial data.

Definition 2.13 (Asymptotically flat initial data)**.** *A triple,* (Σ, h, K) *is called asymptotically flat initial data if and only if the following conditions hold.*

- $\exists$ *compact* $C \subseteq \Sigma$ *such that* $\Sigma \backslash C$ *is diffeomorphic to* $\mathbb{R}^{n-1} \backslash B$*, where* B *is a closed ball.*
- *Let* x^i *be local coordinates on* $\Sigma \backslash C$ *pulled back along the diffeomorphism from Cartesian coordinates on* $\mathbb{R}^{n-1} \backslash B$*. Then, on* $\Sigma \backslash C$*,*

$$h_{ij} = \delta_{ij} + \tilde{h}_{ij} \tag{2.110}$$

  for $\tilde{h}_{ij} = O(r^{-\tau})$*,* $r = \sqrt{x^i x_i}$ *and* $\tau > (n-3)/2$*.*
- $\partial_i \tilde{h}_{jk}$ *and* K_{ij} *are* $O(r^{-\tau-1})$*.*
- *The constraint equations (written in veilbein indices with* $P^a = \delta^{a0}$*) hold, namely*

$$R^{(h)} - K^{IJ} K_{IJ} + K^2 = 16\pi T_{00} \tag{2.111}$$

$$\text{and} \quad D^{(h)}_J K^J{}_I - D^{(h)}_I K = 8\pi T_{I0}. \tag{2.112}$$

Consider equation 2.109 with these particular asymptotics. I am effectively considering variations about Euclidean space with variations of a given decay rate. For the purposes of this derivation, I will assume the stronger condition, $\tau \geq n-3$. Therefore,

$$\delta h_{ij} = O(r^{-\tau}), \;\; \delta K_{ij} = O(r^{-\tau-1}), \;\; N = 1 + O(r^{-\tau}) \;\text{ and }\; N^i = O(r^{-\tau}). \tag{2.113}$$

Furthermore, to leading order, $\mathrm{d}A = r^{n-2} \mathrm{d}^{n-2}\theta$, $Q^i = x^i/r$, covariant derivatives become partial derivatives and I can write all indices downstairs. Hence, with the decay rates assumed only the first two terms in equation 2.109 decay slowly enough to have non-zero integrals, leaving

$$16\pi \delta H = -\int_{S^{n-2}_\infty} Q_i \left(\partial_j \delta h_{ij} - \partial_i (\delta^{jk} \delta h_{jk})\right) \mathrm{d}A = -\delta \left(\int_{S^{n-2}_\infty} Q_i \left(\partial_j h_{ij} - \partial_i h_{jj}\right) \mathrm{d}A \right). \tag{2.114}$$

To have a well-defined variational principle, δH should be zero on-shell. Therefore, an extra term should be added to the Hamiltonian to cancel this variation. Since the RHS of equation 2.99 is just zero on-shell, this extra term is the on-shell value of the Hamiltonian and can be interpreted as the energy.

The N and N^i chosen in this calculation imply $P^a \to \partial_t$, i.e. an asymptotic time translation. However, heuristically, suppose P^a approached an asymptotic space translation. Then, any associated quantity could be interpreted as linear momentum. From equation 2.84, this occurs when N^i dominates N. In that case, only the second integral in equation 2.109 would survive.

In summary, I have sought to motivate the following definition.

Definition 2.14 (ADM energy and linear momentum [2, 6, 26, 27])**.** *Let* (Σ, h, K) *be an asymptotically flat initial data set with* T_{00} $T_{I0} \in L^1(\Sigma)$*. Then, the ADM energy and linear momentum are geometric invariants defined as*

$$E_{ADM} = \frac{1}{16\pi} \int_{S^{n-2}_\infty} Q_i \left(\partial_j h_{ij} - \partial_i h_{jj}\right) \mathrm{d}A \tag{2.115}$$

$$\text{and} \quad \mathbb{P}^i_{ADM} = \frac{1}{8\pi} \int_{S^{n-2}_\infty} \left(K_{ij} Q_j - K Q_i\right) \mathrm{d}A. \tag{2.116}$$

Note that although these integrals are only manifestly convergent when $\tau \geq n-3$, they actually remain finite and unchanged even when $(n-3)/2 < \tau < n-3$ [6, 26].

One of the most remarkable results in general relativity is that this energy is bounded from below.

Theorem 2.15 (Positive energy theorem [107, 108, 125]). *Let (Σ, h, K) be an asymptotically flat initial data set satisfying the dominant energy condition. Then,*

$$E_{ADM} \geq \sqrt{\mathbb{P}^i_{ADM}\mathbb{P}^i_{ADM}}. \tag{2.117}$$

The positive energy theorem was proven by Schoen & Yau [107, 108] and Witten [125] nearly simultaneously, but using strikingly different approaches. While the former relied on minimal surface techniques, the latter made a surprising - and ultimately quite elementary - use of spinors[11], the main ideas of which I sketch in section 2.4. The overarching focus of this thesis will be applying Witten's methods to new contexts. By now, this is a well-trodden path in mathematical relativity. Witten's method is motivated by some insights from supergravity [66] and this suggested some immediate extensions. By viewing it as the bosonic sector of $\mathcal{N} = 2$ supergravity, it was soon proven [51] that 4D Einstein-Maxwell theory satisfies the stronger positive energy theorem,

$$E_{ADM} \geq \sqrt{\mathbb{P}^i_{ADM}\mathbb{P}^i_{ADM} + q_e^2 + q_m^2}, \tag{2.118}$$

where q_e and q_m are the electric and magnetic charges of the spacetime. It was similarly shown that in 5D Einstein-Maxwell-Chern-Simons theory [53],

$$E_{ADM} \geq \sqrt{\mathbb{P}^i_{ADM}\mathbb{P}^i_{ADM} + \frac{3}{4}q_e^2}. \tag{2.119}$$

In between these results, Witten's method was also extended to include the case where Σ has an inner boundary representing an apparent horizon [50].

I will now repeat the analysis of equation 2.109, but for the asymptotically, locally AdS spacetimes of definition 2.11. Since $f_{(1)}, f_{(2)}, \cdots, f_{(n-2)}$ and $\tilde{f}_{(n-1)}$ are determined by $f_{(0)}$ and the Fefferman-Graham expansion always includes an exact $\mathrm{d}r \otimes \mathrm{d}r$ factor, I should fix a background metric and let

$$\delta g = \mathrm{e}^{2r}\left(\mathrm{e}^{-(n-1)r}\delta f_{(n-1)mn} + O(\mathrm{e}^{-nr})\right)\mathrm{d}x^m \otimes \mathrm{d}x^n. \tag{2.120}$$

Furthermore, in Fefferman-Graham coordinates, $Q^i = \delta^{i1}$. Hence, equation 2.109 becomes

$$\begin{aligned}16\pi\delta H = &-\int_{\Sigma_{t,\infty}} N\left(D^{(h)i}\delta h_{1i} - D^{(h)}_1(h^{ij}\delta h_{ij})\right)\mathrm{d}A - 2\int_{\Sigma_{t,\infty}} N^i\delta\left(K_{1i} - Kh_{1i}\right)\mathrm{d}A \\ &+ \int_{\Sigma_{t,\infty}}\left(D^{(h)i}(N)\delta h_{1i} - h^{ij}\delta h_{ij}D^{(h)}_1 N\right)\mathrm{d}A \\ &+ \int_{\Sigma_{t,\infty}}\frac{1}{\sqrt{h}}\left(\delta(h_{ij})N_1 p^{ij} - \delta(h_{ij})N^k p_{k1}h^{ij}\right)\mathrm{d}A.\end{aligned} \tag{2.121}$$

Comparing equations 2.80 and 2.76, it immediately follows that

$$N_i = \mathrm{e}^{2r}\delta^\alpha{}_i f_{0\alpha}, \quad -N^2 + N^i N_i = \mathrm{e}^{2r} f_{00} \text{ and} \tag{2.122}$$

$$h_{ij} = \delta_{i1}\delta_{j1} + \mathrm{e}^{2r}\delta^\alpha{}_i\delta^\beta{}_j f_{\alpha\beta} \equiv \begin{bmatrix}1 & 0 \\ 0 & \mathrm{e}^{2r}f_{\alpha\beta}\end{bmatrix}. \tag{2.123}$$

Since $N_1 = 0$, the seventh term of equation 2.121 is automatically zero. Likewise, from equation 2.120, $\delta h_{1i} = 0$ too, so the fourth and fifth term of 2.121 are zero too.

[11]Note that Witten's approach naturally only works if the manifold is spin, while Schoen & Yau's approach is only agreed to work for $n \leq 8$ due to singularities which can occur for minimal surfaces in higher dimensions.

$\mathrm{d}A = \mathrm{e}^{(n-2)r}\sqrt{\det(f_{(0)\alpha\beta})}\,\mathrm{d}^{n-2}x$ to leading order, so for the remaining terms in equation 2.121, it suffices to keep terms only to $O(\mathrm{e}^{-(n-2)r})$. Since I've chosen $f_{(0)0\alpha} = 0$ in definition 2.11, $N_i = O(\mathrm{e}^r)$ and $N = \mathrm{e}^r\sqrt{-f_{00}} + O(1)$.

Let $f^{\alpha\beta}_{(0)}$ be the inverse metric to $f_{(0)\alpha\beta}$. Then, to leading order

$$ND^{(h)}_1(h^{ij}\delta h_{ij}) = \mathrm{e}^r\sqrt{-f_{(0)00}}\partial_r\left(\mathrm{e}^{-2r}f^{\alpha\beta}_{(0)}\mathrm{e}^{-(n-3)r}\delta f_{(n-1)\alpha\beta}\right) \tag{2.124}$$

$$= -(n-1)\mathrm{e}^{-(n-2)r}\sqrt{-f_{(0)00}}f^{\alpha\beta}_{(0)}\delta f_{(n-1)\alpha\beta}, \tag{2.125}$$

$$h^{ij}\delta h_{ij}D^{(h)}_1 N = \mathrm{e}^{-(n-2)r}\sqrt{-f_{(0)00}}f^{\alpha\beta}_{(0)}\delta f_{(n-1)\alpha\beta} \text{ and} \tag{2.126}$$

$$ND^{(h)i}\delta h_{1i} = Nh^{ij}\left(\partial_j\delta h_{1i} - \Gamma^{(h)k}{}_{1j}\delta h_{ki} - \Gamma^{(h)k}{}_{ij}\delta h_{1k}\right) \tag{2.127}$$

$$= 0 - \frac{1}{2}Nh^{ij}h^{kl}(\partial_r h_{jl} + \partial_j h_{l1} - \partial_l h_{1j})\delta h_{ki} - 0 \tag{2.128}$$

$$= -\frac{1}{2}\sqrt{-f_{(0)00}}\mathrm{e}^{-r}f^{\alpha\beta}_{(0)}f^{\gamma\delta}_{(0)}\partial_r(\mathrm{e}^{2r}f_{(0)\beta\delta})\delta f_{(n-1)\gamma\alpha} + 0 - 0 \tag{2.129}$$

$$= -\mathrm{e}^{-(n-2)r}\sqrt{-f_{(0)00}}f^{\alpha\beta}_{(0)}\delta f_{(n-1)\alpha\beta}. \tag{2.130}$$

It remains to analyse the terms with extrinsic curvature in equation 2.121. For that, I'll need

$$K_{\alpha 1} = \frac{1}{2N}(\partial_t h_{\alpha 1} - D^{(h)}_\alpha N_1 - D^{(h)}_1 N_\alpha) \tag{2.131}$$

$$= 0 - \frac{1}{2N}\left(\partial_\alpha N_1 + \partial_r N_\alpha - 2\Gamma^{(h)i}{}_{\alpha 1}N_i\right) \tag{2.132}$$

$$= 0 - \frac{1}{2N}(\partial_r N_\alpha - 2\Gamma^{(h)\beta}{}_{\alpha 1}N_\beta) \tag{2.133}$$

$$= -\frac{1}{2N}(\partial_r N_\alpha - h^{\beta\gamma}\partial_r(h_{\alpha\gamma})N_\beta). \tag{2.134}$$

Since $N_\alpha = O(\mathrm{e}^r)$, to leading order $K_{\alpha 1}$ is $O(1)$ to leading order. Hence, for the eighth term in equation 2.121, $\frac{1}{\sqrt{h}}\delta(h_{ij})N^k p_{k1}h^{ij} = \delta(h_{ij})N^\alpha(K_{\alpha 1} + 0)h^{ij}$ is $O(\mathrm{e}^{-nr})$, which decays too quickly to give a non-zero integral against the $O(\mathrm{e}^{(n-2)r})$ measure. Thus, only the third term of equation 2.121 remains to be analysed, for which I need the variation of $K_{\alpha 1}$.

$$\delta K_{\alpha 1} = \frac{1}{2N^2}\delta(N)(\partial_r N_\alpha - h^{\beta\gamma}\partial_r(h_{\alpha\gamma})N_\beta) - \frac{1}{2N}\delta(\partial_r N_\alpha - h^{\beta\gamma}\partial_r(h_{\alpha\gamma})N_\beta). \tag{2.135}$$

To leading order,

$$\partial_r N_\alpha - h^{\beta\gamma}\partial_r(h_{\alpha\gamma})N_\beta = \partial_r(\mathrm{e}^r f_{(1)0\alpha}) - \mathrm{e}^{-2r}f^{\beta\gamma}_{(0)}\partial_r(\mathrm{e}^{2r}f_{(0)\alpha\gamma})\mathrm{e}^r f_{(1)0\beta} = -\mathrm{e}^r f_{(1)0\alpha}. \tag{2.136}$$

Since N is $O(\mathrm{e}^r)$, N^i is $O(\mathrm{e}^{-r})$ and δN is $\mathrm{e}^r O(\delta f) = O(\mathrm{e}^{-(n-2)r})$, it then follows that

$$N^\alpha\frac{1}{2N^2}\delta(N)(\partial_r N_\alpha - h^{\beta\gamma}\partial_r(h_{\alpha\gamma})N_\beta) = O(\mathrm{e}^{-(n+1)r}), \tag{2.137}$$

which decays too quickly to give a non-zero integral. Similarly, since $\delta(\partial_r N_\alpha - h^{\beta\gamma}\partial_r(h_{\alpha\gamma})N_\beta)$ is $O(\mathrm{e}^{-(n-3)r})$,

$$N^\alpha\frac{1}{2N}\delta(\partial_r N_\alpha - h^{\beta\gamma}\partial_r(h_{\alpha\gamma})N_\beta) = O(\mathrm{e}^{-(n-1)r}), \tag{2.138}$$

which again decays too quickly. Therefore ultimately, equation 2.121 reduces to saying

$$\delta H = -\frac{n-1}{16\pi}\int_{\Sigma_{t,\infty}} f^{\alpha\beta}_{(0)}\delta(f_{(n-1)\alpha\beta})\sqrt{-f_{(0)00}}\sqrt{\det(f_{(0)\alpha\beta})}\,\mathrm{d}^{n-2}x. \tag{2.139}$$

I'm viewing δg as a perturbation about some background metric, i.e. $\delta f_{(n-1)} = f_{(n-1)} - \bar{f}_{(n-1)}$. In effect, $-\bar{f}_{(n-1)}$ is an integration constant when integrating δH. Following the logic in the asymptotically flat case, it would appear that the energy should therefore be defined as

$$E = \frac{n-1}{16\pi} \int_{\Sigma_{t,\infty}} f_{(0)}^{\alpha\beta} (f_{(n-1)\alpha\beta} - \bar{f}_{(n-1)\alpha\beta}) \sqrt{-f_{(0)00}} \sqrt{\det(f_{(0)\alpha\beta})} \, \mathrm{d}^{n-2}x. \tag{2.140}$$

However, this definition is slightly odd because of the lapse term, $\sqrt{-f_{(0)00}}$, which is typically pure gauge. The resolution is that there is still some gauge freedom. As I'll explain in more detail in section 4.2.1, the boundary metric, $f_{(0)mn}$, can be changed by an arbitrary conformal transformation by choosing an alternative conformal compactification of (M, g). Therefore, there is a distinguished conformal class representative in which $f_{(0)00} = -1$. Alternatively, given any conformal frame, one could instead choose t through Gaussian normal coordinates[12]. Then, $f_{(0)00} = -1$, while still having $f_{(0)0\alpha} = 0$. I could have gauge-fixed fully from the start, but I chose to fix only $f_{(0)0\alpha} = 0$ because that's all that's really needed for a natural definition of Σ_t and to illustrate that different gauge choices lead to different expressions for the energy[13].

Now, barring the choice of "integration constant," $-f_{(0)}^{\alpha\beta} \bar{f}_{(n-1)\alpha\beta}$, this definition matches the results derived using holographic renormalisation [36, 110] in the AdS/CFT literature[14].

In summary, I have derived the following definition of energy in asymptotically, locally AdS spacetimes, which I will use throughout this thesis.

Definition 2.16 (Energy)**.** *In asymptotically, locally AdS spacetimes, the energy relative to a background metric is defined to be*

$$E = \frac{n-1}{16\pi} \int_{\Sigma_{t,\infty}} \hat{f}_{(0)}^{mn} (f_{(n-1)mn} - \bar{f}_{(n-1)mn}) \sqrt{\det(f_{(0)\alpha\beta})} \, \mathrm{d}^{n-2}x, \tag{2.141}$$

where $\hat{f}_{(0)}^{mn} = f_{(0)}^{mn} + P_{(0)}^m P_{(0)}^n$ *is the induced (inverse) metric on* $\Sigma_{t,\infty}$ *times* e^{2r} *and* $P_{(0)}^m$ *is the unit normal to such cross-sections*[15]*.*

Once again, a series of positive energy theorems is known in this context; the negative cosmological constant is accommodated by taking inspiration from gauged supergravity. These theorems have largely been for round sphere cross-sections, i.e. $f_{(0)mn} = -\mathrm{d}t \otimes \mathrm{d}t + g_{S^{n-2}}$. The first of these results [50, 52] appeared even before the concept of Fefferman-Graham expansions, instead relying on the Abbott-Deser definition [1] of energy & asymptotics. More rigorous definitions of energy in asymptotically AdS spacetimes were subsequently given by [121, 29, 31], along with the associated positive energy theorems. So far, the most comprehensive results in the literature are due to [30, 23]. One of the main aims of this thesis will be to build on those results by trying to better quantify how the positive energy theorem changes with changes to the boundary geometry. Like in the $\Lambda = 0$ case, there are also several results for 4D Einstein-Maxwell theory and 5D Einstein-Maxwell-Chern-Simons theory with negative cosmological constant [80, 76, 123, 95]. All of these papers suffer from some flaw or the other in their treatment of magnetic fields; I will discuss them further in section 4.4.

[12] To construct these coordinates, choose a cross-section, $\Sigma_{t,\infty}$, extend the timelike normal, $P_{(0)}^m$, off $\Sigma_{t,\infty}$ by following timelike geodesics and let t be the affine paramenter along the geodesics. Then, it's possible to show the metric must be $f_{(0)mn} = -\mathrm{d}t \otimes \mathrm{d}t + f_{(0)\alpha\beta}(t, x) \, \mathrm{d}x^\alpha \otimes \mathrm{d}x^\beta$.

[13] I could have gone further and left $f_{(0)mn}$ completely arbitrary, but the resulting expression for E turns out to be an impenetrable mess.

[14] See [63] for a wider review on different approaches to defining energy in asymptotically AdS spacetimes.

[15] The e^r factor scalings effectively just remove all e^r factors on the boundary, as would happen in the conformal compactification. Similarly, note that $\sqrt{\det(f_{(0)\alpha\beta})} \, \mathrm{d}^{n-2}x$ is simply the measure/volume form on $\Sigma_{t,\infty}$ after compactification.

2.3.2 Quasilocal definitions

As I've reviewed in the last subsection, in many cases of physical interest, there are canonical, physically well-motivated definitions for the total energy of a spacetime. However, if this was all one could do in general relativity, it would be a deeply unsettling state of affairs. After all, the real Universe is not some homogeneous soup; it has localised areas of substructure such as stars, galaxies or even a donkey in a paddock. While the equivalence principle precludes defining the gravitational energy at any single point, can general relativity still quantify the energy in some subregion of the full spacetime? Or is such a quest doomed to fail at its outset given the existence of extended structures such as gravitational waves? I will not attempt to arbitrate over such philosophical questions in this thesis. Instead, I will take the perspective that we may as well try our best to localise gravitational energy and see how far we get. Scaling this mountain will be unforgiving enough in four spacetime dimensions and hence I will assume $n = 4$ throughout the discussion of quasilocal masses in this thesis.

Therefore ultimately, the task is as follows. Given the set-up in figure 2.1, where Σ is a compact, spacelike, 3D hypersurface with boundary, S, can one define a geometric invariant on S which quantifies the total mass - gravitational or otherwise - contained in Σ, the region bounded by S? Such a geometric invariant is then called the quasilocal mass, $m(S)$, of the surface, S. For the purposes of this subsection, I will also assume $\Lambda = 0$ and that S is topologically a sphere. It will be very convenient in what follows to deploy the NP and GHP formalisms developed in section 2.1. Of course, in mathematics, one can make definitions arbitrarily. The true test of any definition is its utility and any physical meaning which can be ascribed to it. Hence, it will only be possible to distinguish one geometric invariant as mass and another as not based on whether it behaves as a mass should. For a good definition of quasilocal mass, at least the following quantitative properties should hold.

I. $m(S) \geq 0$.

II. $m(S) = 0$ for any surface, S, in Minkowski space.

III. $m(S)$ agrees with the Misner-Sharp mass [89] in spherical symmetry.

IV. $m(S)$ should asymptote to m_{ADM} as S approaches S^2_∞ in an asymptotically flat end.

V. For perturbations of Minkowski space sourced by an infinitesimal energy-momentum tensor, $m(S)$ should agree with a reasonable notion of mass built from that energy-momentum tensor, e.g. $m(S) = \sqrt{-\mathbb{P}^a\mathbb{P}_a}$ where $\mathbb{P}^a = -\int_\Sigma T^a{}_b P^b \mathrm{d}V$.

VI. For a sphere of length scale, r, as $r \to 0$, $m(S)$ approaches $\frac{4}{3}\pi r^3 T_{ab}P^aP^b$ in non-vacuum and an $O(r^5)$ quantity built from the Bel-Robinson tensor in vacuum.

Ideally, it would also have the following qualitative properties.

A. It should be possible to define $m(S)$ for a suitably generic S that is not either (marginally) trapped or anti-trapped.

B. $m(S)$ should be physically well-motivated.

C. $m(S)$ should have a simple, quasilocal expression.

I'll briefly recount some of the most popular definitions in the literature - see [112] for a much more comprehensive review.

As a stepping stone towards the more sophisticated definitions, first consider the simpler case of spherical symmetry. In vacuum gravity, by Birkhoff's theorem, the unique solution is the Schwarzschild metric,

$$g = -\left(1 - \frac{2M}{r}\right) \mathrm{d}t \otimes \mathrm{d}t + \frac{\mathrm{d}r \otimes \mathrm{d}r}{1 - 2M/r} + r^2 g_{S^2}. \tag{2.142}$$

In this case, by comparing with the Newtonian limit, everyone agrees the parameter, M, represents the mass. Furthermore, it's natural to view the mass as being sourced by the singularity at $r = 0$. Hence, one would expect that any viable notion of quasilocal mass should produce M for the quasilocal mass of round spheres in the Schwarzschild spacetime.

As a generalisation of this, Misner and Sharp [89] considered a spherically symmetric spacetime that was instead dynamical due to an ideal fluid. By studying the work done by this fluid they discovered a more covariant and widely applicable way of effectively picking out the M parameter. Rather than just agreement with M in the Schwarzschild spacetime, one would expect that a viable quasilocal mass agrees with this more general, but completely canonical, definition of energy in spherical symmetry.

Definition 2.17 (Misner-Sharp mass [89])**.** *The Misner-Sharp mass for spherically symmetric spacetimes is defined to be*

$$m_{MS}(S_r^2) = \frac{r}{2}\left(1 + (l^a n^b + n^a l^b) D_a(r) D_b(r)\right), \tag{2.143}$$

where r is the area-radius function of a symmetry sphere, S_r^2, and l & n are the null normals to S.

Spherically symmetry is a highly non-generic situation though. We may as well give up on physics if that was all we could do. The simplest quasilocal mass which is generically well-defined is due to Hawking [57]. His definition is underpinned by several physical heuristics:

- The mass contained in S should be measurable by the bending of ingoing and outgoing light rays orthogonal to S. The null directions orthogonal to S are simply l^a and n^a, so $m(S)$ must be constructed from the expansions, μ and ρ.
- The gauge freedom, $l^a \to f l^a$ and $n^a \to n^a/f$ for any non-zero function, f, should leave $m(S)$ unaffected. Hence, $m(S)$ should be constructed from a geometric invariant, such as the mean curvature vector, H^a.
- On an event horizon, the quasilocal mass should coincide with the irreducible mass, $\sqrt{A/16\pi}$. If the event horizon coincides with the apparent horizon, then $\rho = 0$ and thus H^a is null.

Putting these heuristics together naturally suggests the following definition.

Definition 2.18 (Hawking quasilocal mass [57])**.** *Let $A(S)$ denote the area of S. Then, the quasilocal mass of S is defined to be*

$$m_H(S) = \sqrt{\frac{A(S)}{16\pi}}\left(1 - \frac{1}{16\pi}\int_S H^a H_a \mathrm{d}A\right) = \sqrt{\frac{A(S)}{16\pi}}\left(1 - \frac{1}{2\pi}\int_S \mu\rho \,\mathrm{d}A\right). \tag{2.144}$$

Hawking's definition has found numerous applications in general relativity - most famously in Huisken and Ilmanen's proof of the Riemannian Penrose inequality [68]. Despite its many successes, it has one major deficiency; in some situations, it is non-negative [25]. The simplest example is the squashed sphere,

$$S = \{t = 0 \text{ and } (x^2 + y^2)/\xi^2 + z^2 = R^2\} \subseteq (\mathbb{R}^{3,1}, \eta) \tag{2.145}$$

for some constants, $R > 0$ & $\xi > 1$, which has

$$m_H(S) = \frac{1}{12}\left(5 - 2\xi^2 - \frac{3}{\xi\sqrt{\xi^2 - 1}}\cosh^{-1}(\xi)\right) < 0. \tag{2.146}$$

If the Hawking quasilocal mass shows that it's difficult to build a locally defined geometric invariant that maintains physical heuristics whilst not compromising on essential quantitative properties, then a potential alternative approach is to radically rethink the meaning of locality. By doing so, Bartnik was able to define a new quasilocal mass which leveraged the known desirable properties of the global definitions of energy discussed in the last subsection.

Definition 2.19 (Bartnik quasilocal mass [7])**.** *Let h be the metric on Σ and let K_{ij} be a symmetric two-tensor on Σ. Any asymptotically flat initial data set, $(\overline{\Sigma}, \bar{h}, \overline{K})$, is called an admissible extension of (Σ, h, K) if and only if it satisfies the dominant energy condition and $\exists$ an isometric embedding, $\iota : \Sigma \to \overline{\Sigma}$ such that $\iota^*\overline{K} = K$ on $\mathrm{int}(\Sigma)$ and $(\overline{\Sigma}, \bar{h}, \overline{K})$ contains no apparent horizons enclosing $\iota(\Sigma)$. The quasilocal mass of S is then defined to be the infimum of the ADM masses of all possible admissible extensions.*

Typically, the Bartnik quasilocal mass is only studied for time-symmetric data, i.e. K and $\overline{K}$ are both zero - see [87] for a comprehensive review. In this case the dominant energy condition reduces to saying $R^{(h)} \geq 0$ and the apparent horizon condition reduces to requiring Σ to have no closed minimal surfaces enclosing $\iota(\Sigma)$[16]. Very little is actually known in the general, $K \neq 0$, case; the problem is just too hard. Nonetheless, when the Bartnik quasilocal mass can be studied, it has many of the desirable properties I listed earlier. However, these properties are largely inherited from the ADM mass and sometimes hold in a rather tautological way. Furthermore, in all but the very simplest cases, the Bartnik quasilocal mass is absolutely impossible to calculate because one has to find all possible asymptotically flat extensions to data prescribed on some given compact set. A viable quasilocal mass needs to be significantly more practical.

The success of the global notions of energy defined in the last subsection was a tower built over the deep, concrete foundations of the Hamiltonian principle. A popular attempt to define quasilocal mass is to try apply the Hamiltonian principle on a local, rather than global, level. This is typically achieved via the Hamilton-Jacobi equation. In the classical mechanics of particles, given a Lagrangian, L, this equation says

$$H\left(q_f, \frac{\partial S}{\partial q_f}, t_f\right) = -\frac{\partial S}{\partial t_f}, \quad \text{where } S = \int_{t_i}^{t_f} L\left(q, \frac{\mathrm{d}q}{\mathrm{d}t}; q_i, q_f\right)\mathrm{d}t, \tag{2.147}$$

q denotes the set of configuration space coordinates and i & f denote the initial & final values. The Brown-York quasilocal mass [15] seeks to apply this same principle to general relativity by viewing it as a field theory defined by the Einstein-Hilbert action.

Let $\mathbb{V}$ be a 4-volume with non-null boundary, $\partial\mathbb{V}$. Let N_a denote the normal to $\partial\mathbb{V}$ with $N^a N_a = \pm 1$, let h denote its metric and let K denote its extrinsic curvature. Then, the

[16] Instead of this particular "non-degeneracy" condition, there have also been other approaches in the literature, such as only imposing no stable, minimal surfaces or imposing conditions relating the mean curvatures of $\partial\Sigma = S$ and $\partial(\iota(\Sigma)) \subseteq \overline{\Sigma}$.

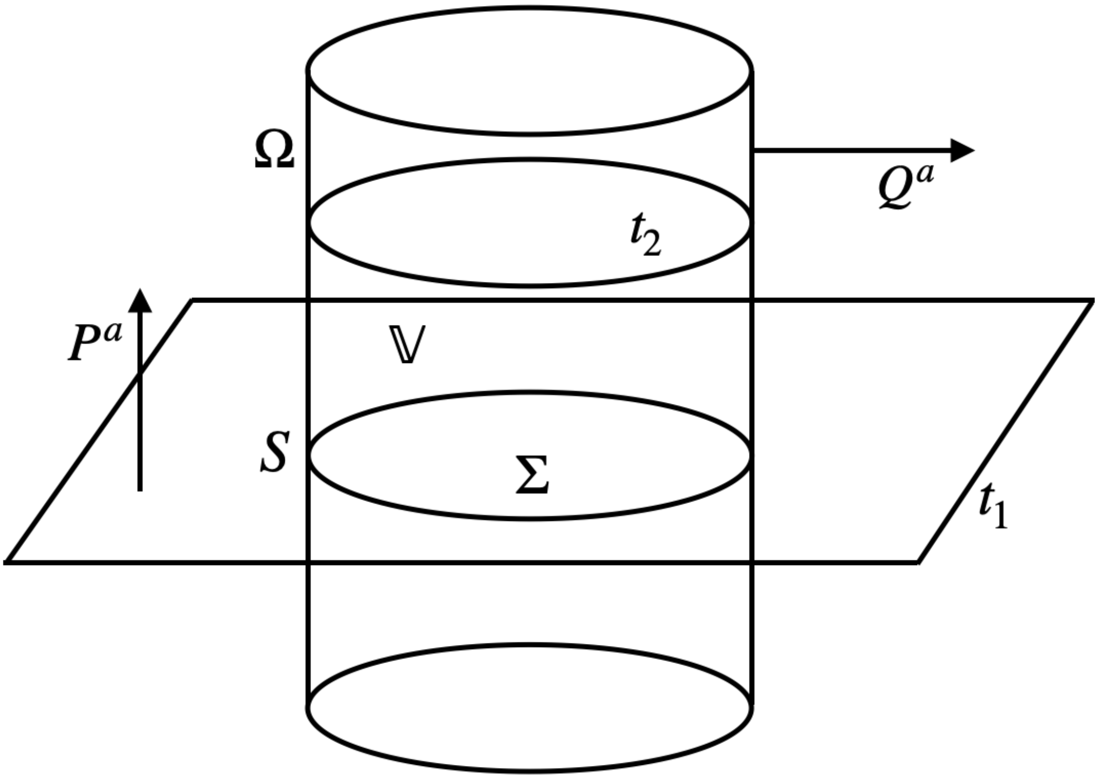

Figure 2.3: The set-up for the Hamilton-Jacobi analysis that goes into defining the Brown-York and Wang-Yau quasilocal masses.

variation of the Einstein-Hilbert action is [15]

$$\delta S_{EH} = \frac{1}{16\pi}\int_{\mathbb{V}} \delta(g^{ab})\left(R_{ab} - \frac{1}{2}Rg_{ab}\right)\mathrm{d}\mu(g) \pm \frac{1}{16\pi}\int_{\partial\mathbb{V}} N_a\left(g^{bc}\delta\Gamma^a{}_{bc} - g^{ab}\delta\Gamma^c{}_{cb}\right)\mathrm{d}V \quad (2.148)$$

$$= \frac{1}{16\pi}\int_{\mathbb{V}} \delta(g^{ab})\left(R_{ab} - \frac{1}{2}Rg_{ab}\right)\mathrm{d}\mu(g) \pm \frac{1}{16\pi}\int_{\partial\mathbb{V}} \left(D_a^{(h)}\delta u^a + (K_{ab} - Kh_{ab})\delta h^{ab}\right)\mathrm{d}V$$

$$\mp \frac{1}{8\pi}\int_{\partial\mathbb{V}} \delta(K\sqrt{h})\mathrm{d}^3x, \quad \text{where} \quad (2.149)$$

$$\delta u^a = \delta N^a + g^{ab}\delta N_b. \quad (2.150)$$

The last term in equation 2.149 is simply the variation of the Gibbons-Hawking term [49, 116],

$$S_{GH} = \mp\frac{1}{8\pi}\int_{\partial\mathbb{V}} K\sqrt{h}\,\mathrm{d}^3x = \mp\frac{1}{8\pi}\int_{\partial\mathbb{V}} K\mathrm{d}V, \quad (2.151)$$

and hence can be removed by changing the action to $S_{EH} + S_{GH}$. Meanwhile, the first integral in equation 2.149 is zero on-shell by the Einstein equation and the first term in the second integral is zero by Stokes' theorem. Finally, the second term of that integral is typically taken to be zero by making the standard variational calculus assumption that the metric variation is zero on the boundary. However, in the Hamilton-Jacobi context, that term will end up being the most important.

As shown in figure 2.3, choose $\mathbb{V}$ to be the region bounded by a timelike cylinder, Ω, and spacelike surfaces, Σ, at time coordinates, t_1 and t_2. Let h and f be metrics on Σ and Ω respectively. Let K and θ be the corresponding extrinsic curvatures. Finally, let c be the mean curvature of S in Σ. Then, by equation 2.149 and the Einstein equation,

$$\delta(S_{EH}+S_{GH}) = \frac{1}{16\pi}\int_{\Sigma_{t_2}} \delta(h_{ij})(K^{ij}-Kh^{ij})\sqrt{h}\mathrm{d}^3x - \frac{1}{16\pi}\int_{\Sigma_{t_1}} \delta(h_{ij})(K^{ij}-Kh^{ij})\sqrt{h}\mathrm{d}^3x \\ - \frac{1}{16\pi}\int_{\Omega\cap\mathbb{V}} \delta(f_{mn})(\theta^{mn}-\theta f^{mn})\sqrt{-f}\mathrm{d}^3x. \tag{2.152}$$

The analogue of $\partial S/\partial t$ in classical mechanics is taking

$$\tau^{mn} = \frac{2}{\sqrt{-f}}\frac{\delta S}{\delta f_{mn}} = \frac{1}{8\pi}(\theta f^{mn} - \theta^{mn}) \tag{2.153}$$

as a surface stress tensor. $\tau^{mn}P_n$ is then a natural energy-momentum current. Write f in a 2+1 split,

$$f_{mn}\mathrm{d}x^m \otimes \mathrm{d}x^n = -N^2\mathrm{d}t\otimes\mathrm{d}t + \beta_{\alpha\beta}(\mathrm{d}x^\alpha + S^\alpha\mathrm{d}t)\otimes(\mathrm{d}x^\beta + S^\beta\mathrm{d}t). \tag{2.154}$$

Then, the observer's trajectory is

$$t^\mu = NP^\mu + S^\mu \equiv \frac{\partial}{\partial t}. \tag{2.155}$$

It's then natural to define the energy density as $t^\mu\tau_{\mu\nu}P^\nu$, which can be shown [15] to equal

$$t^\mu\tau_{\mu\nu}P^\nu = -\frac{1}{8\pi}(Nc - S^\mu Q^\nu K_{\mu\nu}). \tag{2.156}$$

Integrating this energy density typically yields infinity. However, energy is also usually measured against some reference. Subtracting of some reference quantities, denoted by a subscript (0), ultimately yields the Hamilton-Jacobi energy,

$$E_{HJ}(S) = \frac{1}{8\pi}\int_S \left(N_{(0)}c_{(0)} - S^\mu_{(0)}Q^\nu_{(0)}K_{(0)\mu\nu} - Nc + S^\mu Q^\nu K_{\mu\nu}\right)\mathrm{d}A. \tag{2.157}$$

Different versions of the Hamilton-Jacobi quasilocal mass are distinguished by the choices they make for t^μ and the reference.

The simplest prescription is Brown and York's, who chose $t^\mu = P^\mu$, i.e. $N=1$ and $S^\mu = 0$. As for the reference, they invoke the Weyl embedding theorem [102, 94], which states that for any closed, 2D, Riemannian surface, S, with everywhere positive scalar curvature, $\exists$ an isometric embedding, $\iota : S \to \mathbb{R}^3$, that is unique up to Euclidean, rigid motions.

In summary, Brown and York make the following definition.

Definition 2.20 (Brown-York quasilocal mass [15])**.** *Let Σ be a 3D, compact, spacelike manifold with boundary, S. Assume S has positive scalar curvature everywhere. Let c be the mean curvature of S in Σ. Let ι be an isometric embedding of S in $(\mathbb{R}^3,\delta)$ and let $c_{(0)}$ denote the mean curvature of $\iota(S)$ in $\mathbb{R}^3$. Then, the quasilocal mass is*

$$m_{BY}(\Sigma, S) = \frac{1}{8\pi}\int_S (c_{(0)} - c)\,\mathrm{d}A. \tag{2.158}$$

In essence, this definition compares the extrinsic curvature of $S \subseteq \Sigma$ to the extrinsic curvature of a canonical embedding of S into flat space. Any "extra" curvature in the former compared to the latter is effectively interpreted as being due to gravity curving the space. A mild generalisation of this that better takes into account the spacetime character of the set-up and dispenses with the Σ dependence is the Kijowski quasilocal mass [74],

$$m_K(S) = \frac{1}{8\pi}\int_S \left(c_{(0)} - \sqrt{H^a H_a}\right) \mathrm{d}A. \tag{2.159}$$

Despite its great physical motivation, the Brown-York mass has one rather glaring flaw - in a constant time slice of Schwarzschild spacetime, it says

$$m_{BY}(S_r^2) = r\left(1 - \sqrt{1 - 2M/r}\right) \neq M. \tag{2.160}$$

It also has some more subtle flaws. Firstly, not every surface of physical interest has everywhere positive scalar curvature - the best known example being the horizon of a sufficiently rapidly rotating Kerr black hole [40]. Secondly, whereas the Hawking quasilocal mass let itself down by not being consistently non-negative, the Brown-York method fails by having too positive a disposition. As shown in [91], if

$$S = \{t = r = F(\theta, \phi)\} \subseteq (\mathbb{R}^{3,1}, \eta), \tag{2.161}$$

then it is possible to choose F so that the Kijowski mass is positive[17]. Essentially this is because S is not naturally embedded in the flat Euclidean space, but the flat Minkowski space of a dimension higher.

Wang and Yau [118] sought to rectify this problem by trying to embed S directly into $(\mathbb{R}^{3,1}, \eta)$ instead of $(\mathbb{R}^3, \delta)$. This requires a fairly elaborate choice of quantities to input into equation 2.157 and a consideration of all the different ways in which a surface could be embedded into Minkowski space.

Definition 2.21 (Wang-Yau quasilocal mass [118])**.** *Let $\iota : S \to (\mathbb{R}^{3,1}, \eta)$ be an isometric embedding. Choose Cartesian coordinates on $(\mathbb{R}^{3,1}, \eta)$ and let $t^a_{(0)} \equiv \partial/\partial t$. Let $\beta_{(0)ab}$ and $H^a_{(0)}$ be the induced metric and mean curvature vector respectively of $\iota(S)$ in $(\mathbb{R}^{3,1}, \eta)$. The unit normal to $\iota(S)$ sharing the same component normal to $\iota(S)$ as $t^\mu_{(0)}$ is then*

$$P^a_{(0)} = \frac{t^a_{(0)} - \beta^a_{(0)b} t^b_{(0)}}{\sqrt{1 + \beta_{(0)ab} t^a_{(0)} t^b_{(0)}}}. \tag{2.162}$$

Let P^a be the unique vector such that $P^a \perp S$, $P^a P_a = -1$ and $H^a P_a = H^a_{(0)} P_{(0)a}$. Let Q^a then be the unique vector such that $Q^a Q_a = 1$, $Q^a P_a = 0$, $Q^a \perp S$ and Q^a is outward pointing. From equation 2.162, the shift vector is $S^a_{(0)} = \beta^a_{(0)b} t^b_{(0)}$. As $S^a_{(0)}$ is tangent to $\iota(S)$ and ι is an isometric embedding, $S^a_{(0)}$ can be identified with a vector, S^a, tangent to S. On S, choose

$$t^a = \sqrt{1 + \beta_{(0)ab} t^a_{(0)} t^b_{(0)}}\, P^a + S^a. \tag{2.163}$$

Let Σ and $\Sigma_{(0)}$ be the spacelike hypersurfaces orthogonal to P^a and $P^a_{(0)}$ respectively to define K_{ab}, $K_{(0)ab}$, c and $c_{(0)}$. Now, it's possible to define the Wang-Yau quasilocal energy as $E_{WY}(S) = E_{HJ}(S)$, where the $E_{HJ}(S)$ of equation 2.157 is evaluated using all the choices just described. Finally, define the Wang-Yau quasilocal mass as the infimum of the Wang-Yau quasilocal energy over all possible embeddings, i.e.

$$m_{WY}(S) = \inf_{\iota, t_{(0)}} E_{WY}(S) = \inf_{\iota, t_{(0)}} E_{HJ}(S). \tag{2.164}$$

[17] Note that in a sense, Brown-York performs slightly better than Kijowski on this criterion. The Brown-York mass is more naturally defined in the Riemannian setting where Σ is part of a time-symmetric initial data slice. In general though, given S, there are many different fill-ins, $\Sigma \subseteq M$. These will all have the same Kijowksi mass because the mean curvature is defined purely by how S embeds in M. However, they will have different Brown-York mass based on the choice of Σ. For the Brown-York mass, it is known [109] that $m_{BY}(S) = 0 \iff \Sigma \subseteq (\mathbb{R}^3, \delta)$.

Because of the infimum, the Wang-Yau prescription natually picks out an observer who is at rest with respect to the quasilocal energy-momentum of S. This momentum information is effectively thrown away in Kijowski's definition. Hence, the Wang-Yau quasilocal mass always returns zero for surfaces in Minkowski space. However, when a surface is already naturally embedded in a static slice - such as the round spheres of the Schwarzschild metric - it provides no improvement on the Brown-York or Kijowski definitions [103].

The last approach to quasilocal mass I'll discuss is the spinorial method. As I'll outline in section 2.4, Witten's positive energy proof essentially reduces to integrating a very well-chosen divergence identity. The quantity in the bulk turns out to be non-negative and the quantity measured at infinity using Stokes' theorem turns out to be a function of the ADM quantities, including energy. This begs the obvious question - if the same divergence identity is integrated over a compact set, instead of an asymptotically flat end, can the quantity measured on the compact set's boundary be interpreted as a non-negative, quasilocal energy?

It turns out this doesn't quite work - although see [81, 126] for some valiant attempts - but something closely related does yield fantastic results.

Definition 2.22 (Dougan-Mason quasilocal mass [38])**.** *Let φ_α be a two-component spinor satisfying $\bar{\delta}\varphi_\alpha = 0$ on S (or equivalently, $\bar{\eth}\varphi_o + \mu\varphi_\iota = 0$ and $\bar{\eth}\varphi_\iota - \bar{\sigma}\varphi_o = 0$) and let $\{\varphi^A_\alpha\}$ be a basis of solutions. Assume the dominant energy condition holds, $\theta_n < 0$ and S is generic. Normalise φ^A_α so that*

$$\varepsilon^{\alpha\beta}\varphi^A_\alpha\varphi^B_\beta = \sqrt{2}\left(\varphi^A_o\varphi^B_\iota - \varphi^B_o\varphi^A_\iota\right) = \varepsilon^{AB}. \tag{2.165}$$

Then, define the quasilocal energy-momentum vector by

$$\mathbb{P}^a_{DM}(S) = -\frac{1}{2}(\sigma^a)_{A\dot{A}}\mathbb{P}^{\dot{A}A}_{DM}(S) = -\frac{1}{4\sqrt{2}\pi}(\sigma^a)_{A\dot{A}}\int_S \left(\rho\overline{\varphi}^{\dot{A}}_o\varphi^A_o - \mu\overline{\varphi}^{\dot{A}}_\iota\varphi^A_\iota\right)\mathrm{d}A \tag{2.166}$$

and the quasilocal mass by

$$m_{DM}(S) = \sqrt{-\eta_{ab}\mathbb{P}^a_{DM}(S)\mathbb{P}^b_{DM}(S)}. \tag{2.167}$$

It is not all obvious that $m_{DM}(S)$ is well-defined. It needs to be shown that $\varepsilon^{\alpha\beta}\varphi^A_\alpha\varphi^B_\beta$ is constant on S, $-\eta_{ab}\mathbb{P}^a_{DM}(S)\mathbb{P}^b_{DM}(S) \geq 0$, $\bar{\delta}\varphi_\alpha = 0$ generically has a 2D space of solutions and $-\eta_{ab}\mathbb{P}^a_{DM}(S)\mathbb{P}^b_{DM}(S)$ is independent of the choice of linearly independent solutions to $\bar{\delta}\varphi_\alpha = 0$. I will not discuss any of these issues here because analogous questions will arise for my generalisation in chapter 5; simply setting $\Lambda = 0$ there will immediately prove the Dougan-Mason quasilocal mass is also well-defined.

As I've summarised in table 2.1, based on a series of papers [57, 25, 70, 65, 7, 87, 15, 109, 91, 16, 40, 118, 103, 22, 38, 37], the quasilocal masses discussed have been shown to either satisfy or not satisfy the listed properties[18]. From the table, it's clear the Dougan-Mason quasilocal mass is the best known so far. Although, even it isn't fully satisfactory because, just like most other known definitions, on cross-sections of the Kerr event horizon it evaluates to neither M nor $\sqrt{A(S)/16\pi}$, but instead to some seemingly physically meaningless other answer [11]. On this point, I should note that although the Bartnik quasilocal mass flunks the qualitative requirements, it does produce $\sqrt{A(S)/16\pi}$ for any outermost minimal surface in time-symmetric initial data [85].

[18]Note that even for the quantitative properties, I have had to make some subjective assessments. For example, the Hawking quasilocal mass approaches the Bondi mass for round spheres on $\mathcal{I}^+$, but not squashed spheres. Likewise, it agrees with a canonical Hamiltonian on some (physically well-motivated) spheres in linearised gravity, but not for all spheres. There are also some partial results towards some of the unknowns, such as the small-sphere limit of Bartnik's definition.

Quasilocal mass	I	II	III	IV	V	VI	A	B	C
Hawking	×	×	✓	✓	×	✓	✓	✓	✓
Bartnik	✓	✓	✓	✓	?	?	✓	×	×
Brown-York	✓	×	×	✓	×	✓	×	✓	×
Wang-Yau	✓	✓	×	✓	?	✓	×	✓	×
Dougan-Mason	✓	✓	✓	✓	✓	✓	✓	✓	✓

Table 2.1: A summary of the properties of different quasilocal masses. The labels in the top row refer to the lists of properties in the main text.

Nonetheless, if I am seeking to invent a satisfactory definition of quasilocal mass for spacetimes with $\Lambda < 0$, then the Dougan-Mason definition appears to be the most promising one to try generalise; indeed this will be the main point of chapter 5. Furthermore, it seems to be the natural progression of recent research. The proof that $m_{DM}(S) \geq 0$ uses the same techniques as Witten's positive energy theorem proof. As I've reviewed in the last subsection, these same techniques were then adapted to prove global mass-charge inequalities and then global positive energy theorems with $\Lambda < 0$. It's natural to wonder whether a similar progression can be made at the quasilocal level for generalisations of $m_{DM}(S)$. The first step on that path was recently completed by Reall [105], who proved a quasilocal mass-charge inequality by generalising Dougan and Mason's work. It would appear therefore, that a $\Lambda < 0$ version of $m_{DM}(S)$ is ripe for the picking.

In that endeavour, I also drew some inspiration from Penrose's quasilocal mass [99]. Instead of the conditions,

$$\bar{\eth}\varphi_o + \mu\varphi_\iota = 0 \text{ and } \bar{\eth}\varphi_\iota - \bar{\sigma}\varphi_o = 0, \tag{2.168}$$

that Dougan and Mason impose on S, Penrose instead enforces

$$\bar{\eth}\bar{\varphi}_o + \lambda\bar{\varphi}_\iota = 0 \text{ and } \bar{\eth}\varphi_\iota - \bar{\sigma}\varphi_o = 0, \tag{2.169}$$

which originate from considering the components tangent to S in the valence-(1, 0) twistor equation, $D_{\alpha(\dot{\alpha}}\bar{\varphi}_{\dot{\beta})} = 0$. From there, Penrose expands φ_α in a basis of solutions and uses that solution to form a "kinematical twistor," A. The details of that construction are unimportant for my thesis. The only relevant point is that the kinematical twistor is effectively a bilinear form built out of φ_α, and can be represented, much like $\mathbb{P}^{\dot{A}A}_{DM}(S)$, as a matrix, A^{AB}, whose entries are indexed by the different choice of basis solutions, φ^A_α, input into the bilinear form. For Dougan and Mason, their defining equation for φ_α generically admits exactly two linearly independent solutions. This leads to the normalisation of ε^{AB} and the interpretation of $\mathbb{P}^{\dot{A}A}_{DM}(S)$ as a full energy-momentum vector written in an auxiliary two-component spinor space. This is not possible for Penrose's definition, which generically leads to a 4×4 matrix with the energy, linear momentum and angular momentum all mixed together in some unknown way. The situation will be much the same for the quasilocal mass I define in chapter 5 and I will follow the same approach as Penrose to resolve the issue. In particular, I have to find some canonical antisymmetric matrix built out φ^A_α to contract the indices on my 4×4 matrix and produce a single scalar which is interpreted as $m(S)$. In Penrose's case, this antisymmetric matrix is known as the "surface infinity twistor," while for Dougan and Mason it's just ε^{AB}. There isn't always a canonical way to define this matrix in Penrose's set-up, but luckily this is not an issue I will have in my set-up.

I should also note that the conditions imposed on the spinors, φ_α, on S by Dougan & Mason or Penrose are not arbitrary. It turns out [111, 112] there are only four differential operators constructed from the irreducible parts of $\beta^b{}_a D_b$ which are both first order and elliptic when acting on spinors on a compact, 2D, spacelike surface, S. They are the choices made by Dougan-Mason and Penrose, an antiholomorphic version of the Dougan-Mason choice[19] and the Dirac operator.

There have also been some previous attempts at defining a quasilocal mass for spacetimes with negative cosmological constant. The Misner-Sharp and Hawking quasilocal masses have the very simple generalisations (even for positive cosmological constant),

$$m_{MS}(S) = \frac{r}{2}\left(1 - \frac{\Lambda r^2}{3} + (l^a n^b + n^a l^b) D_a(r) D_b(r)\right) \quad \text{and} \tag{2.170}$$

$$m_H(S) = \sqrt{\frac{A(S)}{16\pi}}\left(1 - \frac{\Lambda A(S)}{12\pi} - \frac{1}{16\pi}\int_S H^a H_a \mathrm{d}A\right). \tag{2.171}$$

The Penrose quasilocal mass has also been formulated for $\Lambda < 0$ [72], although the definition in [72] is only ever evaluated at conformal infinity. Finally, there are versions of the Kijowski mass where one embeds in 3D hyperbolic space instead of 3D Euclidean space [117, 40], provided $R^{(S)}$ still has a strict lower bound. These definitions share the virtues and flaws of their $\Lambda = 0$ counterparts. The generalisation of the Dougan-Mason quasilocal mass I will define in chapter 5 is the only known definition that has the following properties.

- $m(S) \geq 0$.
- $m(S) = 0$ for every generic surface in AdS.
- $m(S)$ coincides with the Misner-Sharp mass (including cosmological constant) for spherically symmetric spacetimes.
- For asymptotically AdS spacetimes, $m(S)$ agrees with a global notion of mass as S approaches a sphere on conformal infinity, $\mathcal{I}$.
- For gravity linearised about AdS, $m(S)$ agrees with a reasonable notion of mass built from the energy-momentum tensor, T_{ab}.

2.4 Key ideas of Witten's method

In this section, I'll outline some of the main ideas which will appear frequently in the remainder of the thesis. To avoid unnecessary technical difficulties, I will only sketch the arguments and I will only consider the simplest context in which Witten's method may be applied - namely to prove the ADM energy is non-negative in vacuum, time-symmetric, asymptotically flat initial data or equivalently an asymptotically flat, Riemannian manifold with Ricci scalar everywhere non-negative.

It is sometimes said of Beethoven's 5th symphony that every note he wrote was simply a natural consequence built on the famous four note opening motif. If there is any such analogue for the work of genius that is Witten's positive energy proof, then it is the Lichnerowicz identity.

Lemma 2.23 (Lichnerowicz identity)**.** *On any pseudo-Riemannian manifold, for any Dirac spinor, Ψ,*

$$\gamma^{abc} D_b D_c \Psi = -\frac{1}{2}\left(R^{ab} - \frac{1}{2} g^{ab} R\right)\gamma_b \Psi. \tag{2.172}$$

[19] Dougan and Mason actually consider the antiholomorphic version too in their papers. However, I've chosen to highlight the holomorphic choice because it is the one which is well-defined at $\mathcal{I}^+$.

Crucially, the Lichnerowicz identity links spinors to the Einstein tensor.

Proof. By direct evaluation,

$$\gamma^{abc}D_bD_c\Psi = \frac{1}{2}\gamma^{abc}[D_b, D_c]\Psi \text{ by antisymmetry} \tag{2.173}$$

$$= -\frac{1}{8}R^{de}{}_{bc}\gamma^{abc}\gamma_{de}\Psi \tag{2.174}$$

$$= -\frac{1}{8}R^{de}{}_{bc}\left(\gamma^{abc}{}_{de} - 6\gamma^{[ab}{}_{[e}\delta^{c]}{}_{d]} + 6\gamma^{[a}\delta^{b}{}_{[e}\delta^{c]}{}_{d]}\right)\Psi \tag{2.175}$$

$$= \frac{3}{4}R^{de}{}_{bc}\left(0 + \gamma^{[ab}{}_{e}\delta^{c]}{}_{d} - \gamma^{[a}\delta^{b}{}_{e}\delta^{c]}{}_{d}\right)\Psi \text{ by the Bianchi identity} \tag{2.176}$$

$$= \frac{1}{4}\left(-R_{eb}\gamma^{abe} + R^{a}{}_{ebc}\gamma^{bce} + R_{ec}\gamma^{cae} + R\gamma^{a} - R^{ab}\gamma_b - R^{ac}\gamma_c\right)\Psi \tag{2.177}$$

$$= \frac{1}{4}(0 + 0 + 0 + R\gamma^{a} - 2R^{ab}\gamma_b)\Psi \text{ by Bianchi identity and } R_{ab} = R_{ba} \tag{2.178}$$

$$= -\frac{1}{2}\left(R^{ab} - \frac{1}{2}g^{ab}R\right)\gamma_b\Psi, \tag{2.179}$$

which is the desired result. □

In the Riemannian setting I'll illustrate in this section, it actually suffices to use the slightly simpler result,

$$\gamma^{IJ}D_ID_J\Psi = -\frac{1}{8}R_{KLIJ}\gamma^{IJ}\gamma^{KL}\Psi \tag{2.180}$$

$$= -\frac{1}{8}R_{KLIJ}(\gamma^{IJKL} + \delta^{IK}\gamma^{J}\gamma^{L} - \delta^{IL}\gamma^{J}\gamma^{K} - \delta^{JK}\gamma^{I}\gamma^{L} + \delta^{JL}\gamma^{I}\gamma^{K} + \delta^{IK}\delta^{JL}I - \delta^{IL}\delta^{JK}I)\Psi \tag{2.181}$$

$$= \frac{1}{4}R\Psi. \tag{2.182}$$

For this section only, I will adjust the notation slightly in the interests of simplicity. I will denote the Riemannian manifold as (M, g) instead of $(\Sigma, h, K = 0)$ and likewise I will denote its dimension as n, instead of $n-1$. I will also denote the $\tilde{h}$ in definition 2.13 as h, i.e. asymptotic flatness is equivalent to $g_{ij} = \delta_{ij} + h_{ij}$ with $h_{ij} = O(r^{-\tau})$, $\partial_i h_{jk} = O(r^{-\tau-1})$ and $\tau > (n-2)/2$ near infinity.

The main steps of Witten's proof are then as follows.

- Use equation 2.182 to relate the scalar curvature and Dirac operator, $\gamma^I D_I$, by

$$D_I\left(\Psi^\dagger\gamma^{IJ}D_J\Psi\right) = D_I(\Psi)^\dagger\gamma^{IJ}D_J\Psi + \Psi^\dagger\gamma^{IJ}D_ID_J\Psi \tag{2.183}$$

$$= D_I(\Psi)^\dagger(\gamma^{I}\gamma^{J} + \delta^{IJ}I)D_J\Psi + \frac{1}{4}R\Psi^\dagger\Psi \tag{2.184}$$

$$= D_I(\Psi)^\dagger D^I(\Psi) - (\gamma^I D_I\Psi)^\dagger\gamma^J D_J\Psi + \frac{1}{4}R\Psi^\dagger\Psi. \tag{2.185}$$

- Integrate on M and apply Stokes' theorem to get

$$\int_{S^{n-2}_\infty} Q_I\Psi^\dagger\gamma^{IJ}D_J(\Psi)\mathrm{d}A = \int_M \left(D_I(\Psi)^\dagger D^I(\Psi) - (\gamma^I D_I\Psi)^\dagger\gamma^J D_J\Psi + \frac{1}{4}R\Psi^\dagger\Psi\right)\mathrm{d}V, \tag{2.186}$$

  where $Q_i = x_i/r$ is the unit normal to constant r surfaces in the asymptotic end.

- To best adapt to the flat metric, δ, which g approaches as $r \to \infty$, choose the frame,

$$e_I = \delta^i{}_I \left(\partial_i - \frac{1}{2} h_{ij} \partial_j + o(r^{-\tau}) \right). \tag{2.187}$$

- Choose Ψ so that $\gamma^I D_I \Psi = 0$, thereby making the RHS of equation 2.186 non-negative.
- While ensuring $\gamma^I D_I \Psi = 0$, also choose the boundary condition at infinity that $\Psi \to \Psi_0$, where Ψ_0 is a constant spinor (in the chosen frame).
- Show that this implies the LHS of equation 2.186 equals $4\pi E_{ADM} \Psi^\dagger \Psi$ and therefore $E_{ADM} \geq 0$.

There are essentially two main analytical difficulties in this process:

- Showing the Dirac equation can be solved with the desired boundary conditions.
- Showing that limits work appropriately on both sides of equation 2.186.

These issues ultimately come down to choosing appropriate functional spaces for Ψ and $\gamma^I D_I \Psi$. The original approach for this [6, 98] was to deploy well known weighted Sobolev spaces and then calculate a series of estimates relating the various quantities. However, since then a much simpler approach [9, 8, 27] has been developed where complicated estimates are exchanged for a less familiar Hilbert space. In particular, by using 2.182 itself, one defines an inner product on smooth, compactly supported spinors by

$$\langle \Psi_1, \Psi_2 \rangle_{C_c^\infty} = \int_M \left(D_I(\Psi_1)^\dagger D^I(\Psi_2) + \frac{1}{4} R \Psi_1^\dagger \Psi_2 \right) \mathrm{d}V. \tag{2.188}$$

Hence, the L^2 norm of the Dirac operator is immediately given by

$$||\gamma^I D_I \Psi||_{L^2} = ||\Psi||_{C_c^\infty}. \tag{2.189}$$

This identity suggests natural functional spaces for both a domain and a codomain between which the Dirac operator, $\gamma^I D_I$, is bounded in a very simple way - thereby making further analysis much easier.

Most of chapter 3 and the early parts of chapters 4 & 5 are simply about formalising these key steps and adapting them to the specific contexts I'm studying.

Chapter 3

Elements of analysis

There is a syndrome in sports called "paralysis by analysis."
- Arthur Ashe

In following Witten's method I will frequently need to work with spinors, Ψ, solving $\gamma^I \nabla_I \Psi = 0$ on a spacelike hypersurface, Σ, for certain modified connections, ∇_a. In this chapter, I prove this is always possible given appropriate boundary conditions on $\partial\Sigma$ and given an appropriate functional space for Ψ. The presentation here is heavily based on [8, 9, 27].

For the duration of this chapter, let Σ be either a compact, 3D spacelike hypersurface with boundary or the Σ_t defined in an asymptotically, locally AdS spacetime - these are the situations depicted in figures 2.1 and 2.2 respectively. Let $\partial\Sigma$ denote either S, the boundary of the compact set, or $\Sigma_{t,\infty}$, the boundary at infinity in the non-compact case.

3.1 Lichnerowicz identities

Definition 3.1 (∇_a, $\mathcal{A}_a$, $\mathbb{M}$, Witten-Nester 2-form and $Q(\varepsilon)$)**.** *When acting on any Dirac spinor, Ψ, of the spacetime, define the modified connection, ∇, by*

$$\nabla_a \Psi = D_a \Psi + \mathrm{i}k\gamma_a \Psi + \mathcal{A}_a \Psi \quad \text{and} \tag{3.1}$$

$$\nabla_a \overline{\Psi} = D_a \overline{\Psi} - \mathrm{i}k\overline{\Psi}\gamma_a + \overline{\Psi}\gamma^0 \mathcal{A}_a^\dagger \gamma^0 = (\nabla_a \Psi)^\dagger \gamma^0, \tag{3.2}$$

where $\mathcal{A}_a$ is some Clifford algebra valued one-form[1]*. It will always be assumed that $\gamma^{IJ}\mathcal{A}_J$ is a hermitian matrix and that*

$$\mathbb{M} = 4\pi T^{0a}\gamma_0\gamma_a + \gamma^{IJ} D_I \mathcal{A}_J + \mathrm{i}k(n-2)(\gamma^I \mathcal{A}_I + \mathcal{A}_I^\dagger \gamma^I) - \mathcal{A}_I^\dagger \gamma^{IJ} \mathcal{A}_J \tag{3.3}$$

is a non-negative definite matrix. Furthermore, in the non-compact case it will be assumed $||\mathcal{A}_a||_0$ and $||\mathbb{M}||_0$ decay as $O(\mathrm{e}^{-(n-1)r})$ and $o(\mathrm{e}^{-(n-1)r})$ respectively near $\Sigma_{t,\infty}$, where $||\cdot||_0$ denotes the (pointwise) operator norm of a matrix[2]*. Finally, the Witten-Nester [125, 92] 2-form*[3] *is defined to be*

$$E^{ab}(\varepsilon) = \bar{\varepsilon}\gamma^{abc}\nabla_c\varepsilon + \mathrm{c.c} = \bar{\varepsilon}\gamma^{abc}\nabla_c\varepsilon - \nabla_c(\bar{\varepsilon})\gamma^{abc}\varepsilon. \tag{3.4}$$

[1] In the interests of generality, $\mathcal{A}_a$ is left quite unconstrained for now. But, in the examples, $\mathcal{A}_a$ will either be zero or a function of Maxwell fields such that ∇_a describes the gravitino transformation in a gauged supergravity.

[2] In the latter case, this is equivalent to saying $||\mathbb{M}||_0 \in L^1$.

[3] Some authors would refer to this as the Hodge dual of a Witten-Nester $(n-2)$-form. In that case, one chooses to integrate differential forms, instead of scalars as I will do. A further extension to the Witten-Nester $(n-2)$-form is the Sparling form, which is closed if and only if the Einstein tensor vanishes and allows yet another way of formulating Witten's argument - see [101, 86] for more details.

Given a unit normal, P_a, to Σ, define a functional, $Q(\varepsilon)$, by

$$Q(\varepsilon) = \int_\Sigma P_a D_b(E^{ba}(\varepsilon))\mathrm{d}V. \tag{3.5}$$

Similarly, for a pair of Dirac spinors, Ψ_1 and Ψ_2, define

$$E^{ab}(\Psi_1, \Psi_2) = \overline{\Psi}_1\gamma^{abc}\nabla_c\Psi_2 - \nabla_c(\overline{\Psi}_1)\gamma^{abc}\Psi_2 \text{ and} \tag{3.6}$$

$$Q(\Psi_1, \Psi_2) = \int_\Sigma P_a D_b(E^{ba}(\Psi_1, \Psi_2))\mathrm{d}V. \tag{3.7}$$

It is an interesting fact that although t appears explicitly in $\int_{\Sigma_t}$, $Q(\varepsilon)$ is actually conserved in the non-compact case because Stokes' theorem implies

$$Q(\varepsilon)|_{t_2} - Q(\varepsilon)|_{t_1} = \int_{t_1}^{t_2}\int D_a D_b(E^{ba}(\varepsilon))\mathrm{d}\mu(g) = \int_{t_1}^{t_2}\int R_{ab}(E^{ba}(\varepsilon))\mathrm{d}\mu(g) = 0. \tag{3.8}$$

Theorem 3.2 (Lichnerowicz identity)**.**

$$P_a D_b(E^{ba}(\varepsilon)) = 2\left((\nabla_I\varepsilon)^\dagger\nabla^I\varepsilon - (\gamma^I\nabla_I\varepsilon)^\dagger\gamma^J\nabla_J\varepsilon + \varepsilon^\dagger\mathbb{M}\varepsilon\right). \tag{3.9}$$

Proof. By equation 2.172,

$$\gamma^{abc}D_bD_c\varepsilon = -\frac{1}{2}\left(R^{ab} - \frac{1}{2}g^{ab}R\right)\gamma_b\varepsilon. \tag{3.10}$$

Using that, $-2(n-1)(n-2)k^2 = \Lambda$ and the Einstein equation, $D_b(E^{ba}(\varepsilon))$ can be expanded as

$$\begin{aligned}
D_bE^{ba}(\varepsilon) &= D_b(\bar\varepsilon)\gamma^{bac}\nabla_c\varepsilon + \bar\varepsilon\gamma^{bac}D_b(\nabla_c\varepsilon) - D_b(\nabla_c\bar\varepsilon)\gamma^{bac}\varepsilon - \nabla_c(\bar\varepsilon)\gamma^{bac}D_b\varepsilon && (3.11)\\
&= \nabla_b(\bar\varepsilon)\gamma^{bac}\nabla_c\varepsilon + \mathrm{i}k\bar\varepsilon\gamma_b\gamma^{bac}\nabla_c\varepsilon - \bar\varepsilon\gamma^0\mathcal{A}_b^\dagger\gamma^0\gamma^{bac}\nabla_c\varepsilon + \bar\varepsilon\gamma^{bac}D_b(\nabla_c\varepsilon)\\
&\quad - D_b(\nabla_c\bar\varepsilon)\gamma^{bac}\varepsilon - \nabla_c(\bar\varepsilon)\gamma^{bac}\nabla_b\varepsilon + \mathrm{i}k\nabla_c(\bar\varepsilon)\gamma^{bac}\gamma_b\varepsilon + \nabla_c(\bar\varepsilon)\gamma^{bac}\mathcal{A}_b\varepsilon && (3.12)\\
&= 2\nabla_b(\bar\varepsilon)\gamma^{bac}\nabla_c\varepsilon - \mathrm{i}k(n-2)\bar\varepsilon\gamma^{ab}\nabla_b\varepsilon - \bar\varepsilon\gamma^0\mathcal{A}_b^\dagger\gamma^0\gamma^{bac}\nabla_c\varepsilon + \bar\varepsilon\gamma^{bac}D_b(\nabla_c\varepsilon)\\
&\quad - D_b(\nabla_c\bar\varepsilon)\gamma^{bac}\varepsilon - \mathrm{i}k(n-2)\nabla_b(\bar\varepsilon)\gamma^{ab}\varepsilon + \nabla_c(\bar\varepsilon)\gamma^{bac}\mathcal{A}_b\varepsilon && (3.13)\\
&= 2\nabla_b(\bar\varepsilon)\gamma^{bac}\nabla_c\varepsilon - \mathrm{i}k(n-2)\bar\varepsilon\gamma^{ab}D_b\varepsilon + k^2(n-2)\bar\varepsilon\gamma^{ab}\gamma_b\varepsilon - \mathrm{i}k(n-2)\bar\varepsilon\gamma^{ab}\mathcal{A}_b\varepsilon\\
&\quad - \bar\varepsilon\gamma^0\mathcal{A}_b^\dagger\gamma^0\gamma^{bac}D_c\varepsilon - \mathrm{i}k\bar\varepsilon\gamma^0\mathcal{A}_b^\dagger\gamma^0\gamma^{bac}\gamma_c\varepsilon - \bar\varepsilon\gamma^0\mathcal{A}_b^\dagger\gamma^0\gamma^{bac}\mathcal{A}_c\varepsilon + \bar\varepsilon\gamma^{bac}D_bD_c\varepsilon\\
&\quad + \mathrm{i}k\bar\varepsilon\gamma^{bac}\gamma_cD_b\varepsilon + \bar\varepsilon\gamma^{bac}D_b(\mathcal{A}_c)\varepsilon + \bar\varepsilon\gamma^{bac}\mathcal{A}_cD_b\varepsilon - D_bD_c(\bar\varepsilon)\gamma^{bac}\varepsilon\\
&\quad + \mathrm{i}kD_b(\bar\varepsilon)\gamma_c\gamma^{bac}\varepsilon - D_b(\bar\varepsilon)\gamma^0\mathcal{A}_c^\dagger\gamma^0\gamma^{bac}\varepsilon - \bar\varepsilon\gamma^0D_b(\mathcal{A}_c^\dagger)\gamma^0\gamma^{bac}\varepsilon\\
&\quad - \mathrm{i}k(n-2)D_b(\bar\varepsilon)\gamma^{ab}\varepsilon - k^2(n-2)\bar\varepsilon\gamma_b\gamma^{ab}\varepsilon - \mathrm{i}k(n-2)\bar\varepsilon\gamma^0\mathcal{A}_b^\dagger\gamma^0\gamma^{ab}\varepsilon\\
&\quad + D_c(\bar\varepsilon)\gamma^{bac}\mathcal{A}_b\varepsilon - \mathrm{i}k\bar\varepsilon\gamma_c\gamma^{bac}\mathcal{A}_b\varepsilon + \bar\varepsilon\gamma^0\mathcal{A}_c^\dagger\gamma^0\gamma^{bac}\mathcal{A}_b\varepsilon && (3.14)\\
&= 2\nabla_b(\bar\varepsilon)\gamma^{bac}\nabla_c\varepsilon - \mathrm{i}k(n-2)\bar\varepsilon\gamma^{ab}D_b\varepsilon - k^2(n-2)(n-1)\bar\varepsilon\gamma^a\varepsilon - \mathrm{i}k(n-2)\bar\varepsilon\gamma^{ab}\mathcal{A}_b\varepsilon\\
&\quad - \bar\varepsilon\gamma^0\mathcal{A}_b^\dagger\gamma^0\gamma^{bac}D_c\varepsilon + \mathrm{i}k(n-2)\bar\varepsilon\gamma^0\mathcal{A}_b^\dagger\gamma^0\gamma^{ba}\varepsilon - \bar\varepsilon\gamma^0\mathcal{A}_b^\dagger\gamma^0\gamma^{bac}\mathcal{A}_c\varepsilon\\
&\quad + \frac{1}{2}\bar\varepsilon\left(R^{ab} - \frac{1}{2}g^{ab}R\right)\gamma_b\varepsilon - \mathrm{i}k(n-2)\bar\varepsilon\gamma^{ba}D_b\varepsilon + \bar\varepsilon\gamma^{bac}D_b(\mathcal{A}_c)\varepsilon + \bar\varepsilon\gamma^{bac}\mathcal{A}_cD_b\varepsilon\\
&\quad + \frac{1}{2}\left(R^{ab} - \frac{1}{2}g^{ab}R\right)\bar\varepsilon\gamma_b\varepsilon - \mathrm{i}k(n-2)D_b(\bar\varepsilon)\gamma^{ba}\varepsilon - D_b(\bar\varepsilon)\gamma^0\mathcal{A}_c^\dagger\gamma^0\gamma^{bac}\varepsilon\\
&\quad - \bar\varepsilon\gamma^0D_b(\mathcal{A}_c^\dagger)\gamma^0\gamma^{bac}\varepsilon - \mathrm{i}k(n-2)D_b(\bar\varepsilon)\gamma^{ab}\varepsilon - k^2(n-2)(n-1)\bar\varepsilon\gamma^a\varepsilon\\
&\quad - \mathrm{i}k(n-2)\bar\varepsilon\gamma^0\mathcal{A}_b^\dagger\gamma^0\gamma^{ab}\varepsilon + D_c(\bar\varepsilon)\gamma^{bac}\mathcal{A}_b\varepsilon + \mathrm{i}k(n-2)\bar\varepsilon\gamma^{ba}\mathcal{A}_b\varepsilon\\
&\quad + \bar\varepsilon\gamma^0\mathcal{A}_c^\dagger\gamma^0\gamma^{bac}\mathcal{A}_b\varepsilon && (3.15)\\
&= \bar\varepsilon(8\pi T^{ab}\gamma_b - \mathrm{i}k(n-2)\gamma^{ab}\mathcal{A}_b - \mathrm{i}k(n-2)\gamma^0\mathcal{A}_b^\dagger\gamma^0\gamma^{ab} - 2\gamma^0\mathcal{A}_b^\dagger\gamma^0\gamma^{bac}\mathcal{A}_c\\
&\qquad + \gamma^{bac}D_b\mathcal{A}_c - \gamma^0D_b(\mathcal{A}_c^\dagger)\gamma^0\gamma^{bac})\varepsilon + 2\nabla_b(\bar\varepsilon)\gamma^{bac}\nabla_c\varepsilon\\
&\quad + \bar\varepsilon(\gamma^{bac}\mathcal{A}_c - \gamma^0\mathcal{A}_c^\dagger\gamma^0\gamma^{cab})D_b\varepsilon + D_b(\bar\varepsilon)(\gamma^{cab}\mathcal{A}_c - \gamma^0\mathcal{A}_c^\dagger\gamma^0\gamma^{bac})\varepsilon. && (3.16)
\end{aligned}$$

In vielbein indices $P_a = -\delta_{a0}$. Then, in conjunction with the assumption that $\gamma^{IJ}\mathcal{A}_J$ is hermitian, the previous equation reduces to

$$P_a D_b E^{ba}(\varepsilon) = 2\varepsilon^\dagger \mathbb{M}\varepsilon + 2\nabla_I(\varepsilon)^\dagger \gamma^{IJ}\nabla_J\varepsilon = 2\left((\nabla_I\varepsilon)^\dagger\nabla^I\varepsilon - (\gamma^I\nabla_I\varepsilon)^\dagger\gamma^J\nabla_J\varepsilon + \varepsilon^\dagger\mathbb{M}\varepsilon\right), \quad (3.17)$$

which is the required version of the Lichnerowicz identity. □

Corollary 3.2.1. *The functionals in definition 3.1 can be re-written as*

$$Q(\varepsilon) = 2\int_\Sigma \left((\nabla_I\varepsilon)^\dagger\nabla^I\varepsilon - (\gamma^I\nabla_I\varepsilon)^\dagger\gamma^J\nabla_J\varepsilon + \varepsilon^\dagger\mathbb{M}\varepsilon\right)\mathrm{d}V \text{ and} \quad (3.18)$$

$$Q(\Psi_1,\Psi_2) = 2\int_\Sigma \left((\nabla_I\Psi_1)^\dagger\nabla^I\Psi_2 - (\gamma^I\nabla_I\Psi_1)^\dagger\gamma^J\nabla_J\Psi_2 + \Psi_1^\dagger\mathbb{M}\Psi_2\right)\mathrm{d}V. \quad (3.19)$$

Proof. The expression for $Q(\varepsilon)$ is simply substituting the result of theorem into the definition. $Q(\Psi_1,\Psi_2)$ then follows immediately by the polarisation identity. □

Lemma 3.3. *For any antisymmetric tensor, M^{ab},*

$$P_a D_b M^{ba} = \widetilde{D}_b(P_a M^{ba}), \quad (3.20)$$

where $\widetilde{D}$ is the induced covariant derivative on Σ.

Proof. This result is from [23]. It can be proven as follows. Let h_{ab} be the induced metric on Σ, i.e. $h_{ab} = g_{ab} + P_a P_b$. Then, since $h^a{}_c P_b M^{cb} = P_b M^{ab}$ by M^{ab}'s antisymmetry, the induced covariant derivative acts as

$$\widetilde{D}_b(P_a M^{ba}) = h^c{}_b D_c(P_a M^{ba}) \quad (3.21)$$

$$= K_{ba}M^{ba} + \delta^c{}_b P_a D_c M^{ba} + P^c P_b P_a D_c M^{ba} \quad (3.22)$$

$$= P_a D_b M^{ba}, \quad (3.23)$$

where K_{ab} is the extrinsic curvature of Σ in M and M^{ba}'s antisymmetry has been applied. □

Corollary 3.3.1. *The functionals in definition 3.1 can be re-written as*

$$Q(\varepsilon) = \int_{\partial\Sigma} Q_b P_a E^{ba}(\varepsilon)\mathrm{d}A \text{ and } Q(\Psi_1,\Psi_2) = \int_{\partial\Sigma} Q_b P_a E^{ba}(\Psi_1,\Psi_2)\mathrm{d}A, \quad (3.24)$$

where Q^a is the outward-pointing unit normal to $\partial\Sigma$.

Lemma 3.4. *Let $\partial\Sigma = S$, $\mathcal{A}_a = 0$ and $\Psi = (\psi_\alpha, \bar{\chi}^{\dot\alpha})^T$. Then,*

$$Q(\Psi) = 4\int_S \Big(\psi_\iota\eth\overline{\psi}_o + \overline{\psi}_\iota\bar{\eth}\psi_o - \overline{\chi}_o\eth\chi_\iota - \chi_o\bar{\eth}\overline{\chi}_\iota + \rho|\psi_o|^2 + \mu|\psi_\iota|^2 + \rho|\chi_o|^2 + \mu|\chi_\iota|^2$$
$$+ \mathrm{i}k\sqrt{2}\big(\psi_o\chi_\iota + \psi_\iota\chi_o - \overline{\psi}_o\overline{\chi}_\iota - \overline{\psi}_\iota\overline{\chi}_o\big)\Big)\mathrm{d}A. \quad (3.25)$$

This decomposition into GHP quantities will be essential to the definition and analysis of quasilocal mass later in chapter 5. In particular, it allows $Q(\Psi)$ to be written in terms of scalar functions and derivatives completely intrinsic to the boundary, S.

Proof. By equation 2.24 and corollary 3.3.1,

$$Q(\Psi) = \int_S P_a Q_b E^{ba}(\Psi)\mathrm{d}A = \int_S l_a n_b E^{ab}(\Psi)\mathrm{d}A = \frac{1}{4}\int_S l_{\alpha\dot\alpha} n_{\beta\dot\beta} E^{\alpha\dot\alpha\beta\dot\beta}(\Psi)\mathrm{d}A. \quad (3.26)$$

It's a surprisingly long calculation to find $E_{\alpha\dot{\alpha}\beta\dot{\beta}}(\Psi)$.

$$E_{\alpha\dot{\alpha}\beta\dot{\beta}}(\Psi) = (\sigma_a)_{\alpha\dot{\alpha}}(\sigma_b)_{\beta\dot{\beta}}E^{ab}(\Psi) = (\sigma_a)_{\alpha\dot{\alpha}}(\sigma_b)_{\beta\dot{\beta}}(\overline{\Psi}\gamma^{abc}\nabla_c\Psi - \nabla_c(\overline{\Psi})\gamma^{abc}\Psi). \tag{3.27}$$

It suffices to study only the first term; the second term is just the complex conjugate. By applying the Weyl representation of the gamma matrices, as given in appendix A,

$$\begin{aligned}
&(\sigma_a)_{\alpha\dot{\alpha}}(\sigma_b)_{\beta\dot{\beta}}\overline{\Psi}\gamma^{abc}\nabla_c\Psi \\
&= (\sigma_a)_{\alpha\dot{\alpha}}(\sigma_b)_{\beta\dot{\beta}}\overline{\Psi}\gamma^{abc}(D_c\Psi + \mathrm{i}k\gamma_c\Psi) &(3.28)\\
&= (\sigma_a)_{\alpha\dot{\alpha}}(\sigma_b)_{\beta\dot{\beta}}\overline{\Psi}\gamma^{abc}D_c\Psi - 2\mathrm{i}k(\sigma_a)_{\alpha\dot{\alpha}}(\sigma_b)_{\beta\dot{\beta}}\overline{\Psi}\gamma^{ab}\Psi &(3.29)\\
&= (\sigma_a)_{\alpha\dot{\alpha}}(\sigma_b)_{\beta\dot{\beta}}\left[-\chi^\gamma, -\overline{\psi}_{\dot{\gamma}}\right]\begin{bmatrix}0 & (\sigma^{[a}\tilde{\sigma}^b\sigma^{c]})_{\gamma\dot{\gamma}}\\ (\tilde{\sigma}^{[a}\sigma^b\tilde{\sigma}^{c]})^{\dot{\gamma}\gamma} & 0\end{bmatrix}\begin{bmatrix}D_c\psi_\gamma\\ D_c\bar{\chi}^{\dot{\gamma}}\end{bmatrix}\\
&\quad - 2\mathrm{i}k(\sigma_a)_{\alpha\dot{\alpha}}(\sigma_b)_{\beta\dot{\beta}}\left[-\chi^\gamma, -\overline{\psi}_{\dot{\gamma}}\right]\begin{bmatrix}(\sigma^{[a}\tilde{\sigma}^{b]})_\gamma{}^\delta & 0\\ 0 & (\tilde{\sigma}^{[a}\sigma^{b]})^{\dot{\gamma}}{}_{\dot{\delta}}\end{bmatrix}\begin{bmatrix}\psi_\delta\\ \bar{\chi}^{\dot{\delta}}\end{bmatrix} &(3.30)\\
&= -(\sigma_a)_{\alpha\dot{\alpha}}(\sigma_b)_{\beta\dot{\beta}}\chi^\gamma(\sigma^{[a}\tilde{\sigma}^b\sigma^{c]})_{\gamma\dot{\gamma}}D_c\bar{\chi}^{\dot{\gamma}} - (\sigma_a)_{\alpha\dot{\alpha}}(\sigma_b)_{\beta\dot{\beta}}\overline{\psi}_{\dot{\gamma}}(\tilde{\sigma}^{[a}\sigma^b\tilde{\sigma}^{c]})^{\dot{\gamma}\gamma}D_c\psi_\gamma\\
&\quad + 2\mathrm{i}k(\sigma_a)_{\alpha\dot{\alpha}}(\sigma_b)_{\beta\dot{\beta}}\chi^\gamma(\sigma^{[a}\tilde{\sigma}^{b]})_\gamma{}^\delta\psi_\delta + 2\mathrm{i}k(\sigma_a)_{\alpha\dot{\alpha}}(\sigma_b)_{\beta\dot{\beta}}\overline{\psi}_{\dot{\gamma}}(\tilde{\sigma}^{[a}\sigma^{b]})^{\dot{\gamma}}{}_{\dot{\delta}}\bar{\chi}^{\dot{\delta}}. &(3.31)
\end{aligned}$$

Consider this expression term by term. From the identity,

$$(\sigma_a)_{\alpha\dot{\beta}}(\tilde{\sigma}_b)^{\dot{\beta}\beta}(\sigma_c)_{\beta\dot{\alpha}} = g_{ca}(\sigma_b)_{\alpha\dot{\alpha}} - g_{bc}(\sigma_a)_{\alpha\dot{\alpha}} - g_{ab}(\sigma_c)_{\alpha\dot{\alpha}} + \mathrm{i}\varepsilon_{abcd}(\sigma^d)_{\alpha\dot{\alpha}}, \tag{3.32}$$

it follows that

$$(\sigma_{[a}\tilde{\sigma}_b\sigma_{c]})_{\alpha\dot{\alpha}} = \mathrm{i}\varepsilon_{abcd}(\sigma^d)_{\alpha\dot{\alpha}} = (\sigma_a)_{\alpha\dot{\beta}}(\tilde{\sigma}_b)^{\dot{\beta}\beta}(\sigma_c)_{\beta\dot{\alpha}} - g_{ca}(\sigma_b)_{\alpha\dot{\alpha}} + g_{bc}(\sigma_a)_{\alpha\dot{\alpha}} + g_{ab}(\sigma_c)_{\alpha\dot{\alpha}}. \tag{3.33}$$

In conjunction with the identities in appendix B, the original term then reduces to

$$\begin{aligned}
&-(\sigma_a)_{\alpha\dot{\alpha}}(\sigma_b)_{\beta\dot{\beta}}\chi^\gamma(\sigma^{[a}\tilde{\sigma}^b\sigma^{c]})_{\gamma\dot{\gamma}}D_c\bar{\chi}^{\dot{\gamma}}\\
&= -(\sigma_a)_{\alpha\dot{\alpha}}(\sigma_b)_{\beta\dot{\beta}}\Big((\sigma^a)_{\gamma\dot{\delta}}(\tilde{\sigma}^b)^{\dot{\delta}\delta}(\sigma^c)_{\delta\dot{\gamma}} - g^{ca}(\sigma^b)_{\gamma\dot{\gamma}} + g^{bc}(\sigma^a)_{\gamma\dot{\gamma}} + g^{ab}(\sigma^c)_{\gamma\dot{\gamma}}\Big)\chi^\gamma D_c\bar{\chi}^{\dot{\gamma}} &(3.34)\\
&= \Big(-4\varepsilon_{\alpha\gamma}\varepsilon_{\dot{\alpha}\dot{\delta}}\delta^\delta{}_\beta\delta^{\dot{\delta}}{}_{\dot{\beta}}(\sigma^c)_{\delta\dot{\gamma}} - 2\varepsilon_{\beta\gamma}\varepsilon_{\dot{\beta}\dot{\gamma}}(\sigma^c)_{\alpha\dot{\alpha}} + 2\varepsilon_{\alpha\gamma}\varepsilon_{\dot{\alpha}\dot{\gamma}}(\sigma^c)_{\beta\dot{\beta}} + 2\varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}}(\sigma^c)_{\gamma\dot{\gamma}}\Big)\chi^\gamma D_c\bar{\chi}^{\dot{\gamma}} &(3.35)\\
&= \big(-4\varepsilon_{\alpha\gamma}\varepsilon_{\dot{\alpha}\dot{\beta}}(\sigma^c)_{\beta\dot{\gamma}} - 2\varepsilon_{\beta\gamma}\varepsilon_{\dot{\beta}\dot{\gamma}}(\sigma^c)_{\alpha\dot{\alpha}} + 2\varepsilon_{\alpha\gamma}\varepsilon_{\dot{\alpha}\dot{\gamma}}(\sigma^c)_{\beta\dot{\beta}} + 2\varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}}(\sigma^c)_{\gamma\dot{\gamma}}\big)\chi^\gamma D_c\bar{\chi}^{\dot{\gamma}} &(3.36)\\
&= -4\varepsilon_{\dot{\alpha}\dot{\beta}}\chi_\alpha D_{\beta\dot{\gamma}}\bar{\chi}^{\dot{\gamma}} - 2\chi_\beta D_{\alpha\dot{\alpha}}\bar{\chi}_{\dot{\beta}} + 2\chi_\alpha D_{\beta\dot{\beta}}\bar{\chi}_{\dot{\alpha}} + 2\varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}}\chi^\gamma D_{\gamma\dot{\gamma}}\bar{\chi}^{\dot{\gamma}} &(3.37)\\
&= -4\varepsilon_{\dot{\alpha}\dot{\beta}}\varepsilon^{\dot{\gamma}\dot{\delta}}\chi_\alpha D_{\beta\dot{\gamma}}\bar{\chi}_{\dot{\delta}} - 2\chi_\beta D_{\alpha\dot{\alpha}}\bar{\chi}_{\dot{\beta}} + 2\chi_\alpha D_{\beta\dot{\beta}}\bar{\chi}_{\dot{\alpha}} + 2\varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}}\varepsilon^{\gamma\delta}\varepsilon^{\dot{\gamma}\dot{\delta}}\chi_\delta D_{\gamma\dot{\gamma}}\bar{\chi}_{\dot{\delta}} &(3.38)\\
&= 4(\delta^{\dot{\gamma}}{}_{\dot{\alpha}}\delta^{\dot{\delta}}{}_{\dot{\beta}} - \delta^{\dot{\gamma}}{}_{\dot{\beta}}\delta^{\dot{\delta}}{}_{\dot{\alpha}})\chi_\alpha D_{\beta\dot{\gamma}}\bar{\chi}_{\dot{\delta}} - 2\chi_\beta D_{\alpha\dot{\alpha}}\bar{\chi}_{\dot{\beta}} + 2\chi_\alpha D_{\beta\dot{\beta}}\bar{\chi}_{\dot{\alpha}}\\
&\quad + 2(\delta^\gamma{}_\alpha\delta^\delta{}_\beta - \delta^\gamma{}_\beta\delta^\delta{}_\alpha)(\delta^{\dot{\gamma}}{}_{\dot{\alpha}}\delta^{\dot{\delta}}{}_{\dot{\beta}} - \delta^{\dot{\gamma}}{}_{\dot{\beta}}\delta^{\dot{\delta}}{}_{\dot{\alpha}})\chi_\delta D_{\gamma\dot{\gamma}}\bar{\chi}_{\dot{\delta}} &(3.39)\\
&= 4\chi_\alpha D_{\beta\dot{\alpha}}\bar{\chi}_{\dot{\beta}} - 4\chi_\alpha D_{\beta\dot{\beta}}\bar{\chi}_{\dot{\alpha}} - 2\chi_\beta D_{\alpha\dot{\alpha}}\bar{\chi}_{\dot{\beta}} + 2\chi_\alpha D_{\beta\dot{\beta}}\bar{\chi}_{\dot{\alpha}} + 2\chi_\beta D_{\alpha\dot{\alpha}}\bar{\chi}_{\dot{\beta}} - 2\chi_\alpha D_{\beta\dot{\alpha}}\bar{\chi}_{\dot{\beta}}\\
&\quad - 2\chi_\beta D_{\alpha\dot{\beta}}\bar{\chi}_{\dot{\alpha}} + 2\chi_\alpha D_{\beta\dot{\beta}}\bar{\chi}_{\dot{\alpha}} &(3.40)\\
&= 2\chi_\alpha D_{\beta\dot{\alpha}}\bar{\chi}_{\dot{\beta}} - 2\chi_\beta D_{\alpha\dot{\beta}}\bar{\chi}_{\dot{\alpha}}. &(3.41)
\end{aligned}$$

The other terms in equation 3.31 are handled similarly. For the second term,

$$(\tilde{\sigma}_{[a}\sigma_b\tilde{\sigma}_{c]})^{\dot{\alpha}\alpha} = -\mathrm{i}\varepsilon_{abcd}(\tilde{\sigma}^d)^{\dot{\alpha}\alpha} = (\tilde{\sigma}_a)^{\dot{\alpha}\beta}(\sigma_b)_{\beta\dot{\beta}}(\tilde{\sigma}_c)^{\dot{\beta}\alpha} - g_{ca}(\tilde{\sigma}_b)^{\dot{\alpha}\alpha} + g_{bc}(\tilde{\sigma}_a)^{\dot{\alpha}\alpha} + g_{ab}(\tilde{\sigma}_c)^{\dot{\alpha}\alpha} \tag{3.42}$$

is applied to get

$$
\begin{aligned}
&-(\sigma_a)_{\alpha\dot{\alpha}}(\sigma_b)_{\beta\dot{\beta}}\overline{\psi}_{\dot{\gamma}}(\tilde{\sigma}^{[a}\sigma^b\tilde{\sigma}^{c]})^{\dot{\gamma}\gamma}D_c\psi_\gamma \\
&= -(\sigma_a)_{\alpha\dot{\alpha}}(\sigma_b)_{\beta\dot{\beta}}\Big((\tilde{\sigma}^a)^{\dot{\gamma}\delta}(\sigma^b)_{\delta\dot{\delta}}(\tilde{\sigma}^c)^{\dot{\delta}\gamma} - g^{ca}(\tilde{\sigma}^b)^{\dot{\gamma}\gamma} + g^{bc}(\tilde{\sigma}^a)^{\dot{\gamma}\gamma} + g^{ab}(\tilde{\sigma}^c)^{\dot{\gamma}\gamma}\Big)\overline{\psi}_{\dot{\gamma}}D_c\psi_\gamma && (3.43)\\
&= \Big(-4\delta^\delta{}_\alpha\delta^{\dot{\gamma}}{}_{\dot{\alpha}}\varepsilon_{\beta\delta}\varepsilon_{\dot{\beta}\dot{\delta}}(\tilde{\sigma}^c)^{\dot{\delta}\gamma} - 2\delta^\gamma{}_\beta\delta^{\dot{\gamma}}{}_{\dot{\beta}}(\sigma^c)_{\alpha\dot{\alpha}} + 2\delta^\gamma{}_\alpha\delta^{\dot{\gamma}}{}_{\dot{\alpha}}(\sigma^c)_{\beta\dot{\beta}} + 2\varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}}(\tilde{\sigma}^c)^{\dot{\gamma}\gamma}\Big)\overline{\psi}_{\dot{\gamma}}D_c\psi_\gamma && (3.44)\\
&= -4\varepsilon_{\beta\alpha}\overline{\psi}_{\dot{\alpha}}D^\gamma{}_{\dot{\beta}}\psi_\gamma - 2\overline{\psi}_{\dot{\beta}}D_{\alpha\dot{\alpha}}\psi_\beta + 2\overline{\psi}_{\dot{\alpha}}D_{\beta\dot{\beta}}\psi_\alpha + 2\varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}}\overline{\psi}_{\dot{\gamma}}D^{\gamma\dot{\gamma}}\psi_\gamma && (3.45)\\
&= -4\varepsilon_{\beta\alpha}\varepsilon^{\gamma\delta}\overline{\psi}_{\dot{\alpha}}D_{\delta\dot{\beta}}\psi_\gamma - 2\overline{\psi}_{\dot{\beta}}D_{\alpha\dot{\alpha}}\psi_\beta + 2\overline{\psi}_{\dot{\alpha}}D_{\beta\dot{\beta}}\psi_\alpha + 2\varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}}\varepsilon^{\gamma\delta}\varepsilon^{\dot{\gamma}\dot{\delta}}\overline{\psi}_{\dot{\gamma}}D_{\delta\dot{\delta}}\psi_\gamma && (3.46)\\
&= 4(\delta^\gamma{}_\beta\delta^\delta{}_\alpha - \delta^\gamma{}_\alpha\delta^\delta{}_\beta)\overline{\psi}_{\dot{\alpha}}D_{\delta\dot{\beta}}\psi_\gamma - 2\overline{\psi}_{\dot{\beta}}D_{\alpha\dot{\alpha}}\psi_\beta + 2\overline{\psi}_{\dot{\alpha}}D_{\beta\dot{\beta}}\psi_\alpha \\
&\quad + 2(\delta^\gamma{}_\alpha\delta^\delta{}_\beta - \delta^\gamma{}_\beta\delta^\delta{}_\alpha)(\delta^{\dot{\gamma}}{}_{\dot{\alpha}}\delta^{\dot{\delta}}{}_{\dot{\beta}} - \delta^{\dot{\gamma}}{}_{\dot{\beta}}\delta^{\dot{\delta}}{}_{\dot{\alpha}})\overline{\psi}_{\dot{\gamma}}D_{\delta\dot{\delta}}\psi_\gamma && (3.47)\\
&= 2\overline{\psi}_{\dot{\alpha}}D_{\alpha\dot{\beta}}\psi_\beta - 2\overline{\psi}_{\dot{\beta}}D_{\beta\dot{\alpha}}\psi_\alpha. && (3.48)
\end{aligned}
$$

Finally, the last two terms of equation 3.31 simplify to

$$
\begin{aligned}
&2\mathrm{i}k(\sigma_a)_{\alpha\dot{\alpha}}(\sigma_b)_{\beta\dot{\beta}}\chi^\gamma(\sigma^{[a}\tilde{\sigma}^{b]})_\gamma{}^\delta\psi_\delta + 2\mathrm{i}k(\sigma_a)_{\alpha\dot{\alpha}}(\sigma_b)_{\beta\dot{\beta}}\overline{\psi}_{\dot{\gamma}}(\tilde{\sigma}^{[a}\sigma^{b]})^{\dot{\gamma}}{}_{\dot{\delta}}\bar{\chi}^{\dot{\delta}} \\
&= \mathrm{i}k(\sigma_a)_{\alpha\dot{\alpha}}(\sigma_b)_{\beta\dot{\beta}}\big((\sigma^a)_{\gamma\dot{\delta}}(\tilde{\sigma}^b)^{\dot{\delta}\delta}\chi^\gamma\psi_\delta - (\sigma^b)_{\gamma\dot{\delta}}(\tilde{\sigma}^a)^{\dot{\delta}\delta}\chi^\gamma\psi_\delta + (\tilde{\sigma}^a)^{\dot{\gamma}\delta}(\sigma^b)_{\delta\dot{\delta}}\overline{\psi}_{\dot{\gamma}}\bar{\chi}^{\dot{\delta}} \\
&\quad - (\tilde{\sigma}^b)^{\dot{\gamma}\delta}(\sigma^a)_{\delta\dot{\delta}}\overline{\psi}_{\dot{\gamma}}\bar{\chi}^{\dot{\delta}}\big) && (3.49)\\
&= 4\mathrm{i}k\big(\varepsilon_{\alpha\gamma}\varepsilon_{\dot{\alpha}\dot{\delta}}\delta^\delta{}_\beta\delta^{\dot{\delta}}{}_{\dot{\beta}}\chi^\gamma\psi_\delta - \varepsilon_{\beta\gamma}\varepsilon_{\dot{\beta}\dot{\delta}}\delta^\delta{}_\alpha\delta^{\dot{\delta}}{}_{\dot{\alpha}}\chi^\gamma\psi_\delta + \varepsilon_{\beta\delta}\varepsilon_{\dot{\beta}\dot{\delta}}\delta^\delta{}_\alpha\delta^{\dot{\gamma}}{}_{\dot{\alpha}}\overline{\psi}_{\dot{\gamma}}\bar{\chi}^{\dot{\delta}} - \varepsilon_{\alpha\delta}\varepsilon_{\dot{\alpha}\dot{\delta}}\delta^\delta{}_\beta\delta^{\dot{\gamma}}{}_{\dot{\beta}}\overline{\psi}_{\dot{\gamma}}\bar{\chi}^{\dot{\delta}}\big) && (3.50)\\
&= -4\mathrm{i}k\varepsilon_{\alpha\beta}(\overline{\psi}_{\dot{\alpha}}\bar{\chi}_{\dot{\beta}} + \overline{\psi}_{\dot{\beta}}\bar{\chi}_{\dot{\alpha}}) + 4\mathrm{i}k\varepsilon_{\dot{\alpha}\dot{\beta}}(\chi_\alpha\psi_\beta + \chi_\beta\psi_\alpha). && (3.51)
\end{aligned}
$$

Putting it all together, equation 3.31 reduces to

$$
\begin{aligned}
(\sigma_a)_{\alpha\dot{\alpha}}(\sigma_b)_{\beta\dot{\beta}}\overline{\Psi}\gamma^{abc}\nabla_c\Psi &= 2\chi_\alpha D_{\beta\dot{\alpha}}\bar{\chi}_{\dot{\beta}} - 2\chi_\beta D_{\alpha\dot{\beta}}\bar{\chi}_{\dot{\alpha}} + 2\overline{\psi}_{\dot{\alpha}}D_{\alpha\dot{\beta}}\psi_\beta - 2\overline{\psi}_{\dot{\beta}}D_{\beta\dot{\alpha}}\psi_\alpha \\
&\quad - 4\mathrm{i}k\varepsilon_{\alpha\beta}(\overline{\psi}_{\dot{\alpha}}\bar{\chi}_{\dot{\beta}} + \overline{\psi}_{\dot{\beta}}\bar{\chi}_{\dot{\alpha}}) + 4\mathrm{i}k\varepsilon_{\dot{\alpha}\dot{\beta}}(\chi_\alpha\psi_\beta + \chi_\beta\psi_\alpha). && (3.52)
\end{aligned}
$$

Then, by adding the complex conjugate,

$$
\begin{aligned}
E_{\alpha\dot{\alpha}\beta\dot{\beta}} &= 2\big(\chi_\alpha D_{\beta\dot{\alpha}}\bar{\chi}_{\dot{\beta}} - \chi_\beta D_{\alpha\dot{\beta}}\bar{\chi}_{\dot{\alpha}} + \overline{\psi}_{\dot{\alpha}}D_{\alpha\dot{\beta}}\psi_\beta - \overline{\psi}_{\dot{\beta}}D_{\beta\dot{\alpha}}\psi_\alpha + \bar{\chi}_{\dot{\alpha}}D_{\alpha\dot{\beta}}\chi_\beta - \bar{\chi}_{\dot{\beta}}D_{\beta\dot{\alpha}}\chi_\alpha \\
&\quad + \psi_\alpha D_{\beta\dot{\alpha}}\overline{\psi}_{\dot{\beta}} - \psi_\beta D_{\alpha\dot{\beta}}\overline{\psi}_{\dot{\alpha}}\big) + 8\mathrm{i}k\big(-\varepsilon_{\alpha\beta}(\overline{\psi}_{\dot{\alpha}}\bar{\chi}_{\dot{\beta}} + \overline{\psi}_{\dot{\beta}}\bar{\chi}_{\dot{\alpha}}) + \varepsilon_{\dot{\alpha}\dot{\beta}}(\psi_\alpha\chi_\beta + \psi_\beta\chi_\alpha)\big). && (3.53)
\end{aligned}
$$

Therefore, by equations 2.59 and 2.62, the required integrand is

$$
\begin{aligned}
l^{\alpha\dot{\alpha}}n^{\beta\dot{\beta}}E_{\alpha\dot{\alpha}\beta\dot{\beta}}(\Psi) &= o^\alpha\bar{o}^{\dot{\alpha}}\iota^\beta\bar{\iota}^{\dot{\beta}}E_{\alpha\dot{\alpha}\beta\dot{\beta}} && (3.54)\\
&= 4\sqrt{2}\big(\chi_\iota\bar{\iota}^{\dot{\beta}}\delta\bar{\chi}_{\dot{\beta}} + \chi_o\bar{o}^{\dot{\alpha}}\bar{\delta}\bar{\chi}_{\dot{\alpha}} + \overline{\psi}_\iota\iota^\beta\bar{\delta}\psi_\beta + \overline{\psi}_o o^\alpha\delta\psi_\alpha + \overline{\chi}_\iota\iota^\beta\bar{\delta}\chi_\beta + \overline{\chi}_o o^\alpha\delta\chi_\alpha \\
&\quad + \psi_\iota\bar{\iota}^{\dot{\beta}}\delta\overline{\psi}_{\dot{\beta}} + \psi_o\bar{o}^{\dot{\alpha}}\bar{\delta}\,\overline{\psi}_{\dot{\alpha}}\big) + 16\sqrt{2}\mathrm{i}k\big(-\overline{\psi}_\iota\overline{\chi}_o - \overline{\psi}_o\overline{\chi}_\iota + \psi_\iota\chi_o + \psi_o\chi_\iota\big). && (3.55)
\end{aligned}
$$

Applying definition 2.9 and the GHP form of NP coefficients in appendix B,

$$
\begin{aligned}
\bar{\iota}^{\dot{\beta}}\delta\bar{\chi}_{\dot{\beta}} &= \bar{\iota}^{\dot{\beta}}\delta\left(\overline{\chi}_o\bar{o}_{\dot{\beta}} + \overline{\chi}_\iota\bar{\iota}_{\dot{\beta}}\right) && (3.56)\\
&= \sqrt{2}\delta\overline{\chi}_o + \overline{\chi}_o\bar{\iota}^{\dot{\beta}}\delta\bar{o}_{\dot{\beta}} + 0 + \overline{\chi}_\iota\bar{\iota}^{\dot{\beta}}\delta\bar{\iota}_{\dot{\beta}} && (3.57)\\
&= \sqrt{2}\delta\overline{\chi}_o + \sqrt{2}\overline{\chi}_o\beta + \sqrt{2}\overline{\chi}_\iota\mu && (3.58)\\
&= \sqrt{2}\left(\eth\overline{\chi}_o + \mu\overline{\chi}_\iota\right). && (3.59)
\end{aligned}
$$

Similarly, I also get

$$
\begin{aligned}
\bar{o}^{\dot{\alpha}}\bar{\delta}\bar{\chi}_{\dot{\alpha}} &= -\sqrt{2}\left(\bar{\eth}\overline{\chi}_\iota - \rho\overline{\chi}_o\right), && (3.60)\\
\iota^\beta\bar{\delta}\psi_\beta &= \sqrt{2}\left(\bar{\eth}\psi_o + \mu\psi_\iota\right) \quad \text{and} && (3.61)\\
o^\alpha\delta\psi_\alpha &= -\sqrt{2}\left(\eth\psi_\iota - \rho\psi_o\right). && (3.62)
\end{aligned}
$$

Substituting back, simplifying and taking complex conjugates as needed yields

$$\begin{aligned} l^{\alpha\dot{\alpha}}n^{\beta\dot{\beta}}E_{\alpha\dot{\alpha}\beta\dot{\beta}}(\Psi) &= 8\big(\chi_\iota \eth\overline{\chi}_o - \chi_o\bar{\eth}\overline{\chi}_\iota + \overline{\psi}_\iota\bar{\eth}\psi_o - \overline{\psi}_o\eth\psi_\iota + \overline{\chi}_\iota\bar{\eth}\chi_o - \overline{\chi}_o\eth\chi_\iota + \psi_\iota\eth\overline{\psi}_o - \psi_o\bar{\eth}\overline{\psi}_\iota\big) \\ &+ 16\big(\mu|\chi_\iota|^2 + \rho|\chi_o|^2 + \mu|\psi_\iota|^2 + \rho|\psi_o|^2\big) \\ &+ 16\sqrt{2}\mathrm{i}k\big(-\overline{\psi}_\iota\overline{\chi}_o - \overline{\psi}_o\overline{\chi}_\iota + \psi_\iota\chi_o + \psi_o\chi_\iota\big). \end{aligned} \tag{3.63}$$

Substituting equation 3.63 into equation 3.26 and integrating by parts - which is valid by corollary 2.10.1 - proves the lemma. □

3.2 Properties of the Dirac operator

Definition 3.5 (C_c^∞)**.** *Let C_c^∞ denote the set of Dirac spinors of (M, g) which are smooth and have compact support when restricted to Σ. In the case where Σ itself is 3D and compact with boundary, further impose that the Dirac spinors, $\Psi = (\psi_\alpha, \overline{\chi}^{\dot{\alpha}})^T$, are subject to the boundary conditions, $\psi_o = \chi_\iota = 0$ on $S = \partial\Sigma$.*

Lemma 3.6. *When Σ is non-compact, an inner product can be defined on C_c^∞ by*

$$\langle \Psi, \Psi_2 \rangle_{C_c^\infty} = \int_{\Sigma_t} \Big((\nabla_I \Psi_1)^\dagger \nabla^I \Psi_2 + \Psi_1^\dagger \mathbb{M} \Psi_2 \Big)\, \mathrm{d}V. \tag{3.64}$$

Proof. Linearity and conjugate symmetry are manifest. $\langle \Psi, \Psi \rangle_{C_c^\infty} \geq 0$ is also immediate because $\mathbb{M}$ is assumed to be non-negative definite.

Finally, suppose $\langle \Psi, \Psi \rangle_{C_c^\infty} = 0$. Thus $\nabla_I \Psi = 0$ on Σ_t. In the non-compact case, C_c^∞ only has compactly supported spinors by definition and hence there is a point, say q, where $\Psi = 0$.

Next, choose an arbitrary point, $p \in \Sigma_t$, and a smooth curve from q to p. Let t^I be the tangent to the curve. Then, $t^I \nabla_I \Psi = 0$ along the curve and $\Psi = 0$ at the initial point, q. It follows that $\Psi = 0$ everywhere along the curve since Ψ is also smooth and thus 1st order, linear, homogeneous ODEs will have a unique solution. However, since p is arbitrary, Ψ must be zero everywhere on Σ_t. Therefore, it follows that $\langle \cdot, \cdot \rangle_{C_c^\infty}$ is positive definite. □

Lemma 3.7. *When Σ is 3D and compact, assuming $\mathcal{A}_a$ is zero*[4] *and the null expansions on S satisfy $\theta_l > 0$, $\theta_n < 0$ & $\theta_l\theta_n < -8k^2$, an inner product can be defined on C_c^∞ by*

$$\langle \Psi_1, \Psi_2 \rangle_{C_c^\infty} = \int_{\Sigma} \Big((\nabla_I \Psi_1)^\dagger \nabla^I \Psi_2 + 4\pi T^{0a} \Psi_1^\dagger \gamma_0 \gamma_a \Psi_2 \Big)\, \mathrm{d}V - Q(\Psi_1, \Psi_2). \tag{3.65}$$

Proof. Linearity and conjugate symmetry are again manifest. However, in the compact case, while the Σ integral in $\langle \Psi, \Psi \rangle_{C_c^\infty}$ is non-negative, it remains to analyse $Q(\Psi, \Psi) = Q(\Psi)$. $\Psi \in C_c^\infty$ implies $\psi_o = \chi_\iota = 0$ on S. Then, by lemma 3.4,

$$Q(\Psi) = 4 \int_S \Big(\mu|\psi_\iota|^2 + \rho|\chi_o|^2 + \mathrm{i}k\sqrt{2}\big(\psi_\iota\chi_o - \overline{\psi}_\iota\overline{\chi}_o\big) \Big) \mathrm{d}A. \tag{3.66}$$

For any nowhere vanishing, complex function, z, on S, $Q(\Psi)$ can also be written as[5]

$$Q(\Psi) = 4 \int_S \left(\frac{\mu}{|z|^2}|z\psi_\iota|^2 + \rho|z|^2\left|\frac{\chi_o}{z}\right|^2 + \mathrm{i}k\sqrt{2}\left(z\psi_\iota\frac{\chi_o}{z} - \bar{z}\overline{\psi}_\iota\frac{\overline{\chi}_o}{\bar{z}} \right) \right) \mathrm{d}A \tag{3.67}$$

[4] I could assume $\mathcal{A}_a \neq 0$, but $P_a Q_b \gamma^{bac} \mathcal{A}_a$ is non-positive definite when acting as a bilinear form between spinors on $S = \partial\Sigma$ with $\psi_o = \chi_\iota = 0$. However, since I will only need the case $\mathcal{A}_a = 0$ in chapter 5, I'm assuming that for simplicity here. See appendix B of [88] for a situation when one only has the weaker, non-zero condition on $\mathcal{A}_a$. Also note that assuming $\mathcal{A}_a = 0$ means my assumption on $\mathbb{M}$ reduces to the dominant energy condition for T_{ab}.

[5] The next equation is formally identical to GHP boost invariance.

Let $\mu' = \mu/|z|^2$, $\rho' = |z|^2\rho$, $\psi'_\iota = z\psi_\iota$ and $\chi'_o = \chi_o/z$. Then,

$$Q(\Psi) = 4\int_S \Big(\mu'|\psi'_\iota|^2 + \rho'|\chi'_o|^2 + \mathrm{i}k\sqrt{2}\big(\psi'_\iota\chi'_o - \overline{\psi}'_\iota\overline{\chi}'_o\big)\Big)\mathrm{d}A \tag{3.68}$$

$$= 4\int_S \Big((\mu' + k\sqrt{2})|\psi'_\iota|^2 + (\rho' + k\sqrt{2})|\chi'_o|^2 - k\sqrt{2}|\psi'_\iota + \mathrm{i}\overline{\chi}'_o|^2\Big)\mathrm{d}A \tag{3.69}$$

$$\leq 4\int_S \Big((\mu' + k\sqrt{2})|\psi'_\iota|^2 + (\rho' + k\sqrt{2})|\chi'_o|^2\Big)\mathrm{d}A. \tag{3.70}$$

Choose $z = \sqrt[4]{\mu/\rho}$ so that $\mu' = \rho' = -\sqrt{\mu\rho} = -\frac{1}{2}\sqrt{-\theta_l\theta_n} < -k\sqrt{2}$ by lemma 2.6. Therefore it immediately follows that $Q(\Psi) \leq 0$ and hence $\langle\Psi,\Psi\rangle_{C_c^\infty} \geq 0$.

Finally, suppose $\langle\Psi,\Psi\rangle_{C_c^\infty} = 0$. As before, this means $\nabla_I\Psi = 0$ on Σ. But, in the compact case, $\langle\Psi,\Psi\rangle_{C_c^\infty} = 0$ also implies $Q(\Psi) = 0$. The boundary conditions already imply $\psi_o = \chi_\iota = 0$ on S, equation 3.70 then implies $\chi_o = \psi_\iota = 0$ on S too and therefore $\Psi = 0$ on S. Since $\nabla_I\Psi = 0$ on Σ and there is a point where $\Psi = 0$, $\Psi = 0$ everywhere by the same steps as before. □

The proof of positive-definiteness in the previous lemmas was simplified by invoking smoothness. However, there is an alternative method based on [98] which is applicable to lower regularity metrics. In particular, if σ is the metric on Σ, then $\nabla_i\Psi$ is re-written as $D_i^{(\sigma)}\Psi + \mathcal{F}_i\Psi$ for some Clifford algebra valued one-form, $\mathcal{F}$. Then, letting $||\Psi||_S^2 = \Psi^\dagger\Psi$,

$$\left|\partial_i(\ln(||\Psi||_S^2))\right| = \frac{1}{||\Psi||_S^2}\left|D_i^{(\sigma)}(\Psi^\dagger\Psi)\right| \tag{3.71}$$

$$\leq \frac{1}{||\Psi||_S^2}\left(\left|D_i^{(h)}(\Psi)^\dagger\Psi\right| + \left|\Psi^\dagger D_i^{(h)}(\Psi)\right|\right) \tag{3.72}$$

$$\leq \frac{2||\Psi||_S||D_i^{(h)}\Psi||_S}{||\Psi||_S^2} \tag{3.73}$$

$$= \frac{2\,||\mathcal{F}_i\Psi||_S}{||\Psi||_S} \tag{3.74}$$

$$\leq 2\,||\mathcal{F}_i||_0 \tag{3.75}$$

$$\iff -2\,||\mathcal{F}_i||_0 \leq \partial_i(\ln(||\psi||_S^2)) \leq 2\,||\mathcal{F}_i||_0\,. \tag{3.76}$$

Since Ψ is compactly supported, by the extreme value theorem, $\exists$ a point, $x_1 \in \Sigma$, where $||\Psi||_S$ is maximised. Likewise, there also exists a point in $\overline{\mathrm{supp}(\Psi)}$ where $||\mathcal{F}_i||_0$ is maximised. Let C_i denote the value of that maximum.

Next, Let x_0 be a point on $\overline{\mathrm{supp}(\Psi)} \cap \Sigma$, where $\Psi = 0$ and choose a curve, s, between x_1 and x_0. This curve necessarily has finite length, $l(s)$, as determined by the Riemannian metric, σ, on Σ. Therefore,

$$-2\int_{x_0}^{x_1} ||\mathcal{F}_i||_0\,\mathrm{d}s^i \leq \int_{x_0}^{x_1} \partial_i(\ln(||\Psi||_S^2))\mathrm{d}s^i \leq 2\int_{x_0}^{x_1} ||\mathcal{F}_i||_0\,\mathrm{d}s^i \tag{3.77}$$

$$\implies -2l(s)\sqrt{C_iC_j\sigma^{ij}} \leq \ln(||\Psi||_S^2(x_1)) - \ln(||\Psi||_S^2(x_0)) \leq 2l(s)\sqrt{C_iC_j\sigma^{ij}}. \tag{3.78}$$

$$\implies ||\Psi||_S^2(x_0)\mathrm{e}^{-2l(s)\sqrt{C_iC_j\sigma^{ij}}} \leq ||\Psi||_S^2(x_1) \leq ||\Psi||_S^2(x_0)\mathrm{e}^{2l(s)\sqrt{C_iC_j\sigma^{ij}}}. \tag{3.79}$$

Since Ψ goes to zero as one approaches x_0, both extremes of the inequality are just zero and thus $||\Psi||_S^2(x_1) = 0$. But $||\Psi||_S^2(x_1)$ is maximised at x_1, so it must be that $||\Psi||_S^2 = 0$ everywhere, which finally implies $\Psi = 0$ everywhere.

Definition 3.8 ($\mathfrak{D}$). *Define a linear operator,* $\mathfrak{D} : C_c^\infty \to L^2$, *by* $\mathfrak{D} : \Psi \mapsto \gamma^I\nabla_I\Psi$.

Lemma 3.9. $\langle\Psi_1,\Psi_2\rangle_{C_c^\infty} = \langle\mathfrak{D}(\Psi_1),\mathfrak{D}(\Psi_2)\rangle_{L^2}$.

Proof. Apply corollaries 3.2.1 and 3.3.1. □

This property, which relies crucially on an appropriate Lichnerowicz identity, is the main reason the inner products are defined as they are.

Definition 3.10 ($\mathcal{H}$)**.** *Define $\mathcal{H}$ to be the completion of C_c^∞ under $\langle\cdot,\cdot\rangle_{C_c^\infty}$.*

In general, the completion of a metric space has elements being equivalence classes of Cauchy sequences. However, in this case, the elements, $\Psi \in \mathcal{H}$, can be represented as elements of the more familiar Sobolev space, H^1_{loc}, as follows.

The key technical requirement is a weighted Poincaré inequality. For the chosen asymptotics, Σ_t satisfies definition 9.9 of [8] to be a weakly, asymptotically hyperboloidal end[6]. Then, one can apply proposition 8.3 of [8] to deduce $\exists\, w \in L^1_{\text{loc}}$ such that

$$\int_{\Sigma_t} \Psi^\dagger \Psi w \,\mathrm{d}V \leq \int_{\Sigma_t} (\nabla_I \Psi)^\dagger \nabla^I(\Psi)\mathrm{d}V, \tag{3.80}$$

for any $\Psi \in C_c^\infty$. Thus, for any pair of spinors, $\Psi_m, \Psi_n \in C_c^\infty$,

$$\int_{\Sigma_t} (\Psi_m - \Psi_n)^\dagger(\Psi_m - \Psi_n) w \,\mathrm{d}V \leq \int_{\Sigma_t} \nabla_I(\Psi_m - \Psi_n)^\dagger \nabla^I(\Psi_m - \Psi_n)\mathrm{d}V \tag{3.81}$$

$$\leq ||\Psi_m - \Psi_n||_{C_c^\infty}. \tag{3.82}$$

Therefore, for any Cauchy sequence, $\{\Psi_m\}_{m=0}^\infty \subseteq \mathcal{H}$, $\{\nabla_I \Psi_m\}_{m=0}^\infty$ and $\{\sqrt{w}\Psi_m\}_{m=0}^\infty$ are Cauchy in L^2. Since $w \in L^1_{\text{loc}}$, it finally follows that $\Psi \in H^1_{\text{loc}}$.

When Σ is compact, this is much more straightforward because one can apply the standard methods of elliptic PDEs to get a standard Poincaré inequality (i.e. $w = 1$ in equation 3.80).

Lemma 3.11. *$\mathfrak{D}$ extends to a continuous (i.e. bounded) linear operator from $\mathcal{H}$ to L^2 such that $\langle \Psi_1, \Psi_2\rangle_{\mathcal{H}} = \langle \mathfrak{D}(\Psi_1), \mathfrak{D}(\Psi_2)\rangle_{L^2}$.*

Proof. The points in $\mathcal{H}\backslash C_c^\infty$ are equivalence classes of Cauchy sequences. Let $\{\Psi_m\}_{m=0}^\infty$ be one such Cauchy sequence in C_c^∞ with limit in $\mathcal{H}\backslash C_c^\infty$. Observe that by lemma 3.9,

$$||\mathfrak{D}(\Psi_m) - \mathfrak{D}(\Psi_n)||_{L^2} = ||\mathfrak{D}(\Psi_m - \Psi_n)||_{L^2} = ||\Psi_m - \Psi_n||_{C_c^\infty}. \tag{3.83}$$

Thus $\{\mathfrak{D}(\Psi_m)\}_{m=0}^\infty$ is a Cauchy sequence in L^2 and since L^2 is complete, $\exists \lim_{m\to\infty} \mathfrak{D}(\Psi_m) \in L^2$. Extend the definition of $\mathfrak{D}$ to $\mathcal{H}\backslash C_c^\infty$ by defining[7] $\mathfrak{D}(\lim_{m\to\infty}\Psi_m) = \lim_{m\to\infty}\mathfrak{D}(\Psi_m)$.

Next, observe that this definition implies lemma 3.9 extends to $\mathcal{H}$. In particular, suppose $\Psi = \lim_{m\to\infty}\Psi_m$ and $\Psi' = \lim_{m\to\infty}\Psi'_m$ for Cauchy sequences, $\{\Psi_m\}_{m=0}^\infty, \{\Psi'_m\}_{m=0}^\infty \subseteq C_c^\infty$. Then, by the definitions given so far, continuity of inner products and lemma 3.9,

$$\langle \Psi, \Psi'\rangle_{\mathcal{H}} = \lim_{m,n\to\infty} \langle \Psi_m, \Psi'_n\rangle_{C_c^\infty} = \left\langle \lim_{m\to\infty}\mathfrak{D}(\Psi_m), \lim_{n\to\infty}\mathfrak{D}(\Psi'_n)\right\rangle_{L^2} = \langle \mathfrak{D}(\Psi), \mathfrak{D}(\Psi')\rangle_{L^2}. \tag{3.84}$$

An immediate consequence is

$$||\mathfrak{D}(\Psi)||_{L^2} = ||\Psi||_{\mathcal{H}}, \tag{3.85}$$

which implies that $\mathfrak{D}$ is a continuous/bounded linear operator. □

[6] The x in their definition is e^{-r} here, their h is the pullback of f_{mn} to Σ_t here and their $\mathcal{N}$ is S here.

[7] This definition is independent of the original choice of Cauchy sequence, $\{\Psi_m\}_{m=0}^\infty$, because choosing a different Cauchy sequence with the same "limit," $\{\Psi'_m\}_{m=0}^\infty$, implies $\{\mathfrak{D}(\Psi_m), \mathfrak{D}(\Psi'_n)\}$ is a Cauchy sequence in L^2 by a similar computation to above. Hence, they would have the same limit in L^2.

Theorem 3.12. *$\mathfrak{D}$ is a continuous, linear isomorphism between $\mathcal{H}$ and L^2.*

Most saliently, the theorem implies $(\gamma^I\nabla_I)^{-1} : L^2 \to \mathcal{H}$ exists. The proofs in the compact and non-compact cases have very different flavours and hence I will present them separately.

Proof (compact case). Linearity is by construction and continuity has already been shown by lemma 3.11. Next, suppose $\mathfrak{D}(\Psi) = 0$. By lemma 3.11, $0 = ||\mathfrak{D}(\Psi)||_{L^2} = ||\Psi||_{\mathcal{H}} \implies \Psi = 0$ and therefore $\mathfrak{D}$ is injective. It remains to prove surjectivity.

I'm assuming $\mathcal{A}_a = 0$ in the compact case. Let θ be an arbitrary element of L^2 and define $F_\theta : \mathcal{H} \to \mathbb{C}$ by

$$F_\theta(\Psi) = \langle \theta, \mathfrak{D}(\Psi)\rangle_{L^2}. \tag{3.86}$$

F_θ is manifestly linear. It is also continuous/bounded because the Cauchy-Schwarz inequality and lemma 3.11 imply $|F_\theta(\Psi)| = |\langle \theta, \mathfrak{D}(\Psi)\rangle_{L^2}| \le ||\theta||_{L^2}||\mathfrak{D}(\Psi)||_{L^2} = ||\theta||_{L^2}||\Psi||_{\mathcal{H}}$. Therefore, by the Riesz representation theorem, $\exists \mathcal{Z} \in \mathcal{H}$ such that $F_\theta(\Psi) = \langle \mathcal{Z}, \Psi\rangle_{\mathcal{H}}$. Then, lemma 3.11 and equation 3.86 imply

$$\langle W, \mathfrak{D}(\Psi)\rangle_{L^2} = 0 \ \forall \Psi \in \mathcal{H}, \ \text{where } W = \theta - \mathfrak{D}(\mathcal{Z}). \tag{3.87}$$

Using lemma 3.3, one can perform a formal integration by parts to get

$$0 = \int_\Sigma W^\dagger \mathfrak{D}(\Psi)\mathrm{d}V \tag{3.88}$$

$$= \int_\Sigma \left(-P_a\overline{W}\gamma^{ab}D_b\Psi - 3\mathrm{i}kW^\dagger\Psi\right)\mathrm{d}V \tag{3.89}$$

$$= \int_\Sigma \left(-P_aD_b(\overline{W}\gamma^{ab}\Psi) + P_aD_b(\overline{W})\gamma^{ab}\Psi - 3\mathrm{i}kW^\dagger\Psi\right)\mathrm{d}V \tag{3.90}$$

$$= \int_S l_an_b\overline{W}\gamma^{ab}\Psi\mathrm{d}A + \int_\Sigma \left(\gamma^ID_IW + 3\mathrm{i}kW\right)^\dagger\Psi\mathrm{d}V. \tag{3.91}$$

Let $W = (\phi_\alpha, \bar{\zeta}^{\dot\alpha})^T$ and $\Psi = (\psi_\alpha, \bar{\chi}^{\dot\alpha})^T$ in terms of two-component spinors. Therefore,

$$l_an_b\overline{W}\gamma^{ab}\Psi = l_an_b\begin{bmatrix}-\zeta^\alpha & -\bar{\phi}_{\dot\alpha}\end{bmatrix}\begin{bmatrix}(\sigma^{[a}\tilde{\sigma}^{b]})_\alpha{}^\beta & 0\\ 0 & (\tilde{\sigma}^{[a}\sigma^{b]})^{\dot\alpha}{}_{\dot\beta}\end{bmatrix}\begin{bmatrix}\psi_\beta\\ \bar{\chi}^{\dot\beta}\end{bmatrix} \tag{3.92}$$

$$= -\frac{1}{2}l_an_b\big((\sigma^a)_{\alpha\dot\alpha}(\tilde{\sigma}^b)^{\dot\alpha\beta}\zeta^\alpha\psi_\beta - (\sigma^b)_{\alpha\dot\alpha}(\tilde{\sigma}^a)^{\dot\alpha\beta}\zeta^\alpha\psi_\beta + (\tilde{\sigma}^a)^{\dot\alpha\alpha}(\sigma^b)_{\alpha\dot\beta}\bar{\phi}_{\dot\alpha}\bar{\chi}^{\dot\beta}$$

$$- (\tilde{\sigma}^b)^{\dot\alpha\alpha}(\sigma^a)_{\alpha\dot\beta}\bar{\phi}_{\dot\alpha}\bar{\chi}^{\dot\beta}\big) \tag{3.93}$$

$$= \frac{1}{2}\big(n_{\alpha\dot\alpha}l^{\beta\dot\alpha}\zeta^\alpha\psi_\beta - l_{\alpha\dot\alpha}n^{\beta\dot\alpha}\zeta^\alpha\psi_\beta + n^{\alpha\dot\alpha}l_{\alpha\dot\beta}\bar{\phi}_{\dot\alpha}\bar{\chi}^{\dot\beta} - l^{\alpha\dot\alpha}n_{\alpha\dot\beta}\bar{\phi}_{\dot\alpha}\bar{\chi}^{\dot\beta}\big) \tag{3.94}$$

$$= \sqrt{2}\big(-\zeta_o\psi_\iota - \zeta_\iota\psi_o + \overline{\phi}_o\overline{\chi}_\iota + \overline{\phi}_\iota\overline{\chi}_o\big). \tag{3.95}$$

Since $\psi_o = \chi_\iota = 0$ on S $\forall \Psi \in \mathcal{H}$, the formal integration by parts says

$$0 = \sqrt{2}\int_S \left(\overline{\phi}_\iota\overline{\chi}_o - \zeta_o\psi_\iota\right)\mathrm{d}A + \int_\Sigma \left(\gamma^ID_IW + 3\mathrm{i}kW\right)^\dagger\Psi\mathrm{d}V. \tag{3.96}$$

As $\Psi \in \mathcal{H} \supset C_c^\infty$ is arbitrary, it must be that W is a weak solution to $\gamma^ID_IW + 3\mathrm{i}kW = 0$ on Σ subject to the boundary conditions, $\phi_\iota = \zeta_o = 0$ on S. It is then a technical analytical problem to ascertain whether weak solutions lift to strong solutions in this context. This question was studied in depth by [9, 8]. From the work there, especially theorem 6.4 in [8], one can conclude this is indeed the case.

From here, there is a modified Lichnerowicz identity for W using $\widetilde{\nabla}_a W = D_a W - \mathrm{i}k\gamma_a W$, i.e. $k \mapsto -k$ compared with the original connection, ∇. Then, $\gamma^I D_I W + 3\mathrm{i}kW = 0$ can be re-written as $\gamma^I \widetilde{\nabla}_I W = 0$. The sign of k was never essential in the proof of the Lichnerowicz identity; it merely mattered that $k^2 = -\Lambda/12$. Therefore, from the proofs of theorem 3.2 and lemma 3.4, it immediately follows that

$$0 = \int_\Sigma (\gamma^I \widetilde{\nabla}_I W)^\dagger \gamma^J \widetilde{\nabla}_J (W) \mathrm{d}V \tag{3.97}$$

$$= \int_\Sigma \left((\widetilde{\nabla}_I W)^\dagger \widetilde{\nabla}^I W - 4\pi T^{0a} \overline{W} \gamma_a W \right) \mathrm{d}V - \widetilde{Q}(W), \tag{3.98}$$

$$\text{where } \widetilde{Q}(W) = 2\int_S \Big(\phi_\iota \eth \overline{\phi}_o + \overline{\phi}_\iota \bar{\eth} \phi_o - \overline{\zeta}_o \eth \zeta_\iota - \zeta_o \bar{\eth} \overline{\zeta}_\iota + \rho|\phi_o|^2 + \mu|\phi_\iota|^2 + \rho|\zeta_o|^2 + \mu|\zeta_\iota|^2$$

$$- \mathrm{i}k\sqrt{2}\big(\phi_o\zeta_\iota + \phi_\iota\zeta_o - \overline{\phi}_o\overline{\zeta}_\iota - \overline{\phi}_\iota\overline{\zeta}_o\big)\Big) \mathrm{d}A. \tag{3.99}$$

However, from $\phi_\iota = \zeta_o = 0$ on S,

$$\widetilde{Q}(W) = 2\int_S \left(\rho|\phi_o|^2 + \mu|\zeta_\iota|^2 - \mathrm{i}k\sqrt{2}\big(\phi_o\zeta_\iota - \overline{\phi}_o\overline{\zeta}_\iota\big) \right) \mathrm{d}A. \tag{3.100}$$

As in the proof of lemma 3.7, let $\mu' = \mu/|z|^2$, $\rho' = |z|^2\rho$, $\phi'_o = \phi_o/z$ and $\zeta'_\iota = z\zeta_\iota$. Again, choose $z = \sqrt[4]{\mu/\rho}$ so that $\mu' = \rho' = -\sqrt{\mu\rho} = -\frac{1}{2}\sqrt{-\theta_l\theta_n} < -k\sqrt{2}$. Therefore,

$$\widetilde{Q}(W) = 2\int_S \left(\rho'|\phi'_o|^2 + \mu'|\zeta'_\iota|^2 - \mathrm{i}k\sqrt{2}\big(\phi'_o\zeta'_\iota - \overline{\phi}'_o\overline{\zeta}'_\iota\big) \right) \mathrm{d}A \tag{3.101}$$

$$= 2\int_S \left((\rho' + k\sqrt{2})|\phi'_o|^2 + (\mu' + k\sqrt{2})|\zeta'_\iota|^2 - k\sqrt{2}|\overline{\phi}'_o + \mathrm{i}\zeta'_\iota|^2 \right) \mathrm{d}A \tag{3.102}$$

$$\leq 0. \tag{3.103}$$

Thus, combined with the dominant energy condition, every term on the RHS of equation 3.98 is non-negative.

Therefore $\widetilde{\nabla}_I W = 0$ and $\widetilde{Q}(W) = 0$. The latter implies $\phi_o = \zeta_\iota = 0$ on S by equation 3.102 and consequently $W = 0$ on S since $\phi_\iota = \zeta_o = 0$ on S already. In the proofs of lemmas 3.6 and 3.7 it was shown $\nabla_I \Psi = 0$ on Σ with $\Psi = 0$ on S implies $\Psi = 0$ on Σ. By the same logic used there, it now follows that $W = 0$ on Σ. Equivalently $\theta = \mathfrak{D}(\mathcal{Z})$ and therefore $\mathfrak{D}$ is surjective. □

Proof (non-compact case). Linearity, continuity and injectivity are identical to the compact case. The surjectivity proof follows an index theory argument based on the analysis in [105, 88].

Let $\hat{\nabla}_a = D_a + \mathrm{i}k\gamma_a$, i.e. don't include the $\mathcal{A}_a$ term. Therefore, $\mathfrak{D} - \hat{\mathfrak{D}} = \gamma^I \mathcal{A}_I$. If $\gamma^I \mathcal{A}_I$ were a compact operator, then the index (dimension of kernel minus dimension of cokernel) of $\mathfrak{D}$ and $\hat{\mathfrak{D}}$ would coincide. In particular, if $\hat{\mathfrak{D}}$ was invertible, then it would follow that index($\mathfrak{D}$) = 0. Since $\mathfrak{D}$ has already been shown to have trivial kernel above, it would follow that $\mathfrak{D}$ is invertible.

Even though $\mathcal{A}_a$ is just a (sufficiently regular) matrix valued function, it's not clear in general whether $\gamma^I \mathcal{A}_I$ is actually compact. This would be the case though if the underlying space, Σ_t, were itself compact[8] - a fact leveraged in [105, 88].

[8]This is essentially due to the Rellich-Kondrachov theorem, which in a special case says that H^1 is compactly embedded in L^2. In particular, from lemma 3.6 of [8] and the discussion of Poincaré inequalities earlier, in the compact case $\Psi \in \mathcal{H} \implies \Psi \in H^1$. Then, since constant matrix multiplication is always a compact operator and $\mathcal{A}_a$ is assumed sufficiently regular, $\gamma^I \mathcal{A}_I$ is a compact operator from $\mathcal{H}$ to L^2.

Similarly, given an asymptotic end of the form $\mathbb{R}\times S$, consider the compact subset[9], $\Sigma(r_0) = \Sigma_t\backslash\{r > r_0\}$. With the appropriate boundary conditions for spinors on $\partial\Sigma(r_0) = \{r = r_0\}$, in the previous proof I've shown that $\hat{\mathfrak{D}}$ is invertible on the compact set[10]. Therefore, by the index theory argument above, $\mathfrak{D}$ is invertible on $\Sigma(r_0)$.

Since C_c^∞ is dense in L^2, for any $\Phi \in L^2$, $\exists$ a Cauchy sequence, $\{\Phi_m\}_{m=0}^\infty \subseteq C_c^\infty$, such that $\lim_{m\to\infty}\Phi_m = \Phi$ in L^2. For each Φ_m choose r_m so that $\mathrm{supp}(\Phi_m) \subseteq \mathrm{int}(\Sigma(r_m))$, i.e. choose r_m large enough. Then, the choice of boundary conditions on $\partial\Sigma(r_m)$ doesn't matter and $\exists\Psi_m$ such that $\mathfrak{D}(\Psi_m) = \Phi_m$. Theorem 6.4 in [9, 8] shows $\mathfrak{D}$ has an "elliptic regularity" property on $\Sigma(r_m)$ meaning $\Phi_m \in C_c^\infty$ and the metric's smoothness imply $\Psi_m \in C_c^\infty$[11].

Furthermore, $\{\Psi_m\}_{m=0}^\infty \subseteq C_c^\infty$ is a Cauchy sequence by lemma 3.9. By theorem 3.11, the limit, $\lim_{m\to\infty}\Psi_m = \Psi \in \mathcal{H}$ satisfies

$$\mathfrak{D}(\Psi) = \lim_{m\to\infty}\mathfrak{D}(\Psi_m) = \lim_{m\to\infty}\Phi_m = \Phi, \tag{3.104}$$

thereby proving $\mathfrak{D}$ is surjective. □

The proof of surjectivity had a very different style in the compact and non-compact cases. This is not actually due to the compactness, but instead due to the fact I assumed $\mathcal{A}_a = 0$ in the compact case. I could instead try a similar Lichnerowicz identity based approach in the non-compact case. However, as I'll explain, this will require additional assumptions on $\mathcal{A}_I$.

Once again, define F_θ like in the compact case. Therefore, by the same steps,

$$\langle W, \mathfrak{D}(\Psi)\rangle_{L^2} = 0 \ \forall\Psi \in \mathcal{H}, \text{ where } W = \theta - \mathfrak{D}(\mathcal{Z}), \tag{3.105}$$

and the task is to show W must be zero. This time the formal integration by parts says that $\forall\Psi \in C_c^\infty$,

$$0 = \int_{\Sigma_t}\Psi^\dagger\left(\gamma^I D_I(W) + \mathrm{i}k(n-1)W - \mathcal{A}_I^\dagger\gamma^I W\right)\mathrm{d}V. \tag{3.106}$$

Suppose there exists $\tilde{\mathcal{A}}_a$ such that $(\gamma^I\mathcal{A}_I)^\dagger = \gamma^I\tilde{\mathcal{A}}_I$. Define a new connection on spinors, $\widetilde{\nabla}_a$, by $\widetilde{\nabla}_a = D_a - \mathrm{i}k\gamma_a + \tilde{\mathcal{A}}_a$ and a new Dirac operator, $\widetilde{\mathfrak{D}} = \gamma^I\widetilde{\nabla}_I$. Then, equation 3.106 can be re-written as

$$0 = \int_{\Sigma_t}\Psi^\dagger\widetilde{\mathfrak{D}}(W)\mathrm{d}V. \tag{3.107}$$

Since Ψ could be any compactly supported spinor, it follows that W is a weak solution to $\widetilde{\mathfrak{D}}(W) = 0$. Thus, the surjectivity proof reduces to showing $\widetilde{\mathfrak{D}}$ has trivial kernel. Continuing with the new connection, suppose $\gamma^{IJ}\tilde{\mathcal{A}}_J$ is hermitian and

$$\widetilde{\mathbb{M}} = 4\pi T^{0a}\gamma_0\gamma_a + \gamma^{IJ}D_I\tilde{\mathcal{A}}_J - \mathrm{i}k(n-2)(\gamma^I\tilde{\mathcal{A}}_I + \tilde{\mathcal{A}}_I^\dagger\gamma^I) - \tilde{\mathcal{A}}_I^\dagger\gamma^{IJ}\tilde{\mathcal{A}}_J \tag{3.108}$$

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[9]Here, r is the Fefferman-Graham coordinate, but in general it could be any coordinate for the $\mathbb{R}$ factor in $\mathbb{R}\times S$ such that the conformal boundary is at $r = \infty$.

[10]My invertibility proof in the compact case used the GHP formalism and is hence specific to Σ being 3D. However, as explained in [105, 88], the GHP boundary conditions can be written in a more dimension-agnostic way using eigenspaces of $\gamma^0 \pm \gamma^1$. I could then repeat a similar analysis to show that $\hat{\mathfrak{D}}$ is invertible on the compact set. Even more generally, for the property I need in the non-compact analysis, I don't need to be very specific about the choice of boundary conditions. There are a number of alternative boundary condtions I could choose. Some examples are the chiral boundary condition - i.e. fixing the right-handed or left-handed sector of a spinor - or the Atiyah-Patodi-Singer (APS) boundary condition - expanding the spinor in terms of eigenvectors of the boundary Dirac operator and fixing the spinor's component in half the eigenspaces. It is known [90] that these too lead to the desired invertibility on the compact set.

[11]Theorem 6.4 as stated in [9, 8] only says $\Psi_m \in H^1_{\mathrm{loc}}(\Sigma(r_m))$. However, this is because of the very low regularity assumed in [9, 8] for the metric and the analogue of Φ_m. If one assumes additional regularity - in this case g's smoothness and $\Phi_m \in C_c^\infty$ - then Ψ_m inherits this additional regularity by the same proof.

is non-negative definite. Then, following the same steps as theorem 3.2, if $\Psi \in C_c^\infty$, then

$$||\widetilde{\mathfrak{D}}(\Psi)||_{L^2}^2 = \int_{\Sigma_t} \left((\widetilde{\nabla}_I \Psi)^\dagger \widetilde{\nabla}^I \Psi + \Psi^\dagger \widetilde{\mathbb{M}} \Psi \right) \mathrm{d}V. \tag{3.109}$$

As $\widetilde{\mathbb{M}}$ is assumed to be non-negative definite, this is formally identical to the original Lichnerowicz identity. Then, I could proceed analogously to the analysis of $\mathfrak{D}$ itself by defining an inner product analogous to lemma 3.6 and thereby concluding that $\widetilde{\mathfrak{D}}$ is injective. Hence, $\widetilde{\mathfrak{D}}(W) = 0$ implies $W = \theta - \mathfrak{D}(\mathcal{Z}) = 0$ and therefore $\mathfrak{D}$ is surjective.

While this proof is more in the spirit of the rest of the material because it is built on a Lichnerowicz identity, it has the disadvantage that there may be choices of $\mathcal{A}_a$ for which no simple $\tilde{\mathcal{A}}_a$ exists. Furthermore, even if $\tilde{\mathcal{A}}_a$ exists, it may be that $\widetilde{\mathbb{M}}$ is not non-negative definite.

In fact, this is exactly the situation for the electromagnetic examples of section 4.4. The details of the theories studied there don't matter for the present discussion. In short, given a one-form, A_a, a two-form, $F = \mathrm{d}A$ and its components, $E_I = F_{I0}$ & F_{IJ}, for the 4D and 5D theories studied there, (in $k = 1/2$ units) it makes sense to choose

$$\mathcal{A}_a^{(4)} = -\frac{1}{4} F_{bc} \gamma^{bc} \gamma_a + \mathrm{i} A_a I = \frac{1}{2} E_I \gamma^0 \gamma^I \gamma_a - \frac{1}{4} F_{IJ} \gamma^{IJ} \gamma_a + \mathrm{i} A_a I \;\text{ and} \tag{3.110}$$

$$\mathcal{A}_a^{(5)} = -\frac{1}{4\sqrt{3}} F_{bc} \gamma^{bc} \gamma_a - \frac{1}{2\sqrt{3}} F_{ab} \gamma^b + \mathrm{i}\sqrt{3} A_a I \tag{3.111}$$

$$= -\frac{1}{2\sqrt{3}} E_I \gamma^I \gamma^0 \gamma_a - \frac{1}{4\sqrt{3}} F_{IJ} \gamma^{IJ} \gamma_a - \frac{1}{2\sqrt{3}} F_{ab} \gamma^b + \mathrm{i}\sqrt{3} A_a I. \tag{3.112}$$

Using equations 4.247 and 4.276, one finds $\tilde{\mathcal{A}}^{(4)}$ & $\tilde{\mathcal{A}}^{(5)}$ exist and they are identical to $\mathcal{A}^{(4)}$ & $\mathcal{A}^{(5)}$ except that $F_{IJ} \to -F_{IJ}$. In both cases, by following similar steps to the proof of lemma 4.19, one finds

$$\widetilde{\mathbb{M}} = -2 S_I \gamma^0 \gamma^I, \tag{3.113}$$

where $S^I = F^I{}_J E^J$ is the Poynting vector. Thus, $\widetilde{\mathbb{M}}$ has eigenvalues, $\pm 4\sqrt{S_I S^I}$, and is therefore not non-negative definite[12].

Unlike the proof of theorem 3.12 presented above, it appears the Licherowicz identity based proof can only be rectified in a few specific cases where further assumptions are made. The simplest, but most unsatisfying, assumption would be to restrict to electromagnetic fields which have vanishing Poynting vector.

As the following argument shows, a much less obvious assumption that also works is to restrict to hypersurfaces, Σ_t, which have $K = \delta^{IJ} K_{IJ} = 0$. Instead of the spacetime Levi-Civita connection, D, one could re-write the argument in terms of the connection intrinsic to Σ_t, say $D^{(\sigma)}$, where σ is the metric on Σ_t. Then, for any Dirac spinor, Ψ,

$$D_I \Psi = D_I^{(\sigma)} \Psi + \frac{1}{2} K_{IJ} \gamma^0 \gamma^J \;\text{ and }\; \gamma^I D_I \Psi = \gamma^I D_I^{(\sigma)} \Psi + \frac{1}{2} K \gamma^0. \tag{3.114}$$

Instead of the $\widetilde{\nabla}_a$ defined earlier, one could define another connection, say ∇'_I, differing from ∇_I in not just $F_{IJ} \to -F_{IJ}$ (and a $-\frac{\mathrm{i}}{2}\gamma_I$ term instead of a $+\frac{\mathrm{i}}{2}\gamma_I$ term), but also $K_{IJ} \to -K_{IJ}$. Since $(-K_{IJ}, E_I, -F_{IJ})$ satisfies the constraint equations (encoded in T^{0a}) whenever (K_{IJ}, E_I, F_{IJ}) does, this connection will have $\mathbb{M}' = 0$, which is trivially non-negative definite. $\gamma^I \nabla'_I$ and $\gamma^I \widetilde{\nabla}_I$ differ by a $K\gamma^0$ term, but this goes to zero if K is assumed to vanish.

$K = 0$ is a "maximal gauge" that is sometimes used in the study of the initial value problem [44]. However, it would have to be shown (M, g) admits such a foliation and that the foliation is compatible with the coordinates chosen on $\mathcal{I}$ to get $f_{(0)0\alpha} = 0$.

[12] This issue was also recently pointed out in [105] and exists even when $\Lambda = 0$. An analogous proof to theorem 3.12 presented in this thesis therefore also fixes the mistake in theorem 11.9 of [8], where $\Lambda = 0$.

3.3 Background Killing spinors

In this section, I will only be interested in the non-compact case, i.e. $\Sigma = \Sigma_t$ throughout. In this context, key to Witten's method will be backgrounds admitting imaginary Killing spinors[13]. The specific form of the metric, g, is the Fefferman-Graham expansion given in definition 2.11, while the notion of a background metric, $\bar{g}$, is given in definition 2.12. In particular, the foreground and background metrics agree to $O(\mathrm{e}^{2r}\mathrm{e}^{(n-2)r})$ in the Fefferman-Graham expansion. As with the previous discussions of Fefferman-Graham expansions, it will be convenient to work in units where $k = 1/2$.

Definition 3.13 (Background Killing spinor)**.** *A spinor, ε_k, is called a Killing spinor of the background metric, $\bar{g}$, if and only if it satisfies*

$$\overline{D}_a\varepsilon_k + \frac{\mathrm{i}}{2}\gamma_a\varepsilon_k = 0, \tag{3.115}$$

where $\overline{D}_a$ is the Levi-Civita connection of $\bar{g}$. Similarly, denote the vielbeins associated to $\bar{g}$ as $\bar{e}^a$ and $\bar{e}_a$.

Of course, not every background metric admits a background Killing spinor. However, Witten's method works most naturally when there does exist a non-zero ε_k - see [79, 10] for a broad discussion on the admissible metrics. Furthermore, when no background Killing spinors exist, negative energies are possible - as illustrated by the examples in [28].

Note that ε_k may only be defined in an open neighbourhood of the "boundary" at infinity or equation 3.115 may only have a solution in such a region. This is not a problem because equation 3.115 will only really be required in an open neighbourhood of infinity, say $\overline{M}$, and ε_k can be extended to a spinor on all of Σ_t by multiplying it with a smooth function that's one near infinity but falls to zero within $\overline{M}$.

Given some background metric, $\bar{g}$, with vielbein, $\{\partial_r, \mathrm{e}^{-r}\bar{e}^{(\bar{f})m}_M\partial_m\}$, a natural vielbein for g is ∂_r together with

$$e_M = \mathrm{e}^{-r}\bar{e}^{(\bar{f})m}_M\left(\partial_m - \frac{1}{2}\mathrm{e}^{-(n-1)r}(f_{(n-1)mp} - \bar{f}_{(n-1)mp})\bar{f}^{pn}_{(0)}\partial_n + O(\mathrm{e}^{-nr})\right). \tag{3.116}$$

In particular, this ansatz has $g(e_M, e_N) = \eta_{MN} + O(\mathrm{e}^{-nr})$. The specific $O(\mathrm{e}^{-nr})$ corrections won't be relevant for this thesis. Furthermore, given a background metric, the vielbein for g will be fixed as $\{\partial_r, e_M\}$ throughout[14]. Background Killing spinors interact with this choice of vielbein to yield useful decay properties on Σ_t.

Lemma 3.14. *If a background Killing spinor, ε_k, is $O(\mathrm{e}^{r/2})$ near $\Sigma_{t,\infty}$, then $\nabla_I\varepsilon_k \in L^2$.*

Proof. First note that to be in L^2, an object must decay faster than $O(\mathrm{e}^{-(n-2)r/2})$ because the integration measure over Σ_t is $O(\mathrm{e}^{(n-2)r})$. Next, given a vielbein, $e_a{}^\mu\partial_\mu$, the spin connection coefficients are defined as

$$\omega_{bca} = \frac{1}{2}\left(g(e_a, [e_b, e_c]) - g(e_b, [e_c, e_a]) + g(e_c, [e_b, e_a])\right). \tag{3.117}$$

[13]The subsequent arguments would still work as long as the Killing spinor equation is satisfied up to $O(\mathrm{e}^{-(n-1)r})$ corrections.

[14]Without this, it will not be possible to meaningfully talk about notions such as "constant spinor" later.

Since ε_k is a background Killing spinor, with the vielbein chosen in equation 3.116 (along with $e_1 = \partial_r$),

$$\nabla_M \varepsilon_k = e_M{}^\mu \partial_\mu \varepsilon_k - \frac{1}{4}\omega_{abM}\gamma^{ab}\varepsilon_k + \mathrm{i}k\gamma_M\varepsilon_k + \mathcal{A}_M\varepsilon_k \tag{3.118}$$

$$= \left(e_M{}^\mu - e_M^{(\bar{g})\mu}\right)\partial_\mu\varepsilon_k - \frac{1}{4}(\omega_{abM} - \bar{\omega}_{abM})\gamma^{ab}\varepsilon_k + \mathcal{A}_M\varepsilon_k \tag{3.119}$$

$$= \left(-\frac{1}{2}\mathrm{e}^{-nr}(f_{(n-1)np} - \bar{f}_{(n-1)np})\bar{f}^{pm}\bar{e}_M^{(\bar{f})n} + O\left(\mathrm{e}^{-(n+1)r}\right)\right)\partial_m\varepsilon_k$$
$$- \frac{1}{4}(\omega_{abM} - \bar{\omega}_{abM})\gamma^{ab}\varepsilon_k + \mathcal{A}_M\varepsilon_k \text{ and similarly} \tag{3.120}$$

$$\nabla_1\varepsilon_k = -\frac{1}{4}(\omega_{ab1} - \bar{\omega}_{ab1})\gamma^{ab}\varepsilon_k + \mathcal{A}_1\varepsilon_k. \tag{3.121}$$

$\mathcal{A}_a\varepsilon_k$ is comfortably in L^2 from the decay assumption on $||\mathcal{A}_a||_0$. Likewise, the term proportional to $\mathrm{e}^{-nr}\partial_m\varepsilon_k$ also easily decays fast enough to be in L^2. The connection terms are found by combining equations 3.116 and 3.117. In particular,

$$g(e_M, [\partial_r, e_N])$$
$$= g\Big(\mathrm{e}^{-r}\bar{e}_M^{(\bar{f})m}\partial_m - \frac{1}{2}\mathrm{e}^{-nr}\bar{e}_M^{(\bar{f})m}(f_{(n-1)mp} - \bar{f}_{(n-1)mp})\bar{f}_{(0)}^{pn}\partial_n + O(\mathrm{e}^{-(n+1)r}),$$
$$-\mathrm{e}^{-r}\bar{e}_N^{(\bar{f})q}\partial_q + \mathrm{e}^{-r}\partial_r\left(\bar{e}_N^{(\bar{f})q}\right)\partial_q + \frac{n}{2}\mathrm{e}^{-nr}\bar{e}_N^{(\bar{f})q}(f_{(n-1)qr} - \bar{f}_{(n-1)qr})\bar{f}_{(0)}^{rs}\partial_s + O(\mathrm{e}^{-(n+1)r})\Big) \tag{3.122}$$
$$= \bar{e}_M^{(\bar{f})m}\partial_r\left(\bar{e}_N^{(\bar{f})q}\right)\left(\bar{f}_{mq} + O(\mathrm{e}^{-(n-1)r})\right)$$
$$- \bar{e}_M^{(\bar{f})m}\bar{e}_N^{(\bar{f})q}\Big(\bar{f}_{mq} + \mathrm{e}^{-(n-1)r}(f_{(n-1)mq} - \bar{f}_{(n-1)mq}) - \frac{1}{2}\mathrm{e}^{-(n-1)r}(f_{(n-1)mp} - \bar{f}_{(n-1)mp})\bar{f}_{(0)}^{pn}\bar{f}_{(0)nq}$$
$$- \frac{n}{2}\mathrm{e}^{-(n-1)r}(f_{(n-1)qr} - \bar{f}_{(n-1)qr})\bar{f}_{(0)}^{rs}\bar{f}_{(0)ms} + O(\mathrm{e}^{-nr})\Big) \tag{3.123}$$
$$= \bar{e}_M^{(\bar{f})m}\partial_r\left(\bar{e}_N^{(\bar{f})n}\right)\bar{f}_{mn} + O(\mathrm{e}^{-nr}) - \eta_{MN}$$
$$+ \frac{n-1}{2}\mathrm{e}^{-(n-1)r}\bar{e}_M^{(\bar{f})m}\bar{e}_N^{(\bar{f})n}(f_{(n-1)mn} - \bar{f}_{(n-1)mn}) + O(\mathrm{e}^{-nr}) \tag{3.124}$$

and likewise

$$\bar{g}(\bar{e}_M, [\partial_r, \bar{e}_N]) = \bar{g}\left(\mathrm{e}^{-r}\bar{e}_M^{(\bar{f})m}\partial_m, -\mathrm{e}^{-r}\bar{e}_N^{(\bar{f})n}\partial_n + \mathrm{e}^{-r}\partial_r\left(\bar{e}_N^{(\bar{f})n}\partial_n\right)\right) \tag{3.125}$$
$$= -\eta_{MN} + \bar{e}_M^{(\bar{f})m}\partial_r\left(\bar{e}_N^{(\bar{f})n}\right)\bar{f}_{mn}. \tag{3.126}$$

Putting them together yields

$$g(e_M, [\partial_r, e_N]) - \bar{g}(\bar{e}_M, [\partial_r, \bar{e}_N]) = \frac{n-1}{2}\mathrm{e}^{-(n-1)r}\bar{e}_M^{(\bar{f})m}\bar{e}_N^{(\bar{f})n}(f_{(n-1)mn} - \bar{f}_{(n-1)mn}) \tag{3.127}$$

up to $O(\mathrm{e}^{-nr})$. Now, consider the expressions for the connection coefficients.

$$\omega_{NPM} - \bar{\omega}_{NPM} = \frac{1}{2}\left(g(e_M, [e_N, e_P]) - g(e_N, [e_P, e_M]) + g(e_P, [e_N, e_M])\right)$$
$$- \frac{1}{2}\left(\bar{g}(\bar{e}_M, [\bar{e}_N, \bar{e}_P]) - \bar{g}(\bar{e}_N, [\bar{e}_P, \bar{e}_M]) + \bar{g}(\bar{e}_P, [\bar{e}_N, \bar{e}_M])\right) \tag{3.128}$$
$$= O(\mathrm{e}^{-nr}) \tag{3.129}$$

as e_M & $\bar{e}_N$ are e^{-r}, their difference is a further $O(\mathrm{e}^{-(n-1)r})$ and g & $\bar{g}$ are $O(\mathrm{e}^{2r})$. Next,

$$\omega_{1M1} - \bar{\omega}_{1M1} = \frac{1}{2}(g(\partial_r, [\partial_r, e_M]) - g(\partial_r, [e_M, \partial_r]) + g(e_M, [\partial_r, \partial_r]))$$
$$- \frac{1}{2}(\bar{g}(\partial_r, [\partial_r, \bar{e}_M]) - \bar{g}(\partial_r, [\bar{e}_M, \partial_r]) + \bar{g}(\bar{e}_M, [\partial_r, \partial_r])) \tag{3.130}$$
$$= 0 \tag{3.131}$$

since $[\partial_r, \partial_r] = 0$ and $\partial_r \perp \partial_m$. Finally, by equation 3.127,

$$\begin{aligned}
\omega_{MN1} - \bar{\omega}_{MN1} &= \frac{1}{2}\left(g(\partial_r, [e_M, e_N]) - g(e_M, [e_N, \partial_r]) + g(e_N, [e_M, \partial_r])\right) \\
&\quad - \frac{1}{2}\left(\bar{g}(\partial_r, [\bar{e}_M, \bar{e}_N]) - \bar{g}(\bar{e}_M, [\bar{e}_N, \partial_r]) + \bar{g}(\bar{e}_N, [\bar{e}_M, \partial_r])\right) &(3.132)\\
&= O(\mathrm{e}^{-(n-1)r}) \text{ and} &(3.133)\\
\omega_{1NM} - \bar{\omega}_{1NM} &= \frac{1}{2}(g(e_M, [\partial_r, e_N]) - g(\partial_r, [e_N, e_M]) + g(e_N, [\partial_r, e_M])) \\
&\quad - \frac{1}{2}(\bar{g}(\bar{e}_M, [\partial_r, \bar{e}_N]) - \bar{g}(\partial_r, [\bar{e}_N, \bar{e}_M]) + \bar{g}(\bar{e}_N, [\partial_r, \bar{e}_M])) &(3.134)\\
&= O(\mathrm{e}^{-(n-1)r}). &(3.135)
\end{aligned}$$

In summary, all the terms decay quickly enough to have $\nabla_I \varepsilon_k \in L^2$. □

Theorem 3.15. *Suppose Φ is a spinor such that $\gamma^I \nabla_I \Phi \in L^2$ and Φ grows at most as $O(\mathrm{e}^{r/2})$. By theorem 3.12, let $\Psi \in \mathcal{H}$ be the unique spinor such that $\gamma^I \nabla_I \Psi = \gamma^I \nabla_I \Phi$. Let $\mathcal{Z} = \Phi - \Psi$. Let $\{\Psi_m\}_{m=0}^{\infty} \in C_c^{\infty}$ be a Cauchy sequence whose limit is Ψ and let $\mathcal{Z}_m = \Phi - \Psi_m$. Then, for the functional, Q, in definition 3.1, $\lim_{m\to\infty} Q(\mathcal{Z}_m) = Q(\mathcal{Z})$.*

Proof. As before, theorem 3.2 implies that for any spinor, θ,

$$\begin{aligned}
Q(\Phi - \Theta) = 2\int_{\Sigma_t} \Big(&\nabla_I(\Phi - \Theta)^\dagger \nabla^I(\Phi - \Theta) - (\gamma^I \nabla_I(\Phi - \Theta))^\dagger \gamma^J \nabla_J(\Phi - \Theta) \\
&+ (\Phi - \Theta)^\dagger \mathbb{M}(\Phi - \Theta)\Big)\mathrm{d}V. &(3.136)
\end{aligned}$$

Hence, by lemma 3.6, definition 3.10, definition 3.8, lemma 3.11 and $\gamma^I \nabla_I \Phi \in L^2$,

$$\begin{aligned}
\frac{1}{2}(Q(\mathcal{Z}) - Q(\mathcal{Z}_m)) &= \frac{1}{2}(Q(\Phi - \Psi) - Q(\Phi - \Psi_m)) &(3.137)\\
&= ||\Psi||^2_{\mathcal{H}} - ||\Psi_m||^2_{\mathcal{H}} - ||\mathfrak{D}(\Psi)||^2_{L^2} + ||\mathfrak{D}(\Psi_m)||^2_{L^2} + \langle \mathfrak{D}(\Psi - \Psi_m), \gamma^I \nabla_I \Phi\rangle_{L^2} \\
&\quad + \langle \gamma^I \nabla_I \Phi, \mathfrak{D}(\Psi - \Psi_m)\rangle_{L^2} - \int_{\Sigma_t} (\nabla_I(\Psi - \Psi_m))^\dagger \nabla^I(\Phi)\mathrm{d}V \\
&\quad - \int_{\Sigma_t} \nabla^I(\Phi)^\dagger \nabla_I(\Psi - \Psi_m)\mathrm{d}V - \int_{\Sigma_t} (\Psi - \Psi_m)^\dagger \mathbb{M}\Phi\, \mathrm{d}V \\
&\quad - \int_{\Sigma_t} \Phi^\dagger \mathbb{M}(\Psi - \Psi_m)\, \mathrm{d}V. &(3.138)
\end{aligned}$$

Since inner products and $\mathfrak{D}$ are both continuous, it immediately follows that

$$\begin{aligned}
\lim_{m\to\infty} \frac{1}{2}(Q(\mathcal{Z}) - Q(\mathcal{Z}_m)) = \lim_{m\to\infty} \Big(&- \int_{\Sigma_t} (\nabla_I(\Psi - \Psi_m))^\dagger \nabla^I(\Phi)\mathrm{d}V - \int_{\Sigma_t} \nabla^I(\Phi)^\dagger \nabla_I(\Psi - \Psi_m)\mathrm{d}V \\
&- \int_{\Sigma_t} (\Psi - \Psi_m)^\dagger \mathbb{M}\Phi\, \mathrm{d}V - \int_{\Sigma_t} \Phi^\dagger \mathbb{M}(\Psi - \Psi_m)\, \mathrm{d}V\Big). &(3.139)
\end{aligned}$$

Since the inner product on $\mathcal{H}$ is $\langle \Psi_1, \Psi_2\rangle_{\mathcal{H}} = \int_{\Sigma_t} \left((\nabla_I \Psi_1)^\dagger \nabla^I \Psi_2 + \Psi_1^\dagger \mathbb{M}\Psi_2\right) \mathrm{d}V$ (with limits of Cauchy sequences taken appropriately for Ψ_1 or Ψ_2 in $\mathcal{H}\backslash C_c^{\infty}$) and $\mathbb{M}$ is non-negative definite,

$$\int_{\Sigma_t} (\nabla_I \Psi)^\dagger \nabla^I(\Psi)\, \mathrm{d}V \leq ||\Psi||^2_{\mathcal{H}} < \infty. \tag{3.140}$$

Hence $\nabla_I \Psi \in L^2$ and $\Psi \mapsto \nabla_I \Psi$ is a continuous (i.e. bounded) linear operator. Consequently,

$$\lim_{m\to\infty} \int_{\Sigma_t} (\nabla_I(\Psi - \Psi_m))^\dagger \nabla^I(\Phi) \mathrm{d}V = \lim_{m\to\infty} \langle \nabla_I(\Psi - \Psi_m), \nabla^I \Phi \rangle_{L^2} = 0 \tag{3.141}$$

and likewise for $\int_{\Sigma_t} \nabla^I(\Phi)^\dagger \nabla_I(\Psi - \Psi_m) \mathrm{d}V$. That leaves

$$\begin{aligned} &\lim_{m\to\infty} \frac{1}{2}(Q(\mathcal{Z}) - Q(\mathcal{Z}_m)) \\ &= \lim_{m\to\infty} \Big(- \int_{\Sigma_t} (\Psi - \Psi_m)^\dagger \mathbb{M} \Phi \, \mathrm{d}V - \int_{\Sigma_t} \Phi^\dagger \mathbb{M} (\Psi - \Psi_m) \, \mathrm{d}V \Big). \end{aligned} \tag{3.142}$$

Because it's assumed $\mathbb{M}$ is non-negative definite, $||\mathbb{M}||_0$ decays faster than $O(\mathrm{e}^{-(n-1)r})$ near $\Sigma_{t,\infty}$ and Φ grows at $O(\mathrm{e}^{r/2})$ near $\Sigma_{t,\infty}$,

$$\int_{\Sigma_t} \Phi^\dagger \mathbb{M} \Phi \, \mathrm{d}V \le \int_{\Sigma_t} \Phi^\dagger \Phi ||\mathbb{M}||_0 \, \mathrm{d}V < \infty. \tag{3.143}$$

Therefore, viewed as a real-valued function on Σ_t, $\Phi^\dagger \mathbb{M} \Phi \in L^1$.

Define the function, $\mathcal{B} : \mathcal{H} \to \mathbb{R}$ by

$$\mathcal{B}(\Theta) = \int_{\Sigma_t} \Theta^\dagger \mathbb{M} \Theta \, \mathrm{d}V. \tag{3.144}$$

This function is well-defined because

$$\mathcal{B}(\Theta) \le \int_{\Sigma_t} \big((\nabla_I \Theta)^\dagger \nabla^I \Theta + \Theta^\dagger \mathbb{M} \Theta \big) \, \mathrm{d}V = ||\Theta||^2_{\mathcal{H}} < \infty. \tag{3.145}$$

Equivalently, it means as a real-valued function on Σ_t, $\Theta^\dagger \mathbb{M} \Theta \in L^1 \ \forall \Theta \in \mathcal{H}$. Furthermore, using lemmas 3.2, lemma 3.11 and the result above that $\Psi \mapsto \nabla_I \Psi$ is continuous, it follows that $\mathcal{B}$ is continuous because

$$\begin{aligned} \mathcal{B}(\Theta) - \mathcal{B}(W) = \int_{\Sigma_t} &\big(\mathfrak{D}(\Theta)^\dagger \mathfrak{D}(\Theta) - (\nabla_I \Theta)^\dagger \nabla^I \Theta \\ &+ \mathfrak{D}(W)^\dagger \mathfrak{D}(W) - (\nabla_I W)^\dagger \nabla^I W \big) \mathrm{d}V \end{aligned} \tag{3.146}$$

$$\begin{aligned} = \frac{1}{2} \int_{\Sigma_t} &\big(\mathfrak{D}(\Theta + W)^\dagger \mathfrak{D}(\Theta - W) + \mathfrak{D}(\Theta - W)^\dagger \mathfrak{D}(\Theta + W) \\ &- \nabla_I(\Theta + W)^\dagger \nabla^I(\Theta - W) - \nabla_I(\Theta - W)^\dagger \nabla^I(\Theta + W) \big) \mathrm{d}V \end{aligned} \tag{3.147}$$

implies $\mathcal{B}(W) \to \mathcal{B}(\Theta)$ whenever $W \to \Theta$.

Next, since $\mathbb{M}$ is hermitian and non-negative definite, $\exists$ a "square root" matrix, $\mathbb{S}$, such that $\mathbb{M} = \mathbb{S}^\dagger \mathbb{S}$. Hence, by the Cauchy-Schwarz inequality,

$$|\Phi^\dagger \mathbb{M} (\Psi - \Psi_m)| = |\Phi^\dagger \mathbb{S}^\dagger \mathbb{S} (\Psi - \Psi_m)| \le \sqrt{\Phi^\dagger \mathbb{M} \Phi} \sqrt{(\Psi - \Psi_m)^\dagger \mathbb{M} (\Psi - \Psi_m)}. \tag{3.148}$$

Putting it all together,

$$\left| \lim_{m\to\infty} \int_{\Sigma_t} \Phi^\dagger \mathbb{M} (\Psi - \Psi_m) \, \mathrm{d}V \right| \le \lim_{m\to\infty} \int_{\Sigma_t} \sqrt{\Phi^\dagger \mathbb{M} \Phi} \sqrt{(\Psi - \Psi_m)^\dagger \mathbb{M} (\Psi - \Psi_m)} \mathrm{d}V \tag{3.149}$$

$$= \lim_{m\to\infty} \big\langle \sqrt{\Phi^\dagger \mathbb{M} \Phi}, \sqrt{(\Psi - \Psi_m)^\dagger \mathbb{M} (\Psi - \Psi_m)} \big\rangle_{L^2} \tag{3.150}$$

$$\le \lim_{m\to\infty} ||\sqrt{\Phi^\dagger \mathbb{M} \Phi}||_{L^2} ||\sqrt{(\Psi - \Psi_m)^\dagger \mathbb{M} (\Psi - \Psi_m)}||_{L^2} \tag{3.151}$$

$$= \lim_{m\to\infty} ||\Phi^\dagger \mathbb{M} \Phi||_{L^1} \mathcal{B}(\Psi - \Psi_m) \tag{3.152}$$

$$= 0. \tag{3.153}$$

Analogously, $\lim_{m\to\infty} \int_{\Sigma_t} (\Psi - \Psi_m)^\dagger \mathbb{M} \Phi \, \mathrm{d}V = 0$ too, leaving $\lim_{m\to\infty} Q(\mathcal{Z}_m) = Q(\mathcal{Z})$. □

Chapter 4

Positive energy theorems in asymptotically locally AdS spacetimes

Data! Data! Data! I can't make bricks without clay.
- Sherlock Holmes in *The Adventure of the Copper Beeches* by Arthur Conan Doyle

In this chapter I'll apply the methods developed in the previous chapters to prove positive energy theorems in asymptotically, locally AdS spacetimes. As I'll show, boundary data is the critical information controlling the entire process. After proving a somewhat general theorem, I'll specialise to (conformal) boundaries admitting cross-sections with parallel or real Killing spinors. This will culminate in a positive energy theorem written purely in terms of data intrinsic to cross-sections. I will subsequently apply this theorem to various examples of applicable boundary geometry before extending all the results to include electromagnetic fields. The chapter will conclude by illustrating the theorems with some concrete example metrics. For the duration of this chapter, choose units so that $k = 1/2$, i.e. $\Lambda = -\frac{1}{2}(n-1)(n-2)$.

4.1 General result

The general set-up for this chapter is that of section 2.2. In particular, (M, g) is asymptotically, locally AdS, as defined by a Fefferman-Graham expansion for the metric from the conformal boundary. A background metric, $\bar{g}$, for g is any asymptotically, locally AdS metric with the same boundary metric, $f_{(0)mn}$. The energy of these spacetimes is defined by equation 2.141. One of the central ideas of this chapter will be spacetimes admitting a background possessing a background Killing spinor, as discussed in section 3.3.

Definition 4.1 (p_M)**.** *For future notational convenience, define*

$$p_M = e_M^{(f_{(0)})m} e_0^{(f_{(0)})n} (f_{(n-1)mn} - \bar{f}_{(n-1)mn}) + \delta_{M0} f_{(0)}^{mn} (f_{(n-1)mn} - \bar{f}_{(n-1)mn}) \tag{4.1}$$

$$= \delta_{M0} \hat{f}_{(0)}^{mn} (f_{(n-1)mn} - \bar{f}_{(n-1)mn}) + \delta^A{}_M e_A^{(f_{(0)})m} P_{(0)}^n (f_{(n-1)mn} - \bar{f}_{(n-1)mn}). \tag{4.2}$$

Observe that p_0 is nothing but the integrand in equation 2.141 for the energy.

Theorem 4.2 (Positive energy theorem)**.** *Let (M, g) be an asymptotically, locally AdS spacetime admitting a background metric, $\bar{g}$, which possesses a non-zero background Killing spinor, ε_k. Let $\mathcal{A}_a$ be a Clifford-algebra valued one-form satisfying the conditions in definition 3.1. Assume the Einstein equation holds and ε_k is $O(\mathrm{e}^{r/2})$ near $\Sigma_{t,\infty}$. Then $\exists \epsilon$ such that $\gamma^I \nabla_I \varepsilon = 0$*

and

$$\begin{aligned}
Q(\varepsilon) &= \frac{n-1}{2}\int_{\Sigma_{t,\infty}} \mathrm{e}^{-r} p_M \bar{\varepsilon}_k \gamma^M \varepsilon_k \sqrt{\det(f_{(0)\alpha\beta})}\, \mathrm{d}^{n-2}x \\
&\quad + \int_{\Sigma_{t,\infty}} \mathrm{e}^{(n-2)r} \varepsilon_k^\dagger \left(\gamma^1\gamma^A\mathcal{A}_A + \mathcal{A}_A^\dagger\gamma^A\gamma^1\right)\varepsilon_k \sqrt{\det(f_{(0)\alpha\beta})}\, \mathrm{d}^{n-2}x \qquad (4.3)\\
&= 2\int_{\Sigma_t} \left((\nabla_I\varepsilon)^\dagger\nabla^I\varepsilon + \varepsilon^\dagger \mathbb{M}\varepsilon\right) \mathrm{d}V \qquad (4.4)\\
&\geq 0. \qquad (4.5)
\end{aligned}$$

Note that the growth/decay of ε_k and $||\mathcal{A}_a||_0$ ensure equation 4.3 is convergent.

Proof. From lemma 3.14 and theorem 3.12, $\exists\Psi \in \mathcal{H}$ such that $\gamma^I\nabla_I\Psi = \gamma^I\nabla_I\varepsilon_k$. Choose $\varepsilon = \varepsilon_k - \Psi$. Let $\{\Psi_m\}_{m=0}^{\infty} \subseteq C_c^\infty$ be a Cauchy sequence whose limit is Ψ. Next, let $\varepsilon_m = \varepsilon_k - \Psi_m$ so that $\lim_{m\to\infty}\varepsilon_m = \varepsilon$. Then, since Ψ_m is compactly supported, in a vielbein where $P_a = -\delta_{a0}$ and $\partial_r = e_1$ in the asymptotic end, lemma 3.3 implies

$$Q(\varepsilon_m) = \int_{\Sigma_{t,\infty}} E^{01}(\varepsilon_m)\mathrm{d}A = \int_{\Sigma_{t,\infty}} E^{01}(\varepsilon_k)\mathrm{d}A. \qquad (4.6)$$

Since the RHS doesn't depend on m,

$$\begin{aligned}
\lim_{m\to\infty} Q(\varepsilon_m) &= \int_{\Sigma_{t,\infty}} E^{01}(\varepsilon_k)\mathrm{d}A \qquad (4.7)\\
&= \int_{\Sigma_{t,\infty}} \big(\varepsilon_k^\dagger\gamma^1\gamma^A D_A\varepsilon_k + D_A(\varepsilon_k)^\dagger\gamma^A\gamma^1\varepsilon_k - \mathrm{i}(n-2)\varepsilon_k^\dagger\gamma^1\varepsilon_k \\
&\quad + \varepsilon_k^\dagger\gamma^1\gamma^A\mathcal{A}_A\varepsilon_k + \varepsilon_k^\dagger\mathcal{A}_A^\dagger\gamma^A\gamma^1\varepsilon_k\big)\mathrm{d}A. \qquad (4.8)
\end{aligned}$$

To leading order, the measure, $\mathrm{d}A$, is $\mathrm{e}^{(n-2)r}\sqrt{\det(f_{(0)\alpha\beta})}\,\mathrm{d}^2x\cdots\mathrm{d}^{n-1}x$. This $\mathrm{e}^{(n-2)r}$ growth and the $O(\mathrm{e}^{r/2})$ growth of ε_k means it suffices to keep only terms that decay as $O(\mathrm{e}^{-(n-1)r})$ or slower in the matrices in $E^{01}(\varepsilon_k)$. It's assumed $\mathcal{A}_A$ decays as $O(\mathrm{e}^{-(n-1)r})$, so those terms are kept as they are. Next, consider the derivative terms. Since ε_k is a background Killing spinor,

$$D_A\varepsilon_k = ({e_A}^m - {\bar{e}_A}^m)\partial_m\varepsilon_k - \frac{1}{4}(\omega_{abA} - \bar{\omega}_{abA})\gamma^{ab}\varepsilon_k - \frac{\mathrm{i}}{2}\gamma_A\varepsilon_k. \qquad (4.9)$$

From equation 3.116, $({e_A}^m - {\bar{e}_A}^m)$ is $O(\mathrm{e}^{-nr})$ and hence can be ignored. Likewise, from equations 3.129 and 3.135, only the connection coefficient terms where either a or b is 1 need to be retained. That leaves

$$\varepsilon_k^\dagger\gamma^1\gamma^A D_A\varepsilon_k \to -\frac{1}{2}(\omega_{1MA} - \bar{\omega}_{1MA})\varepsilon_k^\dagger\gamma^A\gamma^M\varepsilon_k + \frac{\mathrm{i}}{2}(n-2)\varepsilon_k^\dagger\gamma^1\varepsilon_k. \qquad (4.10)$$

From equation 3.117, the connection difference is

$$2(\omega_{1MA} - \bar{\omega}_{1MA}) = (g(e_A,[\partial_r,e_M]) + g(e_M,[\partial_r,e_A]) - \bar{g}(\bar{e}_A,[\partial_r,\bar{e}_M]) - \bar{g}(\bar{e}_M,[\partial_r,\bar{e}_A])). \qquad (4.11)$$

This is symmetric in A and M. Hence, it follows that

$$\begin{aligned}
&-\frac{1}{2}(\omega_{1MA} - \bar{\omega}_{1MA})\varepsilon_k^\dagger\gamma^A\gamma^M\varepsilon_k \\
&= \frac{1}{2}\delta^{AB}(g(e_A,[\partial_r,e_B]) - \bar{g}(\bar{e}_A,[\partial_r,\bar{e}_B]))\varepsilon_k^\dagger\varepsilon_k \\
&\quad - \frac{1}{4}(g(e_A,[\partial_r,e_0]) + g(e_0,[\partial_r,e_A]) - \bar{g}(\bar{e}_A,[\partial_r,\bar{e}_0]) - \bar{g}(\bar{e}_0,[\partial_r,\bar{e}_A]))\varepsilon_k^\dagger\gamma^A\gamma^0\varepsilon_k. \qquad (4.12)
\end{aligned}$$

Substituting in equation 3.127 reduces this to

$$
\begin{aligned}
-\frac{1}{2}(\omega_{1MA}-\bar{\omega}_{1MA})\varepsilon_k^\dagger\gamma^A\gamma^M\varepsilon_k &= \delta^{AB}\frac{n-1}{4}\mathrm{e}^{-(n-1)r}\bar{e}_A^{(\bar{f})m}\bar{e}_B^{(\bar{f})n}(f_{(n-1)mn}-\bar{f}_{(n-1)mn})\varepsilon_k^\dagger\varepsilon_k \\
&\quad+\frac{n-1}{4}\mathrm{e}^{-(n-1)r}\bar{e}_A^{(\bar{f})m}\bar{e}_0^{(\bar{f})n}(f_{(n-1)mn}-\bar{f}_{(n-1)mn})\bar{\varepsilon}_k\gamma^A\varepsilon_k \quad (4.13)\\
&=\frac{n-1}{4}\mathrm{e}^{-(n-1)r}\bar{e}_M^{(\bar{f})m}\bar{e}_0^{(\bar{f})n}(f_{(n-1)mn}-\bar{f}_{(n-1)mn})\bar{\varepsilon}_k\gamma^M\varepsilon_k \\
&\quad+\frac{n-1}{4}\mathrm{e}^{-(n-1)r}\bar{f}^{mn}(f_{(n-1)mn}-\bar{f}_{(n-1)mn})\varepsilon_k^\dagger\varepsilon_k \quad (4.14)
\end{aligned}
$$

to leading order. The $\mathrm{e}^{-(n-1)r}$ factor and $\varepsilon_k=O(\mathrm{e}^{r/2})$ mean everything else can be kept to $O(1)$; anything lower order will go to zero in equation 4.8. Therefore ultimately,

$$
-\frac{1}{2}(\omega_{1MA}-\bar{\omega}_{1MA})\varepsilon_k^\dagger\gamma^A\gamma^M\varepsilon_k=\frac{n-1}{4}\mathrm{e}^{-(n-1)r}p_M\bar{\varepsilon}_k\gamma^M\varepsilon_k+O(\mathrm{e}^{-(n-1)r}). \quad (4.15)
$$

Substituting this back implies

$$
\varepsilon_k^\dagger\gamma^1\gamma^A D_A\varepsilon_k\to\frac{n-1}{4}\mathrm{e}^{-(n-1)r}p_M\bar{\varepsilon}_k\gamma^M\varepsilon_k+\frac{\mathrm{i}}{2}(n-2)\varepsilon_k^\dagger\gamma^1\varepsilon_k \quad (4.16)
$$

and consequently equation 4.8 reduces to

$$
\begin{aligned}
\lim_{m\to\infty}Q(\varepsilon_m)&=\frac{n-1}{2}\int_{\Sigma_{t,\infty}}\mathrm{e}^{-r}p_M\bar{\varepsilon}_k\gamma^M\varepsilon_k\sqrt{\det(f_{(0)\alpha\beta})}\,\mathrm{d}^{n-2}x \\
&\quad+\int_{\Sigma_{t,\infty}}\mathrm{e}^{(n-2)r}\varepsilon_k^\dagger\left(\gamma^1\gamma^A\mathcal{A}_A+\mathcal{A}_A^\dagger\gamma^A\gamma^1\right)\varepsilon_k\sqrt{\det(f_{(0)\alpha\beta})}\,\mathrm{d}^{n-2}x. \quad (4.17)
\end{aligned}
$$

Finally, the LHS converges to $Q(\varepsilon)$ by theorem 3.15. □

4.2 Energy bounds and boundary geometry

While theorem 4.2 applies somewhat generally, more progress can be made by restricting the boundary geometry further. The simplest deformation one can make to standard asymptotically AdS spacetimes is to retain a static boundary, but replace the round sphere cross-section with some other compact, Riemannian space.

Definition 4.3 (AdS with cross-section, (S,h))**.** *A metric is defined to be AdS with cross-section, (S,h), if and only if*

$$
g=\mathrm{d}r\otimes\mathrm{d}r+\mathrm{e}^{2r}\left(-\left(1+\frac{c}{4}\mathrm{e}^{-2r}\right)^2\mathrm{d}t\otimes\mathrm{d}t+\left(1-\frac{c}{4}\mathrm{e}^{-2r}\right)^2h\right) \quad (4.18)
$$

and h is a Riemannian metric on a compact, $(n-2)D$ manifold, S, such that $R_{AB}^{(h)}=c(n-3)\delta_{AB}$ for $c=-1,\ 0$ or 1.

It can be checked that any metric of this form satisfies the vacuum Einstein equation. While these metrics may be geodesically incomplete or contain conical singularities when (S,h) is not the round sphere, these global issues don't prevent them from serving as natural background metrics for asymptotically, locally AdS spacetimes with static $\mathbb{R}\times S$ boundary geometry. The main objective of this section is to prove that if h is "symmetric" in some sense, then there is a positive energy theorem for spacetimes with these asymptotics. However, to apply the general result in theorem 4.2, I need Killing spinors for these backgrounds.

Theorem 4.4. *The most general solution to* $D_a\varepsilon_k + \frac{\mathrm{i}}{2}\gamma_a\varepsilon_k = 0$ *for a metric that is AdS with cross-section,* (S, h)*, is*

$$\varepsilon_k = \begin{cases} \mathrm{e}^{r/2}P_1^-\varepsilon_h & \text{for } c = 0 \\ \mathrm{e}^{r/2}P_1^-\left(\mathrm{e}^{\mathrm{i}\gamma^0 t/2} - \mathrm{i}\mathrm{e}^{-\mathrm{i}\gamma^0 t/2}\right)\varepsilon_h + \frac{1}{2}\mathrm{e}^{-r/2}P_1^+\left(\mathrm{e}^{\mathrm{i}\gamma^0 t/2} + \mathrm{i}\mathrm{e}^{-\mathrm{i}\gamma^0 t/2}\right)\varepsilon_h & \text{for } c = 1 \\ 0 & \text{for } c = -1 \end{cases} \tag{4.19}$$

where $P_1^\pm = \frac{1}{2}(I \pm \mathrm{i}\gamma^1)$*,* ε_h *solves* $D_A^{(h)}\varepsilon_h = \frac{c}{2}\gamma_A\varepsilon_h$ *and* $\partial_t\varepsilon_h = 0$*. In the former equation,* $D_A^{(h)}$ *is the formal expression for the Levi-Civita connection of* h*, e.g. if* $\{e^{(h)A}\}_{A=2}^{n-1}$ *is a vielbein for* h *and* $\omega_{BCA}^{(h)}$ *are the corresponding spin coefficients, then*[1] $D_A^{(h)}\varepsilon_h = e_A^{(h)\alpha}\partial_\alpha\varepsilon_h - \frac{1}{4}\omega_{BCA}^{(h)}\gamma^{BC}\varepsilon_h$*.*

Proof. In coordinates, the equation to solve is

$$e_a{}^\mu\partial_\mu\varepsilon_k - \frac{1}{4}\omega_{bca}\gamma^{bc}\varepsilon_k + \frac{\mathrm{i}}{2}\gamma_a\varepsilon_k = 0. \tag{4.20}$$

The most natural vielbein to choose in this context is

$$e^0 = \left(\mathrm{e}^r + \frac{c}{4}\mathrm{e}^{-r}\right)\mathrm{d}t,\ e^1 = \mathrm{d}r \text{ and } e^A = \left(\mathrm{e}^r - \frac{c}{4}\mathrm{e}^{-r}\right)e^{(h)A}, \tag{4.21}$$

where $\{e^{(h)A}\}_{A=2}^{n-1}$ is a vielbein for h. Then, from

$$\mathrm{d}e^0 = \frac{\mathrm{e}^r - \frac{c}{4}\mathrm{e}^{-r}}{\mathrm{e}^r + \frac{c}{4}\mathrm{e}^{-r}}e^1\wedge e^0,\ \ \mathrm{d}e^1 = 0 \text{ and} \tag{4.22}$$

$$\mathrm{d}e^A = \frac{\mathrm{e}^r + \frac{c}{4}\mathrm{e}^{-r}}{\mathrm{e}^r - \frac{c}{4}\mathrm{e}^{-r}}e^1\wedge e^A + \left(\mathrm{e}^r - \frac{c}{4}\mathrm{e}^{-r}\right)\mathrm{d}e^{(h)A}, \tag{4.23}$$

it follows that

$$\omega_{01} = -\frac{\mathrm{e}^r - \frac{c}{4}\mathrm{e}^{-r}}{\mathrm{e}^r + \frac{c}{4}\mathrm{e}^{-r}}e^0,\ \omega_{A1} = \frac{\mathrm{e}^r + \frac{c}{4}\mathrm{e}^{-r}}{\mathrm{e}^r - \frac{c}{4}\mathrm{e}^{-r}}e^A \text{ and } \omega_{AB} = \omega_{AB}^{(h)}, \tag{4.24}$$

where $\omega_{AB}^{(h)}$ are the connection 1-forms of h.

Thus, the $a = 1$ component of equation 4.20 says $0 = \partial_r\varepsilon_k + \frac{\mathrm{i}}{2}\gamma_1\varepsilon_k$, which immediately integrates to $\varepsilon_k = \mathrm{e}^{-\mathrm{i}\gamma^1 r/2}\varepsilon_r$ for some spinor, ε_r, that doesn't depend on r. Projecting ε_r onto eigenspaces of γ^1 using $P_1^\pm = \frac{1}{2}(I \pm \mathrm{i}\gamma^1)$ then yields

$$\varepsilon_k = \mathrm{e}^{-\mathrm{i}\gamma^1 r/2}(P_1^-\varepsilon_- + P_1^+\varepsilon_+) = \mathrm{e}^{r/2}P_1^-\varepsilon_- + \mathrm{e}^{-r/2}P_1^+\varepsilon_+ \tag{4.25}$$

for spinors, $\varepsilon_\pm$, that don't depend on r.

Next, with this expression for ε_k, the $a = 0$ component of equation 4.20 reduces to

$$0 = \frac{1}{\mathrm{e}^r + \frac{c}{4}\mathrm{e}^{-r}}\partial_t\varepsilon_k + \frac{\mathrm{e}^r - \frac{c}{4}\mathrm{e}^{-r}}{2(\mathrm{e}^r + \frac{c}{4}\mathrm{e}^{-r})}\gamma^0\gamma^1 - \frac{\mathrm{i}}{2}\gamma^0\varepsilon_k \tag{4.26}$$

[1] This is only a formal Levi-Civita connection because the gamma matrices used are from the spacetime, not the cross-section. In effect, this is a Levi-Civita connection that results from a reducible representation of the Clifford algebra.

or equivalently

$$\begin{aligned}
0 &= \partial_t(\mathrm{e}^{r/2}P_1^-\varepsilon_- + \mathrm{e}^{-r/2}P_1^+\varepsilon_+) + \frac{1}{2}\left(\mathrm{e}^r - \frac{c}{4}\mathrm{e}^{-r}\right)\gamma^0\gamma^1(\mathrm{e}^{r/2}P_1^-\varepsilon_- + \mathrm{e}^{-r/2}P_1^+\varepsilon_+) \\
&\quad - \frac{\mathrm{i}}{2}\left(\mathrm{e}^r + \frac{c}{4}\mathrm{e}^{-r}\right)\gamma^0(\mathrm{e}^{r/2}P_1^-\varepsilon_- + \mathrm{e}^{-r/2}P_1^+\varepsilon_+) \qquad (4.27) \\
&= \mathrm{e}^{r/2}P_1^-\partial_t\varepsilon_- + \mathrm{e}^{-r/2}P_1^+\partial_t\varepsilon_+ + \frac{\mathrm{i}}{2}\mathrm{e}^{3r/2}\gamma^0P_1^-\varepsilon_- - \frac{\mathrm{i}c}{8}\mathrm{e}^{-r/2}\gamma^0P_1^-\varepsilon_- - \frac{\mathrm{i}}{2}\mathrm{e}^{r/2}\gamma^0P_1^+\varepsilon_+ \\
&\quad + \frac{\mathrm{i}c}{8}\mathrm{e}^{-3r/2}\gamma^0P_1^+\varepsilon_+ - \frac{\mathrm{i}}{2}\mathrm{e}^{3r/2}\gamma^0P_1^-\varepsilon_- - \frac{\mathrm{i}c}{8}\mathrm{e}^{-r/2}\gamma^0P_1^-\varepsilon_- - \frac{\mathrm{i}}{2}\mathrm{e}^{r/2}\gamma^0P_1^+\varepsilon_+ \\
&\quad - \frac{\mathrm{i}c}{8}\mathrm{e}^{-3r/2}\gamma^0P_1^+\varepsilon_+ \qquad (4.28) \\
&= \mathrm{e}^{r/2}P_1^-\left(\partial_t\varepsilon_- - \mathrm{i}\gamma^0\varepsilon_+\right) + \mathrm{e}^{-r/2}P_1^+\left(\partial_t\varepsilon_+ - \frac{\mathrm{i}c}{4}\gamma^0\varepsilon_-\right). \qquad (4.29)
\end{aligned}$$

Since the two γ^1 eigenspaces have no non-trivial intersection, it follows that

$$\partial_t\varepsilon_- = \mathrm{i}\gamma^0\varepsilon_+ \text{ and } \partial_t\varepsilon_+ = \frac{\mathrm{i}c}{4}\gamma^0\varepsilon_-. \qquad (4.30)$$

For the $a = A$ component, following the same process as the $a = 0$ case and also noting $\omega_{AB} = \omega_{AB}^{(h)} \implies \omega_{ABC} = \frac{1}{\mathrm{e}^r - \frac{c}{4}\mathrm{e}^{-r}}\omega_{ABC}^{(h)}$, I get

$$0 = \mathrm{e}^{r/2}P_1^-\left(D_A^{(h)}\varepsilon_- + \mathrm{i}\gamma_A\varepsilon_+\right) + \mathrm{e}^{-r/2}P_1^+\left(D_A^{(h)}\varepsilon_+ - \frac{\mathrm{i}c}{4}\gamma_A\varepsilon_-\right), \qquad (4.31)$$

from which it follows that

$$D_A^{(h)}\varepsilon_- = -\mathrm{i}\gamma_A\varepsilon_+ \text{ and } D_A^{(h)}\varepsilon_+ = \frac{\mathrm{i}c}{4}\gamma_A\varepsilon_-. \qquad (4.32)$$

First consider equations 4.30 and 4.32 for $c = 0$. From the former it immediately follows that $\varepsilon_- = \mathrm{i}t\gamma^0\varepsilon_+ + \varepsilon_h$ for some spinor, ε_h, that (like ε_+) doesn't depend on t. Then, the latter implies $D^{(h)A}D_A^{(h)}\varepsilon_h = -\mathrm{i}\gamma^A D_A^{(h)}\varepsilon_+ - 0 = 0$.

Therefore $D_A^{(h)}\varepsilon_h = 0$ because integrating on the cross-section, S, says

$$0 = \int_S \varepsilon_h^\dagger D^{(h)A}D_A^{(h)}(\varepsilon_h)\mathrm{d}A(h) = -\int_S \left(D^{(h)A}\varepsilon_h\right)^\dagger D_A^{(h)}(\varepsilon_h)\mathrm{d}A(h). \qquad (4.33)$$

Then $-\mathrm{i}\gamma_A\varepsilon_+ = D_A^{(h)}\varepsilon_- = 0 \implies \varepsilon_+ = 0$ and leaves $\varepsilon_- = \varepsilon_h$, completing the $c = 0$ analysis.

Next, consider $c = -1$. Equation 4.32 now implies $D^{(h)A}D_A^{(h)}\varepsilon_- = \frac{n-2}{4}\varepsilon_-$. Therefore,

$$\begin{aligned}
\int_S \varepsilon_-^\dagger\varepsilon_-\mathrm{d}A(h) &= \frac{4}{n-2}\int_S \varepsilon_-^\dagger D^{(h)A}D_A^{(h)}(\varepsilon_-)\mathrm{d}A(h) \qquad (4.34) \\
&= -\frac{4}{n-2}\int_S (D^{(h)A}\varepsilon_-)^\dagger D_A^{(h)}(\varepsilon_-)\mathrm{d}A(h). \qquad (4.35)
\end{aligned}$$

As the LHS is non-negative and the RHS is non-positive, it must be that both are zero. Hence, $\varepsilon_- = 0$ from the LHS and subsequently $\varepsilon_+ = 0$ from $D_A^{(h)}\varepsilon_- = -\mathrm{i}\gamma_A\varepsilon_+$.

Finally, consider $c = 1$. Let

$$\psi = \varepsilon_- + 2\varepsilon_+ \text{ and } \varphi = \varepsilon_- - 2\varepsilon_+ \iff \varepsilon_- = \frac{1}{2}(\psi + \varphi) \text{ and } \varepsilon_+ = \frac{1}{4}(\psi - \varphi). \qquad (4.36)$$

Then, equations 4.30 and 4.32 are

$$\partial_t\psi = \frac{\mathrm{i}}{2}\gamma^0\psi,\ \partial_t\varphi = -\frac{\mathrm{i}}{2}\gamma^0\varphi,\ D^{(h)}_A\psi = \frac{\mathrm{i}}{2}\gamma_A\varphi \text{ and } D^{(h)}_A\varphi = -\frac{\mathrm{i}}{2}\gamma_A\psi. \tag{4.37}$$

The first two immediately integrate to $\psi = 2\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\psi_t$ and $\varphi = 2\mathrm{e}^{-\mathrm{i}\gamma^0 t/2}\varphi_t$ for some spinors, ψ_t and φ_t, that don't depend on t or r. Equivalently,

$$\varepsilon_- = \mathrm{e}^{\mathrm{i}\gamma^0 t/2}\psi_t + \mathrm{e}^{-\mathrm{i}\gamma^0 t/2}\varphi_t \text{ and } \varepsilon_+ = \frac{1}{2}(\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\psi_t - \mathrm{e}^{-\mathrm{i}\gamma^0 t/2}\varphi_t). \tag{4.38}$$

By construction, $P_1^\pm\varepsilon_\pm = \varepsilon_\pm \iff \varepsilon_\pm = \pm\mathrm{i}\gamma^1\varepsilon_\pm$ without loss of generality. Therefore,

$$\varepsilon_- = \mathrm{e}^{\mathrm{i}\gamma^0 t/2}\psi_t + \mathrm{e}^{-\mathrm{i}\gamma^0 t/2}\varphi_t = -\mathrm{i}\gamma^1(\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\psi_t + \mathrm{e}^{-\mathrm{i}\gamma^0 t/2}\varphi_t) = -\mathrm{i}\mathrm{e}^{-\mathrm{i}\gamma^0 t/2}\gamma^1\psi_t - \mathrm{i}\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\gamma^1\varphi_t. \tag{4.39}$$

Setting $t = 0$, π this equation implies

$$\psi_t + \varphi_t = -\mathrm{i}\gamma^1\psi_t - \mathrm{i}\gamma^1\varphi_t \text{ and } \psi_t - \varphi_t = \mathrm{i}\gamma^1\psi_t - \mathrm{i}\gamma^1\varphi_t \tag{4.40}$$

respectively. Putting the two equations together, it follows that $\psi_t = -\mathrm{i}\gamma^1\varphi_t$. Hence,

$$\varepsilon_- = \mathrm{e}^{\mathrm{i}\gamma^0 t/2}\psi_t + \mathrm{e}^{-\mathrm{i}\gamma^0 t/2}\varphi_t = (I - \mathrm{i}\gamma^1)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\psi_t \text{ and} \tag{4.41}$$

$$\varepsilon_+ = \frac{1}{2}(\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\psi_t - \mathrm{e}^{-\mathrm{i}\gamma^0 t/2}\varphi_t) = \frac{1}{2}(I + \mathrm{i}\gamma^1)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\psi_t. \tag{4.42}$$

Let $\varepsilon_h = \frac{1}{2}(I + \gamma^1)\psi_t \iff \psi_t = (I - \gamma^1)\varepsilon_h$. Then, I get

$$D^{(h)}_A\varepsilon_h = \frac{1}{2}(I + \gamma^1)D^{(h)}_A\psi_t = \frac{1}{2}(I + \gamma^1)\frac{1}{2}\gamma_A\gamma^1\psi_t = \frac{1}{4}\gamma_A(\gamma^1 + I)\psi_t = \frac{1}{2}\gamma_A\varepsilon_h, \tag{4.43}$$

$$\varepsilon_- = (I - \mathrm{i}\gamma^1)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}(I - \gamma^1)\varepsilon_h = (I - \mathrm{i}\gamma^1)(\mathrm{e}^{\mathrm{i}\gamma^0 t/2} - \mathrm{i}\mathrm{e}^{-\mathrm{i}\gamma^0 t/2})\varepsilon_h \text{ and} \tag{4.44}$$

$$\varepsilon_+ = \frac{1}{2}(I + \mathrm{i}\gamma^1)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}(I - \gamma^1)\varepsilon_h = \frac{1}{2}(I + \mathrm{i}\gamma^1)(\mathrm{e}^{\mathrm{i}\gamma^0 t/2} + \mathrm{i}\mathrm{e}^{-\mathrm{i}\gamma^0 t/2})\varepsilon_h. \tag{4.45}$$

It can now be checked directly that both equations in 4.32 hold with these $\varepsilon_\pm$, thereby completing the $c = 1$ analysis. □

Solutions to $D^{(h)}_A\varepsilon_h = \frac{c}{2}\gamma_A\varepsilon_h$ are well-studied mathematical problems [119, 5]. However, one subtlety in comparing with the literature is that $\{\gamma^A\}_{A=2}^{n-1}$ don't form an irreducible representation of h's Clifford algebra; an irreducible representation of a (Riemannian) Clifford algebra with $n-2$ elements would have $2^{\lfloor(n-2)/2\rfloor} \times 2^{\lfloor(n-2)/2\rfloor}$ matrices, not $2^{\lfloor n/2\rfloor} \times 2^{\lfloor n/2\rfloor}$ matrices like $\{\gamma^A\}_{A=2}^{n-1} \subset \{\gamma^a\}_{a=0}^{n-1}$. The doubled size means there are effectively two irreducible representations summed in γ^A. This degeneracy can be lifted by choosing the spacetime gamma matrices to be

$$\gamma^0 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix},\ \gamma^1 = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \text{ and } \gamma^A = \begin{bmatrix} 0 & \hat{\gamma}^A \\ \hat{\gamma}^A & 0 \end{bmatrix}, \tag{4.46}$$

where $\hat{\gamma}^A$ are gamma matrices of the Riemannian manifold, (S, h). In this representation, the spacetime spinor space can be viewed as a direct sum of the cross-section spinor space with itself. Now, $D^{(h)}_A\varepsilon_h = \frac{c}{2}\gamma_A\varepsilon_h$ can be written in a form that's truly intrinsic to the cross-section.

Lemma 4.5. *Let $\hat{D}^{(h)}_A = e^{(h)\alpha}_A\partial_\alpha - \frac{1}{4}\omega^{(h)}_{BCA}\hat{\gamma}^{BC}$ be the spin connection intrinsic to h. Then, the most general solution to $D^{(h)}_A\varepsilon_h = 0$ is*

$$\varepsilon_h = \begin{bmatrix} \hat{\psi} \\ \hat{\varphi} \end{bmatrix} \text{ with } \hat{D}^{(h)}_A\hat{\psi} = \hat{D}^{(h)}_A\hat{\varphi} = 0 \tag{4.47}$$

and the most general solution to $D^{(h)}_A\varepsilon_h = \frac{1}{2}\gamma_A\varepsilon_h$ *is*

$$\varepsilon_h = \frac{1}{2}\begin{bmatrix}\hat{\varepsilon}^{(+)}_h + \hat{\varepsilon}^{(-)}_h \\ \hat{\varepsilon}^{(+)}_h - \hat{\varepsilon}^{(-)}_h\end{bmatrix} \text{ with } \hat{D}^{(h)}_A\hat{\varepsilon}^{(\pm)}_h = \pm\frac{1}{2}\hat{\gamma}_A\hat{\varepsilon}^{(\pm)}_h. \tag{4.48}$$

Proof. Let $\varepsilon_h = (\hat{\psi}, \hat{\varphi})^T$ in the chosen representation. Then,

$$D^{(h)}_A\varepsilon_h = \left(e^{(h)\alpha}_A\partial_\alpha - \frac{1}{8}\omega^{(h)}_{BCA}\left(\begin{bmatrix}0 & \hat{\gamma}^A \\ \hat{\gamma}^A & 0\end{bmatrix}\begin{bmatrix}0 & \hat{\gamma}^B \\ \hat{\gamma}^B & 0\end{bmatrix} - \begin{bmatrix}0 & \hat{\gamma}^B \\ \hat{\gamma}^B & 0\end{bmatrix}\begin{bmatrix}0 & \hat{\gamma}^A \\ \hat{\gamma}^A & 0\end{bmatrix}\right)\right)\begin{bmatrix}\hat{\psi} \\ \hat{\varphi}\end{bmatrix} \tag{4.49}$$

$$= \begin{bmatrix}\hat{D}^{(h)}_A\hat{\psi} \\ \hat{D}^{(h)}_A\hat{\varphi}\end{bmatrix}. \tag{4.50}$$

Therefore the claim about $D^{(h)}_A\varepsilon_h = 0$ immediately follows. Meanwhile, since

$$\gamma_A\varepsilon_h = \begin{bmatrix}0 & \hat{\gamma}_A \\ \hat{\gamma}_A & 0\end{bmatrix}\begin{bmatrix}\hat{\psi} \\ \hat{\varphi}\end{bmatrix} = \begin{bmatrix}\hat{\gamma}_A\hat{\varphi} \\ \hat{\gamma}_A\hat{\psi}\end{bmatrix}, \tag{4.51}$$

it also follows that

$$0 = D^{(h)}_A\varepsilon_h - \frac{1}{2}\gamma_A\varepsilon_h \iff \begin{bmatrix}\hat{D}^{(h)}_A\hat{\psi} - \frac{1}{2}\hat{\gamma}_A\hat{\varphi} \\ \hat{D}^{(h)}_A\hat{\varphi} - \frac{1}{2}\hat{\gamma}_A\hat{\psi}\end{bmatrix} = \begin{bmatrix}0 \\ 0\end{bmatrix}. \tag{4.52}$$

Hence, $\hat{\varepsilon}^{(\pm)}_h = \hat{\psi} \pm \hat{\varphi}$ proves the claim for $D^{(h)}_A\varepsilon_h = \frac{1}{2}\gamma_A\varepsilon_h$. □

In summary, ε_k is built from the most general parallel or real Killing spinors (of either + or − orientation) on the cross-section. The simply connected, compact manifolds admitting such spinors have been classified [119, 5]. However, there are also non-simply connected manifolds which satisfy the required conditions [120]; I will consider some such examples in section 4.3.

Meanwhile, observe that if $\hat{D}^{(h)}_A\hat{\varepsilon}_h$ equals $\frac{1}{2}\hat{\gamma}_A\hat{\varepsilon}_h$, $-\frac{1}{2}\hat{\gamma}_A\hat{\varepsilon}_h$ or 0, then

$$\hat{D}^{(h)}_A\left(\hat{\varepsilon}^\dagger_h\hat{\varepsilon}_h\right) = 0 \text{ and } \hat{D}^{(h)}_A\left(-\mathrm{i}\hat{\varepsilon}^\dagger_h\hat{\gamma}^A\hat{\varepsilon}_h\right) = \mathrm{i}\hat{\varepsilon}^\dagger_h\hat{\gamma}_{AB}\hat{\varepsilon}_h,\ -\mathrm{i}\hat{\varepsilon}^\dagger_h\hat{\gamma}_{AB}\hat{\varepsilon}_h \text{ or } 0. \tag{4.53}$$

Therefore $-\mathrm{i}\hat{\varepsilon}^\dagger_h\hat{\gamma}^A\hat{\varepsilon}_h$ is a Killing vector of h and $\hat{\varepsilon}^\dagger_h\hat{\varepsilon}_h$ is constant on S, allowing the following constructions.

Definition 4.6 ("Conserved quantities" on the cross-section)**.** *In an asymptotically AdS spacetime with cross-section, (S, h), given a Killing vector, $\hat{k}$, of h, define a "conserved quantity,"*

$$Q_{\hat{k}} = \frac{n-1}{16\pi}\int_S p_A\hat{k}^A\mathrm{d}A(h) = \frac{n-1}{16\pi}\int_S f_{(n-1)0\alpha}\hat{k}^\alpha\mathrm{d}A(h). \tag{4.54}$$

Theorem 4.7. *For an asymptotically AdS spacetime with cross-section, (S, h), $c = 0$, satisfying the Einstein equation and the dominant energy condition, define $\hat{k}^A = -\mathrm{i}\hat{\psi}^\dagger\hat{\gamma}^A\hat{\psi}$, where $\hat{\psi}$ solves $\hat{D}^{(h)}_A\hat{\psi} = 0$. Without loss of generality, scale $\hat{\psi}$ such that $\hat{\psi}^\dagger\hat{\psi} = 1$. Then,*

$$E + Q_{\hat{k}} \geq 0. \tag{4.55}$$

Proof. Choosing $\mathcal{A}_a = 0$ in theorem 4.2 reduces the non-negativity condition on $\mathbb{M}$ to the dominant energy condition on T_{ab}. Then, substituting the $c = 0$ case of theorem 4.4 yields

$$0 \leq \frac{n-1}{2}\int_S p_M\varepsilon^\dagger_h\gamma^0\gamma^M P^-_1\varepsilon_h\mathrm{d}A(h). \tag{4.56}$$

From lemma 4.5 and the chosen gamma matrices,

$$\varepsilon_h^\dagger\gamma^0\gamma^0 P_1^-\varepsilon_h = \begin{bmatrix}\hat\psi^\dagger & \hat\varphi^\dagger\end{bmatrix}^{\mathrm{T}}\frac{1}{2}\begin{bmatrix} I & \mathrm{i}I \\ -\mathrm{i}I & I\end{bmatrix}\begin{bmatrix}\hat\psi \\ \hat\varphi\end{bmatrix} = \frac{1}{2}\left(\hat\psi^\dagger\hat\psi + \mathrm{i}\hat\psi^\dagger\hat\varphi - \mathrm{i}\hat\varphi^\dagger\hat\psi + \hat\varphi^\dagger\hat\varphi\right) \quad \text{and} \tag{4.57}$$

$$\varepsilon_h^\dagger\gamma^0\gamma^A P_1^-\varepsilon_h = \begin{bmatrix}\hat\psi^\dagger & \hat\varphi^\dagger\end{bmatrix}^{\mathrm{T}}\begin{bmatrix} I & 0 \\ 0 & -I\end{bmatrix}\begin{bmatrix} 0 & \hat\gamma^A \\ \hat\gamma^A & 0\end{bmatrix}\frac{1}{2}\begin{bmatrix} I & \mathrm{i}I \\ -\mathrm{i}I & I\end{bmatrix}\begin{bmatrix}\hat\psi \\ \hat\varphi\end{bmatrix} \tag{4.58}$$

$$= \frac{1}{2}\left(-\mathrm{i}\hat\psi^\dagger\hat\gamma^A\hat\psi + \hat\psi^\dagger\hat\gamma^A\hat\varphi - \hat\varphi^\dagger\hat\gamma^A\hat\psi - \mathrm{i}\hat\varphi^\dagger\hat\gamma^A\hat\varphi\right). \tag{4.59}$$

Putting both parts together,

$$0 \le \frac{n-1}{4}\int_S \left(\hat\psi^\dagger - \mathrm{i}\hat\varphi^\dagger\right)\left(p_0 I - \mathrm{i}p_A\hat\gamma^A\right)\left(\hat\psi + \mathrm{i}\hat\varphi\right)\mathrm{d}A(h). \tag{4.60}$$

Then, defining a new parallel spinor, say $\hat\psi' = \hat\psi + \mathrm{i}\hat\varphi$, and scaling (by a constant) so that $\hat\psi'^\dagger\hat\psi' = 1$ proves the claim. □

Theorem 4.8. *For an asymptotically AdS spacetime with cross-section, (S,h), $c = 1$, satisfying the Einstein equation and the dominant energy condition, let $\hat\varepsilon_h^{(\pm)}$ solve $\hat D_A^{(h)}\hat\varepsilon_h^{(\pm)} = \pm\frac{1}{2}\hat\gamma_A\hat\varepsilon_h^{(\pm)}$ and define $\hat k^{(\pm)A} = -\mathrm{i}\hat\varepsilon_h^{(\pm)\dagger}\hat\gamma^A\hat\varepsilon_h^{(\pm)}$. Without loss of generality, scale $\hat\varepsilon_h^{(\pm)}$ so that $\hat\varepsilon_h^{(\pm)\dagger}\hat\varepsilon_h^{(\pm)} = \hat\delta^{(\pm)}$, where $\hat\delta^{(\pm)} = 1$ if a non-trivial $\hat\varepsilon_h^{(\pm)}$ exists and $\hat\delta^{(\pm)} = 0$ otherwise[2]. Then, if h is not the round metric on a sphere,*

$$E\big(\hat\delta^{(+)} + \hat\delta^{(-)}\big) + Q_{\hat k^{(+)}} + Q_{\hat k^{(-)}} \ge 0. \tag{4.61}$$

Proof. Let $\hat s^{(\pm)} = \hat\varepsilon_h^{(\mp)\dagger}\hat\varepsilon_h^{(\pm)}$ and $\hat\xi^{(\pm)A} = \hat\varepsilon_h^{(\mp)\dagger}\hat\gamma^A\hat\varepsilon_h^{(\pm)}$. From the Killing spinor equation,

$$\hat D_A^{(h)}\hat s^{(\pm)} = \pm\hat\xi_A \quad \text{and} \tag{4.62}$$

$$\hat D_A^{(h)}\hat D_B^{(h)}\hat s^{(\pm)} = \pm\left(\pm\frac{1}{2}\hat\varepsilon_h^{(\mp)\dagger}\hat\gamma_A\right)\hat\gamma_B\hat\varepsilon_h^{(\pm)} \pm \hat\varepsilon_h^{(\mp)\dagger}\hat\gamma_B\left(\pm\frac{1}{2}\hat\gamma_A\hat\varepsilon_h^{(\pm)}\right) = -\delta_{AB}\hat s^{(\pm)}. \tag{4.63}$$

By Obata's theorem [96], the round sphere is the only compact, Riemannian manifold admitting non-trivial solutions to $\hat D_A^{(h)}\hat D_B^{(h)}\hat s^{(\pm)} = -\delta_{AB}\hat s^{(\pm)}$. As that case has been explicitly excluded in this theorem, it must be that $\hat s^{(\pm)} = 0$ and consequently $\hat\xi^{(\pm)A} = 0$.

Now consider theorem 4.2 with $c = 1$, $\mathcal{A}_a = 0$ and the ε_k from theorem 4.4. Since there is a leading e^{-r} factor in equation 4.3, it suffices to retain only the $\mathrm{e}^{r/2}$ term in equation 4.19, i.e.

$$\varepsilon_k \to \mathrm{e}^{r/2}P_1^-\left(\mathrm{e}^{\mathrm{i}\gamma^0 t/2} - \mathrm{i}\mathrm{e}^{-\mathrm{i}\gamma^0 t/2}\right)\varepsilon_h = (1+\mathrm{i})\mathrm{e}^{r/2}P_1^-\left(\cos(t/2)I - \sin(t/2)\gamma^0\right)\varepsilon_h. \tag{4.64}$$

In summary, theorem 4.2 reduces to saying

$$0 \le \int_S p_M\varepsilon_h^\dagger\left(\cos(t/2)I - \sin(t/2)\gamma^0\right)\gamma^0\gamma^M P_1^-(\cos(t/2)I - \sin(t/2)\gamma^0)\varepsilon_h\,\mathrm{d}A(h). \tag{4.65}$$

Consider the coefficient of p_M in the integrand. Using the gamma matrices in equation 4.46

[2]Note that the theorem only applies if at least one of $\hat\delta^{(+)}$ or $\hat\delta^{(-)}$ is non-zero.

and the ε_h from lemma 4.5, when $M=0$,

$$\begin{aligned}
&\varepsilon_h^\dagger\left(\cos(t/2)I-\sin(t/2)\gamma^0\right)P_1^-(\cos(t/2)I-\sin(t/2)\gamma^0)\varepsilon_h\\
&=\frac{1}{8}\begin{bmatrix}\hat\varepsilon_h^{(+)\dagger}+\hat\varepsilon_h^{(-)\dagger} & \hat\varepsilon_h^{(+)\dagger}-\hat\varepsilon_h^{(-)\dagger}\end{bmatrix}\begin{bmatrix}(\cos(t/2)-\sin(t/2))I & 0\\ 0 & (\cos(t/2)+\sin(t/2))I\end{bmatrix}\begin{bmatrix}I & \mathrm{i}I\\ -\mathrm{i}I & I\end{bmatrix}\\
&\quad\times\begin{bmatrix}(\cos(t/2)-\sin(t/2))I & 0\\ 0 & (\cos(t/2)+\sin(t/2))I\end{bmatrix}\begin{bmatrix}\hat\varepsilon_h^{(+)}+\hat\varepsilon_h^{(-)}\\ \hat\varepsilon_h^{(+)}-\hat\varepsilon_h^{(-)}\end{bmatrix} &(4.66)\\
&=\frac{1}{8}\begin{bmatrix}\hat\varepsilon_h^{(+)\dagger}+\hat\varepsilon_h^{(-)\dagger} & \hat\varepsilon_h^{(+)\dagger}-\hat\varepsilon_h^{(-)\dagger}\end{bmatrix}\begin{bmatrix}(\cos(t/2)-\sin(t/2))I & \mathrm{i}(\cos(t/2)-\sin(t/2))I\\ -\mathrm{i}(\cos(t/2)+\sin(t/2))I & (\cos(t/2)+\sin(t/2))I\end{bmatrix}\\
&\quad\times\begin{bmatrix}(\cos(t/2)-\sin(t/2))(\hat\varepsilon_h^{(+)}+\hat\varepsilon_h^{(-)})\\ (\cos(t/2)+\sin(t/2))(\hat\varepsilon_h^{(+)}-\hat\varepsilon_h^{(-)})\end{bmatrix} &(4.67)\\
&=\frac{1}{8}\begin{bmatrix}\hat\varepsilon_h^{(+)\dagger}+\hat\varepsilon_h^{(-)\dagger} & \hat\varepsilon_h^{(+)\dagger}-\hat\varepsilon_h^{(-)\dagger}\end{bmatrix}\\
&\quad\times\begin{bmatrix}(\cos(t/2)-\sin(t/2))^2(\hat\varepsilon_h^{(+)}+\hat\varepsilon_h^{(-)})+\mathrm{i}(\cos^2(t/2)-\sin^2(t/2))(\hat\varepsilon_h^{(+)}-\hat\varepsilon_h^{(-)})\\ -\mathrm{i}(\cos^2(t/2)-\sin^2(t/2))(\hat\varepsilon_h^{(+)}+\hat\varepsilon_h^{(-)})+(\cos(t/2)+\sin(t/2))^2(\hat\varepsilon_h^{(+)}-\hat\varepsilon_h^{(-)})\end{bmatrix} &(4.68)\\
&=\frac{1}{8}\big((\hat\varepsilon_h^{(+)\dagger}+\hat\varepsilon_h^{(-)\dagger})(1-\sin(t))(\hat\varepsilon_h^{(+)}+\hat\varepsilon_h^{(-)})+(\hat\varepsilon_h^{(+)\dagger}+\hat\varepsilon_h^{(-)\dagger})\mathrm{i}\cos(t)(\hat\varepsilon_h^{(+)}-\hat\varepsilon_h^{(-)})\\
&\quad+(\hat\varepsilon_h^{(+)\dagger}-\hat\varepsilon_h^{(-)\dagger})(-\mathrm{i}\cos(t))(\hat\varepsilon_h^{(+)}+\hat\varepsilon_h^{(-)})+(\hat\varepsilon_h^{(+)\dagger}-\hat\varepsilon_h^{(-)\dagger})(1+\sin(t))(\hat\varepsilon_h^{(+)}-\hat\varepsilon_h^{(-)})\big) &(4.69)\\
&=\frac{1}{4}\left(\hat\varepsilon_h^{(+)\dagger}\hat\varepsilon_h^{(+)}-\mathrm{i}\mathrm{e}^{-\mathrm{i}t}\hat\varepsilon_h^{(+)\dagger}\hat\varepsilon_h^{(-)}+\mathrm{i}\mathrm{e}^{\mathrm{i}t}\hat\varepsilon_h^{(-)\dagger}\hat\varepsilon_h^{(+)}+\hat\varepsilon_h^{(-)\dagger}\hat\varepsilon_h^{(-)}\right). &(4.70)
\end{aligned}$$

Similarly, when $M=A$,

$$\begin{aligned}
&\varepsilon_h^\dagger\left(\cos(t/2)I-\sin(t/2)\gamma^0\right)\gamma^0\gamma^A P_1^-(\cos(t/2)I-\sin(t/2)\gamma^0)\varepsilon_h\\
&=\frac{1}{8}\begin{bmatrix}\hat\varepsilon_h^{(+)\dagger}+\hat\varepsilon_h^{(-)\dagger} & \hat\varepsilon_h^{(+)\dagger}-\hat\varepsilon_h^{(-)\dagger}\end{bmatrix}\begin{bmatrix}(\cos(t/2)-\sin(t/2))I & 0\\ 0 & (\cos(t/2)+\sin(t/2))I\end{bmatrix}\begin{bmatrix}0 & \hat\gamma^A\\ -\hat\gamma^A & 0\end{bmatrix}\\
&\quad\times\begin{bmatrix}I & \mathrm{i}I\\ -\mathrm{i}I & I\end{bmatrix}\begin{bmatrix}(\cos(t/2)-\sin(t/2))I & 0\\ 0 & (\cos(t/2)+\sin(t/2))I\end{bmatrix}\begin{bmatrix}\hat\varepsilon_h^{(+)}+\hat\varepsilon_h^{(-)}\\ \hat\varepsilon_h^{(+)}-\hat\varepsilon_h^{(-)}\end{bmatrix} &(4.71)\\
&=\frac{1}{8}\begin{bmatrix}\hat\varepsilon_h^{(+)\dagger}+\hat\varepsilon_h^{(-)\dagger} & \hat\varepsilon_h^{(+)\dagger}-\hat\varepsilon_h^{(-)\dagger}\end{bmatrix}\begin{bmatrix}0 & (\cos(t/2)-\sin(t/2))\hat\gamma^A\\ -(\cos(t/2)+\sin(t/2))\hat\gamma^A & 0\end{bmatrix}\\
&\quad\times\begin{bmatrix}(\cos(t/2)-\sin(t/2))I & \mathrm{i}(\cos(t/2)+\sin(t/2))I\\ -\mathrm{i}(\cos(t/2)-\sin(t/2))I & (\cos(t/2)+\sin(t/2))I\end{bmatrix}\begin{bmatrix}\hat\varepsilon_h^{(+)}+\hat\varepsilon_h^{(-)}\\ \hat\varepsilon_h^{(+)}-\hat\varepsilon_h^{(-)}\end{bmatrix} &(4.72)\\
&=\frac{1}{8}\begin{bmatrix}\hat\varepsilon_h^{(+)\dagger}+\hat\varepsilon_h^{(-)\dagger} & \hat\varepsilon_h^{(+)\dagger}-\hat\varepsilon_h^{(-)\dagger}\end{bmatrix}\begin{bmatrix}-\mathrm{i}(1-\sin(t))\hat\gamma^A & \cos(t)\hat\gamma^A\\ -\cos(t)\hat\gamma^A & -\mathrm{i}(1+\sin(t))\hat\gamma^A\end{bmatrix}\begin{bmatrix}\hat\varepsilon_h^{(+)}+\hat\varepsilon_h^{(-)}\\ \hat\varepsilon_h^{(+)}-\hat\varepsilon_h^{(-)}\end{bmatrix} &(4.73)\\
&=\frac{1}{4}\left(-\mathrm{i}\hat\varepsilon_h^{(+)\dagger}\hat\gamma^A\hat\varepsilon_h^{(+)}-\mathrm{e}^{-\mathrm{i}t}\hat\varepsilon_h^{(+)\dagger}\hat\gamma^A\hat\varepsilon_h^{(-)}+\mathrm{e}^{\mathrm{i}t}\hat\varepsilon_h^{(-)\dagger}\hat\gamma^A\hat\varepsilon_h^{(+)}-\mathrm{i}\hat\varepsilon_h^{(-)\dagger}\hat\gamma^A\hat\varepsilon_h^{(-)}\right). &(4.74)
\end{aligned}$$

Putting both parts together,

$$\begin{aligned}
&p_M\varepsilon_h^\dagger\left(\cos(t/2)I-\sin(t/2)\gamma^0\right)\gamma^0\gamma^M P_1^-(\cos(t/2)I-\sin(t/2)\gamma^0)\varepsilon_h\\
&=\left(\hat\varepsilon_h^{(+)\dagger}+\mathrm{i}\mathrm{e}^{\mathrm{i}t}\hat\varepsilon_h^{(-)\dagger}\right)\left(p_0I-\mathrm{i}p_A\hat\gamma^A\right)\left(\hat\varepsilon_h^{(+)}-\mathrm{i}\mathrm{e}^{-\mathrm{i}t}\hat\varepsilon_h^{(-)}\right). \qquad (4.75)
\end{aligned}$$

Since $\hat s^{(\pm)}=0$ and $\hat\xi^{(\pm)A}=0$, I ultimately get

$$0\le\int_S\left(p_0\hat\varepsilon_h^{(+)\dagger}\hat\varepsilon_h^{(+)}+p_0\hat\varepsilon_h^{(-)\dagger}\hat\varepsilon_h^{(-)}-\mathrm{i}p_A\hat\varepsilon_h^{(+)\dagger}\hat\gamma^A\hat\varepsilon_h^{(+)}-\mathrm{i}p_A\hat\varepsilon_h^{(-)\dagger}\hat\gamma^A\hat\varepsilon_h^{(-)}\right)\mathrm{d}A(h), \qquad (4.76)$$

which is just the desired result. □

4.2.1 Asymptotic symmetries and geometric invariance

To better understand the full physical significance of the inequalities just derived, it's worth considering the geometric invariance of E and $Q_{\hat{k}}$. In particular, $\mathcal{I}$ is defined through a conformal compactification and choosing a different compactification could change $f_{(0)}$ or $f_{(n-1)}$ through a change to the Fefferman-Graham parameter, r. It will be easiest to see this by swapping r for $z = \mathrm{e}^{-r}$. Then,

$$g = \frac{1}{z^2}\left(\mathrm{d}z \otimes \mathrm{d}z + (f_{(0)mn} + z f_{(1)mn} + \cdots)\mathrm{d}x^m \otimes \mathrm{d}x^n\right) \tag{4.77}$$

and the conformal factor for compactification is $\Omega = z$. The most general transformations preserving the Fefferman-Graham form are known as Penrose-Brown-Henneaux (PBH) transformations and were derived in [69]. I will review the construction here.

Following [69], define new coordinates (z', x'), by

$$z = z'\mathrm{e}^{-\sigma(x')} \text{ and } x^m = x'^m + \xi^m(x', z') \tag{4.78}$$

for some $\sigma(x')$ and $\xi^m(x', z')$. Practically, it suffices to take σ and ξ to be infinitesimal, so

$$z = (1 - \sigma(x'))z' \text{ and } x^m = x'^m + \xi^m(x', z'). \tag{4.79}$$

Let σ be arbitrary - i.e. consider an arbitrary alternative conformal compactification. Then, ξ is determined as follows. Since

$$\mathrm{d}z = (1-\sigma)\mathrm{d}z' - z'\partial'_m(\sigma)\mathrm{d}x'^m \text{ and} \tag{4.80}$$

$$\mathrm{d}x'^m = \left(\delta^m{}_n + \partial'_n(\xi^m)\right)\mathrm{d}x'^n + \partial_{z'}(\xi^m)\mathrm{d}z', \tag{4.81}$$

to leading order (in σ and ξ) the metric in the new coordinates is

$$\begin{aligned} g = \frac{1+2\sigma}{z'^2}\Big(& f_{mn}(x,z)\left(\left(\delta^m{}_p + \partial'_p(\xi^m)\right)\mathrm{d}x'^p + \partial_{z'}(\xi^m)\mathrm{d}z'\right) \otimes \left(\left(\delta^n{}_q + \partial'_q(\xi^n)\right)\mathrm{d}x'^q + \partial_{z'}(\xi^n)\mathrm{d}z'\right) \\ & + ((1-\sigma)\mathrm{d}z' - z'\partial'_m(\sigma)\mathrm{d}x'^m) \otimes ((1-\sigma)\mathrm{d}z' - z'\partial'_n(\sigma)\mathrm{d}x'^n)\Big) \end{aligned} \tag{4.82}$$

$$\begin{aligned} = \frac{1+2\sigma}{z'^2}\Big(& f_{mn}(x,z)\big(\mathrm{d}x'^m \otimes \mathrm{d}x'^n + \partial'_p(\xi^m)\mathrm{d}x'^p \otimes \mathrm{d}x'^n + \partial'_p(\xi^n)\mathrm{d}x'^m \otimes \mathrm{d}x'^p \\ & + \partial_{z'}(\xi^m)\mathrm{d}z' \otimes \mathrm{d}x'^n + \partial_{z'}(\xi^n)\mathrm{d}x'^m \otimes \mathrm{d}z'\big) \\ & + (1-2\sigma)\mathrm{d}z' \otimes \mathrm{d}z' - z'\partial'_m(\sigma)\left(\mathrm{d}z' \otimes \mathrm{d}x'^m + \mathrm{d}x'^m \otimes \mathrm{d}z'\right)\Big) \end{aligned} \tag{4.83}$$

$$\begin{aligned} = \frac{1}{z'^2}\Big(& \mathrm{d}z' \otimes \mathrm{d}z' + (f_{mn}(x,z)\partial_{z'}(\xi^n) - z'\partial'_m(\sigma))\left(\mathrm{d}z' \otimes \mathrm{d}x'^m + \mathrm{d}x'^m \otimes \mathrm{d}z'\right) \\ & + \left(f_{mn}(x,z)(1+2\sigma) + f_{mp}(x,z)\partial'_n(\xi^p) + f_{pn}(x,z)\partial'_m(\xi^p)\right)\mathrm{d}x'^m \otimes \mathrm{d}x'^n\Big). \end{aligned} \tag{4.84}$$

Therefore Fefferman-Graham gauge is preserved if and only if $f_{mn}(x,z)\partial_{z'}(\xi^n) - z'\partial'_m(\sigma) = 0$. Since $\partial_{z'}(\xi^n)$ is already first order in the infinitesimal quantities, I can replace $f_{mn}(x,z)$ in this equation with $f_{mn}(x', z')$. Thus, in summary,

$$f_{mn}(x', z')\partial_{z'}(\xi^n) - z'\partial'_m(\sigma) = 0 \iff \partial_{z'}(\xi^m) = z' f^{mn}(x', z')\partial'_n(\sigma). \tag{4.85}$$

Integrating this equation gives the final result for ξ in terms of σ, namely

$$\xi^m(x', z') = \xi^m(x', 0) + \partial'_n(\sigma)\int_0^{z'} \zeta f^{mn}(x', \zeta)\mathrm{d}\zeta. \tag{4.86}$$

Next, consider the transformations of $f_{(k)mn}$. Since ξ and σ are infinitesimal,

$$f_{mn}(x,z) = f_{mn}(x'+\xi, z'-\sigma z') = f_{mn}(x',z') + \xi^p \partial'_p f_{mn}(x',z') - \sigma z' \partial_{z'} f_{mn}(x',z'). \quad (4.87)$$

Substituting this into equation 4.84 and imposing equation 4.85 then yields

$$g = \frac{1}{z'^2}\Big(\big(f_{mn}(x',z') + (\mathcal{L}_\xi f)_{mn}(x',z') + 2\sigma f_{mn}(x',z') - \sigma z' \partial_{z'} f_{mn}(x',z')\big)\, \mathrm{d}x'^m \otimes \mathrm{d}x'^n + \mathrm{d}z' \otimes \mathrm{d}z' \Big). \quad (4.88)$$

The metric on constant z surfaces thus changes as

$$\delta f = \mathcal{L}_\xi f + 2\sigma f - \sigma z \partial_z f. \quad (4.89)$$

Most importantly for the present discussion, the boundary metric changes as

$$\delta f_{(0)} = \mathcal{L}_{\xi|_{z=0}} f_{(0)} + 2\sigma f_{(0)}. \quad (4.90)$$

Therefore, in general, the boundary metric can change in an arbitrary way. Furthermore, by expanding ξ^m in powers of z, one can also find a complicated transformation law for $\delta f_{(n-1)}$ (separately for each value of n). In summary, the boundary data, $(f_{(0)}, f_{(n-1)})$, could change in a somewhat intractable way and therefore affect the expressions for E and $Q_{\hat{k}}$.

However, this is an issue that affects even the standard asymptotically AdS spacetimes (i.e. with round sphere cross-section) and doesn't seem to have been considered in any detail previously in the literature. Firstly, it's unclear to me how physically meaningful this question even is. For example, even an asymptotically flat end can be written in a poor choice of coordinates, which then leads to strange results for ADM mass [27]. Therefore, it would seem reasonable to insist on a particular, canonical conformal class representative for $f_{(0)}$. Indeed, this is what is done in section 5.3 of [97] when studying the first law of black hole mechanics in this context.

A more tractable, and physically well-motivated, problem regarding geometric invariance is to study the question considered in [31] for the energy in asymptotically AdS spacetimes[3]. Specifically, the task is to quantify how E and $Q_{\hat{k}}$ change under asymptotic symmetries preserving both Σ_t and the Fefferman-Graham gauge.

An asymptotic symmetry is determined by a vector field, ξ, such that $(\mathcal{L}_\xi g)|_{r=\infty} = 0$. Meanwhile, following section 3 of [31], to preserve Σ_t, choose $\xi = \xi^i \partial_i = \xi^1 \partial_r + \xi^\alpha \partial_\alpha$. Then,

$$(\mathcal{L}_\xi g)_{\mu\nu} = \xi^1 \partial_r g_{\mu\nu} + g_{1\nu}\partial_\mu \xi^1 + g_{\mu 1}\partial_\nu \xi^1 + \xi^\alpha \partial_\alpha g_{\mu\nu} + g_{\alpha\nu}\partial_\mu \xi^\alpha + g_{\mu\alpha}\partial_\nu \xi^\alpha. \quad (4.91)$$

The various components then reduce to

$$(\mathcal{L}_\xi g)_{11} = 2\partial_r \xi^1, \quad (4.92)$$
$$(\mathcal{L}_\xi g)_{1m} = \partial_m \xi^1 + \mathrm{e}^{2r} f_{m\alpha}\partial_r \xi^\alpha \ \text{ and} \quad (4.93)$$
$$(\mathcal{L}_\xi g)_{mn} = \xi^1 \partial_r(\mathrm{e}^{2r} f_{mn}) + \mathrm{e}^{2r}(\xi^\alpha \partial_\alpha f_{mn} + f_{\alpha n}\partial_m \xi^\alpha + f_{m\alpha}\partial_n \xi^\alpha). \quad (4.94)$$

ξ should be a physical spacetime vector that extends to the boundary. Hence, in the spirit of the Fefferman-Graham expansion, let

$$\xi = \xi_{(0)} + \mathrm{e}^{-r}\xi_{(1)} + \mathrm{e}^{-2r}\xi_{(2)} + \cdots \quad (4.95)$$

[3]The analogous problems for asymptotically hyperbolic Riemannian manifolds and the ADM energy are studied in [121] and [6, 26] respectively.

for $\xi_{(k)}$ being r-independent. Then, for a spacetime that's asymptotically AdS with cross-section, (S, h),

$$(\mathcal{L}_\xi g)_{11} = -2(\mathrm{e}^{-r}\xi_{(1)} + \cdots) \text{ and} \tag{4.96}$$

$$(\mathcal{L}_\xi g)_{1m} = -\mathrm{e}^{r} f_{(0)m\alpha}\xi^{\alpha}_{(1)} + \partial_m(\xi^1_{(0)}) - 2f_{(0)m\alpha}\xi^{\alpha}_{(2)} + \cdots . \tag{4.97}$$

Therefore, $(\mathcal{L}_\xi g)_{11}|_{r=\infty}$ holds automatically. Next, $(\mathcal{L}_\xi g)_{1m}|_{r=\infty} = 0$ implies $f_{(0)m\alpha}\xi^{\alpha}_{(1)} = 0$ and $\partial_m(\xi^1_{(0)}) - 2f_{(0)m\alpha}\xi^{\alpha}_{(2)} = 0$. Since $f_{(0)mn}$ is invertible, the former condition says $\xi^{\alpha}_{(1)} = 0$. Meanwhile, the latter condition says $\partial_t\xi^1_{(0)} = 0$ and $2\xi_{(2)\alpha} = \partial_\alpha\xi^1_{(0)}$, where the index has been lowered using h.

Next, consider $(\mathcal{L}_\xi g)_{mn}|_{r=\infty}$. Using $f_{(1)mn} = 0$ in these asymptotics and $\xi^m_{(1)} = 0$ just proven,

$$\begin{aligned}(\mathcal{L}_\xi g)_{mn} = \mathrm{e}^{2r}\left(2\xi^1_{(0)}f_{(0)mn} + \xi^{\alpha}_{(0)}\partial_\alpha f_{(0)mn} + f_{(0)\alpha n}\partial_m\xi^{\alpha}_{(0)} + f_{(0)m\alpha}\partial_n\xi^{\alpha}_{(0)}\right) + 2\mathrm{e}^{r}\xi^1_{(1)}f_{(0)mn}\\ + \xi^{\alpha}_{(0)}\partial_\alpha f_{(2)mn} + f_{(2)\alpha n}\partial_m\xi^{\alpha}_{(0)} + f_{(2)m\alpha}\partial_n\xi^{\alpha}_{(0)}\\ + \xi^{\alpha}_{(2)}\partial_\alpha f_{(0)mn} + f_{(0)\alpha n}\partial_m\xi^{\alpha}_{(2)} + f_{(0)m\alpha}\partial_n\xi^{\alpha}_{(2)} + \cdots .\end{aligned} \tag{4.98}$$

From the e^r component, it follows that $\xi^1_{(1)} = 0$, meaning $\xi_{(1)} = 0$ when combined with $\xi^m_{(1)} = 0$ from earlier. From the e^{2r} and r-independent components, setting $(m, n) = (\alpha, \beta)$ implies

$$\xi^1_{(0)}h = -\frac{1}{2}\mathcal{L}_{\xi_{(0)}}h \text{ and } -\frac{c}{2}\mathcal{L}_{\xi_{(0)}}h + \mathcal{L}_{\xi_{(2)}}h = 0, \tag{4.99}$$

where the vector field argument of the Lie derivatives only considers the part tangential to S. Re-writing the Lie derivatives in terms of $\hat{D}^{(h)}$ and applying $2\xi_{(2)\alpha} = \partial_\alpha\xi^1_{(0)}$ from above, it then follows that

$$\hat{D}^{(h)}_\alpha\hat{D}^{(h)}_\beta\xi^1_{(0)} = -c\xi^1_{(0)}h_{\alpha\beta}. \tag{4.100}$$

Since (S, h) is assumed to not be the round sphere[4] in theorems 4.7 and 4.8, by the Obata theorem [96], it must be that $\xi^1_{(0)} = 0$. Substituting back into equation 4.99 implies $\mathcal{L}_{\xi_{(0)}}h = 0$, i.e. $\xi_{(0)}$ is a just a Killing vector of h.

Therefore $\delta f_{(0)\alpha\beta} = \delta h_{\alpha\beta} = 0$ (by virtue of being an asymptotic symmetry) and $\mathcal{L}_{\xi_{(0)}}h = 0$. Hence, $\sigma = 0$ by equation 4.90. Being an asymptotic symmetry only determines the leading order behaviour of ξ. However, there is still the additional requirement of staying in Fefferman-Graham gauge. From equation 4.78 with $\sigma = 0$, it follows that $\xi = \xi^{\alpha}_{(0)}\partial_\alpha$.

In summary, the only asymptotic symmetries preserving Σ_t and the Fefferman-Graham expansion when (S, h) is Einstein, but not the round sphere, are simply coordinate transformations generated by the Killing vectors of (S, h). These don't affect $f_{(0)}$ by definition. Moreover, since there is no z dependence, they also don't affect the Taylor series split for f_{mn}; in particular $f_{(n-1)mn}$ just transforms as a tensor under coordinate transformations. Since the integrands of E and $Q_{\hat{k}}$ are scalars, they remain invariant under these transformations.

4.3 Example boundaries

In this section I will present various examples of (S, h) which satisfy the requirements in section 4.2 and thus allow applications of the various versions of the positive energy theorem discussed there. These examples are not exhaustive. While the simply-connected, compact Riemannian manifolds with parallel or Killing spinors have been classified [119, 5], the general list of non-simply-connected examples remains open [120]. The examples below include both simply-connected and non-simply-connected (S, h) and have been ordered by computational complexity. Throughout this section the assumed decay on $\mathbb{M}$ in definition 3.1 translates to a running assumption that T^{0a} decays quicker than $O(\mathrm{e}^{-(n-1)r})$.

[4] The case of the round sphere will be considered later in subsection 4.3.3.

4.3.1 Squashed S^7

The simplest metric that admits Killing spinors is the round sphere. However, it comes with additional subtleties and is thereby postponed to subsection 4.3.3. The simplest deformation to a sphere is squashing, yet this typically destroys all Killing spinors. A rare exception is a particular 7D squashed sphere [39] with metric,

$$h = \frac{9}{20}\left(\mathrm{d}a \otimes \mathrm{d}a + \frac{1}{4}\sin^2(a) b_i \otimes b_i + \frac{1}{20}(c_i + \cos(a) b_i) \otimes (c_i + \cos(a) b_i)\right), \tag{4.101}$$

$$\text{where } b_i = \sigma_i - \Sigma_i, \quad c_i = \sigma_i + \Sigma_i, \tag{4.102}$$

$$\sigma_1 = \cos(\psi)\mathrm{d}\theta + \sin(\psi)\sin(\theta)\mathrm{d}\phi, \quad \sigma_2 = -\sin(\psi)\mathrm{d}\theta + \cos(\psi)\sin(\theta)\mathrm{d}\phi,$$
$$\sigma_3 = \mathrm{d}\psi + \cos(\theta)\mathrm{d}\phi \tag{4.103}$$

and Σ_i are defined identically to σ_i, but with (ψ, θ, ϕ) replaced by some analogous coordinates, (ψ', θ', ϕ'). The squashing comes from the factor of $1/20$ in equation 4.101; if that factor were also $1/4$, then h would be the usual round metric on the sphere.

From [39], h satisfies $R^{(h)}_{AB} = 6\delta_{AB}$ and admits exactly one linearly independent Killing spinor, namely

$$\hat{\varepsilon}^{(+)}_h = \frac{1}{\sqrt{2}}\begin{bmatrix}0 & 1 & -1 & 0 & 0 & 0 & 0 & 0\end{bmatrix}^{\mathrm{T}}, \tag{4.104}$$

where the components are in the basis chosen for $\hat{\gamma}^A$ in [39]. Furthermore, from [5], this metric only admits $\hat{\varepsilon}^{(-)}_h = 0$.

Theorem 4.9. *For spacetimes asymptotically AdS with squashed S^7 cross-section, if the Einstein equation and dominant energy condition hold, then $E \geq 0$.*

Proof. The proof is simply applying theorem 4.8 with $\hat{\varepsilon}^{(-)}_h = 0$ and equation 4.104. The former means $\hat{\delta}^{(-)} = 0$ and $Q_{\hat{k}^{(-)}} = 0$. Meanwhile, it can be checked that for this particular $\hat{\varepsilon}^{(+)}_h$, $\hat{\varepsilon}^{(+)\dagger}_h \hat{\gamma}^A \hat{\varepsilon}^{(+)}_h = 0$ for all A. Thus $Q_{\hat{k}^{(+)}}$ is also zero and theorem 4.8 reduces to $E \geq 0$. □

4.3.2 Torus

The simplest cross-section with $c = 0$ is the torus, $\mathbb{T}^{n-2} = S^1 \times \cdots \times S^1$, which has metric,

$$h = \mathrm{d}\theta^2 \otimes \mathrm{d}\theta^2 + \cdots + \mathrm{d}\theta^{n-1} \otimes \mathrm{d}\theta^{n-1}, \tag{4.105}$$

where $\theta^2, \cdots, \theta^{n-1}$ are the angles on each S^1 factor. It follows immediately that $k_\alpha = \partial_{\theta^\alpha}$ is a Killing vector for each angle. As per definition 4.6, define

$$\mathbb{J}_A = \frac{n-1}{16\pi}\int_{\mathbb{T}^{n-2}} p_A \mathrm{d}^{n-2}\theta \tag{4.106}$$

as the associated "conserved quantities."

Theorem 4.10. *For spacetimes asymptotically AdS with torus cross-section, if the Einstein equation and dominant energy condition hold, then*

$$E \geq \sqrt{\mathbb{J}_A \mathbb{J}^A}. \tag{4.107}$$

Proof. h being locally flat implies the parallel spinors are just constant spinors, $\hat{\psi} = \hat{\psi}_0$. Therefore, theorem 4.7 can be re-written as

$$0 \leq E + Q_{\hat{k}} = E + \frac{n-1}{16\pi}\int_{\mathbb{T}^{n-2}} p_A \hat{k}^A \mathrm{d}(h) = \hat{\psi}_0^\dagger \left(EI - \mathrm{i}\mathbb{J}_A \hat{\gamma}^A\right) \hat{\psi}_0. \tag{4.108}$$

Since $EI - \mathrm{i}\mathbb{J}_A\hat{\gamma}^A$ has eigenvalues, $E \pm \sqrt{\mathbb{J}_A \mathbb{J}^A}$, and $\hat{\psi}_0$ is arbitrary, the result follows. □

This reproduces equation 4.14 of [30] - the only difference is the Fefferman-Graham expansion here is Lorentzian, as opposed to the Riemannian asymptotics on Σ_t required by the initial data point of view adopted by [30].

4.3.3 Sphere

With a round sphere cross-section, the problem reduces to finding a positive energy theorem for asymptotically AdS spacetimes, essentially reproducing the results of [23, 30, 122]. In this case, it will be easier to apply theorem 4.2 directly, instead of the cross-section treatment in section 4.2; although I will connect the two approaches later. Furthermore, it will be easier to represent AdS in the Poincaré ball model, rather than a Fefferman-Graham expansion. In particular,

$$\bar{g}_{\text{AdS}} = -\left(\frac{1+\rho^2}{1-\rho^2}\right)^2 \mathrm{d}t\otimes\mathrm{d}t + \frac{4}{(1-\rho^2)^2}\delta_{IJ}\mathrm{d}x^I\otimes\mathrm{d}x^J, \tag{4.109}$$

where x^I are Cartesian coordinates inside the unit ball, $\rho=\sqrt{x^I x_I}$ and the indices on x^I are understood to be raised and lowered by δ. The natural vielbein is

$$e'_0 = \frac{1-\rho^2}{1+\rho^2}\partial_t \;\text{ and }\; e'_I = \frac{1-\rho^2}{2}\partial_I. \tag{4.110}$$

It can be checked that in this frame,

$$\mathrm{d}e'^0 = \frac{2}{1+\rho^2}x_I e'^I\wedge e'^0, \tag{4.111}$$

$$\mathrm{d}e'^I = x_J e'^J\wedge e'^I, \tag{4.112}$$

$$\omega'_{0I} = -\frac{2}{1+\rho^2}x_I e'^0, \tag{4.113}$$

$$\omega'^{IJ} = x^J e'^I - x^I e'^J \tag{4.114}$$

and that subsequently the Killing spinors are

$$\varepsilon'_k = \frac{1}{\sqrt{1-\rho^2}}\left(I - \mathrm{i}x_I\gamma^I\right)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0, \tag{4.115}$$

for any constant spinor, ε_0. The coordinates and frame chosen are related to the Fefferman-Graham version of equation 2.77 by

$$r = \ln(R+\sqrt{1+R^2}) - \ln(2), \quad R = \frac{2\rho}{1-\rho^2}, \tag{4.116}$$

$$e_0 = \frac{\mathrm{e}^{-r}}{1+\frac{1}{4}\mathrm{e}^{-2r}}\partial_t, \quad e_1 = \partial_r, \quad e_A = \frac{\mathrm{e}^{-r}}{1-\frac{1}{4}\mathrm{e}^{-2r}}e^{(s)\alpha}_A\partial_\alpha, \tag{4.117}$$

$$e'_0 = e_0 \;\text{ and }\; e'_I = \hat{x}_I e_1 + \rho\frac{\partial\theta^\alpha}{\partial x^I}e^{(s)A}_\alpha e_A, \tag{4.118}$$

where $\hat{x}^I$ are unit vectors, i.e. $x^I=\rho\hat{x}^I$, θ^α are coordinates on S^{n-2} and $h=s$ is the round metric on S^{n-2}. Hence, the local Lorentz transformation relating e_a and e'_a, i.e. $e'_a = \Lambda^b{}_a(x)e_b$, is given by

$$\Lambda^a{}_0 = \delta^a{}_0 \;\text{ and }\; \Lambda^a{}_I = \delta^a{}_1\hat{x}_I + \delta^a{}_A\rho\frac{\partial\theta^\alpha}{\partial x^I}e^{(s)A}_\alpha. \tag{4.119}$$

Definition 4.11 ("Conserved" quantities on the sphere). *Define the linear momentum, angular momentum and centre of mass position as*

$$P_I = \frac{n-1}{16\pi}\int_{S^{n-2}} \hat{f}^{mn}_{(0)}(f_{(n-1)mn} - \bar{f}_{(n-1)mn})\hat{x}_I \mathrm{d}A(s) = \frac{n-1}{16\pi}\int_{S^{n-2}} p_0\hat{x}_I\mathrm{d}A(s), \tag{4.120}$$

$$J_{IJ} = \frac{n-1}{16\pi}\int_{S^{n-2}} f_{(n-1)0\alpha}\left(\hat{x}_I \frac{\partial\theta^\alpha}{\partial x^J}\bigg|_{\rho=1} - \hat{x}_J\frac{\partial\theta^\alpha}{\partial x^I}\bigg|_{\rho=1}\right)\mathrm{d}A(s) \text{ and} \tag{4.121}$$

$$K_I = \frac{n-1}{16\pi}\int_{S^{n-2}} f_{(n-1)0\alpha}\frac{\partial\theta^\alpha}{\partial x^J}\bigg|_{\rho=1}\left(\delta^J{}_I - \hat{x}^J\hat{x}_I\right)\mathrm{d}A(s). \tag{4.122}$$

The conserved quantities are more complicated now than in definition 4.6, although some comparison will be provided later. These definitions are based off [30] - especially their equations 3.5 and 3.6 - but written in terms of Fefferman-Graham expansions/coordinates instead. Their exact form is motivated by terms that appear in the positive energy theorem about to be proven. However, some heuristics can be discussed now. It can be shown [121] the Riemannian analogue of (E, P_I) transforms as a Lorentz vector when one chooses a different conformal class representative to define $\mathcal{I}$. Hence, P_I naturally behaves like linear momentum. Next, observe that the vector, $(\hat{x}_I\frac{\partial\theta^\alpha}{\partial x^J}|_{\rho=1} - \hat{x}_J\frac{\partial\theta^\alpha}{\partial x^A}|_{\rho=1})\partial_\alpha$ equals $\hat{x}_I\partial_J - \hat{x}_J\partial_I$, which is the generator of rotations. Hence, it's natural to expect the J_{IJ} above to behave like angular momentum. Likewise, $\frac{\partial\theta^\alpha}{\partial x^J}|_{\rho=1}\left(\delta^J{}_I - \hat{x}^J\hat{x}_I\right)\partial_\alpha = \left(\delta^J{}_I - \hat{x}^J\hat{x}_I\right)\partial_J$ can be seen as a generator of boosts when AdS is viewed as a hyperboloid in $\mathbb{R}^{3,2}$ [30], suggesting the K_I above should be interpreted as a centre of mass position.

It's now possible to state the main result of this section.

Theorem 4.12. *In an asymptotically AdS spacetime (i.e. with round sphere cross-section), if the Einstein equation and the dominant energy condition hold, then*

$$EI - \mathrm{i}P_I\gamma^I + \frac{\mathrm{i}}{2}J_{IJ}\gamma^0\gamma^{IJ} + K_I\gamma^0\gamma^I \tag{4.123}$$

is a non-negative definite matrix.

Proof. Theorem 4.2 was derived in a vielbein adapted to the Fefferman-Graham expansion. However, since $\bar{\varepsilon}_k\gamma^a\varepsilon_k$ in theorem 4.2 transforms as a Lorentz vector, it suffices to apply $\bar{\varepsilon}_k\gamma^a\varepsilon_k = \Lambda^a{}_b\bar{\varepsilon}'_k\gamma^b\varepsilon'_k$. From equation 4.115,

$$\bar{\varepsilon}'_k\gamma^0\varepsilon'_k = \frac{1}{1-\rho^2}\varepsilon_0^\dagger \mathrm{e}^{-\mathrm{i}\gamma^0 t/2}(I - \mathrm{i}x_I\gamma^I)(I-\mathrm{i}x_J\gamma^J)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0 \tag{4.124}$$

$$= \frac{1}{1-\rho^2}\varepsilon_0^\dagger \mathrm{e}^{-\mathrm{i}\gamma^0 t/2}(I - 2\mathrm{i}x_I\gamma^I - x_Ix_J\gamma^I\gamma^J)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0 \tag{4.125}$$

$$= \frac{1}{1-\rho^2}\varepsilon_0^\dagger \mathrm{e}^{-\mathrm{i}\gamma^0 t/2}((1+\rho^2)I - 2\mathrm{i}x_I\gamma^I)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0 \text{ and} \tag{4.126}$$

$$\bar{\varepsilon}'_k\gamma^I\varepsilon'_k = \frac{1}{1-\rho^2}\varepsilon_0^\dagger \mathrm{e}^{-\mathrm{i}\gamma^0 t/2}(I - \mathrm{i}x_J\gamma^J)\gamma^0\gamma^I(I-\mathrm{i}x_K\gamma^K)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0 \tag{4.127}$$

$$= \frac{1}{1-\rho^2}\varepsilon_0^\dagger \mathrm{e}^{-\mathrm{i}\gamma^0 t/2}(\gamma^0\gamma^I - \mathrm{i}x_J\gamma^J\gamma^0\gamma^I - \mathrm{i}x_J\gamma^0\gamma^I\gamma^J - x_Jx_K\gamma^J\gamma^0\gamma^I\gamma^K)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0 \tag{4.128}$$

$$= \frac{1}{1-\rho^2}\varepsilon_0^\dagger \mathrm{e}^{-\mathrm{i}\gamma^0 t/2}(\gamma^0\gamma^I - 2\mathrm{i}x_J\gamma^0\gamma^{IJ} - 2x^Ix_J\gamma^0\gamma^J + \rho^2\gamma^0\gamma^I)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0 \tag{4.129}$$

$$= \frac{1}{1-\rho^2}\varepsilon_0^\dagger \mathrm{e}^{-\mathrm{i}\gamma^0 t/2}((1+\rho^2)\gamma^0\gamma^I - 2\mathrm{i}x_J\gamma^0\gamma^{IJ} - 2x^Ix_J\gamma^0\gamma^J)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0. \tag{4.130}$$

By equation 4.116, as $r \to \infty$, $\rho \to 1$ and $\frac{1}{1-\rho^2} \to \frac{1}{2}\mathrm{e}^r$. Therefore,

$$\bar{\varepsilon}'_k \gamma^0 \varepsilon'_k \to \mathrm{e}^r \varepsilon_0^\dagger \mathrm{e}^{-\mathrm{i}\gamma^0 t/2}(I - \mathrm{i}\hat{x}_I \gamma^I)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0 \text{ and} \tag{4.131}$$

$$\bar{\varepsilon}'_k \gamma^I \varepsilon'_k \to \mathrm{e}^r \varepsilon_0^\dagger \mathrm{e}^{-\mathrm{i}\gamma^0 t/2}(\gamma^0\gamma^I - \mathrm{i}\hat{x}_J \gamma^0\gamma^{IJ} - \hat{x}^I \hat{x}_J \gamma^0 \gamma^J)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0. \tag{4.132}$$

Substituting these into theorem 4.2, applying equation 4.119, definition 4.11 and

$$p_A = e_A^{(f_{(0)})m} P^n_{(0)}(f_{(n-1)mn} - \bar{f}_{(n-1)mn}) = e_A^{(s)\alpha} f_{(n-1)0\alpha} \tag{4.133}$$

ultimately yields

$$0 \le Q(\varepsilon) = \frac{n-1}{2}\int_{\Sigma_{t,\infty}} \mathrm{e}^{-r}\left(p_0 \Lambda^0{}_a \bar{\varepsilon}'_k \gamma^a \varepsilon'_k + p_A \Lambda^A{}_a \bar{\varepsilon}'_k \gamma^a \varepsilon'_k\right) \mathrm{d}A(s) \tag{4.134}$$

$$= \frac{n-1}{2}\int_{\Sigma_{t,\infty}} \Bigg(p_0 \varepsilon_0^\dagger \mathrm{e}^{-\mathrm{i}\gamma^0 t/2}(I - \mathrm{i}\hat{x}_I\gamma^I)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0$$

$$+ \mathrm{e}^{-r} e_A^{(s)\alpha} f_{(n-1)0\alpha} \frac{\partial \theta^\beta}{\partial x^I}\bigg|_{\rho=1} e_\beta^{(s)A} \bar{\varepsilon}'_k \gamma^I \varepsilon'_k\Bigg)\mathrm{d}A(s) \tag{4.135}$$

$$= \frac{n-1}{2}\int_{\Sigma_{t,\infty}} \Bigg(p_0 \varepsilon_0^\dagger \mathrm{e}^{-\mathrm{i}\gamma^0 t/2}(I - \mathrm{i}\hat{x}_I\gamma^I)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0$$

$$+ f_{(n-1)0\alpha}\frac{\partial \theta^\alpha}{\partial x^I}\bigg|_{\rho=1} \varepsilon_0^\dagger \mathrm{e}^{-\mathrm{i}\gamma^0 t/2}(\gamma^0\gamma^I - \mathrm{i}\hat{x}_J\gamma^0\gamma^{IJ} - \hat{x}^I\hat{x}_J\gamma^0\gamma^J)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0\Bigg)\mathrm{d}A(s) \tag{4.136}$$

$$= 8\pi \varepsilon_0^\dagger \mathrm{e}^{-\mathrm{i}\gamma^0 t/2}\left(EI - \mathrm{i}P_I\gamma^I + K_I\gamma^0\gamma^I + \frac{\mathrm{i}}{2}J_{IJ}\gamma^0\gamma^{IJ}\right)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0. \tag{4.137}$$

Since ε_0 is an arbitrary constant spinor and $\mathrm{e}^{\mathrm{i}\gamma^0 t/2}$ is a constant, unitary matrix on Σ_t, this inequality is satisfied if and only if $EI - \mathrm{i}P_I\gamma^I + K_I\gamma^0\gamma^I + \frac{\mathrm{i}}{2}J_{IJ}\gamma^0\gamma^{IJ}$ is non-negative definite. □

The result produced is formally identical to [30] - in particular, see the unnamed equation directly above their equation 3.12 on their page 11 - the only difference being that [30] considered Riemannian asymptotics on an initial data slice. As such, they don't have any t dependence in their Killing spinors. Adding this dependence was essentially the main result of [122]. Theorem 4.12 differs from their work in that the t dependence has been extracted from the physical quantities; in contrast, their analogues of P_I and K_I have explicit t dependence in their definition. Of course, my definitions have implicit t dependence through the Σ_t in $\int_{\Sigma_t}$ and this means it's not necessary that P_I or K_I are conserved in the sense that $\partial_t P_I = \partial_t K_I = 0$. even though there is some overall conservation because of equation 3.8.

The eigenvalues of the matrix in theorem 4.12 can't be found analytically in general and thus there is no concrete inequality such as theorem 4.10. However, more progress can be made in specific examples. For example, if $n = 4$ and $K_I = P_I = 0$, then[5] the eigenvalues of $EI + \frac{\mathrm{i}}{2}J_{IJ}\gamma^0\gamma^{IJ}$ are $E \pm \sqrt{\frac{1}{2}J_{IJ}J^{IJ}} = E \pm |J|$, leading to the familiar 4D BPS inequality [13],

$$E \ge |J|. \tag{4.138}$$

Similarly, consider the case where $n = 5$ and $K_I = P_I = 0$. There are two independent rotation planes in 4 space dimensions. Without loss of generality, suppose the coordinates are oriented so that the angular momentum is in the 1-2 and 3-4 planes. Then, since the eigenvalues of

[5] As explained later, it is typically possible to choose a frame in which $P_I = 0$ by performing a conformal isometry of the round sphere boundary [30].

$EI + \mathrm{i}J_{12}\gamma^0\gamma^1\gamma^2 + \mathrm{i}J_{34}\gamma^0\gamma^3\gamma^4$ are $E \pm J_{12} \pm J_{34}$ and $E \pm J_{12} \mp J_{34}$, I get the corresponding 5D BPS inequality,

$$E \geq |J_1| + |J_2|, \tag{4.139}$$

where J_1 and J_2 describe the independent rotations.

For a different type of example, suppose $K_I = 0$ and $J_{IJ} = 0$. Then, the eigenvalues of $EI - \mathrm{i}P_I\gamma^I$ are $E \pm \sqrt{P_IP^I} = E \pm |P|$, leading to $E \geq |P|$, which was the result in [50].

Many more permutations can be chosen like this - see [30] for a detailed analysis.

Theorem 4.12 can also be derived from the cross-section point of view as follows. To deal with S^{n-2} for arbitrary n, the only practical coordinates are the "nested spheres." In particular,

$$x^I = \rho \begin{bmatrix} \cos(\theta_2) \\ \sin(\theta_2)\cos(\theta_3) \\ \vdots \\ \sin(\theta_2)\cdots\sin(\theta_{n-2})\cos(\theta_{n-1}) \\ \sin(\theta_2)\cdots\sin(\theta_{n-2})\sin(\theta_{n-1}) \end{bmatrix} \quad \text{and} \tag{4.140}$$

$$h = \rho^2(\mathrm{d}\theta_2 \otimes \mathrm{d}\theta_2 + \sin^2(\theta_2)\mathrm{d}\theta_3 \otimes \mathrm{d}\theta_3 + \cdots + \sin^2(\theta_2)\cdots\sin^2(\theta_{n-2})\mathrm{d}\theta_{n-1} \otimes \mathrm{d}\theta_{n-1}). \tag{4.141}$$

The natural vielbein to use on the unit sphere is thus

$$e^{(s)2} = \mathrm{d}\theta_2,\ e^{(s)3} = \sin(\theta_2)\mathrm{d}\theta_3,\ \cdots,\ e^{(s)n-1} = \sin(\theta_2)\cdots\sin(\theta_{n-2})\mathrm{d}\theta_{n-1}. \tag{4.142}$$

In this frame, the most general solution to $D^{(h)}_A\varepsilon_h = \frac{1}{2}\gamma_A\varepsilon_h$ on the unit sphere is [82]

$$\varepsilon_h = \mathrm{e}^{\theta_2\gamma^2/2}\mathrm{e}^{\theta_3\gamma^3\gamma^2/2}\cdots\mathrm{e}^{\theta_{n-1}\gamma^{n-1}\gamma^{n-2}/2}\varepsilon_0 \tag{4.143}$$

for a constant spinor, ε_0. Now, defining ε_k as per theorem 4.4 and applying theorem 4.2 should give the same result as the method based on ε'_k used in theorem 4.12. However, seeing this requires performing a spinorial change of frame.

Define $\mathfrak{o}(\mathfrak{n}-\mathfrak{1})$ through the generators,

$$(M_{IJ})_{KL} = \delta_{IK}\delta_{JL} - \delta_{IL}\delta_{JK}. \tag{4.144}$$

Lemma 4.13. *The Lorentz transformation matrix, $\Lambda^I{}_J$, of equation 4.119 is*

$$\begin{bmatrix} \cos(\theta_2) & \sin(\theta_2)\cos(\theta_3) & \cdots & \sin(\theta_2)\cdots\sin(\theta_{n-2})\cos(\theta_{n-1}) & \sin(\theta_2)\cdots\sin(\theta_{n-2})\sin(\theta_{n-1}) \\ -\sin(\theta_2) & \cos(\theta_2)\cos(\theta_3) & \cdots & \cos(\theta_2)\sin(\theta_3)\cdots\sin(\theta_{n-2})\cos(\theta_{n-1}) & \cos(\theta_2)\sin(\theta_3)\cdots\sin(\theta_{n-1}) \\ 0 & -\sin(\theta_3) & \cdots & \cos(\theta_3)\sin(\theta_4)\cdots\sin(\theta_{n-2})\cos(\theta_{n-1}) & \cos(\theta_3)\sin(\theta_4)\cdots\sin(\theta_{n-1}) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -\sin(\theta_{n-1}) & \cos(\theta_{n-1}) \end{bmatrix} \tag{4.145}$$

$$= \mathrm{e}^{\theta_2 M_{12}}\cdots\mathrm{e}^{\theta_{n-1}M_{n-2,n-1}}. \tag{4.146}$$

Proof. From equation 4.119,

$$\Lambda^I{}_J = \delta^I{}_1\hat{x}_J + \delta^I{}_A\rho\frac{\partial\theta^\alpha}{\partial x^J}e^{(s)A}_\alpha. \tag{4.147}$$

Therefore the first row of Λ can be read-off from equation 4.140. For $I > 1$, I require $\frac{\partial\theta^\alpha}{\partial x^J}$, which can be calculated from

$$\frac{\partial}{\partial x^I}\frac{x^2_{\alpha-1}}{x^2_\alpha + \cdots + x^2_{n-1}} = \frac{\partial}{\partial x^I}\cot^2(\theta_\alpha) = -\frac{2\cos(\theta_\alpha)}{\sin^3(\theta_\alpha)}\frac{\partial\theta_\alpha}{\partial x^I}. \tag{4.148}$$

$$\iff \frac{\partial\theta_\alpha}{\partial x^I} = -\frac{\sin^3(\theta_\alpha)}{2\cos(\theta_\alpha)}\frac{\partial}{\partial x^I}\frac{x^2_{\alpha-1}}{x^2_\alpha + \cdots + x^2_{n-1}}. \tag{4.149}$$

Hence, $\frac{\partial \theta_\alpha}{\partial x^I} = 0$ when $I < \alpha - 1$. Next, when $I = \alpha - 1$,

$$\frac{\partial \theta_\alpha}{\partial x^I} = -\frac{\sin^3(\theta_\alpha)}{2\cos(\theta_\alpha)} \frac{2x_{\alpha-1}}{x_\alpha^2 + \cdots + x_{n-1}^2} = -\frac{\sin(\theta_\alpha)}{\rho \sin(\theta_2) \cdots \sin(\theta_{\alpha-1})} \tag{4.150}$$

and when $I \geq \alpha$ (taking $\cos(\theta_n)$ to mean 1 in one of the equations below),

$$\frac{\partial \theta_\alpha}{\partial x^I} = -\frac{\sin^3(\theta_\alpha)}{2\cos(\theta_\alpha)} \left(-\frac{x_{\alpha-1}^2}{(x_\alpha^2 + \cdots + x_{n-1}^2)^2} \right) 2x_I \tag{4.151}$$

$$= \frac{\sin^3(\theta_\alpha)}{\cos(\theta_\alpha)} \frac{\rho^2 \sin^2(\theta_2) \cdots \sin^2(\theta_{\alpha-1}) \cos^2(\theta_\alpha)}{\rho^4 \sin^4(\theta_2) \cdots \sin^4(\theta_\alpha)} \rho \sin(\theta_2) \cdots \sin(\theta_I) \cos(\theta_{I+1}) \tag{4.152}$$

$$= \frac{\cos(\theta_\alpha) \sin(\theta_{\alpha+1}) \cdots \sin(\theta_I) \cos(\theta_{I+1})}{\rho \sin(\theta_2) \cdots \sin(\theta_{\alpha-1})}. \tag{4.153}$$

Since $e^{(s)A} = \delta^A{}_\alpha \sin(\theta_2) \cdots \sin(\theta_{\alpha-1}) \mathrm{d}\theta^\alpha$, I get the matrix in equation 4.145. The exponential product in equation 4.146 then follows from

$$\exp\left(\theta \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}, \tag{4.154}$$

$M_{I,I+1}$, $M_{I+1,I+2}, \cdots$ only acting non-trivially on rows & columns with index $\geq I$ and induction (on n). □

Corollary 4.13.1. *The ε_k' from equation 4.115 agrees with the ε_k from equations 4.19 and 4.143.*

Proof. For ε_k' to be a well-define spinor, x^I should be viewed as a vector. Hence, upon the change of frame, x^I should go to $\Lambda^I{}_J x^J$. From equations 4.145 and 4.140, it follows by inspection that $\Lambda^I{}_J x^J = \rho(1, 0, \cdots, 0)^T$. As for the spinor, ε_0, as with any spinor,

$$\varepsilon_0 \to S[\Lambda]\varepsilon_0 = \mathrm{e}^{-\frac{1}{4}\omega_{IJ}\gamma^{IJ}} \varepsilon_0 \tag{4.155}$$

under a Lorentz transformation defined by $\Lambda = \mathrm{e}^{\frac{1}{2}\omega_{IJ}M^{IJ}}$. Therefore, upon the change of frame,

$$\varepsilon_k' = \frac{1}{\sqrt{1-\rho^2}} \left(I - \mathrm{i}x_I\gamma^I\right) \mathrm{e}^{\mathrm{i}\gamma^0 t/2} \varepsilon_0 \tag{4.156}$$

$$\implies \varepsilon_k = \frac{1}{\sqrt{1-\rho^2}} \left(I - \mathrm{i}\Lambda^I{}_J x^J \gamma_I\right) \mathrm{e}^{\mathrm{i}\gamma^0 t/2} \mathrm{e}^{\theta_2\gamma^2\gamma^1/2} \cdots \mathrm{e}^{\theta_{n-1}\gamma^{n-1}\gamma^{n-2}/2} \varepsilon_0 \tag{4.157}$$

$$= \frac{1}{\sqrt{1-\rho^2}} \left(I - \mathrm{i}\rho\gamma^1\right) \mathrm{e}^{\mathrm{i}\gamma^0 t/2} \mathrm{e}^{\theta_2\gamma^2\gamma^1/2} \cdots \mathrm{e}^{\theta_{n-1}\gamma^{n-1}\gamma^{n-2}/2} \varepsilon_0. \tag{4.158}$$

ε_0 is an arbitrary spinor, so it can be re-defined without loss of generality as $\frac{1}{\sqrt{2}}(I - \gamma^1)\varepsilon_0$. Since I and γ^1 commute with γ^{AB}, I get

$$\varepsilon_k = \frac{1}{\sqrt{2(1-\rho^2)}} \left(I - \mathrm{i}\rho\gamma^1\right) \mathrm{e}^{\mathrm{i}\gamma^0 t/2} \mathrm{e}^{\theta_2\gamma^2\gamma^1/2} (I - \gamma^1) \mathrm{e}^{\theta_3\gamma^3\gamma^2/2} \cdots \mathrm{e}^{\theta_{n-1}\gamma^{n-1}\gamma^{n-2}/2} \varepsilon_0. \tag{4.159}$$

Next, observe that

$$\mathrm{e}^{\theta_2\gamma^2\gamma^1/2}(I - \gamma^1) = (\cos(\theta_2/2)I - \sin(\theta_2/2)\gamma^2\gamma^1)(I - \gamma^1) \tag{4.160}$$

$$= (I - \gamma^1)(\cos(\theta_2/2)I - \sin(\theta_2/2)\gamma^2) \tag{4.161}$$

$$= (I - \gamma^1)\mathrm{e}^{\theta_2\gamma^2/2}. \tag{4.162}$$

Substituting back and applying equation 4.143 says

$$\varepsilon_k = \frac{1}{\sqrt{2(1-\rho^2)}} \left(I - \mathrm{i}\rho\gamma^1\right) \mathrm{e}^{\mathrm{i}\gamma^0 t/2} (I-\gamma^1) \mathrm{e}^{\theta_2\gamma^2/2} \mathrm{e}^{\theta_3\gamma^3\gamma^2/2} \cdots \mathrm{e}^{\theta_{n-1}\gamma^{n-1}\gamma^{n-2}/2} \varepsilon_0 \tag{4.163}$$

$$= \frac{1}{\sqrt{2(1-\rho^2)}} \left(I - \mathrm{i}\rho\gamma^1\right) \mathrm{e}^{\mathrm{i}\gamma^0 t/2} (I-\gamma^1) \varepsilon_h. \tag{4.164}$$

From equation 4.116,

$$\mathrm{e}^r = \frac{1}{2}\left(\frac{2\rho}{1-\rho^2} + \sqrt{1+\frac{4\rho^2}{(1-\rho^2)^2}}\right) = \frac{(1+\rho)^2}{2(1-\rho^2)}, \tag{4.165}$$

$$\rho = \frac{1-\frac{1}{2}\mathrm{e}^{-r}}{1+\frac{1}{2}\mathrm{e}^{-r}} \quad \text{and} \quad \frac{1}{\sqrt{1-\rho^2}} = \frac{1}{\sqrt{2}}\mathrm{e}^{r/2}\left(1+\frac{1}{2}\mathrm{e}^{-r}\right). \tag{4.166}$$

Therefore, I ultimately get

$$\varepsilon_k = \frac{1}{2}\mathrm{e}^{r/2}\left(1+\frac{1}{2}\mathrm{e}^{-r}\right)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}(I-\gamma^1)\varepsilon_h - \frac{\mathrm{i}}{2}\mathrm{e}^{r/2}\left(1-\frac{1}{2}\mathrm{e}^{-r}\right)\gamma^1\mathrm{e}^{\mathrm{i}\gamma^0 t/2}(I-\gamma^1)\varepsilon_h \tag{4.167}$$

$$= \mathrm{e}^{r/2}P_1^-\mathrm{e}^{\mathrm{i}\gamma^0 t/2}(I-\gamma^1)\varepsilon_h + \frac{1}{2}\mathrm{e}^{-r/2}P_1^+\mathrm{e}^{\mathrm{i}\gamma^0 t/2}(I-\gamma^1)\varepsilon_h \tag{4.168}$$

$$= \mathrm{e}^{r/2}P_1^-\left(\mathrm{e}^{\mathrm{i}\gamma^0 t/2} - \mathrm{i}\mathrm{e}^{-\mathrm{i}\gamma^0 t/2}\right)\varepsilon_h + \frac{1}{2}\mathrm{e}^{-r/2}P_1^+\left(\mathrm{e}^{\mathrm{i}\gamma^0 t/2} + \mathrm{i}\mathrm{e}^{-\mathrm{i}\gamma^0 t/2}\right)\varepsilon_h, \tag{4.169}$$

which is exactly the result in theorem 4.4. □

Theorem 4.12 could have been derived by a method even more intrinsic to the cross-section by deploying $\hat{\varepsilon}_h^{(\pm)}$, like in theorem 4.8. From [82],

$$\hat{\varepsilon}_h^{(\pm)} = \mathrm{e}^{\pm\theta_2\hat{\gamma}^2/2}\mathrm{e}^{\theta_3\hat{\gamma}^3\hat{\gamma}^2/2}\cdots\mathrm{e}^{\theta_{n-1}\hat{\gamma}^{n-1}\hat{\gamma}^{n-2}/2}\hat{\varepsilon}_0. \tag{4.170}$$

While theorem 4.8 doesn't apply to round sphere cross-sections, the proof still holds until

$$0 \leq \int_{\Sigma_{t,\infty}} \left(\hat{\varepsilon}_h^{(+)\dagger} + \mathrm{i}\mathrm{e}^{\mathrm{i}t}\hat{\varepsilon}_h^{(-)\dagger}\right)\left(p_0 I - \mathrm{i}p_A\hat{\gamma}^A\right)\left(\hat{\varepsilon}_h^{(+)} - \mathrm{i}\mathrm{e}^{-\mathrm{i}t}\hat{\varepsilon}_h^{(-)}\right)\mathrm{d}A(h), \tag{4.171}$$

which is effectively the result that would arise by combining appendix F and section 8 of [23].

However, the individual terms in the integrand can be analysed further. When $h = s$, the cross-terms between $\hat{\varepsilon}_h^{(+)}$ and $\hat{\varepsilon}_h^{(-)}$ are no longer simply guaranteed to be zero by the Obata theorem. The $p_0(\hat{\varepsilon}_h^{(+)\dagger}\hat{\varepsilon}_h^{(+)} + \hat{\varepsilon}_h^{(-)\dagger}\hat{\varepsilon}_h^{(-)})$ terms produce energy as before, but this time the $p_0(\mathrm{i}\mathrm{e}^{\mathrm{i}t}\hat{\varepsilon}_h^{(-)\dagger}\hat{\varepsilon}_h^{(+)} - \mathrm{i}\mathrm{e}^{-\mathrm{i}t}\hat{\varepsilon}_h^{(+)\dagger}\hat{\varepsilon}_h^{(-)})$ terms produce linear momentum, the $-\mathrm{i}p_A(\hat{\varepsilon}_h^{(+)\dagger}\hat{\varepsilon}_h^{(+)} + \hat{\varepsilon}_h^{(-)\dagger}\hat{\varepsilon}_h^{(-)})$ terms produce angular momentum and the $p_A(\mathrm{e}^{\mathrm{i}t}\hat{\varepsilon}_h^{(-)\dagger}\hat{\gamma}^A\hat{\varepsilon}_h^{(+)} - \mathrm{e}^{-\mathrm{i}t}\hat{\varepsilon}_h^{(+)\dagger}\hat{\gamma}^A\hat{\varepsilon}_h^{(-)})$ terms produce the centre of mass position. This approach with $\hat{\varepsilon}_h^{(\pm)}$ wasn't followed earlier for computational ease, but also because the interpretation of the physical quantities is much easier in the approach taken earlier.

Finally, consider asymptotic symmetries for these metrics, following the discussion at the end of section 4.2. Again, with the round sphere cross-section the Obata theorem can no longer be applied to equation 4.100 to conclude $\xi^1_{(0)} = 0$. Consequently, equation 4.99 implies $\xi^\alpha_{(0)}\partial_\alpha$ is a conformal Killing vector of the sphere with $\hat{D}^{(h)}_\alpha\xi^\alpha_{(0)} = -(n-2)\xi^1_{(0)}$. The net effect is the asymptotic symmetry group is equal to the conformal group of S^{n-2}, namely $O(n-1,1)$. This reproduces the result of [31]. Furthermore, [31] goes on to show that under these asymptotic symmetries, E & P_I can be combined into an object, $P_a \equiv (E, P_I)$, transforming as a vector under $O(n-1,1)$ and likewise K_I & J_{IJ} can be combined into an object, J_{ab} with $J_{0I} = K_I$, transforming as a 2-form under $O(n-1,1)$. Therefore, E, P_I, J_{IJ} and K_I are not individually geometric invariants when the cross-section is a round sphere - for example, if P_a is timelike, one could always perform an $O(n-1,1)$ transformation to set P_I to zero. As explained in [31, 30], this is fine because various different combinations of E, P_I, J_{IJ} and K_I are invariants, e.g. $E^2 - P_I P^I$, and theorem 4.12 remains true despite the action of asymptotic symmetries.

4.3.4 $L(p,1)$

View S^3 as the level set,

$$\{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}. \tag{4.172}$$

Then, the lens space, $L(p,1)$, is defined as the quotient of S^3 by the $\mathbb{Z}_p$ action,

$$(z_1, z_2) \to (z_1 \mathrm{e}^{2\pi \mathrm{i}/p}, z_2 \mathrm{e}^{2\pi \mathrm{i}/p}). \tag{4.173}$$

It will be easiest to work in coordinates, $(\theta, \phi_1, \phi_2) \in [0,\pi] \times [0, 2\pi) \times [0, 2\pi)$, defined by

$$x_1 = \cos(\theta/2)\cos(\phi_1), \quad x_2 = \cos(\theta/2)\sin(\phi_1), \quad z_1 = x_1 + \mathrm{i}x_2, \tag{4.174}$$

$$x_3 = \sin(\theta/2)\cos(\phi_2), \quad x_4 = \sin(\theta/2)\sin(\phi_2) \text{ and } z_2 = x_3 + \mathrm{i}x_4, \tag{4.175}$$

where x_I are the Cartesian coordinates from the standard embedding of S^3 in $\mathbb{R}^4$.

The metric on $L(p,1)$ is locally isometric to the round metric on S^3; in the chosen coordinates it reads

$$h = \frac{1}{4}\mathrm{d}\theta \otimes \mathrm{d}\theta + \cos^2(\theta/2)\mathrm{d}\phi_1 \otimes \mathrm{d}\phi_1 + \sin^2(\theta/2)\mathrm{d}\phi_2 \otimes \mathrm{d}\phi_2. \tag{4.176}$$

Meanwhile, the $\mathbb{Z}_p$ quotient in these coordinates is

$$(\theta, \phi_1, \phi_2) \sim (\theta, \phi_1 + 2\pi/p, \phi_2 + 2\pi/p). \tag{4.177}$$

Lemma 4.14. *The Killing vectors of $L(p,1)$ are spanned by*

$$k_1 = \frac{\partial}{\partial\phi_1} + \frac{\partial}{\partial\phi_2}, \quad k_2 = \frac{\partial}{\partial\phi_1} - \frac{\partial}{\partial\phi_2}, \tag{4.178}$$

$$k_3 = \tan(\theta/2)\sin(\phi_1 - \phi_2)\frac{\partial}{\partial\phi_1} + 2\cos(\phi_1 - \phi_2)\frac{\partial}{\partial\theta} + \cot(\theta/2)\sin(\phi_1 - \phi_2)\frac{\partial}{\partial\phi_2} \text{ and} \tag{4.179}$$

$$k_4 = \tan(\theta/2)\cos(\phi_1 - \phi_2)\frac{\partial}{\partial\phi_1} - 2\sin(\phi_1 - \phi_2)\frac{\partial}{\partial\theta} + \cot(\theta/2)\cos(\phi_1 - \phi_2)\frac{\partial}{\partial\phi_2}. \tag{4.180}$$

Proof. $L(p,1)$ is locally isometric to S^3 and hence $L(p,1)$'s Killing vectors are the subspace of S^3's Killing vectors which survive the $\mathbb{Z}_p$ quotient. The Killing vectors of a sphere are known to be spanned by

$$v_{IJ} = \left(\hat{x}_I \frac{\partial\theta^\alpha}{\partial x^J}\bigg|_{\rho=1} - \hat{x}_J \frac{\partial\theta^\alpha}{\partial x^I}\bigg|_{\rho=1} \right)\partial_\alpha. \tag{4.181}$$

Re-writing everything in terms of (θ, ϕ_1, ϕ_2) and taking the following invertible linear transformation of Killing vector basis, it can be checked that

$$k_1 = v_{12} + v_{34} = \frac{\partial}{\partial\phi_1} + \frac{\partial}{\partial\phi_2}, \tag{4.182}$$

$$k_2 = v_{12} - v_{34} = \frac{\partial}{\partial\phi_1} - \frac{\partial}{\partial\phi_2}, \tag{4.183}$$

$$k_3 = v_{24} + v_{13} \tag{4.184}$$

$$= \tan(\theta/2)\sin(\phi_1 - \phi_2)\frac{\partial}{\partial\phi_1} + 2\cos(\phi_1 - \phi_2)\frac{\partial}{\partial\theta} + \cot(\theta/2)\sin(\phi_1 - \phi_2)\frac{\partial}{\partial\phi_2}, \tag{4.185}$$

$$k_4 = v_{14} - v_{23} \tag{4.186}$$

$$= \tan(\theta/2)\cos(\phi_1 - \phi_2)\frac{\partial}{\partial\phi_1} - 2\sin(\phi_1 - \phi_2)\frac{\partial}{\partial\theta} + \cot(\theta/2)\cos(\phi_1 - \phi_2)\frac{\partial}{\partial\phi_2}, \tag{4.187}$$

$$k_5 = v_{13} - v_{24} \tag{4.188}$$

$$= \tan(\theta/2)\sin(\phi_1 + \phi_2)\frac{\partial}{\partial\phi_1} + 2\cos(\phi_1 + \phi_2)\frac{\partial}{\partial\theta} - \cot(\theta/2)\sin(\phi_1 + \phi_2)\frac{\partial}{\partial\phi_2} \quad \text{and} \tag{4.189}$$

$$k_6 = v_{14} + v_{23} \tag{4.190}$$

$$= -\tan(\theta/2)\cos(\phi_1 + \phi_2)\frac{\partial}{\partial\phi_1} + 2\sin(\phi_1 + \phi_2)\frac{\partial}{\partial\theta} + \cot(\theta/2)\cos(\phi_1 + \phi_2)\frac{\partial}{\partial\phi_2}. \tag{4.191}$$

k_1, k_2, k_3 and k_4 are manifestly well-defined on $L(p,1)$ while any linear combination involving k_5 and k_6 is not well-defined on $L(p,1)$. □

As the lens space is 3D, it will be convenient to choose the cross-section gamma matrices as $\hat{\gamma}^2 = \mathrm{i}\sigma_1$, $\hat{\gamma}^3 = \mathrm{i}\sigma_2$ and $\hat{\gamma}^4 = \mathrm{i}\sigma_3$.

Lemma 4.15. *The most general solution to* $\hat{D}^{(h)}_A \hat{\varepsilon}^{(\pm)}_h = \pm\frac{1}{2}\hat{\gamma}_A\hat{\varepsilon}^{(\pm)}_h$ *on* $L(p,1)$ *is*

$$\hat{\varepsilon}^{(+)}_h = 0 \;\text{ and }\; \hat{\varepsilon}^{(-)}_h = \mathrm{e}^{-\mathrm{i}\theta\sigma_2/4}\mathrm{e}^{-\mathrm{i}(\phi_1-\phi_2)\sigma_1/2}\hat{\varepsilon}_0, \tag{4.192}$$

where $\hat{\varepsilon}_0$ *is an arbitrary constant spinor and the chosen vielbein on* $L(p,1)$ *is*

$$e^2 = \cos(\theta/2)\mathrm{d}\phi_1,\; e^3 = \frac{1}{2}\mathrm{d}\theta \text{ and } e^4 = \sin(\theta/2)\mathrm{d}\phi_2. \tag{4.193}$$

Observe that the quotienting has halved the number of Killing spinors from the sphere.

Proof. For this tetrad, the connection one-forms are

$$\omega^{(h)}_{32} = \tan(\theta/2)e^2,\; \omega^{(h)}_{34} = -\cot(\theta/2)e^4 \text{ and } \omega^{(h)}_{24} = 0. \tag{4.194}$$

Therefore, $0 = e^{(h)\alpha}_A\partial_\alpha\hat{\varepsilon}^{(+)}_h - \frac{1}{4}\omega^{(h)}_{BCA}\hat{\gamma}^{BC}\hat{\varepsilon}^{(+)}_h - \frac{1}{2}\hat{\gamma}_A\hat{\varepsilon}^{(+)}_h$ reduces to the three equations,

$$\partial_\theta\hat{\varepsilon}^{(+)}_h = \frac{\mathrm{i}}{4}\sigma_2\hat{\varepsilon}^{(+)}_h, \tag{4.195}$$

$$\partial_{\phi_1}\hat{\varepsilon}^{(+)}_h = \frac{\mathrm{i}}{2}\sin(\theta/2)\sigma_3\hat{\varepsilon}^{(+)}_h + \frac{\mathrm{i}}{2}\cos(\theta/2)\sigma_1\hat{\varepsilon}^{(+)}_h \quad\text{and} \tag{4.196}$$

$$\partial_{\phi_2}\hat{\varepsilon}^{(+)}_h = \frac{\mathrm{i}}{2}\cos(\theta/2)\sigma_1\hat{\varepsilon}^{(+)}_h + \frac{\mathrm{i}}{2}\sin(\theta/2)\sigma_3\hat{\varepsilon}^{(+)}_h. \tag{4.197}$$

The first equation immediately integrates to $\hat{\varepsilon}_h^{(+)} = \mathrm{e}^{\mathrm{i}\theta\sigma_2/4}\hat{\varepsilon}_\theta$, for a spinor, $\hat{\varepsilon}_\theta$, that doesn't depend on θ. Using $\mathrm{e}^{\mathrm{i}\theta\sigma_2} = \cos(\theta)I + \mathrm{i}\sin(\theta)\sigma_2$, the other two equations simplify to

$$\partial_{\phi_1}\hat{\varepsilon}_\theta = \frac{\mathrm{i}}{2}\sin(\theta/2)\mathrm{e}^{-\mathrm{i}\theta\sigma_2/2}\sigma_3\hat{\varepsilon}_\theta + \frac{\mathrm{i}}{2}\cos(\theta/2)\mathrm{e}^{-\mathrm{i}\theta\sigma_2/2}\sigma_1\hat{\varepsilon}_\theta \tag{4.198}$$

$$= \frac{\mathrm{i}}{2}\sin(\theta/2)(\cos(\theta/2)I - \mathrm{i}\sin(\theta/2)\sigma_2)\sigma_3\hat{\varepsilon}_\theta + \frac{\mathrm{i}}{2}\cos(\theta/2)(\cos(\theta/2)I - \mathrm{i}\sin(\theta/2)\sigma_2)\sigma_1\hat{\varepsilon}_\theta \tag{4.199}$$

$$= \frac{\mathrm{i}}{2}\sigma_1\hat{\varepsilon}_\theta \tag{4.200}$$

$$\text{and similarly } \partial_{\phi_2}\hat{\varepsilon}_\theta = \frac{\mathrm{i}}{2}\sigma_1\hat{\varepsilon}_\theta. \tag{4.201}$$

These equations simultaneously integrate to $\hat{\varepsilon}_\theta = \mathrm{e}^{\mathrm{i}(\phi_1+\phi_2)\sigma_1/2}\hat{\varepsilon}_0$ for some constant spinor, $\hat{\varepsilon}_0$.

Proceeding completely analogously for $\hat{\varepsilon}_h^{(-)}$ yields

$$\hat{\varepsilon}_h^{(\pm)} = \mathrm{e}^{\pm\mathrm{i}\theta\sigma_2/4}\mathrm{e}^{\pm\mathrm{i}(\phi_1\pm\phi_2)\sigma_1/2}\hat{\varepsilon}_0^{(\pm)}. \tag{4.202}$$

While these spinors are well-defined on S^3, to be well-defined on $L(p,1)$, they must be invariant under the $\mathbb{Z}_p$ quotient. Thus,

$$\hat{\varepsilon}_h^{(\pm)} \to \mathrm{e}^{\pm\mathrm{i}\theta\sigma_2/4}\mathrm{e}^{\pm\mathrm{i}((\phi_1+2\pi/p)\pm(\phi_2+2\pi/p))\sigma_1/2}\hat{\varepsilon}_0^{(\pm)}. \tag{4.203}$$

Choosing the $-$ in $\pm$ means the $2\pi/p$ factors immediately cancel and the spinor is left invariant. Hence, every $\hat{\varepsilon}_h^{(-)}$ of S^3 is also a $\hat{\varepsilon}_h^{(-)}$ of $L(p,1)$. Meanwhile, in the $+$ case,

$$\hat{\varepsilon}_h^{(+)} \to \mathrm{e}^{\mathrm{i}\theta\sigma_2/4}\mathrm{e}^{\mathrm{i}(\phi_1+\phi_2+4\pi/p)\sigma_1/2}\hat{\varepsilon}_0 = \mathrm{e}^{\mathrm{i}\theta\sigma_2/4}\mathrm{e}^{\mathrm{i}(\phi_1+\phi_2)\sigma_1/2}\mathrm{e}^{2\pi\mathrm{i}\sigma_1/p}\hat{\varepsilon}_0. \tag{4.204}$$

Since $\mathrm{e}^{\mathrm{i}\theta\sigma_2/4}\mathrm{e}^{\mathrm{i}(\phi_1+\phi_2)\sigma_1/2}$ is invertible, $\hat{\varepsilon}_h^{(+)}$ remains invariant if and only if $\mathrm{e}^{2\pi\mathrm{i}\sigma_1/p}\hat{\varepsilon}_0 = \hat{\varepsilon}_0$. However, $\mathrm{e}^{2\pi\mathrm{i}\sigma_1/p}$ has eigenvalues $\cos(2\pi/p) \pm \mathrm{i}\sin(2\pi/p)$, meaning $\hat{\varepsilon}_h^{(+)}$ is never invariant. □

One could consider $L(p,q)$ more generally, i.e. the quotient,

$$(z_1,\, z_2) \sim (z_1\mathrm{e}^{2\pi\mathrm{i}/p},\, z_2\mathrm{e}^{2\pi\mathrm{i}q/p}). \tag{4.205}$$

However, a similar procedure to the previous lemma shows non-trivial Killing spinors only exist for $q = \pm 1$. $L(p,-1)$ differs from $L(p,1)$ only by a discrete isometry though.

Definition 4.16 (Angular momenta on $L(p,1)$)**.** *For each Killing vector, k_I, on $L(p,1)$, define an "angular momentum,"*

$$J_I = \frac{1}{4\pi}\int_{L(p,1)} f_{(4)0\alpha}\, k_I^\alpha\, \mathrm{d}A(h). \tag{4.206}$$

Note that these J_Is are identical to the "conserved quantities" of definition 4.6.

Theorem 4.17. *For spacetimes asymptotically AdS with $L(p,1)$ cross-section, if the Einstein equation and the dominant energy condition hold, then*

$$E \geq \sqrt{J_2^2 + J_3^2 + J_4^2}. \tag{4.207}$$

Observe that J_1 does not appear in the theorem. J_1 is distinguished because its generator, k_1, is the symmetry along the circle direction when S^3 is viewed as a Hopf fibration over S^2.

Proof. Since $\hat{\varepsilon}_h^{(+)} = 0$ by lemma 4.15, theorem 4.8 reduces to $0 \leq E + Q_{\hat{k}^{(-)}}$. Furthermore, from lemma 4.15, by direct evaluation one finds

$$\begin{aligned} p_A \hat{k}^{(-)A} &= -\mathrm{i} p_A \hat{\varepsilon}_h^{(-)\dagger} \hat{\gamma}^A \hat{\varepsilon}_h^{(-)} \qquad (4.208) \\ &= \hat{\varepsilon}_0^\dagger \big((p_2 \sin(\theta/2) \sin(\phi_1 - \phi_2) + p_3 \cos(\phi_1 - \phi_2) + p_4 \cos(\theta/2) \sin(\phi_1 - \phi_2))\sigma_2 \\ &\quad + (p_2 \sin(\theta/2) \cos(\phi_1 - \phi_2) - p_3 \sin(\phi_1 - \phi_2) + p_4 \cos(\theta/2) \cos(\phi_1 - \phi_2))\sigma_3 \\ &\quad + (p_2 \cos(\theta/2) - p_4 \sin(\theta/2))\sigma_1 \big) \hat{\varepsilon}_0. \qquad (4.209) \end{aligned}$$

From lemma 4.14 and the choice of vielbein, this expression can be re-written as

$$p_A \hat{k}^{(-)A} = \hat{\varepsilon}_0^\dagger \left(f_{(4)0\alpha} k_2^\alpha \sigma_1 + f_{(4)0\alpha} k_3^\alpha \sigma_2 + f_{(4)0\alpha} k_4^\alpha \sigma_3 \right) \hat{\varepsilon}_0. \qquad (4.210)$$

Since $\hat{\varepsilon}_0$ and σ_i are both constants, it follows that

$$Q_{\hat{k}^{(-)}} = \hat{\varepsilon}_0^\dagger \left(J_2 \sigma_1 + J_3 \sigma_2 + J_4 \sigma_3 \right) \hat{\varepsilon}_0. \qquad (4.211)$$

Then, given the normalisation of $\hat{\varepsilon}_0$, the positive energy inequality reduces to

$$0 \leq \hat{\varepsilon}_0^\dagger \left(EI + J_2 \sigma_1 + J_3 \sigma_2 + J_4 \sigma_3 \right) \hat{\varepsilon}_0. \qquad (4.212)$$

The eigenvalues of the matrix inbetween $\hat{\varepsilon}_0^\dagger$ and $\hat{\varepsilon}_0$ are $E \pm \sqrt{J_2^2 + J_3^2 + J_4^2}$ and hence the theorem follows. □

4.3.5 Compatibility of spin structures

The formalism I'm applying to explore asymptotically, locally AdS spacetimes relies on the existence of background Killing spinors near $\mathcal{I}$. By construction, all the examples in this section satisfy that requirement.

However, there are more subtle issues which may arise. Let $\overline{M}$ be an open neighbourhood of $\mathcal{I}$ and let $\bar{g}$ be the background metric. The main problem is that $(\overline{M}, \bar{g})$ may admit multiple spin structures and the spin structure which admits a non-zero solution, ε_k, may not be compatible with the spin structure on (M, g). The classic example of this is the AdS soliton [64], which is a vacuum solution of the Einstein equation with $\Lambda < 0$ and has metric,

$$\begin{aligned} g = &-r^2 \mathrm{d}t \otimes \mathrm{d}t + \frac{\mathrm{d}r \otimes \mathrm{d}r}{r^2(1 - 1/r^{n-1})} \\ &+ r^2 \left(\left(1 - \frac{1}{r^{n-1}} \right) \mathrm{d}\phi_2 \otimes \mathrm{d}\phi_2 + \mathrm{d}\phi_3 \otimes \mathrm{d}\phi_3 + \cdots \mathrm{d}\phi_{n-1} \otimes \mathrm{d}\phi_{n-1} \right). \qquad (4.213) \end{aligned}$$

The angles are identified by $\phi_2 \sim \phi_2 + \frac{4\pi}{n-1}$ and $\phi_3 \sim \phi_3 + a_3, \cdots, \phi_{n-1} \sim \phi_{n-1} + a_{n-1}$ for arbitrary $a_3, \cdots, a_{n-1}$. The manifold is therefore asymptotically AdS with toroidal cross-section. However, its energy is

$$E = -\frac{1}{4} a_3 \cdots a_{n-1} < 0, \qquad (4.214)$$

seemingly contradicting theorem 4.10. The reason for this is that a circle admits two inequivalent spin structures - periodic and anti-periodic. Thus, $(\overline{M}, \bar{g})$ admits 2^{n-2} inequivalent spin structures. However, the constant spinor, $\hat{\psi}_0$, used in the proof of theorem 4.10 is manifestly periodic around each circle of T^{n-2}. Therefore, for theorem 4.10 to apply, this particular spin structure must extend beyond $(\overline{M}, \bar{g})$ to all of (M, g). It turns out the global topology of the AdS soliton, namely $\mathbb{R}^3 \times T^{n-3}$, requires anti-periodicity around ϕ_2; since that particular torus direction bounds a disk, the spin structure must agree with the unique spin structure of $\mathbb{R}^2$, which is anti-periodic around the disk. Hence, the AdS soliton's spin structure is incompatible with the spin structure required for theorem 4.10. There doesn't appear to be any spinorial remedy to this issue; however the modified positive energy conjecture of [64] has recently been confirmed by [14] using very different means.

Similar issues can arise for the lens space cross-section used in theorem 4.17. Firstly, $L(p,1)$ admits a unique spin structure for odd p and two inequivalent spin structures for even p [45]. It happens that there is an explicitly known soliton solution with $L(p,1)$ asymptotics [32]. However, it has negative energy, in violation of theorem 4.17. The situation is similar to the AdS soliton. This time, depending on p the solution in [32] either has no spin structure or a different spin structure to the one required by theorem 4.17.

4.4 BPS inequalities

In this section I will discuss BPS inequalities for the bosonic sectors of minimal, gauged supergravity in four and five dimensions. Although the final results are in line with previous partial results in the literature, I will discuss a number of subtleties regarding magnetic fields. The theories considered are described by the actions,

$$S = \frac{1}{16\pi}\int_M \left(R - 2\Lambda - F_{ab}F^{ab}\right) \mathrm{d}\mu(g) \tag{4.215}$$

in 4D and

$$S = \frac{1}{16\pi}\int_M \left(R - 2\Lambda - F_{ab}F^{ab} - \frac{2}{3\sqrt{3}}\varepsilon^{abcde}F_{ab}F_{cd}A_e\right) \mathrm{d}\mu(g) \tag{4.216}$$

in 5D. In both cases, $F = \mathrm{d}A$ is an electromagnetic field. Hence, the 4D theory is simply Einstein-Maxwell theory with a negative cosmological constant, while the 5D theory has an additional Chern-Simons term[6]. The equations of motion are

$$R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = 2\left(F_a{}^cF_{bc} - \frac{1}{4}g_{ab}F^{cd}F_{cd}\right), \tag{4.217}$$

$$D_bF^{ba} = 0 \ \ \& \ \ D_{[a}F_{bc]} = 0 \tag{4.218}$$

in the 4D case and

$$R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = 2\left(F_a{}^cF_{bc} - \frac{1}{4}g_{ab}F^{cd}F_{cd}\right), \tag{4.219}$$

$$D_bF^{ba} = -\frac{1}{2\sqrt{3}}\varepsilon^{abcde}F_{bc}F_{de} \ \ \& \ \ D_{[a}F_{bc]} = 0 \tag{4.220}$$

in the 5D case. These equations are assumed to hold throughout this section.

It will be very natural in what follows to split F_{ab} into separate electric and magnetic fields.

Definition 4.18 (Electric & magnetic components and electric charge)**.** *Given a Maxwell field, F_{ab}, the electric and magnetic components with respect to Σ_t will be defined as $E_I = F_{I0}$ and F_{IJ} respectively. In Fefferman-Graham form, the electric charge is then*

$$q_e = \frac{1}{4\pi}\int_{\Sigma_{t,\infty}} \star F = \frac{1}{4\pi}\int_{\Sigma_{t,\infty}} \varepsilon_{012\cdots n-1}F^{01}e^2\wedge\cdots\wedge e^{n-1} = \frac{1}{4\pi}\int_{\Sigma_{t,\infty}} E_1 \mathrm{d}A. \tag{4.221}$$

[6] In both cases, one can also add additional matter terms linearly coupled to A_a and use them to introduce source charges and currents in the Maxwell equations. The results below go through in almost identical fashion with the main modification being to energy conditions.

Unlike sections 4.2 and 4.3, $\mathcal{A}_a$ can no longer be chosen as 0. Instead, $\mathcal{A}_a$ is chosen so that $\nabla_a\Psi$ is the gravitino transformation in the gauged supergravity. Therefore,

$$\mathcal{A}_a^{(4)} = -\frac{1}{4}F_{bc}\gamma^{bc}\gamma_a + \mathrm{i}A_a I = \frac{1}{2}E_I\gamma^0\gamma^I\gamma_a - \frac{1}{4}F_{IJ}\gamma^{IJ}\gamma_a + \mathrm{i}A_a I \tag{4.222}$$

for the 4D theory and

$$\mathcal{A}_a^{(5)} = -\frac{1}{4\sqrt{3}}F_{bc}\gamma^{bc}\gamma_a - \frac{1}{2\sqrt{3}}F_{ab}\gamma^b + \mathrm{i}\sqrt{3}A_a I \tag{4.223}$$
$$= -\frac{1}{2\sqrt{3}}E_I\gamma^I\gamma^0\gamma_a - \frac{1}{4\sqrt{3}}F_{IJ}\gamma^{IJ}\gamma_a - \frac{1}{2\sqrt{3}}F_{ab}\gamma^b + \mathrm{i}\sqrt{3}A_a I \tag{4.224}$$

for the 5D theory [46, 56].

Theorem 4.19. *Let (M, g) be an asymptotically, locally AdS spacetime admitting a background metric, $\bar{g}$, which possesses a non-zero background Killing spinor, ε_k. Assume the Einstein-Maxwell equations or Einstein-Maxwell-Chern-Simons equations hold in 4D or 5D respectively. Assume ε_k is $O(\mathrm{e}^{r/2})$ near $\Sigma_{t,\infty}$, while the electric field, E_I, magnetic field, F_{IJ}, and (the spacelike part of) the gauge potential, A_I, decay as $O(\mathrm{e}^{-(n-2)r})$, $O(\mathrm{e}^{-(n-1)r})$ and $O(\mathrm{e}^{-(n-1)r})$ respectively[7]. Then, in the 4D theory, theorem 4.2 implies*

$$Q(\varepsilon) = \frac{3}{2}\int_{\Sigma_{t,\infty}} \mathrm{e}^{-r}p_M\bar{\varepsilon}_k\gamma^M\varepsilon_k\sqrt{\det(f_{(0)\alpha\beta})}\,\mathrm{d}^{n-2}x - 8\pi q_e\bar{\varepsilon}_k\varepsilon_k$$
$$- 2\int_{\Sigma_{t,\infty}} F_{23}\varepsilon_k^\dagger\gamma^1\gamma^2\gamma^3\varepsilon_k\mathrm{d}A + 2\mathrm{i}\int_{\Sigma_{t,\infty}} A_A\varepsilon_k^\dagger\gamma^1\gamma^A\varepsilon_k\mathrm{d}A \tag{4.225}$$
$$\geq 0 \tag{4.226}$$

while in the 5D theory it implies

$$Q(\varepsilon) = 2\int_{\Sigma_{t,\infty}} \mathrm{e}^{-r}p_M\bar{\varepsilon}_k\gamma^M\varepsilon_k\sqrt{\det(f_{(0)\alpha\beta})}\,\mathrm{d}^{n-2}x - 4\pi\sqrt{3}q_e\bar{\varepsilon}_k\varepsilon_k$$
$$- \frac{\sqrt{3}}{2}\int_{\Sigma_{t,\infty}} F_{AB}\varepsilon_k^\dagger\gamma^1\gamma^{AB}\varepsilon_k\mathrm{d}A + 2\mathrm{i}\sqrt{3}\int_{\Sigma_{t,\infty}} A_A\varepsilon_k^\dagger\gamma^1\gamma^A\varepsilon_k\mathrm{d}A \tag{4.227}$$
$$\geq 0. \tag{4.228}$$

Proof. To apply theorem 4.2, it must first be checked that the assumptions in definition 3.1 hold. First consider $\gamma^{IJ}\mathcal{A}_J$. In the 4D case,

$$\gamma^{IJ}\mathcal{A}_J^{(4)} = \frac{1}{2}E_K\gamma^{IJ}\gamma^0\gamma^K\gamma_J - \frac{1}{4}F_{KL}\gamma^{IJ}\gamma^{KL}\gamma_J + \mathrm{i}A_J\gamma^{IJ} \tag{4.229}$$

The first term simplifies as

$$\frac{1}{2}E_K\gamma^{IJ}\gamma^0\gamma^K\gamma_J = -\frac{1}{2}E_K\gamma^{IJ}\gamma^0\gamma_J\gamma^K - E_K\gamma^{IJ}\gamma^0\delta^K{}_J \tag{4.230}$$
$$= \frac{1}{2}E_K\gamma^{IJ}\gamma_J\gamma^0\gamma^K - E_J\gamma^{IJ}\gamma^0 \tag{4.231}$$
$$= -E_J\gamma^I\gamma^0\gamma^J - E_J\gamma^{IJ}\gamma^0 \tag{4.232}$$
$$= E_J\gamma^0(\gamma^I\gamma^J - \gamma^{IJ}) \tag{4.233}$$
$$= -E^I\gamma^0, \tag{4.234}$$

[7] These assumptions are there to ensure convergence of the integrals to follow.

while the second term simplifies as

$$-\frac{1}{4}F_{KL}\gamma^{IJ}\gamma^{KL}\gamma_J = -\frac{1}{4}F_{KL}\gamma^{IJ}(\gamma_J\gamma^{KL} - 2\delta^L{}_J\gamma^K + 2\delta^K{}_J\gamma^L) \tag{4.235}$$

$$= \frac{1}{2}F_{JK}\gamma^I\gamma^{JK} - F_{JK}\gamma^{IJ}\gamma^K \tag{4.236}$$

$$= \frac{1}{2}F_{JK}(\gamma^{IJK} - \delta^{IJ}\gamma^K + \delta^{IK}\gamma^J - 2\gamma^{IJK} + 2\delta^{KJ}\gamma^I - 2\delta^{KI}\gamma^J) \tag{4.237}$$

$$= -\frac{1}{2}F_{JK}\gamma^{IJK}. \tag{4.238}$$

In summary,

$$\gamma^{IJ}\mathcal{A}_J^{(4)} = -E^I\gamma^0 - \frac{1}{2}F_{JK}\gamma^{IJK} + \mathrm{i}A_J\gamma^{IJ}, \tag{4.239}$$

which is manifestly hermitian, as required. Proceeding similarly, in the 5D case, the result is

$$\gamma^{IJ}\mathcal{A}_J^{(5)} = -\frac{\sqrt{3}}{2}E^I\gamma^0 + \frac{\sqrt{3}}{4}F_{JK}\gamma^I\gamma^{JK} - \frac{\sqrt{3}}{2}F_{JK}\gamma^{IJ}\gamma^K + \mathrm{i}\sqrt{3}A_J\gamma^{IJ}, \tag{4.240}$$

which is also hermitian. Next, with the present energy-momentum tensor,

$$\mathbb{M} = 4\pi T^{0a}\gamma_0\gamma_a + \gamma^{IJ}D_I\mathcal{A}_J + \frac{\mathrm{i}(n-2)}{2}(\gamma^I\mathcal{A}_I + \mathcal{A}_I^\dagger\gamma^I) - \mathcal{A}_I^\dagger\gamma^{IJ}\mathcal{A}_J \tag{4.241}$$

$$= \frac{1}{2}E^IE_I I + \frac{1}{4}F^{IJ}F_{IJ}I - F_{IJ}E^J\gamma^0\gamma^I + \gamma^{IJ}D_I\mathcal{A}_J + \frac{\mathrm{i}(n-2)}{2}(\gamma^I\mathcal{A}_I + \mathcal{A}_I^\dagger\gamma^I)$$
$$- \mathcal{A}_I^\dagger\gamma^{IJ}\mathcal{A}_J. \tag{4.242}$$

Consider the 4D theory first. Using the equations of motion,

$$\gamma^{IJ}D_I\mathcal{A}_J^{(4)} = -D_I(E^I)\gamma^0 - \frac{1}{2}D_{[I}F_{JK]}\gamma^{IJK} + \mathrm{i}D_{[I}A_{J]}\gamma^{IJ} = 0 - 0 + \frac{\mathrm{i}}{2}F_{IJ}\gamma^{IJ}. \tag{4.243}$$

For the next term,

$$\gamma^I\mathcal{A}_I^{(4)} = \frac{1}{2}E_J\gamma^I\gamma^0\gamma^J\gamma_I - \frac{1}{4}F_{JK}\gamma^I\gamma^{JK}\gamma_I + \mathrm{i}A_I\gamma^I \tag{4.244}$$

$$= -\frac{1}{2}E_J\gamma^0\gamma^I\gamma^J\gamma_I - \frac{1}{4}F_{JK}(\gamma^{JK}\gamma^I - 2\delta^{IJ}\gamma^K + 2\delta^{IK}\gamma^J)\gamma_I + \mathrm{i}A_I\gamma^I \tag{4.245}$$

$$= \frac{1}{2}E_J\gamma^0\gamma^J\gamma^I\gamma_I + E_J\gamma^0\delta^{IJ}\gamma_I + \frac{3}{4}F_{IJ}\gamma^{IJ} - F_{IJ}\gamma^I\gamma^J + \mathrm{i}A_I\gamma^I \tag{4.246}$$

$$= -\frac{1}{2}E_I\gamma^0\gamma^I - \frac{1}{4}F_{IJ}\gamma^{IJ} + \mathrm{i}A_I\gamma^I \text{ and therefore} \tag{4.247}$$

$$\gamma^I\mathcal{A}_I^{(4)} + \mathcal{A}_I^{(4)\dagger}\gamma^I = \gamma^I\mathcal{A}_I^{(4)} - (\gamma^I\mathcal{A}_I^{(4)})^\dagger = -\frac{1}{2}F_{IJ}\gamma^{IJ}. \tag{4.248}$$

The most tedious to simplify is

$$\mathcal{A}_I^{(4)\dagger}\gamma^{IJ}\mathcal{A}_J^{(4)}$$
$$= \left(\frac{1}{2}E_J\gamma_I\gamma^J\gamma^0 + \frac{1}{4}F_{JK}\gamma_I\gamma^{KJ} - \mathrm{i}A_I I\right)\left(-E^I\gamma^0 - \frac{1}{2}F_{LM}\gamma^{ILM} + \mathrm{i}A_L\gamma^{IL}\right) \tag{4.249}$$

$$= -\frac{1}{2}E_JE^I\gamma_I\gamma^J\gamma^0\gamma^0 - \frac{1}{4}E_JF_{LM}\gamma_I\gamma^J\gamma^0\gamma^{ILM} + \frac{\mathrm{i}}{2}E_Ja_L\gamma_I\gamma^J\gamma^0\gamma^{IL} - \frac{1}{4}E^IF_{JK}\gamma_I\gamma^{KJ}\gamma^0$$
$$- \frac{1}{8}F_{JK}F_{LM}\gamma_I\gamma^{KJ}\gamma^{ILM} + \frac{\mathrm{i}}{4}F_{JK}A_L\gamma_I\gamma^{KJ}\gamma^{IL} + \mathrm{i}A_IE^I\gamma^0 + \frac{\mathrm{i}}{2}A_IF_{LM}\gamma^{ILM} + 0. \tag{4.250}$$

Consider each set of similar terms in this expression individually.

$$-\frac{1}{2}E_J E^I \gamma_I \gamma^J \gamma^0 \gamma^0 = -\frac{1}{2}E_I E_J \gamma^I \gamma^J = \frac{1}{2}E^I E_I I. \tag{4.251}$$

$$\begin{aligned}
&-\frac{1}{4}E_J F_{LM}\gamma_I\gamma^J\gamma^0\gamma^{ILM} - \frac{1}{4}E^I F_{JK}\gamma_I\gamma^{KJ}\gamma^0 && (4.252)\\
&= \frac{1}{4}E_I F_{JK}(-\gamma_L\gamma^I\gamma^0\gamma^{LJK} - \gamma^I\gamma^{KJ}\gamma^0) && (4.253)\\
&= \frac{1}{4}E_I F_{JK}\gamma^0(\gamma^I\gamma_L\gamma^{LJK} + 2\delta^I{}_L\gamma^{LJK} - \gamma^I\gamma^{JK}) && (4.254)\\
&= \frac{1}{4}E_I F_{JK}\gamma^0(-\gamma^I\gamma^{JK} + 2\gamma^{IJK} - \gamma^I\gamma^{JK}) && (4.255)\\
&= \frac{1}{2}E_I F_{JK}\gamma^0(-\gamma^I\gamma^{JK} + \gamma^I\gamma^{JK} + \delta^{IJ}\gamma^K - \delta^{IK}\gamma^J) && (4.256)\\
&= E^I F_{IJ}\gamma^0\gamma^J. && (4.257)
\end{aligned}$$

$$\begin{aligned}
\frac{\mathrm{i}}{2}E_J A_L\gamma_I\gamma^J\gamma^0\gamma^{IL} + \mathrm{i}A_I E^I\gamma^0 &= -\frac{\mathrm{i}}{2}E_J A_K\gamma^J\gamma_I\gamma^{IK}\gamma^0 - \mathrm{i}E_J A_K\delta^J{}_I\gamma^{IK}\gamma^0 + \mathrm{i}A_I E^I\gamma^0 && (4.258)\\
&= \mathrm{i}E_I A_J(\gamma^I\gamma^J - \gamma^{IJ})\gamma^0 + \mathrm{i}A_I E^I\gamma^0 && (4.259)\\
&= -\mathrm{i}E_I A_J\delta^{IJ}\gamma^0 + +\mathrm{i}A_I E^I\gamma^0 && (4.260)\\
&= 0. && (4.261)
\end{aligned}$$

$$\begin{aligned}
&-\frac{1}{8}F_{JK}F_{LM}\gamma_I\gamma^{KJ}\gamma^{ILM} && (4.262)\\
&= -\frac{1}{8}F_{JK}F_{LM}(\gamma^{KJ}\gamma_I - 2\delta^K{}_I\gamma^J + 2\delta^J{}_I\gamma^K)\gamma^{ILM} && (4.263)\\
&= -\frac{1}{8}F_{IJ}F_{KL}\gamma^{IJ}\gamma^{KL} - \frac{1}{2}F_{IJ}F_{KL}\gamma^J\gamma^{IKL} && (4.264)\\
&= -\frac{1}{8}F_{IJ}F_{KL}(\gamma^{IJKL} + \delta^{IK}\gamma^J\gamma^L - \delta^{IL}\gamma^J\gamma^K - \delta^{JK}\gamma^I\gamma^L + \delta^{JL}\gamma^I\gamma^K + \delta^{IK}\delta^{JL}I - \delta^{IL}\delta^{JK}I\\
&\quad + 4\gamma^{JIKL} - 4\delta^{JI}\gamma^{KL} + 4\delta^{JK}\gamma^{IL} - 4\delta^{JL}\gamma^{IK}) && (4.265)\\
&= 0 - \frac{1}{8}F_{IJ}F^I{}_L\gamma^J\gamma^L + \frac{1}{8}F_{IJ}F_K{}^I\gamma^J\gamma^K + \frac{1}{8}F_{IJ}F^J{}_L\gamma^I\gamma^L - \frac{1}{8}F_{IJ}F_K{}^J\gamma^I\gamma^K - \frac{1}{8}F^{IJ}F_{IJ}I\\
&\quad + \frac{1}{8}F^{IJ}F_{JI}I - 0 + 0 - \frac{1}{2}F_{IJ}F^J{}_L\gamma^{IL} + \frac{1}{2}F_{IJ}F_K{}^J\gamma^{IK} \quad \text{as } \gamma^{IJKL} = 0 \text{ when } n = 4 && (4.266)\\
&= \frac{1}{2}F_{IJ}F^{IJ}I - \frac{1}{4}F^{IJ}F_{IJ}I - 0 - 0 && (4.267)\\
&= \frac{1}{4}F^{IJ}F_{IJ}I. && (4.268)
\end{aligned}$$

$$\begin{aligned}
&\frac{\mathrm{i}}{4}F_{JK}A_L\gamma_I\gamma^{KJ}\gamma^{IL} + \frac{\mathrm{i}}{2}A_I F_{LM}\gamma^{ILM}\\
&= \frac{\mathrm{i}}{4}A_I F_{JK}(-\gamma_L\gamma^{JK}\gamma^{LI} + 2\gamma^{IJK}) && (4.269)\\
&= \frac{\mathrm{i}}{4}A_I F_{JK}(-\gamma^{JK}\gamma_L\gamma^{LI} + 2\delta^J{}_L\gamma^K\gamma^{LI} - 2\delta^K{}_L\gamma^J\gamma^{LI} + 2\gamma^{IJK}) && (4.270)\\
&= \frac{\mathrm{i}}{2}A_I F_{JK}(\gamma^{JK}\gamma^I + 2\gamma^K\gamma^{JI} + \gamma^{IJK}) && (4.271)\\
&= \frac{\mathrm{i}}{2}A_I F_{JK}(\gamma^{JKI} - \delta^{KI}\gamma^J + \delta^{IJ}\gamma^K + 2\gamma^{KJI} - 2\delta^{KJ}\gamma^I + 2\delta^{KI}\gamma^J + \gamma^{IJK}) && (4.272)\\
&= 0. && (4.273)
\end{aligned}$$

Substituting these results back up, I get

$$\mathcal{A}_I^{(4)\dagger}\gamma^{IJ}\mathcal{A}_J^{(4)} = \frac{1}{2}E^I E_I I + E^I F_{IJ}\gamma^0\gamma^J + \frac{1}{4}F^{IJ}F_{IJ}I. \tag{4.274}$$

Therefore equation 4.242 reduces to $\mathbb{M} = 0$, which trivially satisfies all required assumptions.

Proceeding completely analogously, in the 5D theory the individual terms are

$$\gamma^{IJ}D_I\mathcal{A}_J^{(5)} = -\frac{1}{4}\varepsilon^{IJKL}F_{IJ}F_{KL}\gamma^0 + \frac{\mathrm{i}\sqrt{3}}{2}F_{IJ}\gamma^{IJ}, \tag{4.275}$$

$$\gamma^I\mathcal{A}_I^{(5)} = \frac{1}{2\sqrt{3}}E_I\gamma^I\gamma^0 - \frac{1}{2\sqrt{3}}F_{IJ}\gamma^{IJ} + \mathrm{i}\sqrt{3}A_I\gamma^I, \tag{4.276}$$

$$\gamma^I\mathcal{A}_I^{(5)} + \mathcal{A}_I^{(5)\dagger}\gamma^I = -\frac{1}{\sqrt{3}}F_{IJ}\gamma^{IJ} \quad \text{and} \tag{4.277}$$

$$\begin{aligned}\mathcal{A}_I^{(5)\dagger}\gamma^{IJ}\mathcal{A}_J^{(5)} = {} & \frac{1}{4}E_J E^I\gamma_I\gamma^0\gamma^J\gamma^0 - \frac{1}{8}E_J F_{LM}\gamma_I\gamma^0\gamma^J\gamma^I\gamma^{LM} + \frac{1}{4}E_J F_{LM}\gamma_I\gamma^0\gamma^J\gamma^{IL}\gamma^M \\ & - \frac{\mathrm{i}}{2}E_J A_L\gamma_I\gamma^0\gamma^J\gamma^{IL} - \frac{1}{8}E^I F_{JK}\gamma_I\gamma^{KJ}\gamma^0 + \frac{1}{16}F_{LM}F_{JK}\gamma_I\gamma^{KJ}\gamma^I\gamma^{LM} \\ & - \frac{1}{8}F_{LM}F_{JK}\gamma_I\gamma^{KJ}\gamma^{IL}\gamma^M + \frac{\mathrm{i}}{4}A_L F_{JK}\gamma_I\gamma^{KJ}\gamma^{IL} + \frac{1}{4}E^I E_I(\gamma^0)^2 \\ & - \frac{1}{8}E_I F_{LM}\gamma^0\gamma^I\gamma^{LM} + \frac{1}{4}E_I F_{LM}\gamma^0\gamma^{IL}\gamma^M - \frac{\mathrm{i}}{2}E_I A_L\gamma^0\gamma^{IL} - \frac{1}{4}E^I F_{IJ}\gamma^J\gamma^0 \\ & + \frac{1}{8}F_{LM}F_{IJ}\gamma^J\gamma^I\gamma^{LM} - \frac{1}{4}F_{LM}F_{IJ}\gamma^J\gamma^{IL}\gamma^M + \frac{\mathrm{i}}{2}A_L F_{IJ}\gamma^J\gamma^{IL} + \frac{3\mathrm{i}}{2}A_I E^I\gamma^0 \\ & - \frac{3\mathrm{i}}{4}A_I F_{LM}\gamma^I\gamma^{LM} + \frac{3\mathrm{i}}{2}A_I F_{LM}\gamma^{IL}\gamma^M + 0.\end{aligned} \tag{4.278}$$

This time, the different types of terms in $\mathcal{A}_I^{(5)\dagger}\gamma^{IJ}\mathcal{A}_J^{(5)}$ simplify to

$$-\frac{\mathrm{i}}{2}E_J a_L\gamma_I\gamma^0\gamma^J\gamma^{IL} - \frac{\mathrm{i}}{2}E_I a_L\gamma^0\gamma^{IL} + \frac{3\mathrm{i}}{2}a_I E^I\gamma^0 = 0, \tag{4.279}$$

$$\frac{1}{4}E_J E^I\gamma_I\gamma^0\gamma^J\gamma^0 + \frac{1}{4}E^I E_I(\gamma^0)^2 = \frac{1}{2}E^I E_I I, \tag{4.280}$$

$$\frac{\mathrm{i}}{4}a_L F_{JK}\gamma_I\gamma^{KJ}\gamma^{IL} + \frac{\mathrm{i}}{2}a_L F_{IJ}\gamma^J\gamma^{IL} - \frac{3\mathrm{i}}{4}a_I F_{LM}\gamma^I\gamma^{LM} + \frac{3\mathrm{i}}{2}a_I F_{LM}\gamma^{IL}\gamma^M = 0, \tag{4.281}$$

$$\begin{aligned}& \frac{1}{16}F_{LM}F_{JK}\gamma_I\gamma^{KJ}\gamma^I\gamma^{LM} - \frac{1}{8}F_{LM}F_{JK}\gamma_I\gamma^{KJ}\gamma^{IL}\gamma^M + \frac{1}{8}F_{LM}F_{IJ}\gamma^J\gamma^I\gamma^{LM} - \frac{1}{4}F_{LM}F_{IJ}\gamma^J\gamma^{IL}\gamma^M \\ & \quad = \frac{1}{4}F_{IJ}F_{JK}\varepsilon^{IJKL}\gamma^1\gamma^2\gamma^3\gamma^4 + \frac{1}{4}F^{IJ}F_{IJ}I \quad \text{and}\end{aligned} \tag{4.282}$$

$$\begin{aligned}& -\frac{1}{8}E_J F_{LM}\gamma_I\gamma^0\gamma^J\gamma^I\gamma^{LM} + \frac{1}{4}E_J F_{LM}\gamma_I\gamma^0\gamma^J\gamma^{IL}\gamma^M - \frac{1}{8}E^I F_{JK}\gamma_I\gamma^{KJ}\gamma^0 - \frac{1}{8}E_I F_{LM}\gamma^0\gamma^I\gamma^{LM} \\ & \quad + \frac{1}{4}E_I F_{LM}\gamma^0\gamma^{IL}\gamma^M - \frac{1}{4}E^I F_{IJ}\gamma^J\gamma^0 = E^I F_{IJ}\gamma^0\gamma^J,\end{aligned} \tag{4.283}$$

which implies

$$\mathcal{A}_I^{(5)\dagger}\gamma^{IJ}\mathcal{A}_J^{(5)} = \frac{1}{2}E^I E_I I + E^I F_{IJ}\gamma^0\gamma^J + \frac{1}{4}F_{IJ}F_{JK}\varepsilon^{IJKL}\gamma^1\gamma^2\gamma^3\gamma^4 + \frac{1}{4}F^{IJ}F_{IJ}I. \tag{4.284}$$

In 5D, there are two inequivalent, irreducible representations of the Clifford algebra; they have $\gamma^4 = \pm\gamma^0\gamma^1\gamma^2\gamma^3$ respectively. Choosing the representation in which $\gamma^4 = +\gamma^0\gamma^1\gamma^2\gamma^3$ once again leads to $\mathbb{M} = 0$.

The only remaining assumptions are on $\mathcal{A}$'s decay rate. These transfer to decay rates on the fields; they ensure all boundary integrals are convergent and are stronger than the decay rates required for results in chapter 3.

Having established that theorem 4.2 is valid in the present context, all that remains is to evaluate the $\mathcal{A}$ dependent boundary terms in the theorem. In the 4D case,

$$
\begin{aligned}
&\gamma^1\gamma^A\mathcal{A}^{(4)}_A \\
&= \frac{1}{2}E_J\gamma^0\gamma^1\gamma^A\gamma^J\gamma_A - \frac{1}{4}F_{JK}\gamma^1\gamma^A\gamma^{JK}\gamma_A + \mathrm{i}a_A\gamma^1\gamma^A && (4.285)\\
&= -\frac{1}{2}E_J\gamma^0\gamma^1\gamma^J\gamma^A\gamma_A - E_J\gamma^0\gamma^1\delta^{AJ}\gamma_A - \frac{1}{4}F_{JK}\gamma^1(\gamma^{JK}\gamma^A - 2\delta^{JA}\gamma^K + 2\delta^{KA}\gamma^J)\gamma_A && (4.286)\\
&= E_J\gamma^0\gamma^1\gamma^J - E_A\gamma^0\gamma^1\gamma^A + \frac{1}{2}F_{IJ}\gamma^1\gamma^{IJ} + F_{AI}\gamma^1\gamma^I\gamma^A && (4.287)\\
&= E_1\gamma^0\gamma^1\gamma^1 + E_A\gamma^0\gamma^1\gamma^A - E_A\gamma^0\gamma^1\gamma^A + F_{1A}\gamma^1\gamma^1\gamma^A + \frac{1}{2}F_{AB}\gamma^1\gamma^{AB} + F_{A1}\gamma^1\gamma^1\gamma^A \\
&\quad + F_{AB}\gamma^1\gamma^B\gamma^A && (4.288)\\
&= -E_1\gamma^0 - \frac{1}{2}F_{AB}\gamma^1\gamma^{AB} && (4.289)\\
&= -E_1\gamma^0 - F_{23}\gamma^1\gamma^2\gamma^3. && (4.290)
\end{aligned}
$$

Proceeding analogously, in the 5D case I get

$$\gamma^1\gamma^A\mathcal{A}^{(5)}_A = -\frac{\sqrt{3}}{2}E_1\gamma^0 - \frac{\sqrt{3}}{4}F_{AB}\gamma^1\gamma^{AB} + \mathrm{i}\sqrt{3}A_A\gamma^1\gamma^A. \tag{4.291}$$

These are hermitian already, so $\gamma^1\gamma^A\mathcal{A}_A + \mathcal{A}^\dagger_A\gamma^A\gamma^1 = 2\gamma^1\gamma^A\mathcal{A}_A$. In theorem 4.2, this matrix is inbetween $\varepsilon^\dagger_k$ and ε_k. In the case of the electric field term, that produces $E_1\bar{\varepsilon}_k\varepsilon_k$ as an integrand. However, the Killing spinor equation implies that $\overline{D}_a(\bar{\varepsilon}_k\varepsilon_k) = 0$. Since $\bar{\varepsilon}_k\varepsilon_k$ is a scalar (and all derivatives act identically on a scalar), it must be that $\bar{\varepsilon}_k\varepsilon_k$ is constant. Hence, $\bar{\varepsilon}_k\varepsilon_k$ can be pulled out of the integral, leaving a term proportional to $\int_{\Sigma_{t,\infty}} E_1\mathrm{d}A = 4\pi q_e$ and thus the claimed result. □

Corollary 4.19.1. *If the extrinsic curvature, K_{IJ}, of Σ_t is $o(\mathrm{e}^{-r})$ near $\Sigma_{t,\infty}$, then*

$$Q(\varepsilon) = \frac{3}{2}\int_{\Sigma_{t,\infty}} \mathrm{e}^{-r}p_M\bar{\varepsilon}_k\gamma^M\varepsilon_k\sqrt{\det(f_{(0)\alpha\beta})}\,\mathrm{d}^{n-2}x - 8\pi q_e\bar{\varepsilon}_k\varepsilon_k \geq 0 \tag{4.292}$$

in the 4D theory, while in the 5D theory

$$Q(\varepsilon) = 2\int_{\Sigma_{t,\infty}} \mathrm{e}^{-r}p_M\bar{\varepsilon}_k\gamma^M\varepsilon_k\sqrt{\det(f_{(0)\alpha\beta})}\,\mathrm{d}^{n-2}x - 4\pi\sqrt{3}q_e\bar{\varepsilon}_k\varepsilon_k \geq 0. \tag{4.293}$$

Proof. The objective here is to show the magnetic and gauge field integrals in the theorem cancel. By construction, the t coordinate is chosen such that the boundary has $f_{(0)0\alpha} = 0$. Therefore, to leading order, $e_I{}^\mu = \delta^\mu{}_i e^{(\sigma)i}_I$, where σ is the metric on Σ_t. Thus,

$$F_{JK} \to e^{(\sigma)j}_J e^{(\sigma)k}_K F_{jk} = e^{(\sigma)j}_J e^{(\sigma)k}_K(\partial_j A_k - \partial_k A_j) \to D^{(\sigma)}_J A_K - D^{(\sigma)}_K A_J. \tag{4.294}$$

The decay conditions assumed mean only the leading order contributions survive the integral[8]. Let $Q^I \equiv \partial_r$ be the normal to constant r surfaces. Then,

$$
\begin{aligned}
\frac{1}{2}\int_{\Sigma_{t,\infty}} Q_I F_{JK}\varepsilon^\dagger_k\gamma^{IJK}\varepsilon_k\mathrm{d}A &= \int_{\Sigma_{t,\infty}} Q_I D^{(\sigma)}_J(A_K)\varepsilon^\dagger_k\gamma^{IJK}\varepsilon_k\mathrm{d}A && (4.295)\\
&= \int_{\Sigma_{t,\infty}} Q_I D^{(\sigma)}_J(A_K\varepsilon^\dagger_k\gamma^{IJK}\varepsilon_k)\mathrm{d}A \\
&\quad - \int_{\Sigma_{t,\infty}} Q_I A_K\left(D^{(\sigma)}_J(\varepsilon_k)^\dagger\gamma^{IJK}\varepsilon_k + \varepsilon^\dagger_k\gamma^{IJK}D^{(\sigma)}_J\varepsilon_k\right)\mathrm{d}A. && (4.296)
\end{aligned}
$$

[8]One could relax the decay rate assumed on F_{IJ} if more information is known about $f_{(k)0\alpha}$ for $k > 0$.

The first term in equation 4.296 vanishes by applying Stokes' theorem in the same way as lemma 3.3. As for the other two terms, the Levi-Civita connection of Σ_t and M are related by

$$D_I^{(\sigma)}\varepsilon_k = D_I\varepsilon_k + \frac{1}{2}K_{IJ}\gamma^J\gamma^0\varepsilon_k \tag{4.297}$$

when acting on spinors. Since K_{IJ} is assumed to decay quicker than $O(\mathrm{e}^{-r})$, $D_I^{(\sigma)}\varepsilon_k \to D_I\varepsilon_k$ to leading order. Then, since the metric also approaches the background to leading order, $D_I^{(\sigma)}\varepsilon_k \to -\frac{\mathrm{i}}{2}\gamma_I\varepsilon_k$. Hence, equation 4.296 reduces to

$$\frac{1}{2}\int_{\Sigma_{t,\infty}} Q_I F_{JK}\varepsilon_k^\dagger\gamma^{IJK}\varepsilon_k \mathrm{d}A = \frac{\mathrm{i}}{2}\int_{\Sigma_{t,\infty}} Q_I A_K\left(\varepsilon_k^\dagger\gamma_J\gamma^{IJK}\varepsilon_k + \varepsilon_k^\dagger\gamma^{IJK}\gamma_J\varepsilon_k\right)\mathrm{d}A \tag{4.298}$$

$$= \mathrm{i}(n-3)\int_{\Sigma_{t,\infty}} A_A\varepsilon_k^\dagger\gamma^1\gamma^A\varepsilon_k \mathrm{d}A, \tag{4.299}$$

which means the magnetic and gauge field integrals do indeed cancel. □

The most subtle point in the previous proofs is the implicit assumption that $F = \mathrm{d}A$ everywhere in an open neighbourhood of $\Sigma_{t,\infty}$ in corollary 4.19.1. If $\Sigma_{t,\infty}$ has a topology where H^2_{dR} is trivial, then this assumption is fine. However, there are many examples - including the most standard example of the S^2 cross-section - where this is not true. As such, it becomes problematic to incorporate magnetic charge into the discussion.

Even if F wasn't assumed exact, magnetic charge doesn't arise as naturally from the equations as it does when $\Lambda = 0$. In the 4D Einstein-Maxwell theory [51], one still gets a term,

$$-2\int_{\Sigma_{t,\infty}} F_{23}\varepsilon_k^\dagger\gamma^1\gamma^2\gamma^3\varepsilon_k \mathrm{d}A, \tag{4.300}$$

in the analogue of $Q(\varepsilon)$. However, in that case ε_k is just a constant, meaning $\varepsilon_k^\dagger\gamma^1\gamma^2\gamma^3\varepsilon_k$ can be pulled out of the integral, leaving the standard magnetic charge integral. But, in the present situation, $\varepsilon_k^\dagger\gamma^1\gamma^2\gamma^3\varepsilon_k$ is non-constant. Furthermore, it grows as e^r. Hence, a convergent integral requires F_{23} to decay as $O(\mathrm{e}^{-3r})$, which is faster than the $O(\mathrm{e}^{-2r})$ decay required to get non-zero magnetic charge. To some extent, this reflects the breakdown in electric-magnetic duality when $\Lambda \neq 0$.

Subtleties of F's exactness also arise tacitly when solving the Dirac equation, $\gamma^I\nabla_I\varepsilon = 0$. The connection, ∇_a, is constructed to be gauge covariant; in particular, under $A_a \to A_a + D_a\lambda$, if $\varepsilon \to \mathrm{e}^{\mathrm{i}\lambda}\varepsilon$, then $\nabla_a\varepsilon \to \mathrm{e}^{\mathrm{i}\lambda}\nabla_a\varepsilon$. However, on a manifold, changing from one coordinate patch to another also transforms A_a in a formally identical way. Thus, ε must also transform by a phase to keep ∇_a covariant. Hence, in principle one could get merely a spin^c structure, rather than a spin structure[9]. This appears to be incompatible with Witten's method though because $\gamma^I\nabla_I\varepsilon = 0$ is solved subject to the boundary condition, $\varepsilon \to \varepsilon_k$, in which ε_k is a true spinor, not a section of a non-trivial spin^c bundle. Therefore, imposing the required boundary condition breaks the gauge covariance. If F were exact though, the issue is avoided. It remains open whether Witten's method can be adjusted in any way to accommodate spin^c structures.

[9]The *Spin*^{c} group is defined to be $(Spin \times U(1))/\mathbb{Z}_2$, where *Spin* is the spin group of the appropriate signature, $\mathbb{Z}_2 = \{(I,1),(-I,-1)\}$ and I is the identity element of the spin group. A spin^c structure on a manifold is tantamount to the existence of a principal bundle with structure group, *Spin*^{c}. In particular, because of the $\mathbb{Z}_2$ quotienting, the cocycle condition for the transition functions can fail in each of the *Spin* and $U(1)$ factors individually, but hold overall as long as the discrepancy is only a -1 factor in each of *Spin* and $U(1)$. Furthermore, this means every manifold with a spin structure also has a spin^c structure - by tensoring with a line bundle. However, the converse is not true.

The only previous work on classical BPS inequalities in these two gauged supergravity theories appears to be in [80, 76, 123, 95]. [80] doesn't consider magnetic fields in their positive energy theorem at all, so the issues discussed don't arise.

Meanwhile, my result disagrees with [76], who explicitly have magnetic charge in their equation 27. However, as explained in [88], this is likely due to some error relating to the fact their analogue of ∇_a - see their equation 24 - omits the $\mathrm{i}A_a I$ term and is therefore not gauge covariant. Furthermore, my results are consistent with subsequent work in [19, 67], where the BPS limit is explicitly required to have zero magnetic charge. Another subtlety explained by [67] is that the 4D minimal, gauged supergravity has two distinct vacuum states, one of which has a non-zero magnetic charge determined by the cosmological constant.

[123] uses the same connection as [76] and studies it in much greater detail. Despite using the same connection, their main result - their theorem 1.1 - differs from the main result of [76] - their equation 27. Since $\mathrm{i}A_a I$ is omitted, the analogue of $\mathbb{M}$ found in [123] is only non-negative definite when a modified dominant energy condition holds - their equation 1.2. However, their condition is very unnatural; for example, it can be seen from the constraint equations that in a purely electrovacuum spacetime with non-zero magnetic field, their modified dominant energy condition never holds.

The closest paper to my thesis is [95] - see their section 3.1 in particular. They have the same connection as me and make similar observations about the decay rates in equation 4.300. However, their main result - their equation 22 - does not include the fourth term in lemma 4.19 because they relied on [76] to get their result. A heuristic argument is given describing corrections to [76], leading to the same conclusions as corollary 4.19.1 (for S^2 cross-section topology) and equation 4.311 below, but the precise steps required - including the Dirac equation analysis in chapter 3 - are not given in [95].

Anyhow, bearing in mind all these subtleties, since the boundary geometries considered in section 4.3 are time symmetric, they all satisfy the extra assumptions in corollary 4.19.1. Hence, borrowing from the work there, the following results hold.

Theorem 4.20. *In an asymptotically AdS spacetime (i.e. with round sphere cross-section),*

$$EI - \mathrm{i}P_I\gamma^I + \frac{\mathrm{i}}{2}J_{IJ}\gamma^0\gamma^{IJ} + K_I\gamma^0\gamma^I - q_e\gamma^0 \tag{4.301}$$

is a non-negative definite matrix in the 4D theory and

$$EI - \mathrm{i}P_I\gamma^I + \frac{\mathrm{i}}{2}J_{IJ}\gamma^0\gamma^{IJ} + K_I\gamma^0\gamma^I - \frac{\sqrt{3}}{2}q_e\gamma^0 \tag{4.302}$$

is a non-negative definite matrix in the 5D theory.

Proof. With the ε_k chosen in section 4.3.3,

$$\begin{aligned}
\bar{\varepsilon}_k\varepsilon_k &= \bar{\varepsilon}'_k\varepsilon'_k && (4.303)\\
&= \frac{1}{1-\rho^2}\varepsilon_0^\dagger \mathrm{e}^{-\mathrm{i}\gamma^0 t/2}(I - \mathrm{i}x_I\gamma^I)\gamma^0(I - \mathrm{i}x_J\gamma^J)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0 && (4.304)\\
&= \frac{1}{1-\rho^2}\varepsilon_0^\dagger \mathrm{e}^{-\mathrm{i}\gamma^0 t/2}\gamma^0(I + \mathrm{i}x_I\gamma^I)(I - \mathrm{i}x_J\gamma^J)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0 && (4.305)\\
&= \frac{1}{1-\rho^2}\varepsilon_0^\dagger \mathrm{e}^{-\mathrm{i}\gamma^0 t/2}\gamma^0(I + x_Ix_J\gamma^I\gamma^J)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0 && (4.306)\\
&= \frac{1}{1-\rho^2}\varepsilon_0^\dagger \mathrm{e}^{-\mathrm{i}\gamma^0 t/2}\gamma^0(I - \rho^2 I)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0 && (4.307)\\
&= \varepsilon_0^\dagger\gamma^0\varepsilon_0. && (4.308)
\end{aligned}$$

Meanwhile, the first term in corollary 4.19.1 was already calculated in theorem 4.12. Borrowing from the calculation there, corollary 4.19.1 says

$$0 \leq 8\pi\varepsilon_0^\dagger e^{-i\gamma^0 t/2}\left(EI - iP_I\gamma^I + K_I\gamma^0\gamma^I + \frac{i}{2}J_{IJ}\gamma^0\gamma^{IJ} - q_e\gamma^0\right)e^{i\gamma^0 t/2}\varepsilon_0 \text{ in 4D and} \tag{4.309}$$

$$0 \leq 8\pi\varepsilon_0^\dagger e^{-i\gamma^0 t/2}\left(EI - iP_I\gamma^I + K_I\gamma^0\gamma^I + \frac{i}{2}J_{IJ}\gamma^0\gamma^{IJ} - \frac{\sqrt{3}}{2}q_e\gamma^0\right)e^{i\gamma^0 t/2}\varepsilon_0 \text{ in 5D.} \tag{4.310}$$

Since t is constant on $\Sigma_{t,\infty}$ and ε_0 is an arbitrary constant spinor, the conclusion follows. □

Like theorem 4.12, the matrix in theorem 4.20 doesn't have closed form eigenvalues in general. Again, something more concrete can be said in specific cases. For example, if $n = 4$ and $K_I = P_I = 0$, the eigenvalues are $E \pm q_e \pm |J|$ and $E \pm q_e \mp |J|$, leading to the BPS inequality,

$$E \geq |q_e| + |J|. \tag{4.311}$$

Likewise, suppose $n = 5$ and $K_I = P_I = 0$. Like in the derivation of equation 4.139, suppose the independent rotations are in the 1-2 and 3-4 planes. Then, from the eigenvalues of $EI + iJ_1\gamma^0\gamma^1\gamma^2 + iJ_2\gamma^0\gamma^3\gamma^4 - \frac{\sqrt{3}}{2}q_e\gamma^0$, it follows that

$$E - \frac{\sqrt{3}}{2}q_e \geq |J_1 + J_2| \text{ and } E + \frac{\sqrt{3}}{2}q_e \geq |J_1 - J_2|. \tag{4.312}$$

The Chern-Simons term means the equations of motion are not invariant under $F \to -F$ in the 5D theory and this example illustrates that one must in fact keep track of the relative signs between the charge and angular momentum. Inequalities 4.312 agree with the BPS relations derived using the supersymmetry algebra in section 3 of [34][10]. Furthermore, these inequalities are saturated by the supersymmetric solutions in [55, 24] and [75] respectively.

For a different type of example, suppose $n = 4$, $K_I = 0$ and $J_{IJ} = 0$; then the eigenvalues are $E \pm \sqrt{P_I P^I + q_e^2}$, which implies $\sqrt{E^2 - P^2} \geq |q_e|$.

Theorem 4.21. *For spacetimes asymptotically AdS with $L(p,1)$ cross-section,*

$$E \geq -\frac{\sqrt{3}}{2}q_e + \sqrt{J_2^2 + J_3^2 + J_4^2} \tag{4.313}$$

Proof. ε_k can be chosen identically to section 4.3.4, namely

$$\varepsilon_k = e^{r/2}P_1^-\left(e^{i\gamma^0 t/2} - ie^{-i\gamma^0 t/2}\right)\varepsilon_h + \frac{1}{2}e^{-r/2}P_1^+\left(e^{i\gamma^0 t/2} + ie^{-i\gamma^0 t/2}\right)\varepsilon_h, \tag{4.314}$$

$$\text{with } \varepsilon_h = \frac{1}{2}\begin{bmatrix}\hat{\varepsilon}_h^{(-)} \\ -\hat{\varepsilon}_h^{(-)}\end{bmatrix} \text{ and } \hat{\varepsilon}_h^{(-)} = e^{-i\theta\sigma_2/4}e^{-i(\phi_1-\phi_2)\sigma_1/2}\hat{\varepsilon}_0. \tag{4.315}$$

Then, from the proof of theorem 4.17, it follows the first term in corollary 4.19.1 is

$$4\pi\hat{\varepsilon}_0^\dagger(EI + J_2\sigma_1 + J_3\sigma_2 + J_4\sigma_3)\hat{\varepsilon}_0. \tag{4.316}$$

[10]Also note that inequalities 4.312 are not equivalent to the $E \geq |J_1| + |J_2| + \frac{\sqrt{3}}{2}|q_e|$ that [55] claim results from modifying the results of [80]; in fact, the intended modification would produce exactly inequalities 4.312. Moreover, $E \geq |J_1| + |J_2| + \frac{\sqrt{3}}{2}|q_e|$ can only be concluded from a matrix with eight eigenvalues (to cover all possible combinations of $\pm$), which is impossible to achieve using the 4×4 gamma matrices in Witten's method when $n = 5$.

Meanwhile, direct evaluation shows $\bar{\varepsilon}_k \varepsilon_k = -\hat{\varepsilon}_0 \hat{\varepsilon}_0$ for the present Killing spinor. Hence, corollary 4.19.1 reduces to saying

$$0 \leq 4\pi \hat{\varepsilon}_0^\dagger \left(EI + J_2\sigma_1 + J_3\sigma_2 + J_4\sigma_3 + \frac{\sqrt{3}}{2} q_e I \right) \hat{\varepsilon}_0. \tag{4.317}$$

As in theorem 4.17, the eigenvalues of the matrix in between $\hat{\varepsilon}_0^\dagger$ and $\hat{\varepsilon}_0$ prove the theorem. □

It turns out an explicit locally supersymmetric solution is known to 5D minimal gauged supergravity with $\mathbb{R} \times L(p,1)$ conformal infinity - see section 2.3.1 of [41] or appendix B of [83]. However, that solution satisfies a different BPS-like equation, namely

$$E = \frac{\sqrt{3}}{2} q_e + J_1. \tag{4.318}$$

The solution evades theorem 4.21 in much the same way as the AdS soliton evaded theorem 4.10. It turns out for even p, the spin structure required is the opposite to the spin structure required for the Killing spinors in theorem 4.21. Meanwhile for odd p, the solution turns out to have no spin structure at all, but merely a spin^c structure.

Finally, the only other relevant boundary considered in section 4.3 is the torus. In this case, theorem 4.10 is unchanged by the electromagnetic fields in either theory because $\bar{\varepsilon}_k \varepsilon_k$ is simply zero for the required ε_k.

4.5 Example metrics

So far, the discussion has been quite abstract. In this section I'll illustrate some of the main ideas of this chapter and compute some of main geometric quantities for a pair of particularly simple, concrete metrics.

These examples are black hole metrics, contrary to the assumption I made at the start of chapter 3. However, the following argument from [50, 59] allows Σ_t to have a marginally outer trapped inner boundary. In particular, if Σ_t has an inner boundary, then I need to choose boundary conditions for the Dirac equation, $\gamma^I \nabla_I \varepsilon = 0$, on that inner boundary. By choosing the one direction in the vielbein to be an inward-pointing, unit normal to $\partial_{\text{inner}}\Sigma_t$ and the boundary condition, $\gamma^0\gamma^1\varepsilon = \varepsilon$ on $\partial_{\text{inner}}\Sigma_t$, it turns out that the Lichnerowicz identity yields the boundary terms,

$$\int_{\partial_{\text{inner}}\Sigma_t} \varepsilon^\dagger \varepsilon\, \theta_l \,\mathrm{d}A - \int_{\partial_{\text{inner}}\Sigma_t} \varepsilon^\dagger \left(\gamma^1 \gamma^A \mathcal{A}_A + \mathcal{A}_A^\dagger \gamma^A \gamma^1 \right) \varepsilon \,\mathrm{d}A. \tag{4.319}$$

In this expression $l = \frac{1}{\sqrt{2}}(e_0 + e_1)$ and θ_l is the expansion, $l^a H_a$ (where H^a is the mean curvature vector of $\partial_{\text{inner}}\Sigma_t$). But, on a marginally outer trapped surface $\theta_l = 0$. Hence, that term makes no contribution to the results. The vanishing of the second term must be checked on a case by case. The trivial case is just $\mathcal{A}_a = 0$, which is all I'll need for the first example in this section. Meanwhile, for the second example I will consider, $\mathcal{A}_a$ is defined by equation 4.224. It can be checked in that case that if $\gamma^0\gamma^1\varepsilon = \varepsilon$, then $\varepsilon^\dagger(\gamma^1\gamma^A\mathcal{A}_A + \mathcal{A}_A^\dagger\gamma^A\gamma^1)\varepsilon$ is just zero.

Having established the validity of the relevant theorems, it's now time to consider the concrete examples. First, I'll study the 5D, equal angular momenta Myers-Perry solution (with cosmological constant). Following [78], this solution can be expressed as

$$g = -S^2 \mathrm{d}t \otimes \mathrm{d}t + f^2 \mathrm{d}R \otimes \mathrm{d}R + \frac{1}{4}h^2(\mathrm{d}\psi + \cos(\theta)\mathrm{d}\phi - \Omega \mathrm{d}t) \otimes (\mathrm{d}\psi + \cos(\theta)\mathrm{d}\phi - \Omega \mathrm{d}t) + \frac{1}{4}R^2(\mathrm{d}\theta \otimes \mathrm{d}\theta + \sin^2(\theta)\mathrm{d}\phi \otimes \mathrm{d}\phi), \tag{4.320}$$

$$\text{where } \frac{1}{f^2} = 1 + R^2 - \frac{2MZ}{R^2} + \frac{2Ma^2}{R^4}, \quad h^2 = R^2\left(1 + \frac{2Ma^2}{R^4}\right), \quad \Omega = \frac{4Ma}{R^2h^2},$$
$$Z = 1 - a^2, \quad S = \frac{R}{fh} \quad \text{and } M \text{ and } a \text{ are constants with } -1 < a < 1. \tag{4.321}$$

For my purposes, it will be more convenient to swap (ψ, ϕ) for (ϕ_1, ϕ_2), where

$$\phi_1 = \frac{1}{2}(\psi + \phi) \text{ and } \phi_2 = \frac{1}{2}(\psi - \phi). \tag{4.322}$$

Then t is a "time coordinate" taking values in $\mathbb{R}$, $\phi_1, \phi_2 \in [0, 2\pi]$, $R \geq R_0$ where R_0 is the event horizon radius,

$$\mathrm{d}\psi + \cos(\theta)\mathrm{d}\phi = (1 + \cos(\theta))\mathrm{d}\phi_1 + (1 - \cos(\theta))\mathrm{d}\phi_2 \tag{4.323}$$

and spacelike coordinates would parameterise $\mathbb{R}^4$ as

$$x_1 = R\cos(\theta/2)\cos(\phi_1), \quad x_2 = R\cos(\theta/2)\sin(\phi_1),$$
$$x_3 = R\sin(\theta/2)\cos(\phi_2) \;\&\; x_4 = R\sin(\theta/2)\sin(\phi_2). \tag{4.324}$$

To calculate the various geometric quantities, I need to start by writing equation 4.320 in Fefferman-Graham form for an asymptotically AdS space[11]. Since the $f^2\mathrm{d}R \otimes \mathrm{d}R$ in equation 4.320 depends only on R and $R \to \infty$ heuristically looks like the asymptotic end, it is natural to try $r \equiv r(R)$ as the Fefferman-Graham coordinate. Therefore, try

$$\frac{\mathrm{d}r}{\mathrm{d}R} = f \iff r = \int \frac{1}{\sqrt{1 + R^2 - \frac{2MZ}{R^2} + \frac{2Ma^2}{R^4}}}\mathrm{d}R. \tag{4.325}$$

This integral cannot be done explicitly. However, it only needs to be done perturbatively to generate a Fefferman-Graham expansion. For AdS, the square root in the expression above would have just $1 + R^2$, so it makes sense to perturb around that. Thus, to leading order,

$$r = \int \frac{1}{\sqrt{1+R^2}} \frac{1}{\sqrt{1 - \frac{2MZ}{R^2(1+R^2)} + \frac{2Ma^2}{R^4(1+R^2)}}}\mathrm{d}R \tag{4.326}$$

$$\to \int \left(\frac{1}{\sqrt{1+R^2}} + \frac{MZ}{R^5}\right)\mathrm{d}R \tag{4.327}$$

$$= \ln(R + \sqrt{1+R^2}) - \frac{MZ}{4R^4} + C \tag{4.328}$$

$$\implies \mathrm{e}^r \to C(R + \sqrt{1+R^2})\mathrm{e}^{-MZ/4R^4} \to C(R + \sqrt{1+R^2})\left(1 - \frac{MZ}{4R^4}\right). \tag{4.329}$$

To match the AdS solution asymptotically, where $M = 0$, I should choose $C = \frac{1}{2}$, yielding

$$\mathrm{e}^r \to \frac{1}{2}(R + \sqrt{1+R^2})\left(1 - \frac{MZ}{4R^4}\right). \tag{4.330}$$

[11] Note that being able to do so is proof the metric is indeed asymptotically AdS.

To leading order, the AdS metric implies $R^2 = \mathrm{e}^{2r}\left(1 - \frac{1}{4}\mathrm{e}^{-2r}\right)^2$. The corrections to this come from tracking the terms involving M. Hence,

$$\begin{aligned}
\mathrm{e}^{2r}\left(1-\frac{1}{4}\mathrm{e}^{-2r}\right)^2 &\to \frac{1}{4}(R+\sqrt{1+R^2})^2\left(1-\frac{MZ}{2R^4}\right) \\
&\quad \times \left(1 - \frac{1}{(R+\sqrt{1+R^2})^2}\left(1+\frac{MZ}{2R^4}\right)\right)^2 . && (4.331) \\
&\to R^2 - \frac{MZ}{2R^2}. && (4.332)
\end{aligned}$$

Since $R = \mathrm{e}^r$ to leading order, I also immediately get

$$R^2 = \mathrm{e}^{2r}\left(\left(1-\frac{1}{4}\mathrm{e}^{-2r}\right)^2 + \frac{MZ}{2}\mathrm{e}^{-4r}\right) = \mathrm{e}^{2r}\left(\left(1-\frac{1}{4}\mathrm{e}^{-2r}\right)^2 + \frac{M(1-a^2)}{2}\mathrm{e}^{-4r}\right). \quad (4.333)$$

Next, consider the various functions in equation 4.320 perturbatively. To leading order

$$\begin{aligned}
S^2 &= R^2\left(1+R^2-\frac{2MZ}{R^2}+\frac{2Ma^2}{R^4}\right)\frac{1}{R^2(1+2Ma^2/R^4)} && (4.334) \\
&\to \left(1+R^2-\frac{2M}{R^2}\right) && (4.335) \\
&\to \mathrm{e}^{2r}\left(\left(1+\frac{1}{4}\mathrm{e}^{-2r}\right)^2 - \frac{M(a^2+3)}{2}\mathrm{e}^{-4r}\right), && (4.336) \\
h^2 &= R^2\left(1+\frac{2Ma^2}{R^4}\right) \to \mathrm{e}^{2r}\left(\left(1-\frac{1}{4}\mathrm{e}^{-2r}\right)^2 + \frac{M(1+3a^2)}{2}\mathrm{e}^{-4r}\right) \quad \text{and} && (4.337) \\
h^2\Omega &= \frac{4Ma}{R^2} \to 4Ma\mathrm{e}^{-2r} && (4.338)
\end{aligned}$$

Substituting these expressions back into equation 4.320 yields

$$\begin{aligned}
g = \mathrm{d}r\otimes\mathrm{d}r + \mathrm{e}^{2r}\Bigg(&-\left(\left(1+\frac{1}{4}\mathrm{e}^{-2r}\right)^2 - \frac{M(a^2+3)}{2}\mathrm{e}^{-4r}\right)\mathrm{d}t\otimes\mathrm{d}t \\
&+\frac{1}{4}\left(\left(1-\frac{1}{4}\mathrm{e}^{-2r}\right)^2 + \frac{M(1+3a^2)}{2}\mathrm{e}^{-4r}\right)(\mathrm{d}\psi+\cos(\theta)\mathrm{d}\phi)\otimes(\mathrm{d}\psi+\cos(\theta)\mathrm{d}\phi) \\
&- Ma\mathrm{e}^{-4r}(\mathrm{d}t\otimes(\mathrm{d}\psi+\cos(\theta)\mathrm{d}\phi)+(\mathrm{d}\psi+\cos(\theta)\mathrm{d}\phi)\otimes\mathrm{d}t) \\
&+\frac{1}{4}\left(\left(1-\frac{1}{4}\mathrm{e}^{-2r}\right)^2 + \frac{M(1-a^2)}{2}\mathrm{e}^{-4r}\right)(\mathrm{d}\theta\otimes\mathrm{d}\theta+\sin^2(\theta)\mathrm{d}\phi\otimes\phi) + O(\mathrm{e}^{-6r})\Bigg). \quad (4.339)
\end{aligned}$$

Therefore, the metric is indeed in the form of definition 2.11. Choosing AdS as the background metric, one can immediately read off

$$\begin{aligned}
f_{(0)} = \bar{f}_{(0)} &= -\mathrm{d}t\otimes\mathrm{d}t + \frac{1}{4}(\mathrm{d}\psi+\cos(\theta)\mathrm{d}\phi)\otimes(\mathrm{d}\psi+\cos(\theta)\mathrm{d}\phi) \\
&\quad + \frac{1}{4}(\mathrm{d}\theta\otimes\mathrm{d}\theta+\sin^2(\theta)\mathrm{d}\phi\otimes\mathrm{d}\phi) \quad \text{and} && (4.340) \\
f_{(4)} - \bar{f}_{(4)} &= \frac{M(a^3+3)}{2}\mathrm{d}t\otimes\mathrm{d}t + \frac{M(1+3a^2)}{8}(\mathrm{d}\psi+\cos(\theta)\mathrm{d}\phi)\otimes(\mathrm{d}\psi+\cos(\theta)\mathrm{d}\phi) \\
&\quad - Ma(\mathrm{d}t\otimes(\mathrm{d}\psi+\cos(\theta)\mathrm{d}\phi)+(\mathrm{d}\psi+\cos(\theta)\mathrm{d}\phi)\otimes\mathrm{d}t) \\
&\quad + \frac{M(1-a^2)}{8}(\mathrm{d}\theta\otimes\mathrm{d}\theta+\sin^2(\theta)\mathrm{d}\phi\otimes\mathrm{d}\phi). && (4.341)
\end{aligned}$$

The $f_{(0)}$ expression also implies $P^m_{(0)} \equiv \delta^{m0}$ and

$$f^{mn}_{(0)}\partial_m \otimes \partial_n = -\partial_t \otimes \partial_t + 4\partial_\psi \otimes \partial_\psi + 4\partial_\theta \otimes \partial_\theta + \frac{4}{\sin^2(\theta)}(-\cos(\theta)\partial_\psi + \partial_\phi) \otimes (-\cos(\theta)\partial_\psi + \partial_\phi). \tag{4.342}$$

A direct calclation now yields

$$E = \frac{4}{16\pi}\int_{S^3} \hat{f}^{mn}_{(0)}(f_{(4)mn} - \bar{f}_{(4)mn})\mathrm{d}(g_{S^3}) = \frac{\pi M(a^2+3)}{4}, \tag{4.343}$$

which matches the result quoted in [78], but calculated via a different method. Similarly,

$$P_I = \frac{4}{16\pi}\int_{S^3} \hat{f}^{mn}_{(0)}(f_{(4)mn} - \bar{f}_{(4)mn})\hat{x}_I\mathrm{d}(g_{S^3}) = 0. \tag{4.344}$$

When calculating K_I and J_{IJ}, the $\frac{\partial\theta^\alpha}{\partial x^I}$ terms in definition 4.11 are more easily calculated when using the (θ, ϕ_1, ϕ_2) coordinates on S^3, as opposed to the (ψ, θ, ϕ) coordinates used to calculate E and P_I. For both K_I and J_{IJ}, I need to first calculate $f_{(4)0\alpha}\frac{\partial\theta^\alpha}{\partial x^I}|_{\rho=1}$. From equation 4.341,

$$f_{(4)0\alpha}\mathrm{d}x^\alpha = -Ma(\mathrm{d}\psi + \cos(\theta)\mathrm{d}\phi) \tag{4.345}$$

$$= -Ma((1+\cos(\theta))\mathrm{d}\phi_1 + (1-\cos(\theta))\mathrm{d}\phi_2). \tag{4.346}$$

Using $\phi_1 = \tan^{-1}(x_2/x_1)$ and $\phi_2 = \tan^{-1}(x_4/x_3)$. one finds that on the unit three sphere

$$\frac{\partial\phi_1}{\partial x_1} = -\frac{\sin(\phi_1)}{\cos(\theta/2)},\ \frac{\partial\phi_1}{\partial x_2} = \frac{\cos(\phi_1)}{\cos(\theta/2)},\ \frac{\partial\phi_1}{\partial x_3} = \frac{\partial\phi_1}{\partial x_4} = 0, \tag{4.347}$$

$$\frac{\partial\phi_2}{\partial x_1} = \frac{\partial\phi_2}{\partial x_2} = 0,\ \frac{\partial\phi_2}{\partial x_3} = -\frac{\sin(\phi_2)}{\sin(\theta/2)} \text{ and } \frac{\partial\phi_2}{\partial x_4} = \frac{\cos(\phi_2)}{\sin(\theta/2)}. \tag{4.348}$$

Then, it can immediately be calculated that

$$K_I = \frac{4}{16\pi}\int_{S^3} f_{(4)0\alpha}\frac{\partial\theta^\alpha}{\partial x^J}\bigg|_{\rho=1}\left(\delta^J{}_I - \hat{x}^J\hat{x}_I\right)\mathrm{d}(g_{S^3}) = 0 \text{ and} \tag{4.349}$$

$$J_{IJ} = \frac{4}{16\pi}\int_{S^{n-2}} f_{(4)0\alpha}\left(\hat{x}_I\frac{\partial\theta^\alpha}{\partial x^J}\bigg|_{\rho=1} - \hat{x}_J\frac{\partial\theta^\alpha}{\partial x^I}\bigg|_{\rho=1}\right)\mathrm{d}(g_{S^3}) \equiv \frac{\pi Ma}{2}\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{4.350}$$

This result justifies interpreting the original metric - equation 4.320 - as containing two equal, independent angular momenta, $\pi Ma/2$. If one measures angular momenta with respect to $\frac{\partial}{\partial\psi}$ and $\frac{\partial}{\partial\phi}$ instead, then since $\frac{\partial}{\partial\psi} = \frac{1}{2}(\frac{\partial}{\partial\phi_1} + \frac{\partial}{\partial\phi_2})$ and $\frac{\partial}{\partial\phi} = \frac{1}{2}(\frac{\partial}{\partial\phi_1} - \frac{\partial}{\partial\phi_2})$, the angular momenta would be $\pi Ma/2$ and 0 respectively - matching the result in [78] up to a factor of two, which is only a matter of conventions.

From equation 4.139, the positive energy theorem reduces to saying

$$E \geq |J_{12}| + |J_{34}| \iff \frac{\pi M(a^2+3)}{4} \geq \pi Ma \iff (a-1)(a-3) \geq 0. \tag{4.351}$$

Therefore, a supersymmetric limit is reached by taking $a \to 1^-$. The supersymmetric limit is singular, as observed in [33], but this is because of the lack of any further matter fields; for example, even minimal gauged supergravity has a Maxwell-Chern-Simons field in 5D.

There is also a charged version of the equal angular momentum Myers-Perry solution and it can be used to illustrate some of the results in section 4.4. In the form presented in [84, 77],

$$g = -\frac{R^2W}{4b^2}\mathrm{d}t\otimes\mathrm{d}t + \frac{1}{W}\mathrm{d}R\otimes\mathrm{d}R + \frac{1}{4}R^2(\mathrm{d}\theta\otimes\mathrm{d}\theta + \sin^2(\theta)\mathrm{d}\phi\otimes\mathrm{d}\phi)$$
$$+ b^2(\mathrm{d}\psi + \cos(\theta)\mathrm{d}\phi + f\mathrm{d}t)\otimes(\mathrm{d}\psi + \cos(\theta)\mathrm{d}\phi + f\mathrm{d}t) \quad (4.352)$$

$$\text{and } A = -\frac{Q\sqrt{3}}{2R^2}\left(\mathrm{d}t - \frac{j}{2}(\mathrm{d}\psi + \cos(\theta)\mathrm{d}\phi)\right), \quad (4.353)$$

$$\text{where } W = 1 + 4b^2 - \frac{2P-2Q}{R^2} + \frac{Q^2+2Pj^2}{R^4}, \quad (4.354)$$

$$f = -\frac{j}{2b^2}\left(\frac{2P-Q}{R^2} - \frac{Q^2}{R^4}\right), \quad (4.355)$$

$$b^2 = \frac{1}{4}R^2\left(1 + \frac{2j^2P}{R^4} - \frac{j^2Q^2}{R^6}\right) \quad (4.356)$$

and P, Q & j are constants. For the Fefferman-Graham coordinate, $\frac{1}{W}$ plays the same role here as f^2 in the previous example.

$$W = 1 + R^2\left(1 + \frac{2j^2P}{R^4} - \frac{j^2Q^2}{R^6}\right) - \frac{2P-2Q}{R^2} + \frac{Q^2+2Pj^2}{R^4} \quad (4.357)$$

$$= 1 + R^2 + \frac{2((j^2-1)P+Q)}{R^2} + \frac{(1-j^2)Q + 2j^2P}{R^4}. \quad (4.358)$$

Comparing with the previous example, the analogue of MZ in equation 4.325 is $(1-j^2)P - Q$. Hence, from that analysis, I immediately get

$$\mathrm{e}^r \to \frac{1}{2}(R + \sqrt{1+R^2})\left(1 - \frac{(1-j^2)P-Q}{4R^4}\right) \quad \text{and} \quad (4.359)$$

$$R^2 \to \mathrm{e}^{2r}\left(\left(1 - \frac{1}{4}\mathrm{e}^{-2r}\right)^2 + \frac{1}{2}\left((1-j^2)P - Q\right)\mathrm{e}^{-4r}\right). \quad (4.360)$$

These expansions fully determine the other coefficients in the metric. In particular, I find

$$b^2 \to \frac{1}{4}\mathrm{e}^{2r}\left(\left(1 - \frac{1}{4}\mathrm{e}^{-2r}\right)^2 + \frac{1}{2}\left((1+3j^2)P - Q\right)\mathrm{e}^{-4r}\right), \quad (4.361)$$

$$\frac{R^2W}{4b^2} \to \mathrm{e}^{2r}\left(\left(1 + \frac{1}{4}\mathrm{e}^{-2r}\right)^2 + \frac{1}{2}\left(3Q - (j^2+3)P\right)\mathrm{e}^{-4r}\right) \quad \text{and} \quad (4.362)$$

$$b^2 f = -\frac{j}{2}\left(\frac{2P-Q}{R^2} - \frac{Q^2}{R^4}\right) \to -\frac{j(2P-Q)}{2}\mathrm{e}^{-2r}. \quad (4.363)$$

Substituting these into equation 4.352 gives

$$g = \mathrm{e}^{2r}\Bigg(-\left(1 + \frac{1}{4}\mathrm{e}^{-2r}\right)^2 \mathrm{d}t\otimes\mathrm{d}t + \frac{1}{4}\left(1 - \frac{1}{4}\mathrm{e}^{-2r}\right)^2 \big(\mathrm{d}\theta\otimes\mathrm{d}\theta + \sin^2(\theta)\mathrm{d}\phi\otimes\mathrm{d}\phi$$
$$+ (\mathrm{d}\psi + \cos(\theta)\mathrm{d}\phi)\otimes(\mathrm{d}\psi + \cos(\theta)\mathrm{d}\phi)\big)$$
$$+ \mathrm{e}^{-4r}\Bigg(-\frac{1}{2}(3Q - (J^2+3)P)\mathrm{d}t\otimes\mathrm{d}t + \frac{1}{8}((1-j^2)P - Q)(\mathrm{d}\theta\otimes\mathrm{d}\theta + \sin^2(\theta)\mathrm{d}\phi\otimes\mathrm{d}\phi)$$
$$+ \frac{1}{8}((1+3j^2)P - Q)(\mathrm{d}\psi + \cos(\theta)\mathrm{d}\phi)\otimes(\mathrm{d}\psi + \cos(\theta)\mathrm{d}\phi)$$
$$- \frac{1}{2}j(2P-Q)(\mathrm{d}t\otimes(\mathrm{d}\psi + \cos(\theta)\mathrm{d}\phi) + (\mathrm{d}\psi + \cos(\theta)\mathrm{d}\phi)\otimes\mathrm{d}t)\Bigg)\Bigg) + \mathrm{d}r\otimes\mathrm{d}r. \quad (4.364)$$

Therefore the metric is indeed asymptotically AdS; it has

$$f_{(0)} = \bar{f}_{(0)} = -\mathrm{d}t \otimes \mathrm{d}t + \frac{1}{4}(\mathrm{d}\psi + \cos(\theta)\mathrm{d}\phi) \otimes (\mathrm{d}\psi + \cos(\theta)\mathrm{d}\phi)$$
$$+ \frac{1}{4}(\mathrm{d}\theta \otimes \mathrm{d}\theta + \sin^2(\theta)\mathrm{d}\phi \otimes \mathrm{d}\phi) \text{ and} \tag{4.365}$$
$$f_{(4)} - \bar{f}_{(4)} = -\frac{1}{2}(3Q - (J^2+3)P)\mathrm{d}t \otimes \mathrm{d}t + \frac{1}{8}((1-j^2)P - Q)(\mathrm{d}\theta \otimes \mathrm{d}\theta + \sin^2(\theta)\mathrm{d}\phi \otimes \mathrm{d}\phi)$$
$$+ \frac{1}{8}((1+3j^2)P - Q)(\mathrm{d}\psi + \cos(\theta)\mathrm{d}\phi) \otimes (\mathrm{d}\psi + \cos(\theta)\mathrm{d}\phi)$$
$$- \frac{1}{2}j(2P - Q)(\mathrm{d}t \otimes (\mathrm{d}\psi + \cos(\theta)\mathrm{d}\phi) + (\mathrm{d}\psi + \cos(\theta)\mathrm{d}\phi) \otimes \mathrm{d}t). \tag{4.366}$$

These are the same form as equations 4.340 and 4.341. Analysing them the same way yields

$$E = \frac{\pi}{4}((j^2+3)P - 3Q), \quad P_I = K_1 = 0 \text{ and} \tag{4.367}$$
$$J_{IJ} \equiv \frac{\pi j(2P - Q)}{4} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{4.368}$$

My result for E matches the result calculated in [77] using the completely different methods of [4], while the angular momentum matches up to a convention-dependent factor of -2. The only remaining quantity in equation 4.302 is the electric charge. For that,

$$F = \mathrm{d}A = \frac{Q\sqrt{3}}{R^3}\mathrm{d}R \wedge \left(\mathrm{d}t - \frac{j}{2}(\mathrm{d}\psi + \cos(\theta)\mathrm{d}\phi)\right) - \frac{jQ\sqrt{3}}{4R^2}\sin(\theta)\mathrm{d}\theta \wedge \mathrm{d}\phi \tag{4.369}$$
$$\to Q\sqrt{3}\mathrm{e}^{-2r}\mathrm{d}r \wedge \left(\mathrm{d}t - \frac{j}{2}(\mathrm{d}\psi + \cos(\theta)\mathrm{d}\phi)\right) - \frac{jQ\sqrt{3}}{4}\mathrm{e}^{-2r}\sin(\theta)\mathrm{d}\theta \wedge \mathrm{d}\phi \tag{4.370}$$
$$\implies E_1 = F_{10} = Q\sqrt{3}\mathrm{e}^{-3r} \text{ and therefore} \tag{4.371}$$
$$q_e = \frac{1}{4\pi}\int_{S^3_\infty} E_1 \mathrm{d}A = \frac{\pi Q\sqrt{3}}{2}. \tag{4.372}$$

Substituting all these quantities into equation 4.302 implies

$$\frac{\pi}{4}((j^2+3)P - 3Q)I + \frac{\mathrm{i}\pi j(2P-Q)}{4}\gamma^0\left(\gamma^2\gamma^1 + \gamma^4\gamma^3\right) - \frac{3\pi Q}{4}\gamma^0 \tag{4.373}$$

is non-negative definite. The eigenvalues of this matrix are

$$\frac{\pi}{4}((j^2+3)P - 3Q) + \frac{3\pi Q}{4}, \quad \frac{\pi}{4}((j^2+3)P - 3Q) - \frac{3\pi Q}{4} + \frac{\pi j(2P-Q)}{2}$$
$$\text{and } \frac{\pi}{4}((j^2+3)P - 3Q) - \frac{3\pi Q}{4} - \frac{\pi j(2P-Q)}{2}. \tag{4.374}$$

Which of these is the lowest eigenvalue depends on the choices of j, P and Q. Nonetheless, they all have to be non-negative. Therefore,

$$\frac{\pi}{4}((j^2+3)P - 3Q) + \frac{3\pi Q}{4} \geq 0 \iff P \geq 0, \tag{4.375}$$
$$\frac{\pi}{4}((j^2+3)P - 3Q) - \frac{3\pi Q}{4} + \frac{\pi j(2P-Q)}{2} \geq 0 \iff P \geq \frac{2Q}{j+1} \text{ and} \tag{4.376}$$
$$\frac{\pi}{4}((j^2+3)P - 3Q) - \frac{3\pi Q}{4} - \frac{\pi j(2P-Q)}{2} \geq 0 \iff P \geq -\frac{2Q}{j-1}. \tag{4.377}$$

From [84], each inequality is saturated by a known supersymmetric solution. In particular, $P = 0$ is the Klemm-Sabra solution [75] and $P = \pm\frac{2Q}{j\pm 1}$ are the Gutowski-Reall solutions [55] with their $\epsilon = \mp 1$ respectively.

As an extension to this example, consider a $\mathbb{Z}_p$ quotient of each constant t and R sphere so that the asymptotic end becomes $\mathbb{R}^2 \times L(p,1)$ instead of $\mathbb{R}^2 \times S^3$. Note that the metric components have no explicit dependence on ψ and hence this quotient is well-defined from that point of view[12]. Furthermore, since Σ_t only extends up to the event horizon, any orbifold singularities or other issues which may arise from the quotient are shielded. Therefore, theorem 4.21 should apply.

From equations 4.182 to 4.191, 4.181 & 4.368, definitions 4.11 & 4.16 and the fact the components in equations 4.365 & 4.366 don't depend on ψ or ϕ, the angular momenta on the lens space are related to the angular momenta on the sphere by

$$\begin{aligned} J_1^{L(p,1)} &= \frac{1}{p}\left(J_{12}^{S^3} + J_{34}^{S^3}\right) = -\frac{\pi j(2P-Q)}{2p}, \; J_2^{L(p,1)} = \frac{1}{p}\left(J_{12}^{S^3} - J_{34}^{S^3}\right) = 0, \\ J_3^{L(p,1)} &= \frac{1}{p}\left(J_{24}^{S^3} + J_{13}^{S^3}\right) = 0 \text{ and } J_4^{L(p,1)} = \frac{1}{p}\left(J_{14}^{S^3} - J_{23}^{S^3}\right) = 0. \end{aligned} \tag{4.378}$$

Hence, theorem 4.21 reduces to saying

$$\frac{\pi}{4p}((j^2+3)P - 3Q) + \frac{3\pi Q}{4p} + 0 \geq 0 \iff P \geq 0. \tag{4.379}$$

This time, saturating the theorem only yields the (quotiented) Klemm-Sabra solution, even though both the Klemm-Sabra and Gutowski-Reall solutions have well-defined metrics on the domains of outer communication after the quotient. The obstruction instead comes from the Killing spinors, i.e. the solutions to $\nabla_a \varepsilon = 0$, not surviving the quotient. This is much the same as lemma 4.15, where the quotient removed half the Killing spinors on the sphere.

From equations 5.2 and 4.2 of [75], the Killing spinors for the Klemm-Sabra solution are independent of ψ and are therefore unaffected by the quotient. Thus, the $\mathbb{Z}_p$ quotient of the Klemm-Sabra solution remains a supersymmetric solution - this time with $L(p,1)$ cross-sections - as shown by the application of theorem 4.21 above. On the other hand, setting $a = b$ in equation A.12 of [18] gives the Killing spinors of the Gutowski-Reall solution. Since these Killing spinors have an overall $\mathrm{e}^{\mathrm{i}\psi/2}$ factor, they do not survive the quotient. Hence, although the metric is well-defined after the quotient, it is no longer supersymmetric.

[12]From equations 4.322 and 4.177, the $\mathbb{Z}_p$ quotient acts by $\psi \sim \psi + \frac{4\pi}{p}$ in these coordinates. A tensor or spinor must be invariant under this identification to survive the quotient.

Chapter 5

Quasilocal mass for spacetimes with negative cosmological constant

"Roads?" Where we're going, we don't need "roads."
- Dr. Emmett Brown in *Back to the Future*

In this chapter, I leave behind the well-driven road of asymptotics and global definitions of energy to embark on a journey towards quasilocal mass. The main result of this chapter will be a new definition of quasilocal mass for spacetimes with negative cosmological constant based on the Dougan-Mason definition for $\Lambda = 0$. I will subsequently prove that my definition satisfies a number of physically desirable properties. $\mathcal{A}_a$ is set to zero throughout this chapter.

5.1 New quasilocal mass and its positivity

The set-up for this chapter is the same as section 2.3.2 and figure 2.1. In particular, let S be a compact, closed, 2D spacelike surface in a 4D spacetime with negative cosmological constant. Let $\{P, Q, X, Y\}$ be orthonormal with X & Y tangent to S, P being a future-directed, timelike normal to S and Q a spacelike normal to S. Let Σ denote a 3D, spacelike region bounded by S with Q tangent to Σ. Given this set-up and the use of spinors, it will be very natural to use the GHP formalism adapted to S, as reviewed in section 2.1. The main task in this chapter is to define a geometric invariant, $m(S)$, which quantifies the mass within S.

A key element of the construction to follow will be Dirac spinors, $\Phi = (\varphi_\alpha,\, \bar{\xi}^{\dot{\alpha}})^T$, satisfying $\overline{m}^a \nabla_a \Phi = 0$ on S. A basis for the solution space will be denoted $\{\Phi^A = (\varphi^A_\alpha,\, \bar{\xi}^{A\dot{\alpha}})^T\}$ and $A, B, \ldots$ will be indices on this space[1].

Lemma 5.1. *Applying the GHP formalism and NP coefficients, $\overline{m}^a \nabla_a \Phi = 0$ is equivalent to*

$$0 = \bar{\eth}\varphi_o + \mu\varphi_\iota - \mathrm{i}k\sqrt{2}\,\bar{\xi}_o, \tag{5.1}$$

$$0 = \bar{\eth}\bar{\xi}_\iota - \rho\bar{\xi}_o - \mathrm{i}k\sqrt{2}\,\varphi_\iota, \tag{5.2}$$

$$0 = \bar{\eth}\varphi_\iota - \bar{\sigma}\varphi_o \text{ and} \tag{5.3}$$

$$0 = \bar{\eth}\bar{\xi}_o + \lambda\bar{\xi}_\iota. \tag{5.4}$$

Proof. In terms of two component spinors,

$$\overline{m}^a \nabla_a \Phi = \begin{bmatrix} \overline{m}^a D_a \varphi_\alpha \\ \overline{m}^a D_a \bar{\xi}^{\dot{\alpha}} \end{bmatrix} + \mathrm{i}k\overline{m}^a \begin{bmatrix} 0 & (\sigma_a)_{\alpha\dot{\alpha}} \\ (\tilde{\sigma}_a)^{\dot{\alpha}\alpha} & 0 \end{bmatrix} \begin{bmatrix} \varphi_\alpha \\ \bar{\xi}^{\dot{\alpha}} \end{bmatrix} = \begin{bmatrix} \bar{\delta}\varphi_\alpha - \mathrm{i}k\sqrt{2}\,\bar{\xi}_o o_\alpha \\ \bar{\delta}\bar{\xi}^{\dot{\alpha}} - \mathrm{i}k\sqrt{2}\varphi_\iota \bar{\iota}^{\dot{\alpha}} \end{bmatrix}. \tag{5.5}$$

[1] Note that $A, B, \ldots$ are not vielbein indices running 2, 3 in this context. More fundamentally, this labeling implicitly assumes the solution space has countable dimension. As I will explain later, this is expected to be the case for generic S.

Contracting the top half with o^α, applying the NP coefficients from appendix B and equations 2.65 & 2.66 yields

$$0 = o^\alpha \left(\bar{\delta}\varphi_\alpha - \mathrm{i}k\sqrt{2}\,\bar{\xi}_o o_\alpha\right) \tag{5.6}$$

$$= o^\alpha \left(\bar{\delta}\left(\varphi_o o_\alpha + \varphi_\iota \iota_\alpha\right) - \mathrm{i}k\sqrt{2}\,\bar{\xi}_o o_\alpha\right) \tag{5.7}$$

$$= 0 + \varphi_o o^\alpha \bar{\delta} o_\alpha - \sqrt{2}\bar{\delta}\varphi_\iota + \varphi_\iota o^\alpha \bar{\delta}\iota_\alpha - 0 \tag{5.8}$$

$$= \sqrt{2}\bar{\sigma}\varphi_o - \sqrt{2}\bar{\delta}\varphi_\iota + \sqrt{2}\varphi_\iota\bar{\beta} \tag{5.9}$$

$$= \sqrt{2}\left(\bar{\sigma}\varphi_o - \bar{\eth}\varphi_\iota\right), \tag{5.10}$$

which proves equation 5.3. Similarly,

$$0 = \iota^\alpha \left(\bar{\delta}\varphi_\alpha - \mathrm{i}k\sqrt{2}\,\bar{\xi}_o o_\alpha\right) = \sqrt{2}\left(\bar{\eth}\varphi_o + \mu\varphi_\iota - \mathrm{i}k\sqrt{2}\,\bar{\xi}_o\right), \tag{5.11}$$

$$0 = \bar{o}_{\dot{\alpha}} \left(\delta\bar{\xi}^{\dot{\alpha}} - \mathrm{i}k\sqrt{2}\varphi_\iota\bar{\iota}^{\dot{\alpha}}\right) = \sqrt{2}\left(\bar{\eth}\bar{\xi}_\iota - \rho\bar{\xi}_o - \mathrm{i}k\sqrt{2}\varphi_\iota\right) \text{ and} \tag{5.12}$$

$$0 = \bar{\iota}_{\dot{\alpha}} \left(\delta\bar{\xi}^{\dot{\alpha}} - \mathrm{i}k\sqrt{2}\varphi_\iota\bar{\iota}^{\dot{\alpha}}\right) = -\sqrt{2}\left(\bar{\eth}\bar{\xi}_o + \lambda\bar{\xi}_\iota\right) \tag{5.13}$$

prove the remaining three equations. □

Definition 5.2 (Q^{AB})**.** *Define the hermitian matrix, Q^{AB}, by*

$$Q^{AB} = 4\int_S \Big(\rho\overline{\varphi}_o^A\varphi_o^B + \mu\xi_\iota^A\overline{\xi}_\iota^B - \rho\xi_o^A\overline{\xi}_o^B - \mu\overline{\varphi}_\iota^A\varphi_\iota^B + \mathrm{i}k\sqrt{2}\big(\xi_\iota^A\varphi_o^B - \overline{\varphi}_o^A\overline{\xi}_\iota^B - \xi_o^A\varphi_\iota^B + \overline{\varphi}_\iota^A\overline{\xi}_o^B\big)\Big)\mathrm{d}A. \tag{5.14}$$

Theorem 5.3. *If the dominant energy condition holds on Σ and the null expansions on S satisfy $\theta_l > 0$, $\theta_n < 0$ & $\theta_l\theta_n < -8k^2$, then Q^{AB} is a non-negative definite matrix.*

Proof. From lemma 3.4,

$$Q(\Phi) = 4\int_S \Big(\varphi_\iota\eth\overline{\varphi}_o + \overline{\varphi}_\iota\bar{\eth}\varphi_o - \overline{\xi}_o\eth\xi_\iota - \xi_o\bar{\eth}\overline{\xi}_\iota + \rho|\varphi_o|^2 + \mu|\varphi_\iota|^2 + \rho|\xi_o|^2 + \mu|\xi_\iota|^2 + \mathrm{i}k\sqrt{2}\big(\varphi_o\xi_\iota + \varphi_\iota\xi_o - \overline{\varphi}_o\overline{\xi}_\iota - \overline{\varphi}_\iota\overline{\xi}_o\big)\Big)\mathrm{d}A. \tag{5.15}$$

From equations 5.1 and 5.2, this reduces to

$$Q(\Phi) = 4\int_S \Big(\rho|\varphi_o|^2 - \mu|\varphi_\iota|^2 - \rho|\xi_o|^2 + \mu|\xi_\iota|^2 + \mathrm{i}k\sqrt{2}\big(\varphi_o\xi_\iota - \xi_o\varphi_\iota - \overline{\varphi}_o\overline{\xi}_\iota + \overline{\xi}_o\overline{\varphi}_\iota\big)\Big)\mathrm{d}A. \tag{5.16}$$

Let $\mathcal{Z} = (\phi_\alpha, \bar{\zeta}^{\dot{\alpha}})^T$ be any Dirac spinor on Σ with sufficient regularity so that $\gamma^I\nabla_I\mathcal{Z} \in L^2$. Furthermore, choose $\mathcal{Z}$ to have $\phi_o = \varphi_o$ and $\zeta_\iota = \xi_\iota$ on S. Therefore, by theorem 3.12, $\exists\Psi' \in \mathcal{H}$ such that $\mathfrak{D}(\Psi') = -\gamma^I\nabla_I\mathcal{Z}$. Thus $\Psi = \Psi' + \mathcal{Z}$ satisfies $\gamma^I\nabla_I\Psi = 0$ and by corollary 3.2.1,

$$Q(\Psi) = \int_\Sigma \left(\nabla_I(\Psi)^\dagger\nabla^I\Psi - 4\pi T^{0a}\overline{\Psi}\gamma^a\Psi\right)\mathrm{d}V \geq 0, \tag{5.17}$$

where the first term is manifestly non-negative and the second term is non-negative by the dominant energy condition. Furthermore, since every element, $\Psi' \in \mathcal{H}$, has $\psi'_o = \chi'_\iota = 0$ on S by construction, it follows that Ψ has $\psi_o = \varphi_o$ and $\chi_\iota = \xi_\iota$ on S.

Therefore, by lemma 3.4 and the fact all the derivatives in the lemma are tangent to S, $Q(\Psi)$ can also be written as

$$Q(\Psi) = 4\int_S \Big(\psi_\iota \eth\overline{\varphi}_o + \overline{\psi}_\iota \bar{\eth}\varphi_o - \overline{\chi}_o \eth\xi_\iota - \chi_o \bar{\eth}\overline{\xi}_\iota + \rho|\varphi_o|^2 + \mu|\psi_\iota|^2 + \rho|\chi_o|^2 + \mu|\xi_\iota|^2 + \mathrm{i}k\sqrt{2}\big(\varphi_o\xi_\iota + \psi_\iota\chi_o - \overline{\varphi}_o\overline{\xi}_\iota - \overline{\psi}_\iota\overline{\chi}_o\big)\Big)\mathrm{d}A. \tag{5.18}$$

Then, from equations 5.1 and 5.2,

$$Q(\Psi) = 4\int_S \Big(\mu\big(-\psi_\iota\overline{\varphi}_\iota - \overline{\psi}_\iota\varphi_\iota + |\psi_\iota|^2 + |\xi_\iota|^2\big) + \rho\big(-\overline{\chi}_o\xi_o - \chi_o\overline{\xi}_o + |\varphi_o|^2 + |\chi_o|^2\big) + \mathrm{i}k\sqrt{2}\big(-\psi_\iota\xi_o + \overline{\psi}_\iota\overline{\xi}_o + \overline{\chi}_o\overline{\varphi}_\iota - \chi_o\varphi_\iota + \varphi_o\xi_\iota + \psi_\iota\chi_o - \overline{\varphi}_o\overline{\xi}_\iota - \overline{\psi}_\iota\overline{\chi}_o\big)\Big)\mathrm{d}A. \tag{5.19}$$

Therefore re-writing equation 5.16 in terms of $Q(\Psi)$ yields

$$Q(\Phi) = 4\int_S \Big(-\mu\big(|\varphi_\iota|^2 - \psi_\iota\overline{\varphi}_\iota - \overline{\psi}_\iota\varphi_\iota + |\psi_\iota|^2\big) - \rho\big(|\xi_o|^2 - \overline{\chi}_o\xi_o - \chi_o\overline{\xi}_o + |\chi_o|^2\big) - \mathrm{i}k\sqrt{2}\big(-\psi_\iota\xi_o + \overline{\psi}_\iota\overline{\xi}_o + \overline{\chi}_o\overline{\varphi}_\iota - \chi_o\varphi_\iota + \psi_\iota\chi_o - \overline{\psi}_\iota\overline{\chi}_o + \xi_o\varphi_\iota - \overline{\xi}_o\overline{\varphi}_\iota\big)\Big)\mathrm{d}A + Q(\Psi) \tag{5.20}$$

$$= 4\int_S \Big(-\mathrm{i}k\sqrt{2}\big((\xi_o - \chi_o)(\varphi_\iota - \psi_\iota) - (\overline{\xi}_o - \overline{\chi}_o)(\overline{\varphi}_\iota - \overline{\psi}_\iota)\big) - \mu|\varphi_\iota - \psi_\iota|^2 - \rho|\xi_o - \chi_o|^2\Big)\mathrm{d}A + Q(\Psi). \tag{5.21}$$

As done previously in section 3.2, let $\mu' = \mu/|z|^2$, $\rho' = |z|^2\rho$, $\xi_o' = \xi_o/z$, $\chi_o' = \chi_o/z$, $\varphi_\iota' = z\varphi_\iota$ and $\psi_\iota' = z\psi_\iota$. Again, choose $z = \sqrt[4]{\mu/\rho}$ so that $\mu' = \rho' = -\sqrt{\mu\rho} = -\frac{1}{2}\sqrt{-\theta_l\theta_n} < -k\sqrt{2}$. Hence,

$$Q(\Phi) = 4\int_S \Big(-\mathrm{i}k\sqrt{2}\big((\xi_o' - \chi_o')(\varphi_\iota' - \psi_\iota') - (\overline{\xi}_o' - \overline{\chi}_o')(\overline{\varphi}_\iota' - \overline{\psi}_\iota')\big) - \mu'|\varphi_\iota' - \psi_\iota'|^2 - \rho'|\xi_o' - \chi_o'|^2\Big)\mathrm{d}A + Q(\Psi) \tag{5.22}$$

$$= 4\int_S \Big(\sqrt{2}k|\xi_o' - \chi_o' + \mathrm{i}\overline{\varphi}_\iota' - \mathrm{i}\overline{\psi}_\iota'|^2 - (\mu' + \sqrt{2}k)|\varphi_\iota' - \psi_\iota'|^2 - (\rho' + \sqrt{2}k)|\xi_o' - \chi_o'|^2\Big)\mathrm{d}A + Q(\Psi) \tag{5.23}$$

$$\geq 0. \tag{5.24}$$

Since $\{\Phi^A\}$ is a basis for the solution space to $\overline{m}^a\nabla_a\Phi = 0$, I can let $\Phi = c_A\Phi^A$ for any constants, c_A. Hence, $\varphi_o = c_A\varphi_o^A$, $\varphi_\iota = c_A\varphi_\iota^A$, $\overline{\xi}_o = c_A\overline{\xi}_o^A$ and $\overline{\xi}_\iota = c_A\overline{\xi}_\iota^A$. Finally, definition 5.2, equation 5.16 and equation 5.24 imply

$$0 \leq Q(\Phi) = \bar{c}_A Q^{AB} c_B. \tag{5.25}$$

Since c_A are arbitrary, it must be that Q^{AB} is non-negative definite. □

While this theorem achieves a manifestly non-negative object, some auxiliary constructions are still required to extract a mass from Q^{AB}.

Definition 5.4 (T^{AB})**.** *Define the matrix, T^{AB}, by*

$$T^{AB} = \varepsilon^{\alpha\beta}\varphi_\alpha^A\varphi_\beta^B - \varepsilon^{\dot{\alpha}\dot{\beta}}\bar{\xi}_{\dot{\alpha}}^A\bar{\xi}_{\dot{\beta}}^B = \sqrt{2}\left(\varphi_o^A\varphi_\iota^B - \varphi_o^B\varphi_\iota^A + \overline{\xi}_o^B\overline{\xi}_\iota^A - \overline{\xi}_o^A\overline{\xi}_\iota^B\right). \tag{5.26}$$

The notion of a surface, S, being "generic" can now finally be stated precisely.

Definition 5.5 (Generic - T^{AB} form)**.** *The surface, S, is said to be generic if and only if T^{AB} is invertible.*

Definition 5.6 (Generic - Φ^A form)**.** *The surface, S, is said to be generic if and only if the solution space to $\overline{m}^a\nabla_a\Phi = 0$ on S is four (complex) dimensional and the basis, $\{\Phi^A\}_{A=1}^4$, is pointwise linearly independent at least at one point of S.*

I will show later that the Φ^A version of generic implies the T^{AB} version, although only the T^{AB} version will be needed for defining $m(S)$. More importantly though, surfaces generic in name should be generic in practice too. For the T^{AB} form, it could be argued that since the set of singular $n \times n$ matrices are measure zero in the set of all $n \times n$ matrices, this is indeed a valid notion of generic. However, it's not obvious the solution space is finite dimensional and this argument doesn't consider the possibility there is something specific to this situation precluding T^{AB}'s invertibility. Furthermore, the examples considered in sections 5.2 and 5.3 either satisfy both notions of generic or neither notion of generic. Hence, it's unclear whether $m(S)$ constructed on a surface satisfying the T^{AB} form, but not the Φ^A form, of generic has physical meaning beyond simple mathematical validity. Finally, from a practical point of view, one would like to know what size of matrix to expect for T^{AB} - and for that matter, Q^{AB}. As defined so far, they could be of arbitrarily large size, maybe even infinitely large. Fortunately, at least for topologically spherical S, there are reasons to believe the Φ^A form is also a valid notion of generic, implying T^{AB} is only a 4×4 matrix.

It is known - e.g. from section 8.2.2 of [112] - that $\bar{\delta}$ is an elliptic operator and the compactness of S then guarantees $\bar{\delta}$ has finite dimensional kernel. Then, it is also known [112] that $\bar{\delta}$'s index (dimension of kernel minus dimension of cokernel) is $4(1-g)$ when S has genus, g. The difference between $\overline{m}^a\nabla_a$ - the actual operator of interest - and $\bar{\delta}$ is $\mathrm{i}k\overline{m}^a\gamma_a$, which is a compact operator since S is compact and $\mathrm{i}k\overline{m}^a\gamma_a$ is just a smooth, 4×4 matrix[2]. Therefore by Fredholm theory, $\mathrm{index}(\overline{m}^a\nabla_a) = \mathrm{index}(\bar{\delta}) = 4(1-g)$. Thus, if S is diffeomorphic to a sphere, then $\overline{m}^a\nabla_a\Phi = 0$ must have at least four linearly independent solutions.

In the spherical examples of sections 5.2.1 and 5.3, there happen to be precisely four linearly independent solutions. From a similar situation, Penrose then argues [99] as long as S is not too far from "canonical" situations - such as the examples to be considered - there would still remain precisely four linearly independent solutions. At least for spherical S, this justifies the first half of the generic definition in Φ^A form. For non-spherical S, the situation is far less constrained and I can't immediately say whether either definition of generic is actually realistic. In section 5.2.2 I study two examples with toroidal S. In the first, it will be shown $\overline{m}^a\nabla_a\Phi = 0$ has two linearly independent solutions and the corresponding T^{AB} is just zero, while in the second, $\overline{m}^a\nabla_a\Phi = 0$ won't even have a single non-zero solution. Hence both definitions of generic fail. However, the wider implications of those examples are unclear to me.

The second half of definition 5.6 is motivated by a possibility that occurs in the Dougan-Mason definition, where one needs to solve the analogous equation, $\bar{\delta}\varphi_\alpha = 0$. It turns out there exist "exceptional" surfaces - bifurcate Killing surfaces are one example - where there are two solutions to $\bar{\delta}\varphi_\alpha = 0$ (the number expected and desired in that context) which are linearly independent as functions despite being pointwise linearly dependent at every point of S. The Dougan-Mason mass cannot be defined on such surfaces because the analogue of T^{AB} just becomes zero. However, based on considerations of holomorphic spin bundles, Dougan and Mason argue such surfaces really are exceptional and not generic. Similarly, definition 5.6 insists $\{\Phi^A\}_{A=1}^4$ are pointwise linearly independent at least at one point of S for S to be called generic in the Φ^A sense.

[2]Compactness is again essentially due to the Rellich-Kondrachov theorem, as explained in the proof of theorem 3.12 in the non-compact case.

Lemma 5.7. *T^{AB} is antisymmetric and constant on S. Furthermore, the notion of generic in definition 5.6 implies the notion of generic in definition 5.5.*

Proof. Antisymmetry follows directly from the definition. Next, observe that by equation 5.5,

$$\begin{aligned}\bar{\delta}T^{AB} &= \varepsilon^{\alpha\beta}\bar{\delta}(\varphi^A_\alpha)\varphi^B_\beta + \varepsilon^{\alpha\beta}\varphi^A_\alpha\bar{\delta}\varphi^B_\beta - \varepsilon^{\dot{\alpha}\dot{\beta}}\bar{\delta}(\bar{\xi}^A_{\dot{\alpha}})\bar{\xi}^B_{\dot{\beta}} - \varepsilon^{\dot{\alpha}\dot{\beta}}\bar{\xi}^A_{\dot{\alpha}}\bar{\delta}\bar{\xi}^B_{\dot{\beta}} &(5.27)\\ &= \varepsilon^{\alpha\beta}\mathrm{i}k\sqrt{2}\bar{\xi}^A_o o_\alpha\varphi^B_\beta + \varepsilon^{\alpha\beta}\varphi^A_\alpha\mathrm{i}k\sqrt{2}\bar{\xi}^B_o o_\beta - \varepsilon^{\dot{\alpha}\dot{\beta}}\mathrm{i}k\sqrt{2}\varphi^A_\iota\bar{\iota}_{\dot{\alpha}}\bar{\xi}^B_{\dot{\beta}} - \varepsilon^{\dot{\alpha}\dot{\beta}}\bar{\xi}^A_{\dot{\alpha}}\mathrm{i}k\sqrt{2}\varphi^B_\iota\bar{\iota}_{\dot{\beta}} &(5.28)\\ &= 2\mathrm{i}k\left(\bar{\xi}^A_o\varphi^B_\iota - \bar{\xi}^B_o\varphi^A_\iota + \varphi^A_\iota\bar{\xi}^B_o - \varphi^B_\iota\bar{\xi}^A_o\right) &(5.29)\\ &= 0. &(5.30)\end{aligned}$$

Therefore for each A and B, T^{AB} is a holomorphic function on S. Then, since S is compact, Liouville's theorem implies T^{AB} is constant on S.

To prove invertibility, it's easier to work in Dirac spinor notation. With the charge conjugation matrix given in appendix A, observe that

$$(\Phi^A)^T C^{-1}\Phi^B = \begin{bmatrix}\varphi^A_\alpha & \bar{\xi}^{A\dot{\alpha}}\end{bmatrix}\begin{bmatrix}\varepsilon^{\alpha\beta} & 0\\ 0 & \varepsilon_{\dot{\alpha}\dot{\beta}}\end{bmatrix}\begin{bmatrix}\varphi^B_\beta\\ \bar{\xi}^{B\dot{\beta}}\end{bmatrix} = \varphi^A_\alpha\varepsilon^{\alpha\beta}\varphi^B_\beta + \bar{\xi}^{A\dot{\alpha}}\varepsilon_{\dot{\alpha}\dot{\beta}}\bar{\xi}^{B\dot{\beta}} = T^{AB}. \qquad (5.31)$$

Let v_A be a vector in the nullspace of T^{AB}, i.e. $T^{AB}v_B = 0$. Let $\mathcal{Z} = v_A\Phi^A$. Then,

$$T^{AB}v_B = 0 \iff (\Phi^A)^T C^{-1}\mathcal{Z} = 0 \iff w_A(\Phi^A)^T C^{-1}\mathcal{Z} = 0 \qquad (5.32)$$

for any vector, w_A. Definition 5.6 says there are four different Φ^A and they are pointwise linearly independent at least at one point, say p, on S. Since Dirac spinors also have four components, $\{\Phi^A\}_{A=1}^4$ must form a pointwise basis at p. Hence, $w_A\Phi^A$ can be any Dirac spinor at p, which then implies $C^{-1}\mathcal{Z}|_p = 0$. But C^{-1} is invertible, so it must be that $\mathcal{Z}|_p = 0$. However, then $v_A = 0$ by the linear independence of $\{\Phi^A\}_{A=1}^4$ at p, which is equivalent to T^{AB} having trivial nullspace. ☐

Before T^{AB} can be put to use in extracting information from Q^{AB}, one more auxiliary result is required.

Lemma 5.8. *For any non-negative definite, hermitian matrix, H, and antisymmetric matrix, A, $\mathrm{tr}(HA\overline{H}\bar{A})$ is real and $\mathrm{tr}(HA\overline{H}\bar{A}) \leq 0$.*

Proof. In this proof I will write all indices downstairs and all summations explicitly. Then,

$$\mathrm{tr}(HA\overline{H}\bar{A}) = \sum_{A,B,C,D} H_{AB}A_{BC}\overline{H}_{CD}\bar{A}_{DA} = -\sum_{A,B,C,D} H_{AB}A_{BC}\overline{H}_{CD}\bar{A}_{AD} \qquad (5.33)$$

is non-positive because H is non-negative definite, H_{AB} acts like a metric on the first index of A and $\overline{H}_{CD}$ acts like a metric on the second of A, i.e. the expression is just the (negative of the) norm of A with respect to the metric, H. The result is real because the cyclic property of the trace implies $\overline{\mathrm{tr}(HA\overline{H}\bar{A})} = \mathrm{tr}(\overline{H}\bar{A}HA) = \mathrm{tr}(HA\overline{H}\bar{A})$.

As an alternative proof, first note that every hermitian matrix is orthogonally diagonalisable and has real eigenvalues. Therefore, $\exists$ vectors, $\{v_{(A)}\}$, such that $v^\dagger_{(A)}v_{(B)} = \delta_{AB}$, $Hv_{(A)} = \lambda_A v_{(A)}$ for some $\lambda_A \in \mathbb{R}$ and $U_{AB} = v_{(B)A}$ is the change of basis matrix that leads to diagonalisation. Then, it can be checked that

$$\mathrm{tr}(HA\overline{H}\bar{A}) = -\sum_{A,B}\lambda_A\lambda_B|v^\dagger_{(A)}A\bar{v}_{(B)}|^2, \qquad (5.34)$$

which is manifestly real and non-positive. ☐

Definition 5.9 (Quasilocal mass). *Suppose the dominant energy condition holds on Σ, the null expansions on S satisfy $\theta_l > 0$, $\theta_n < 0$ & $\theta_l\theta_n < -8k^2$ and S is generic (either definition). Then, construct Q^{AB} and T^{AB} by definitions 5.2 & 5.4 and define the quasilocal mass, $m(S)$, to be*

$$m(S) = \frac{1}{16\pi}\sqrt{-\mathrm{tr}(QT^{-1}\overline{Q}\overline{T}^{-1})}. \tag{5.35}$$

Theorem 5.3, lemma 5.7 and lemma 5.8 ensure $m(S)$ is well-defined and manifestly non-negative. Furthermore, $m(S)$ is independent of the choice of basis, $\{\Phi^A\}$, as follows. Define $\Phi'^A = B^A{}_B\Phi^B$ for a constant, invertible matrix, B. Then, by definitions 5.2 and 5.4,

$$Q'^{AB} = \bar{B}^A{}_C Q^{CD} B^B{}_D \iff Q' = \bar{B}QB^T \text{ and} \tag{5.36}$$

$$T'^{AB} = B^A{}_C T^{CD} B^B{}_D \iff T' = BTB^T. \tag{5.37}$$

Thus, the object in $m(S)$ transforms as

$$\mathrm{tr}(Q'(T')^{-1}\overline{Q}'(\overline{T}')^{-1}) = \mathrm{tr}(\bar{B}QB^TB^{-T}T^{-1}B^{-1}B\overline{Q}\bar{B}^T\bar{B}^{-T}\overline{T}^{-1}\bar{B}^{-1}) = \mathrm{tr}(QT^{-1}\overline{Q}\overline{T}^{-1}). \tag{5.38}$$

Lemma 5.10. *$m(S) = 0$ for every surface, S, in AdS that is generic in the Φ^A sense.*

Proof. With this notion of generic $\exists$ exactly four linearly independent solutions to $\overline{m}^a\nabla_a\Phi = 0$. However, AdS already has a four dimensional space of Killing spinors, i.e. solutions to $\nabla_a\varepsilon_k = 0$. Therefore, since $\nabla_a\varepsilon_k = 0$ is a stronger condition, one can use the Killing spinors of AdS as $\{\Phi^A\}_{A=1}^4$. Then, $\Phi = \varepsilon_k$ and $\nabla_a\varepsilon_k = 0 \implies E^{ab}(\Phi) = 0 \implies Q(\Phi) = 0 \implies m(S) = 0$. □

It's natural to consider the converse, i.e. study the implications of $m(S) = 0$. This problem is considerably harder even in the context of an asymptotic end - see [61, 60, 62] for recent progress - and I will not consider it in this thesis.

My new definition of quasilocal mass is perhaps closest in spirit to Penrose's definition, albeit there is no need for twistors. In particular, Q^{AB} is analogous to Penrose's "kinematical twistor" - see the material around equation 23 in [99] - while T^{AB} is analogous to his surface "infinity twistor" - see the discussion between equations 25 and 26 in [99]. Meanwhile, my definition is also closely related to the Dougan-Mason mass. When $\Lambda = 0$, the left-handed and right-handed sectors of all the equations decouple, meaning it suffices to simply set the right-handed sector to zero. Then, $A, B, \cdots$ only run $1, 2$. Thus, T^{AB} can be normalised to ε^{AB} and one can use it to manipulate two-component spinors with Q^{AB} now viewed as $P^{\dot{A}A}$, a 4-momentum converted to two-component spinors. Then, my definition reduces to

$$-256\pi^2 m(S)^2 = \mathrm{tr}(QT^{-1}\overline{Q}\overline{T}^{-1}) = Q^{AB}T_{BC}\overline{Q}^{CD}\overline{T}_{DA} \equiv P^{\dot{A}A}\varepsilon_{AB}P^{B\dot{B}}\varepsilon_{\dot{B}\dot{A}}, \tag{5.39}$$

which is the Dougan-Mason mass (up to normalisation). However, since Dougan and Mason have a full energy-momentum vector, $P^{\dot{A}A}$, they are able to further decompose $m(S)$ into a quasilocal energy and quasilocal linear momentum. This decomposition is lost in my definition - as it is in Penrose's definition when S is away from $\mathcal{I}$. While the technical reason is simply that $A, B, \cdots$ run over four indices, instead of two, a more physical reason could be the difference between the Casimir operators of $\mathfrak{o}(3,2)$ and $\mathfrak{iso}(3,1)$, as I will discuss further at the end of section 5.3.

5.2 Highly symmetric examples

For an arbitrary surface, S, the quasilocal mass of definition 5.9 will likely be very difficult, if not impossible, to calculate analytically. However, if the surface has a high degree of symmetry, then more progress can be made. In section 5.2.1 I will focus on spherically symmetric spacetimes, where I will show definition 5.9 reduces to the Misner-Sharp mass[3] [89] of such

[3]The Misner-Sharp mass is usually taken as the standard mass for spherically symmetric spacetimes [112].

spacetimes[4]. Likewise, section 5.2.2 explores two examples with toroidal symmetry, where it will turn out that a number of assumptions required for definition 5.9 don't hold. The canonical examples of spacetimes with such high symmetry are the Schwarzschild spacetime and its variations, described by the metric,

$$g = -\left(c - \frac{2M}{r} + 4k^2r^2\right) \mathrm{d}t \otimes \mathrm{d}t + \frac{\mathrm{d}r \otimes \mathrm{d}r}{c - 2M/r + 4k^2r^2} + r^2 g^{(c)}, \tag{5.40}$$

where $c = 1$, 0 or -1 and $g^{(c)}$ is the standard metric on the round 2-sphere, the 2-torus or a compactified 2D hyperbolic space respectively.

5.2.1 Spherical symmetry

Given there is a heavy reliance on null normals in the NP and GHP formalisms, it will be be easiest to study a general, spherically symmetric spacetime by deploying double null coordinates. In particular, for any spherically symmetric spacetime, let r be the area-radius function and let u & v be null coordinates normal to the symmetry spheres, S^2_r. Then, in such "double null" coordinates, spherical symmetry dictates the metric is

$$g = -\Omega(u,v)^2(\mathrm{d}u \otimes \mathrm{d}v + \mathrm{d}v \otimes \mathrm{d}u) + r(u,v)^2 g_{S^2}, \tag{5.41}$$

for some function, $\Omega(u,v)$. Without loss of generality assume u is outgoing and v is ingoing, i.e. $\partial_u r > 0$ and $\partial_v r < 0$.

For any S^2_r in this spacetime, a natural NP tetrad is

$$l = \frac{1}{\Omega}\frac{\partial}{\partial u},\ n = \frac{1}{\Omega}\frac{\partial}{\partial v} \text{ and } m = \frac{1}{r\sqrt{2}}\left(\frac{\partial}{\partial \theta} + \frac{\mathrm{i}}{\sin(\theta)}\frac{\partial}{\partial \phi}\right). \tag{5.42}$$

For this tetrad, it can be checked by directly evaluating any necessary Christoffel symbols that

$$\sigma = \lambda = 0,\ \rho = -\frac{\partial_u r}{\Omega r},\ \mu = \frac{\partial_v r}{\Omega r} \text{ and } \beta = -\alpha = \frac{1}{2\sqrt{2}r}\cot(\theta). \tag{5.43}$$

Theorem 5.11. *The general solution to $\overline{m}^a \nabla_a \Phi$ on S^2_r is*

$$\varphi_o = -\left(\frac{\sqrt{2}}{\Omega}\partial_v(r)c_3 + 2\mathrm{i}krc_2\right)\left({}_{1/2}Y_{1/2,-1/2}\right) - \left(\frac{\sqrt{2}}{\Omega}\partial_v(r)c_4 - 2\mathrm{i}krc_1\right)\left({}_{1/2}Y_{1/2,1/2}\right), \tag{5.44}$$

$$\xi_\iota = \left(\frac{\sqrt{2}}{\Omega}\partial_u(r)\bar{c}_1 + 2\mathrm{i}kr\bar{c}_4\right)\left({}_{-1/2}Y_{1/2,-1/2}\right) + \left(\frac{\sqrt{2}}{\Omega}\partial_u(r)\bar{c}_2 - 2\mathrm{i}kr\bar{c}_3\right)\left({}_{-1/2}Y_{1/2,1/2}\right), \tag{5.45}$$

$$\xi_o = \bar{c}_1\left({}_{1/2}Y_{1/2,-1/2}\right) + \bar{c}_2\left({}_{1/2}Y_{1/2,1/2}\right) \text{ and} \tag{5.46}$$

$$\varphi_\iota = c_3\left({}_{-1/2}Y_{1/2,-1/2}\right) + c_4\left({}_{-1/2}Y_{1/2,1/2}\right), \tag{5.47}$$

where c_A are arbitrary constants and $({}_sY_{jm})$ are spin-weighted spherical harmonics[5].

Proof. Let $\eth_s$ and $\bar{\eth}_s$ be differential operators that act on functions on the sphere, F, by

$$\eth_s F = s\cot(\theta)F - \left(\frac{\partial}{\partial\theta} + \frac{\mathrm{i}}{\sin(\theta)}\frac{\partial}{\partial\phi}\right)F \text{ and} \tag{5.48}$$

$$\bar{\eth}_s F = -s\cot(\theta)F - \left(\frac{\partial}{\partial\theta} - \frac{\mathrm{i}}{\sin(\theta)}\frac{\partial}{\partial\phi}\right)F. \tag{5.49}$$

[4] While this could appear to be merely a sanity check, in fact it is non-trivial. For example, the Brown-York mass [15] does not agree with the Misner-Sharp mass and in fact produces $m(S^2_r) = r(1 - \sqrt{1 - 2M/r})$ in the Schwarzschild spacetime (with $\Lambda = 0$) despite being physically very well-motivated.

[5] The exact expressions for the four spin-weighted spherical harmonics used are listed in appendix A.

Then, since φ_o & ξ_o are type-$(0,-1)$ and φ_ι & ξ_ι are type-$(0,1)$ in the GHP formalism, the chosen tetrad and the NP coefficients in equation 5.43 imply the equations of lemma 5.1 can be written as (after multiplying by $-r\sqrt{2}$)

$$0 = \bar{\eth}_{1/2}\varphi_o - \frac{\sqrt{2}}{\Omega}\partial_v(r)\varphi_\iota + 2\mathrm{i}rk\overline{\xi}_o, \tag{5.50}$$

$$0 = \eth_{-1/2}\xi_\iota - \frac{\sqrt{2}}{\Omega}\partial_u(r)\xi_o - 2\mathrm{i}rk\overline{\varphi}_\iota, \tag{5.51}$$

$$0 = \bar{\eth}_{-1/2}\varphi_\iota \text{ and} \tag{5.52}$$

$$0 = \eth_{1/2}\xi_o. \tag{5.53}$$

The spin-weighted spherical harmonics, $({}_sY_{jm})$, are known [100] to be eigenfunctions of $\eth_s$ and $\bar{\eth}_s$; in particular

$$\eth_s\left({}_sY_{jm}\right) = \sqrt{(j-s)(j+s+1)}\left({}_{s+1}Y_{jm}\right), \tag{5.54}$$

$$\bar{\eth}_s\left({}_sY_{jm}\right) = -\sqrt{(j+s)(j-s+1)}\left({}_{s-1}Y_{jm}\right) \text{ and} \tag{5.55}$$

$$\overline{\left({}_sY_{jm}\right)} = (-1)^{s+m}\left({}_{-s}Y_{j(-m)}\right). \tag{5.56}$$

Furthermore, they form a complete basis for expanding functions on the round sphere. Hence, it immediately follows that the solutions to equations 5.52 and 5.53 are

$$\varphi_\iota = c_3\left({}_{-1/2}Y_{1/2,-1/2}\right) + c_4\left({}_{-1/2}Y_{1/2,1/2}\right) \text{ and} \tag{5.57}$$

$$\xi_o = \bar{c}_1\left({}_{1/2}Y_{1/2,-1/2}\right) + \bar{c}_2\left({}_{1/2}Y_{1/2,1/2}\right) \tag{5.58}$$

for some constants, c_1, c_2, c_3 and c_4.

Substituting these into equations 5.50 and 5.51 then says

$$\begin{aligned}\bar{\eth}_{1/2}\varphi_o = &\left(\frac{\sqrt{2}}{\Omega}\partial_v(r)c_3 + 2\mathrm{i}krc_2\right)\left({}_{-1/2}Y_{1/2,-1/2}\right)\\ &+\left(\frac{\sqrt{2}}{\Omega}\partial_v(r)c_4 - 2\mathrm{i}krc_1\right)\left({}_{-1/2}Y_{1/2,1/2}\right) \text{ and}\end{aligned} \tag{5.59}$$

$$\eth_{-1/2}\xi_\iota = \left(\frac{\sqrt{2}}{\Omega}\partial_u(r)\bar{c}_1 + 2\mathrm{i}kr\bar{c}_4\right)\left({}_{1/2}Y_{1/2,-1/2}\right) + \left(\frac{\sqrt{2}}{\Omega}\partial_u(r)\bar{c}_2 - 2\mathrm{i}kr\bar{c}_3\right)\left({}_{1/2}Y_{1/2,1/2}\right). \tag{5.60}$$

The claimed expressions for φ_o and ξ_ι then follow by once again applying the completeness and eigenfunction properties (under $\eth_s$ and $\bar{\eth}_s$) of spin-weighted spherical harmonics. □

The mass definition 5.9 will be compared against is the Misner-Sharp mass, as given in equation 2.170.

Definition 5.12 (Misner-Sharp mass)**.** *Including a cosmological constant, the Misner-Sharp mass for spherically symmetric spacetimes is defined to be*

$$m_{MS}(S_r^2) = \frac{r}{2}\left(1 + 4k^2r^2 - (g^{ab} - \beta^{ab})D_a(r)D_b(r)\right), \tag{5.61}$$

where β_{ab} is the induced metric on each S_r^2.

Theorem 5.13. *$m(S_r^2)$ agrees with the Misner-Sharp mass (with cosmological constant) for spherically symmetric spacetimes.*

Proof. Taking the four c_A to be the coefficients multiplying the four linearly independent solutions, it follows from theorem 5.11 that

$$Q^{AB} \equiv \frac{4r(2\partial_u(r)\partial_v(r) + \Omega^2(1+4k^2r^2))}{\Omega^3}\begin{bmatrix} \partial_u r & 0 & 0 & -\mathrm{i}k\Omega r\sqrt{2} \\ 0 & \partial_u r & \mathrm{i}k\Omega r\sqrt{2} & 0 \\ 0 & -\mathrm{i}k\Omega r\sqrt{2} & -\partial_v r & 0 \\ \mathrm{i}k\Omega r\sqrt{2} & 0 & 0 & -\partial_v r \end{bmatrix}, \quad (5.62)$$

$$T^{AB} \equiv \frac{1}{\pi\Omega}\begin{bmatrix} 0 & -\partial_u r & -\mathrm{i}k\Omega r\sqrt{2} & 0 \\ \partial_u r & 0 & 0 & -\mathrm{i}k\Omega r\sqrt{2} \\ \mathrm{i}k\Omega r\sqrt{2} & 0 & 0 & -\partial_v r \\ 0 & \mathrm{i}k\Omega r\sqrt{2} & \partial_v r & 0 \end{bmatrix} \text{ and hence} \quad (5.63)$$

$$T^{-1} = \frac{\pi\Omega}{\partial_u(r)\partial_v(r) + 2k^2\Omega^2r^2}\begin{bmatrix} 0 & \partial_v r & -\mathrm{i}k\Omega r\sqrt{2} & 0 \\ -\partial_v r & 0 & 0 & -\mathrm{i}k\Omega r\sqrt{2} \\ \mathrm{i}k\Omega r\sqrt{2} & 0 & 0 & \partial_u r \\ 0 & \mathrm{i}k\Omega r\sqrt{2} & -\partial_u r & 0 \end{bmatrix}. \quad (5.64)$$

Then, one finds

$$m(S_r^2) = \frac{1}{16\pi}\sqrt{-\mathrm{tr}(QT^{-1}\overline{Q}\overline{T}^{-1})} = \frac{r}{2}\left(\frac{2}{\Omega^2}\partial_u(r)\partial_v(r) + 1 + 4k^2r^2\right), \quad (5.65)$$

which is the Misner-Sharp mass in double null coordinates (note the Misner-Sharp mass is manifestly coordinate independent). □

Corollary 5.13.1. *For the Schwarzschild-AdS spacetime, $m(S_r^2)$ coincides with the mass parameter, M, in the metric.*

Proof. The Misner-Sharp mass for Schwarzschild-AdS is most easily calculated in the standard (t, r, θ, ϕ) coordinates instead of double null coordinates. Hence,

$$m(S_r^2) = \frac{r}{2}\left(1 + 4k^2r^2 + \frac{1}{1+4k^2r^2-2M/r}\partial_t(r)^2 - (1+4k^2r^2-2M/r)\partial_r(r)^2\right) = M \quad (5.66)$$

as expected. □

In theorem 5.3 it was assumed that $\theta_l\theta_n < -8k^2$. However, that assumption never came up in the preceding spherical symmetry discussion. In the standard (t, r, θ, ϕ) coordinates, the most natural tetrad to choose is

$$l = \frac{1}{\sqrt{2}}\left(\frac{1}{f}\frac{\partial}{\partial t} + f\frac{\partial}{\partial r}\right), \quad n = \frac{1}{\sqrt{2}}\left(\frac{1}{f}\frac{\partial}{\partial t} - f\frac{\partial}{\partial r}\right) \text{ and } m = \frac{1}{r\sqrt{2}}\left(\frac{\partial}{\partial\theta} + \frac{\mathrm{i}}{\sin(\theta)}\frac{\partial}{\partial\phi}\right), \quad (5.67)$$

where $f = \sqrt{1 - 2M/r + 4k^2r^2}$. Then, one finds

$$\rho = \mu = -\frac{f}{r\sqrt{2}} \text{ and hence } \theta_l\theta_n = -\frac{2f^2}{r^2} = -\frac{2}{r^2} + \frac{4M}{r^3} - 8k^2. \quad (5.68)$$

Therefore $\theta_l\theta_n < -8k^2 \iff r > 2M$. This result is quite mysterious to me because $r = 2M$ is no longer a special radius when a cosmological constant is added to the Schwarzschild metric.

5.2.2 Toroidal symmetry

In this section, the round spheres in section 5.2.1 are replaced with tori. Because the torus is flat, it serves as the simplest - and most practically tractable - example where the constructions of section 5.1 can be studied on a non-spherical surface, S. Since $\text{index}(\overline{m}^a\nabla_a) = 4(1-g)$ for a genus, g, surface from earlier, there might not be any solutions to $\overline{m}^a\nabla_a\Phi = 0$ in general. Furthermore, it's possible the number of solutions varies with sufficient deformations of the torus in question. The aim of this section is to illustrate these possibilities via counterexamples to the assumptions underpinning definition 5.9. First consider the $c = 0$ case in equation 5.40. To summarise, the domain of outer communication of the toroidal Schwarzschild-AdS spacetime is $\mathbb{R} \times ((M/2k^2)^{1/3}, \infty) \times \mathbb{T}^2$ with the metric,

$$g = -f(r)^2 \mathrm{d}t \otimes \mathrm{d}t + \frac{\mathrm{d}r \otimes \mathrm{d}r}{f(r)^2} + r^2(\mathrm{d}\theta \otimes \mathrm{d}\theta + \mathrm{d}\phi \otimes \mathrm{d}\phi), \tag{5.69}$$

$$\text{where } f(r) = \sqrt{-\frac{2M}{r} + 4k^2r^2} \tag{5.70}$$

and (θ, ϕ) are coordinates on each $\mathbb{T}^2 = S^1 \times S^1$.

Then, one can follow the same steps as section 5.2.1. In particular, with

$$l = \frac{1}{\sqrt{2}}\left(\frac{1}{f}\frac{\partial}{\partial t} + f\frac{\partial}{\partial r}\right), \; n = \frac{1}{\sqrt{2}}\left(\frac{1}{f}\frac{\partial}{\partial t} - f\frac{\partial}{\partial r}\right) \text{ and } m = \frac{1}{r\sqrt{2}}\left(\frac{\partial}{\partial \theta} + \mathrm{i}\frac{\partial}{\partial \phi}\right) \tag{5.71}$$

as a NP tetrad for $\mathbb{T}^2_r$, one finds

$$\sigma = \lambda = \alpha = \beta = 0 \text{ and } \rho = \mu = -\frac{f}{r\sqrt{2}}. \tag{5.72}$$

Consequently $\overline{m}^a\nabla_a\Phi = 0$ is equivalent to

$$0 = 2\partial_{\bar{z}}\varphi_o - f\varphi_\iota - 2\mathrm{i}kr\overline{\xi}_o, \tag{5.73}$$
$$0 = 2\partial_z\xi_\iota + f\xi_o + 2\mathrm{i}kr\overline{\varphi}_\iota, \tag{5.74}$$
$$0 = \partial_{\bar{z}}\varphi_\iota \text{ and} \tag{5.75}$$
$$0 = \partial_z\xi_o, \tag{5.76}$$

where $z = \theta - \mathrm{i}\phi$. Note that ∂_z is globally well defined even if θ and ϕ are not.

Liouville's theorem immediately implies φ_ι and ξ_o are constants, say c_3 and $\bar{c}_4$. Substituting these into the other two equations says

$$\partial_{\bar{z}}\varphi_o = \frac{1}{2}(fc_3 + 2\mathrm{i}c_4kr) \text{ and } \partial_z\xi_\iota = -\frac{1}{2}(f\bar{c}_4 + 2\mathrm{i}\bar{c}_3kr). \tag{5.77}$$

Since r is also just a constant on $\mathbb{T}^2_r$, the equations can be immediately integrated to

$$\varphi_o = \frac{1}{2}(fc_3 + 2\mathrm{i}c_4kr)\bar{z} + c_1(z) \text{ and } \xi_\iota = -\frac{1}{2}(f\bar{c}_4 + 2\mathrm{i}\bar{c}_3kr)z + \bar{c}_2(\bar{z}) \tag{5.78}$$

for some holomorphic functions, c_1 and c_2. However, by Liouville's theorem, c_1 and c_2 must actually be constants. Furthermore, $\mathbb{T}^2_r$ has 2π periodicity in the θ and ϕ coordinates which neither $(fc_3 + 2\mathrm{i}c_4kr)\bar{z}$ nor $(f\bar{c}_4 + 2\mathrm{i}\bar{c}_3kr)z$ do. Hence, φ_o and ξ_ι can only be well-defined if $c_3 = c_4 = 0$. In summary,

$$\varphi_o = c_1, \;\; \xi_o = 0, \;\; \varphi_\iota = 0 \text{ and } \xi_\iota = \bar{c}_2 \tag{5.79}$$

for constants, c_1 and c_2. Note that the Φ^A form of generic immediately fails because there are only two linearly independent solutions, not four. Furthermore, $\theta_l\theta_n < -8k^2$ fails to hold at any radius. By taking c_1 and c_2 to parameterise the two linearly independent solutions, direct calculation shows

$$Q^{AB} = 8\sqrt{2}\pi^2 r \begin{bmatrix} -f & -2\mathrm{i}kr \\ 2\mathrm{i}kr & -f \end{bmatrix} \quad \text{and} \quad T^{AB} = 0. \tag{5.80}$$

Hence, the T^{AB} form of generic also fails and $m(\mathbb{T}^2_r)$ cannot be formed via definition 5.9. Furthermore, even Q^{AB} is not non-negative definite - an effect of the $\theta_{l,n}$ conditions failing.

It's unclear to me exactly what conclusions can be drawn more generally from this example. The failure of the Φ^A form of generic is unsurprising given the earlier discussion of index$(\overline{m}^a\nabla_a)$. However, the failure of the T^{AB} form of generic could potentially be explained by the high degree of symmetry and flatness of the chosen surface. While deforming $\mathbb{T}^2_r$ slightly is unlikely to change the number of solutions to $\overline{m}^a\nabla_a\Phi = 0$, it may yet result in an invertible T^{AB} and a well-defined $m(S)$. It's also unclear to me whether there is any relation between the $\theta_{l,n}$ conditions and the generic conditions.

As the next example shows though, optimism needs to be tempered because the situation can in fact be even worse. Consider an "AdS solition," constructed from equation 5.69 via the procedure in [64]. In particular, define new coordinates, $\tau = \mathrm{i}\theta$ and $\omega = \mathrm{i}t$. Analytically continue the coordinates so that τ & ω are real and the metric is

$$g = -r^2\mathrm{d}\tau\otimes\mathrm{d}\tau + \frac{\mathrm{d}r\otimes\mathrm{d}r}{f(r)^2} + f(r)^2\mathrm{d}\omega\otimes\mathrm{d}\omega + r^2\mathrm{d}\phi\otimes\mathrm{d}\phi. \tag{5.81}$$

Unwrap the τ coordinate so $\tau\in\mathbb{R}$ and compactify the ω coordinate so that (ω,ϕ) are coordinates on a torus. Avoiding a conical singularity as $r\to r_0 = (M/2k^2)^{1/3}$ forces the periodicity,

$$\omega \sim \omega + \frac{\pi}{3k^2 r_0}, \tag{5.82}$$

although the period won't actually matter for what follows.

Once again consider constant-r tori, $\mathbb{T}^2_r$. With the NP tetrad,

$$l = \frac{1}{\sqrt{2}}\left(\frac{1}{r}\frac{\partial}{\partial\tau} + f\frac{\partial}{\partial r}\right),\ n = \frac{1}{\sqrt{2}}\left(\frac{1}{r}\frac{\partial}{\partial\tau} - f\frac{\partial}{\partial r}\right) \text{ and } m = \frac{1}{\sqrt{2}}\left(\frac{1}{f}\frac{\partial}{\partial\omega} + \frac{\mathrm{i}}{r}\frac{\partial}{\partial\phi}\right), \tag{5.83}$$

one finds

$$\alpha=\beta=0,\ \sigma=\lambda=-\frac{3M}{2\sqrt{2}r^2 f} \text{ and } \rho=\mu=-\frac{8k^2r^3-M}{2\sqrt{2}r^2f} = -\frac{f^2+12k^2r^2}{4\sqrt{2}rf}. \tag{5.84}$$

Package the GHP components of Φ into a vector, $v = (\varphi_o, \varphi_\iota, \bar{\xi}_o, \bar{\xi}_\iota)^T$. Then, with these NP coefficients, the equations of lemma 5.1 become $\overline{m}^\mu\partial_\mu v = Av$, where

$$A = \begin{bmatrix} 0 & -\mu & \mathrm{i}k\sqrt{2} & 0 \\ \bar{\sigma} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\lambda \\ 0 & \mathrm{i}k\sqrt{2} & \rho & 0 \end{bmatrix} = \frac{1}{2\sqrt{2}r^2f}\begin{bmatrix} 0 & 8k^2r^3-M & 4\mathrm{i}kr^2f & 0 \\ -3M & 0 & 0 & 0 \\ 0 & 0 & 0 & 3M \\ 0 & 4\mathrm{i}kr^2f & -(8k^2r^3-M) & 0 \end{bmatrix} \tag{5.85}$$

is effectively a constant matrix on $\mathbb{T}^2_r$.

$\{m,\overline{m}\}$ induces a complex structure on $\mathbb{T}^2_r$. Choose a corresponding complex coordinate on the torus, $z = \frac{1}{\sqrt{2}}(f\omega - \mathrm{i}r\phi)$, so that $m^\mu\partial_\mu = \partial_z$ and $\overline{m}^\mu\partial_\mu = \partial_{\bar{z}}$ on $\mathbb{T}^2_r$. Then, the equation to solve is $\partial_{\bar{z}}v = Av$. Integrating immediately yields

$$v = \mathrm{e}^{\bar{z}A}c(z) \tag{5.86}$$

for some holomorphic vector, $c(z)$. However, by Liouville's theorem, $c(z)$ must be a constant vector, c. But, then v would be a globally defined, non-constant, antiholomorphic vector on the compact space, $\mathbb{T}^2_r$, contradicting Liouville's theorem. The only way around this is to have $c \in \text{nullspace}(A)$, so that the $\bar{z}$ dependence falls out[6] in $v = \mathrm{e}^{\bar{z}A}c$. However, it turns out

$$\det(A) = (\mu^2 - 2k^2)\lambda^2. \tag{5.87}$$

Hence, by $\mu = \rho$ and lemma 2.6, A is invertible whenever the $\theta_l\theta_n < -8k^2$ assumption holds. In this example, this inequality always holds since

$$\begin{aligned} \mu^2 > 2k^2 &\iff (f^2 + 12k^2r^2)^2 > 64k^2r^2f^2 && (5.88)\\ &\iff f^4 + 144k^4r^4 > 40k^2r^2f^2 && (5.89)\\ &\iff 64k^2rM + \frac{4M^2}{r^2} > 0. && (5.90) \end{aligned}$$

Therefore, the only solution is $v = 0$ and $\overline{m}^a\nabla_a\Phi = 0$ has no non-trivial solutions.

As explained in section 4.3.5, the AdS soliton famously has negative energy [64] and avoids spinorial positive energy theorems because the torus has two inequivalent spin structures and the soliton's spin structure is incompatible with the one required to apply Witten's method. It's unclear to me if any of these properties is linked to $\overline{m}^a\nabla_a\Phi = 0$ having only trivial solution.

5.3 Asymptotic limit

The next property of $m(S)$ I'll study is the large sphere limit. In this section, it will once again be convenient to set the AdS length scale to 1, i.e. choose units such that $\Lambda = -3$ and $k = 1/2$, like I did in chapter 4. The length scales can always be restored on dimensional grounds. My main aim here is to connect the quasilocal mass defined in this chapter with the asymptotically AdS analysis in section 4.3.3.

Theorem 5.14. *If $S = S^2_\infty$, i.e. the sphere at infinity in an asymptotically AdS spacetime, then $Q(\Phi) = Q(\varepsilon_k)$, where ε_k is a Killing spinor of AdS.*

Proof. First note the Φ^A form of generic holds in this instance because S^2_r becomes increasingly round as $r \to \infty$ and spherically symmetric spacetimes are known to satisfy this notion of generic from section 5.2. AdS itself has four globally defined, linearly independent solutions to $\nabla_a\Phi = 0$, namely the the 4D space of Killing spinors, ε_k. Therefore in AdS, the space of solutions to $\overline{m}^a\nabla_a\Phi = 0$ on any generic surface can be spanned by simply restricting the Killing spinors to the surface.

By definition 2.11, the difference between g and g_{AdS} is $O(\mathrm{e}^{-3r})$. Hence, in the asymptotic region of (M, g), $\Phi = \varepsilon_k + \mathcal{Z}$ for some $\mathcal{Z}$ that is $O(\mathrm{e}^{-3r})$ below leading order. From theorem 4.4, ε_k is $O(\mathrm{e}^{r/2})$ and thus $\mathcal{Z}$ must be $O(\mathrm{e}^{-5r/2})$.

In vierbein indices $P_a = -\delta_{a0}$ and $Q_a = \delta_{a1} \equiv \mathrm{d}r$. Therefore,

$$\begin{aligned} Q(\Phi) &= \int_{S^2_\infty} E^{01}(\Phi)\mathrm{d}A && (5.91)\\ &= Q(\varepsilon_k) + \int_{S^2_\infty} \big(\mathcal{Z}^\dagger\gamma^1\gamma^A\nabla_A\varepsilon_k + \varepsilon_k^\dagger\gamma^1\gamma^A\nabla_A\mathcal{Z} + \mathcal{Z}^\dagger\gamma^1\gamma^A\nabla_A\mathcal{Z} + \nabla_A(\varepsilon_k)^\dagger\gamma^A\gamma^1\mathcal{Z}\\ &\qquad + \nabla_A(\mathcal{Z})^\dagger\gamma^A\gamma^1\varepsilon_k + \nabla_A(\mathcal{Z})^\dagger\gamma^A\gamma^1\mathcal{Z}\big)\mathrm{d}A. && (5.92) \end{aligned}$$

[6] In fact, the toroidal Schwarzschild-AdS example earlier can be analysed in exactly this way. Since $\sigma = \lambda = 0$ in that example, the analogue of A has two rows of zeroes, which then yield a 2D nullspace and the two constant solutions in equation 5.79.

From equation 2.76, $\mathrm{d}A$ is $O(\mathrm{e}^{2r})$. Consequently, $\mathcal{Z}^\dagger\gamma^1\gamma^A\nabla_A(\mathcal{Z})\mathrm{d}A$ and $\nabla_A(\mathcal{Z})^\dagger\gamma^A\gamma^1\mathcal{Z}\mathrm{d}A$ are both $O(\mathrm{e}^{-3r})$ and go to zero as $r\to\infty$. Meanwhile, from equations 3.116, 3.115, 3.129 and 3.135, $\nabla_A\varepsilon_k$ is also $O(\mathrm{e}^{-5r/2})$ in the asymptotic region, implying $\mathcal{Z}^\dagger\gamma^1\gamma^A\nabla_A(\varepsilon_k)\mathrm{d}A$ and $\nabla_A(\varepsilon_k)^\dagger\gamma^A\gamma^1\mathcal{Z}\mathrm{d}A$ also go to zero. That leaves

$$Q(\Phi)=Q(\varepsilon_k)+\int_{S^2_\infty}\left(\varepsilon_k^\dagger\gamma^1\gamma^A\nabla_A\mathcal{Z}+\nabla_A(\mathcal{Z})^\dagger\gamma^A\gamma^1\varepsilon_k\right)\mathrm{d}A. \tag{5.93}$$

The 2nd term is the complex conjugate of the first so it suffices to prove the 1st term integrates to zero. This term can be re-written as

$$\begin{aligned}\varepsilon_k^\dagger\gamma^1\gamma^A\nabla_A\mathcal{Z}&=\varepsilon_k^\dagger\gamma^1\gamma^AD_A\mathcal{Z}+\frac{\mathrm{i}}{2}\varepsilon_k^\dagger\gamma^1\gamma^A\gamma_A\mathcal{Z} &(5.94)\\ &=D_A(\varepsilon_k^\dagger\gamma^1\gamma^A\mathcal{Z})-D_A(\varepsilon_k)^\dagger\gamma^1\gamma^A\mathcal{Z}-\mathrm{i}\varepsilon_k^\dagger\gamma^1\mathcal{Z} &(5.95)\\ &=D_A(\varepsilon_k^\dagger\gamma^1\gamma^A\mathcal{Z})-\nabla_A(\varepsilon_k)^\dagger\gamma^1\gamma^A\mathcal{Z}-\left(-\frac{\mathrm{i}}{2}\gamma_A\varepsilon_k\right)^\dagger\gamma^1\gamma^A\mathcal{Z}-\mathrm{i}\varepsilon_k^\dagger\gamma^1\mathcal{Z} &(5.96)\\ &=D_A(\varepsilon_k^\dagger\gamma^1\gamma^A\mathcal{Z})-\nabla_A(\varepsilon_k)^\dagger\gamma^1\gamma^A\mathcal{Z}+\frac{\mathrm{i}}{2}\varepsilon_k^\dagger\gamma_A\gamma^1\gamma^A\mathcal{Z}-\mathrm{i}\varepsilon_k^\dagger\gamma^1\mathcal{Z} &(5.97)\\ &=D_A(\varepsilon_k^\dagger\gamma^1\gamma^A\mathcal{Z})-\nabla_A(\varepsilon_k)^\dagger\gamma^1\gamma^A\mathcal{Z}. &(5.98)\end{aligned}$$

From above, $\nabla_A(\varepsilon_k)^\dagger\gamma^1\gamma^A\mathcal{Z}$ contributes nothing to the integral, leaving

$$\int_{S^2_\infty}\varepsilon_k^\dagger\gamma^1\gamma^A\nabla_A(Z)\mathrm{d}A=\int_{S^2_\infty}D_A(\varepsilon_k^\dagger\gamma^1\gamma^A\mathcal{Z})\mathrm{d}A. \tag{5.99}$$

Let $D_A^{(S)}$ be the intrinsic Levi-Civita connection of S, let K_{IJ} be the extrinsic curvature of Σ in M and let c_{AB} be the extrinsic curvature of S in Σ. Then,

$$\begin{aligned}D_A(\varepsilon_k^\dagger\gamma^1\gamma^A\mathcal{Z})&=\left(D_A^{(S)}\varepsilon_k-\frac{1}{2}K_{AI}\gamma^I\gamma^0\varepsilon_k-\frac{1}{2}c_{AB}\gamma^B\gamma^1\varepsilon_k\right)^\dagger\gamma^1\gamma^A\mathcal{Z}\\ &\quad+\varepsilon_k^\dagger\gamma^1\gamma^A\left(D_A^{(S)}\mathcal{Z}-\frac{1}{2}K_{AI}\gamma^I\gamma^0\mathcal{Z}-\frac{1}{2}c_{AB}\gamma^B\gamma^1\mathcal{Z}\right) &(5.100)\\ &=D_A^{(S)}(\varepsilon_k^\dagger\gamma^1\gamma^A\mathcal{Z})+\frac{1}{2}K_{AI}(\varepsilon_k^\dagger\gamma^0\gamma^I\gamma^1\gamma^A\mathcal{Z}-\varepsilon_k^\dagger\gamma^1\gamma^A\gamma^I\gamma^0\mathcal{Z})\\ &\quad-\frac{1}{2}c_{AB}(\varepsilon_k^\dagger\gamma^1\gamma^B\gamma^1\gamma^A\mathcal{Z}+\varepsilon_k^\dagger\gamma^1\gamma^A\gamma^B\gamma^1\mathcal{Z}) &(5.101)\\ &=D_A^{(S)}(\varepsilon_k^\dagger\gamma^1\gamma^A\mathcal{Z})+\frac{1}{2}K_{AI}(\varepsilon_k^\dagger\gamma^0\gamma^I\gamma^1\gamma^A\mathcal{Z}-\varepsilon_k^\dagger\gamma^1\gamma^A\gamma^I\gamma^0\mathcal{Z})+0. &(5.102)\end{aligned}$$

The measure, $\mathrm{d}A$, is $O(\mathrm{e}^{2r})$ while the ε_k-$\mathcal{Z}$ products are already $O(\mathrm{e}^{-2r})$. Since AdS is time-symmetric, $K_{IJ}=0$ to leading order, meaning the second term contributes nothing to the integral. That leaves

$$\int_{S^2_\infty}\varepsilon_k^\dagger\gamma^1\gamma^A\nabla_A(Z)\mathrm{d}A=\int_{S^2_\infty}D_A^{(S)}(\varepsilon_k^\dagger\gamma^1\gamma^A\mathcal{Z})\mathrm{d}A=0 \tag{5.103}$$

by Stokes' theorem. Equation 5.93 then implies $Q(\Phi)=Q(\varepsilon_k)$. □

Corollary 5.14.1. *When $S=S^2_\infty$, $m(S)=\sqrt{E^2-||P||^2+||J||^2-||K||^2}$.*

Proof. From theorem 4.12,

$$Q(\varepsilon_k)=8\pi\varepsilon_0^\dagger\mathrm{e}^{-\mathrm{i}\gamma^0t/2}\left(EI-\mathrm{i}P_I\gamma^I+K_I\gamma^0\gamma^I+\frac{\mathrm{i}}{2}J_{IJ}\gamma^0\gamma^{IJ}\right)\mathrm{e}^{\mathrm{i}\gamma^0t/2}\varepsilon_0. \tag{5.104}$$

The four components of the constant spinor, ε_0, can be used to parameterise the four linearly independent solutions, Φ^A. Hence, as a matrix,

$$Q^{AB} \equiv 8\pi e^{-i\gamma^0 t/2}\left(EI - iP_I\gamma^I + K_I\gamma^0\gamma^I + \frac{i}{2}J_{IJ}\gamma^0\gamma^{IJ}\right)e^{i\gamma^0 t/2}. \tag{5.105}$$

The other matrix required for $m(S^2_\infty)$ is T^{AB}. In the present context, it will be easiest to use the alternative expression, $T^{AB} = (\Phi^A)^T C^{-1}\Phi^B$, of equation 5.31. In the conventions chosen, $(\gamma_a)^T = -C^{-1}\gamma_a C$. Therefore,

$$\begin{aligned}
T^{AB} &= (\varepsilon_k^A)^T C^{-1}\varepsilon_k^B && (5.106)\\
&= \frac{1}{1-\rho^2}(\varepsilon_0^A)^T\left(e^{i\gamma^0 t/2}\right)^T\left(I - ix_I(\gamma^I)^T\right)C^{-1}\left(I - ix_J\gamma^J\right)e^{i\gamma^0 t/2}\varepsilon_0^B && (5.107)\\
&= \frac{1}{1-\rho^2}(\varepsilon_0^A)^T C^{-1}e^{-i\gamma^0 t/2}C\left(I + ix_I C^{-1}\gamma^I C\right)C^{-1}\left(I - ix_J\gamma^J\right)e^{i\gamma^0 t/2}\varepsilon_0^B && (5.108)\\
&= \frac{1}{1-\rho^2}(\varepsilon_0^A)^T C^{-1}e^{-i\gamma^0 t/2}\left(I + ix_I\gamma^I\right)\left(I - ix_J\gamma^J\right)e^{i\gamma^0 t/2}\varepsilon_0^B && (5.109)\\
&= \frac{1}{1-\rho^2}(\varepsilon_0^A)^T C^{-1}e^{-i\gamma^0 t/2}\left(I + x_I x_J\gamma^I\gamma^J\right)e^{i\gamma^0 t/2}\varepsilon_0^B && (5.110)\\
&= \frac{1}{1-\rho^2}(\varepsilon_0^A)^T C^{-1}e^{-i\gamma^0 t/2}(1-\rho^2)Ie^{i\gamma^0 t/2}\varepsilon_0^B && (5.111)\\
&= (C^{-1})^{AB}. && (5.112)
\end{aligned}$$

Finally, with the Q^{AB} and T^{AB} derived, definition 5.9 reduces to

$$m(S^2_\infty) = \frac{1}{16\pi}\sqrt{-\mathrm{tr}(QT^{-1}\overline{Q}\overline{T}^{-1})} = \sqrt{E^2 - P_I P^I + J_I J^I - K_I K^I}, \tag{5.113}$$

where $J_I = \frac{1}{2}\varepsilon_{IJK}J^{JK}$. □

The question naturally arises whether $\sqrt{E^2 - ||P||^2 + ||J||^2 - ||K||^2}$ is an appropriate notion of mass in asymptotically AdS spacetimes. For example, from special relativity, one thinks of mass as just $\sqrt{E^2 - ||P||^2}$, without any contributions from angular momenta, J_{IJ}, or boost charges, K_I. However, I argue this is an artefact of Minkowski space's symmetry group, namely the Poincaré group. As in quantum field theory, one could define m^2 to be proportional to a quadratic Casimir operator of (the Lie algebra of) the symmetry group [17] in an irreducible complex representation. Therefore, in the AdS context, one should seek a quadratic Casimir for $\mathfrak{o}(3,2)$.

Choose generators, $\{J_{MN} = -J_{NM}\}_{M,N=1}^5$, for $\mathfrak{o}(3,2)$ such that the defining Lie bracket is[7]

$$\left[J^{MN}, J^{PQ}\right] = i\left(\eta^{MP}J^{NQ} - \eta^{MQ}J^{NP} - \eta^{NP}J^{MQ} + \eta^{NQ}J^{MP}\right), \tag{5.114}$$

where $\eta_{MN} \equiv \mathrm{diag}(-1,1,1,1,-1)$ and all $M, N, \cdots$ indices are raised/lowered by η^{-1}/η. Then, it immediately follows that $C = \frac{1}{2}J^{MN}J_{MN}$ is a quadratic Casimir[8] for $\mathfrak{o}(3,2)$, i.e. $[C, J_{MN}] = 0$.

Interpret J^{5a} as a 4-momentum generator, $\mathbb{P}^a$, J^{0I} as boost generators, $\mathbb{K}^I$, and J^{IJ} as angular momentum generators, $\mathbb{J}_I = \frac{1}{2}\varepsilon_{IJK}J^{JK}$, in line with [30] and the heuristics accompanying definition 4.11. Then,

$$C = J^{5a}J_{5a} + \frac{1}{2}J^{IJ}J_{IJ} + J^{0I}J_{0I} = \mathbb{P}^0\mathbb{P}^0 - \mathbb{P}^I\mathbb{P}_I + \mathbb{J}^I\mathbb{J}_I - \mathbb{K}^I\mathbb{K}_I, \tag{5.115}$$

suggesting that the limit in corollary 5.14.1 is physically reasonable.

[7] The fact such a basis exists can be seen immediately by following the analogous steps in [124] for $\mathfrak{o}(3,1)$.

[8] Assume we are working with a faithful matrix representation of the Lie algebra so that multiplying two Lie algebra elements is well-defined.

5.4 Linearised gravity

This section concerns perturbations of AdS sourced by a matter field. In particular, the metric is assumed to be

$$g_{ab} = B_{ab} + \eta h_{ab}, \tag{5.116}$$

where $B = g_{\text{AdS}}$ is the background metric, h is the perturbation and η is assumed to be an infinitesimal parameter. Furthermore, the energy-momentum tensor, T_{ab}, is assumed to be $O(\eta)$. The aim is to show that definition 5.9 captures a mass associated with T_{ab}. Throughout this section, the coordinates and tetrad will be the same as in equations 4.109 to 4.115. It will once again be convenient to set the AdS length scale to one, i.e. choose units where $\Lambda = -3$ and $k = 1/2$.

A natural way to construct physical quantities, like mass, out of the energy-momentum tensor is to contract T_{ab} with the Killing vectors of the background metric. It can be checked that the Killing vectors of AdS are spanned by

$$\tau = \partial_t, \tag{5.117}$$

$$j_{IJ} = x_I \partial_J - x_J \partial_I, \tag{5.118}$$

$$p_I = \frac{2x_I}{1+\rho^2}\cos(t)\partial_t + \frac{1}{2}\left((1+\rho^2)\delta^J{}_I - 2x^J x_I\right)\sin(t)\partial_J \quad \text{and} \tag{5.119}$$

$$k_I = -\frac{2x_I}{1+\rho^2}\sin(t)\partial_t + \frac{1}{2}\left((1+\rho^2)\delta^J{}_I - 2x^J x_I\right)\cos(t)\partial_J. \tag{5.120}$$

In analogy with definition 4.11, define the following "matter charges."

Definition 5.15 (Matter charges). *Let matter charges on Σ be defined as*

$$E = \int_\Sigma T_{0a}\tau^a \mathrm{d}V, \quad P_I = \int_\Sigma T_{0a}p_I^a \mathrm{d}V, \quad J_{IJ} = \int_\Sigma T_{0a}j_{IJ}^a \mathrm{d}V \quad \text{and} \quad K_I = \int_\Sigma T_{0a}k_I^a \mathrm{d}V. \tag{5.121}$$

Theorem 5.16. *For gravity linearised about AdS, if S is generic in the Φ^A sense, then*

$$m(S) = \sqrt{E^2 - ||P||^2 + ||J||^2 - ||K||^2}. \tag{5.122}$$

This expression is formally identical to corollary 5.14.1 and therefore the result can once again be thought of as a Casimir mass, but this time for T_{ab}.

Proof. AdS already has four linearly independent Killing spinors, i.e. global solutions to $\nabla^{(B)}_a \varepsilon_k = 0$. Hence, in AdS, if S is generic in the Φ^A sense, then solutions to $\overline{m}^a \nabla_a \Phi = 0$ can be spanned by simply restricting ε_k to S. Since $g_{ab} = B_{ab} + \eta h_{ab}$ though, one can therefore let $\Phi = \varepsilon_k + \eta \mathcal{Z}$ for some Dirac spinor, $\mathcal{Z}$.

Extend $\mathcal{Z}$'s definition off S in an arbitrary, but sufficiently regular, way so that $\Phi = \varepsilon_k + \eta\mathcal{Z}$ is defined on all of Σ. Then, by corollary 3.2.1,

$$Q(\Phi) = 2\int_\Sigma \left(\nabla_I(\Phi)^\dagger \nabla^I \Phi - 4\pi T^{0a}\overline{\Phi}\gamma_a \Phi - (\gamma^I \nabla_I \Phi)^\dagger \gamma^J \nabla_J \Phi\right) \mathrm{d}V. \tag{5.123}$$

$\nabla^{(B)}_a \varepsilon_k = 0$ implies $\nabla_a \Phi = O(\eta)$ and thus the first and third terms in equation 5.123 are both $O(\eta^2)$. Meanwhile, since T_{ab} is assumed to be $O(\eta)$, the second term is $-4\pi T^{0a}\bar{\varepsilon}_k \gamma_a \varepsilon_k + O(\eta^2)$. In summary, the linearised limit yields,

$$Q(\Phi) \to 8\pi \int_\Sigma T_{0a}\bar{\varepsilon}_k \gamma^a \varepsilon_k \mathrm{d}V. \tag{5.124}$$

Equations 4.126 and 4.130 say

$$\bar{\varepsilon}_k\gamma^0\varepsilon_k = \frac{1}{1-\rho^2}\varepsilon_0^\dagger \mathrm{e}^{-\mathrm{i}\gamma^0 t/2}((1+\rho^2)I - 2\mathrm{i}x_I\gamma^I)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0 \text{ and} \tag{5.125}$$

$$\bar{\varepsilon}_k\gamma^I\varepsilon_k = \frac{1}{1-\rho^2}\varepsilon_0^\dagger \mathrm{e}^{-\mathrm{i}\gamma^0 t/2}((1+\rho^2)\gamma^0\gamma^I - 2\mathrm{i}x_J\gamma^0\gamma^{IJ} - 2x^I x_J\gamma^0\gamma^J)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0. \tag{5.126}$$

Substituting back into equation 5.124, applying definition 5.15 and converting to vierbein indices where required then gives

$$\begin{aligned}
Q(\Phi) \to{}& 8\pi\varepsilon_0^\dagger\Bigg(\int_\Sigma \frac{1+\rho^2}{1-\rho^2}T_{00}\mathrm{d}V\, I - 2\mathrm{i}\int_\Sigma \frac{x_I}{1-\rho^2}T_{00}\mathrm{e}^{-\mathrm{i}\gamma^0 t}\mathrm{d}V\,\gamma^I - 2\mathrm{i}\int_\Sigma \frac{x_J}{1-\rho^2}T_{0I}\mathrm{d}V\,\gamma^0\gamma^{IJ} \\
&+ \int_\Sigma \frac{1}{1-\rho^2}T_{0I}\left((1+\rho^2)\delta^I{}_J - 2x^I x_J\right)\mathrm{e}^{-\mathrm{i}\gamma^0 t}\mathrm{d}V\,\gamma^0\gamma^J\Bigg)\varepsilon_0
\end{aligned} \tag{5.127}$$

$$\begin{aligned}
={}& 8\pi\varepsilon_0^\dagger\Bigg(\int_\Sigma \frac{1+\rho^2}{1-\rho^2}T_{00}\mathrm{d}V\, I + \mathrm{i}\int_\Sigma \frac{1}{1-\rho^2}(x_I T_{0J} - x_J T_{0I})\mathrm{d}V\,\gamma^0\gamma^{IJ} \\
&- \mathrm{i}\int_\Sigma\left(\frac{2x_I\cos(t)}{1-\rho^2}T_{00} + \frac{\sin(t)}{1-\rho^2}T_{0I}\left((1+\rho^2)\delta^I{}_J - 2x^I x_J\right)\right)\mathrm{d}V\,\gamma^I \\
&+ \int_\Sigma\left(-\frac{2x_I\sin(t)}{1-\rho^2}T_{00} + \frac{\cos(t)}{1-\rho^2}T_{0I}\left((1+\rho^2)\delta^I{}_J - 2x^I x_J\right)\right)\mathrm{d}V\,\gamma^0\gamma^I\Bigg)\varepsilon_0
\end{aligned} \tag{5.128}$$

$$= 8\pi\varepsilon_0^\dagger\left(EI + \frac{\mathrm{i}}{2}J_{IJ}\gamma^0\gamma^{IJ} - \mathrm{i}P_I\gamma^I + K_I\gamma^0\gamma^I\right)\varepsilon_0. \tag{5.129}$$

Taking the four components of the constant spinor, ε_0, to parameterise the four linearly independent solutions, Φ^A, then gives

$$Q^{AB} \to 8\pi\left(EI + \frac{\mathrm{i}}{2}J_{IJ}\gamma^0\gamma^{IJ} - \mathrm{i}P_I\gamma^I + K_I\gamma^0\gamma^I\right). \tag{5.130}$$

Since this Q^{AB} is already $O(\eta)$, for the linearised limit it suffices to take T^{AB} to $O(1)$ in definition 5.9. Thus, $T^{AB} \to (\varepsilon_k^A)^T C^{-1}\varepsilon_k^B = (C^{-1})^{AB}$, borrowing the calculation from the proof of corollary 5.14.1.

Finally, evaluating $m(S)$ for this Q^{AB} and T^{AB} results in

$$m(S) = \frac{1}{16\pi}\sqrt{-\mathrm{tr}(QT^{-1}\overline{Q}\overline{T}^{-1})} \to \sqrt{E^2 + J_I J^I - P_I P^I - K_I K^I}, \tag{5.131}$$

where $J_I = \frac{1}{2}\varepsilon_{IJK}J^{JK}$. □

To better understand the matter charges in this section, an instructive example to consider is a collection of point particle test masses moving in an AdS background.

Let $\{M_\alpha\}_\alpha$ be a set of point particles moving along arbitrary (timelike) trajectories, $x_\alpha^\mu(s)$. For each particle, introduce its proper time by

$$\mathrm{d}\tau_\alpha = \sqrt{-g_{\mu\nu}(x_\alpha(s))\frac{\mathrm{d}x_\alpha^\mu(s)}{\mathrm{d}s}\frac{\mathrm{d}x_\alpha^\nu(s)}{\mathrm{d}s}}\mathrm{d}s. \tag{5.132}$$

Given the matter action of the point particles is

$$S_m = -\sum_\alpha M_\alpha\int\sqrt{-g_{\mu\nu}(x_\alpha(s))\frac{\mathrm{d}x_\alpha^\mu(s)}{\mathrm{d}s}\frac{\mathrm{d}x_\alpha^\nu(s)}{\mathrm{d}s}}\mathrm{d}s = -\sum_\alpha M_\alpha\int\mathrm{d}\tau_\alpha, \tag{5.133}$$

the energy-momentum tensor is well-known to be

$$T^{\mu\nu}(x) = \frac{1}{\sqrt{-g(x)}} \sum_\alpha M_\alpha \int \frac{\mathrm{d}x^\mu_\alpha(\tau_\alpha)}{\mathrm{d}\tau_\alpha} \frac{\mathrm{d}x^\nu_\alpha(\tau_\alpha)}{\mathrm{d}\tau_\alpha} \delta^4(x - x_\alpha(\tau_\alpha))\mathrm{d}\tau_\alpha. \tag{5.134}$$

Introduce a "3+1 split" adapted to Σ so that

$$g = -N^2 \mathrm{d}t \otimes \mathrm{d}t + h_{ij}(\mathrm{d}x^i + N^i \mathrm{d}t) \otimes (\mathrm{d}x^j + N^j \mathrm{d}t). \tag{5.135}$$

Then, $P_\mu \equiv -N\mathrm{d}t$, $\sqrt{-g} = N\sqrt{h}$ and $\mathrm{d}V = \sqrt{h}\,\mathrm{d}^3x$. This implies that for any vector, v^μ,

$$\int_\Sigma T_{0a} v^a \mathrm{d}V \tag{5.136}$$

$$= -\int_\Sigma T^{\mu\nu} P_\mu v_\nu \mathrm{d}V$$

$$= -\sum_\alpha M_\alpha \int_\Sigma \int \frac{1}{N\sqrt{h}} \frac{\mathrm{d}x^\mu_\alpha(\tau_\alpha)}{\mathrm{d}\tau_\alpha} \frac{\mathrm{d}x^\nu_\alpha(\tau_\alpha)}{\mathrm{d}\tau_\alpha} \delta^4(x - x_\alpha(\tau_\alpha))(-N\delta_{\nu 0}) v_\mu \sqrt{h}\,\mathrm{d}^3x\,\mathrm{d}\tau_\alpha \tag{5.137}$$

$$= \sum_\alpha M_\alpha \int_\Sigma \int \frac{\mathrm{d}x^\mu_\alpha(\tau_\alpha)}{\mathrm{d}\tau_\alpha} \delta^3(x - x_\alpha(\tau_\alpha))\delta(t - t_\alpha) v_\mu \mathrm{d}^3x\,\mathrm{d}t_\alpha \tag{5.138}$$

$$= \sum_{\alpha\in\Sigma} M_\alpha \frac{\mathrm{d}x^\mu(\tau_\alpha)}{\mathrm{d}\tau_\alpha} v_\mu. \tag{5.139}$$

As a special case, if $x^\mu_\alpha(\tau_\alpha)$ is a geodesic and v^μ is a Killing vector (like τ, p_I, j_{IJ} or k_I in definition 5.15), then $\frac{\mathrm{d}x^\mu(\tau_\alpha)}{\mathrm{d}\tau_\alpha} v_\mu$ is exactly the conserved quantity associated to v^μ along the geodesic. Meanwhile, if Σ contains a single particle (travelling along an arbitrary timelike curve, $x^\mu(\tau)$), then equation 5.139 says

$$m(S) = M\sqrt{\frac{\mathrm{d}x^\mu(\tau)}{\mathrm{d}\tau} \frac{\mathrm{d}x^\nu(\tau)}{\mathrm{d}\tau} \left(\tau_\mu\tau_\nu - p_{I\mu}p^I_\nu + \frac{1}{2} j_{IJ\mu} j^{IJ}_\nu - k_{I\mu}k^I_\nu\right)}. \tag{5.140}$$

It can be checked that in fact

$$\tau_\mu\tau_\nu - p_{I\mu}p^I_\nu + \frac{1}{2} j_{IJ\mu} j^{IJ}_\nu - k_{I\mu}k^I_\nu = -(g_{\mathrm{AdS}})_{\mu\nu}. \tag{5.141}$$

Since τ is proper time, it follows that $m(S) = M$, i.e. the quasilocal mass exactly captures the mass of the particle contained within the surface. However, if there was more than one particle within Σ, then $m(S)$ would be some complicated function of the different M_α. This is as expected though, because even in special relativity, $P^\mu = P^\mu_1 + P^\mu_2$ does not mean $M = M_1 + M_2$.

5.5 Kerr-AdS horizon

As a more complicated example where the new quasilocal mass can be calculated, I will consider the horizon of the Kerr-AdS spacetime. This calculation will be analogous to the $\Lambda = 0$ calculation for the Dougan-Mason quasilocal mass in [11]. In Kerr coordinates, the Kerr-AdS metric is [20]

$$\begin{aligned} g = &-\frac{\Delta - \vartheta a^2 \sin^2\theta}{\Sigma} \mathrm{d}v \otimes \mathrm{d}v + \mathrm{d}v \otimes \mathrm{d}r + \mathrm{d}r \otimes \mathrm{d}v + \frac{\Sigma}{\vartheta} \mathrm{d}\theta \otimes \mathrm{d}\theta \\ &+ \frac{a\sin^2(\theta)(\Delta - \vartheta(r^2 + a^2))}{\Sigma \mathcal{Z}} (\mathrm{d}v \otimes \mathrm{d}\chi + \mathrm{d}\chi \otimes \mathrm{d}v) - \frac{a\sin^2(\theta)}{\mathcal{Z}} (\mathrm{d}r \otimes \mathrm{d}\chi + \mathrm{d}\chi \otimes \mathrm{d}r) \\ &+ \frac{\sin^2(\theta)(\vartheta(r^2 + a^2) - \Delta a^2 \sin^2(\theta))}{\Sigma \mathcal{Z}^2} \mathrm{d}\chi \otimes \mathrm{d}\chi, \end{aligned} \tag{5.142}$$

where (θ, χ) are angular coordinates on a sphere, $a^2 < \frac{|\Lambda|}{3}$ is a constant rotation parameter and the various functions are

$$\Delta = (r^2 + a^2)\left(1 - \frac{\Lambda r^2}{3}\right) - 2Mr \text{ for a constant, } M, \tag{5.143}$$

$$\Sigma = r^2 + a^2\cos^2(\theta), \tag{5.144}$$

$$\mathcal{Z} = 1 + \frac{\Lambda a^2}{3} \text{ and} \tag{5.145}$$

$$\vartheta = 1 + \frac{\Lambda a^2}{3}\cos^2(\theta). \tag{5.146}$$

The event horizon is the constant r surface where $\Delta = 0$. The most natural NP tetrad to use is the Kinnersley tetrad,

$$l_K = 2(r^2 + a^2)\partial_v + \Delta\partial_r + 2\mathcal{Z}a\partial_\chi, \tag{5.147}$$

$$n_K = -\frac{1}{2\Sigma}\partial_r \text{ and} \tag{5.148}$$

$$m_K = \frac{1}{\sqrt{2\vartheta}(r + \mathrm{i}a\cos(\theta))}\left(\mathrm{i}a\sin(\theta)\partial_v + \vartheta\partial_\theta + \frac{\mathrm{i}\mathcal{Z}}{\sin(\theta)}\partial_\chi\right), \tag{5.149}$$

because l_K and n_K are the principal null vectors. In this tetrad, it can be checked that the relevant NP coefficients are

$$\kappa = \lambda = \nu = \sigma = 0,\ \rho = -\frac{\Delta}{r - \mathrm{i}a\cos(\theta)},\ \mu = -\frac{1}{2\Sigma(r - \mathrm{i}a\cos(\theta))},\ \pi = \frac{\mathrm{i}a\sin(\theta)}{(r - \mathrm{i}a\cos(\theta))^2}\sqrt{\frac{\vartheta}{2}},$$
$$\varepsilon = \frac{1}{2}\frac{\mathrm{d}\Delta}{\mathrm{d}r},\ \alpha = \frac{r\cos(\theta)(\mathcal{Z} - 2\vartheta) + \mathrm{i}a(\sin^2(\theta) + \vartheta)}{2\sqrt{2\vartheta}(r - \mathrm{i}a\cos(\theta))^2\sin(\theta)} \text{ and } \beta = \frac{(2\vartheta - \mathcal{Z})\cot(\theta)}{2\sqrt{2\vartheta}(r + \mathrm{i}a\cos(\theta))}. \tag{5.150}$$

However, since μ and ρ are not real, by lemma 2.7, this tetrad is not surface forming.

Finding the NP coefficients for a general surface-forming tetrad is very difficult for this spacetime. However, in this section I am only interested in one surface, S_{v,r_0}, i.e. a constant v and r surface where $r = r_0 \implies \Delta = 0$. Therefore, I only need μ and ρ to be real on this one surface for lemma 2.7 to guarantee the tetrad is admissible for the calculation.

Following [11], perform what Chandrasekhar defines as a "type-I" tetrad transformation [21]. In particular, for a function, F,

$$l = l_K, \tag{5.151}$$

$$n = n_K + |F|^2 l_K + \bar{F}m_K + F\overline{m}_K \text{ and} \tag{5.152}$$

$$m = m_K + Fl_K. \tag{5.153}$$

m^a is tangent to S_{v,r_0} if and only if $m^a \in \operatorname{span}(\{\partial_\theta, \partial_\chi\})$. Therefore, one must choose

$$F = -\frac{\mathrm{i}a\sin(\theta)}{2\sqrt{2\vartheta}(r^2 + a^2)(r + \mathrm{i}a\cos(\theta))}. \tag{5.154}$$

Under this transformation the new NP coefficients (denoted with primes) are related to the old ones by [21]

$$\rho' = \rho + \kappa\bar{F}, \tag{5.155}$$

$$\mu' = \mu + 2\beta\bar{F} + \pi F + 2\varepsilon|F|^2 + \kappa|F|^2\bar{F} + \sigma\bar{F}^2 + \delta\bar{F} + FD\bar{F}, \tag{5.156}$$

$$\sigma' = \sigma + \kappa F, \tag{5.157}$$

$$\lambda' = \lambda + (2\alpha + \pi)\bar{F} + (2\varepsilon + \rho)\bar{F}^2 + \kappa\bar{F}^3 + \bar{\delta}\bar{F} + \bar{F}D\bar{F}, \tag{5.158}$$

$$\alpha' = \alpha + (\rho + \varepsilon)\bar{F} + \kappa\bar{F}^2 \text{ and} \tag{5.159}$$

$$\beta' = \beta + \varepsilon F + \sigma\bar{F} + \kappa|F|^2. \tag{5.160}$$

Since $\mathbb{A} = 0$ on the horizon and $\kappa = 0$, $\rho' = 0$ on the horizon. Luckily, while μ' is very complicated, it nonetheless turns out to be manifestly real when $\mathbb{A} = 0$. Hence, $\{l, n, m, \overline{m}\}$ is a valid tetrad for defining the quasilocal mass, $m(S_{v,r_0})$.

Like ρ', σ' remains zero after the transformation. However, λ' is no longer zero. Hence, dropping primes from hereon, the $\overline{m}^a \nabla_a \Phi = 0$ equations to solve are

$$0 = \bar{\eth}\varphi_o + \mu\varphi_\iota - \mathrm{i}k\sqrt{2}\,\overline{\xi}_o, \tag{5.161}$$

$$0 = \bar{\eth}\overline{\xi}_\iota - \mathrm{i}k\sqrt{2}\,\varphi_\iota, \tag{5.162}$$

$$0 = \bar{\eth}\varphi_\iota \text{ and} \tag{5.163}$$

$$0 = \bar{\eth}\overline{\xi}_o + \lambda\overline{\xi}_\iota. \tag{5.164}$$

Conveniently, the third equation has only one variable. Therefore, one might hope to solve the third equation for φ_ι, use that to solve the second equation for $\overline{\xi}_\iota$, use that to solve the fourth equation for $\overline{\xi}_o$ and then finally solve the first equation for φ_0. At each stage, the equation to solve is a first order linear PDE in one variable. Therefore, it is natural to look for solutions by separation of variables.

The simplest solution to $\bar{\eth}\varphi_\iota = 0$ is $\varphi_\iota = 0$. Hence, the next equation to solve is $\bar{\eth}\overline{\xi}_\iota = 0$. With the separated variable ansatz,

$$\overline{\xi}_\iota = \bar{B}_\xi(\theta)\tilde{\bar{B}}_\xi(\chi), \tag{5.165}$$

the PDE becomes

$$\frac{1}{\tilde{\bar{B}}_\xi(\chi)}\frac{\mathrm{d}\tilde{\bar{B}}_\xi(\chi)}{\mathrm{d}\chi} = \frac{1}{\overline{m}^3}\left(\alpha - \overline{m}^2\frac{1}{\bar{B}_\xi(\theta)}\frac{\mathrm{d}\bar{B}_\xi(\theta)}{\mathrm{d}\theta}\right). \tag{5.166}$$

Since the LHS depends only on χ and the RHS depends only on θ, both sides must be constants. The LHS is then immediately integrable, meaning

$$\overline{\xi}_\iota = \bar{B}_\xi(\theta)\mathrm{e}^{\mathrm{i}s\chi} \tag{5.167}$$

for some constant, s. For $\overline{\xi}_\iota$ to be a well-defined spinor component, it must be that $s \in \frac{1}{2}\mathbb{Z}$. Then, the ODE for $\bar{B}_\xi(\theta)$ immediately integrates to

$$\bar{B}_\xi = \exp\left(\int \frac{1}{\overline{m}^2}(\alpha - \mathrm{i}s\overline{m}^3)\mathrm{d}\theta\right). \tag{5.168}$$

Miraculously, there is a closed-form expression for the RHS. While the actual expression is a mess, it schematically takes the form of $\sin^{-s-1/2}(\theta/2)\cos^{s-1/2}(\theta/2)$ multiplied by something manifestly regular.

However, since $\sin^{-s-1/2}(\theta/2)\cos^{s-1/2}(\theta/2)$ is not regular on the sphere for any value of s, it must be that in fact $\overline{\xi}_\iota = 0$ too.

Thus, the next equation becomes $\bar{\eth}\overline{\xi}_o = 0$. This time a separated variables ansatz implies

$$\overline{\xi}_o = \bar{A}_\xi(\theta)\mathrm{e}^{\mathrm{i}s\chi} \text{ for } \bar{A}_\xi = \exp\left(\int \frac{1}{\overline{m}^2}(\alpha + \mathrm{i}s\overline{m}^3)\mathrm{d}\theta\right). \tag{5.169}$$

This is the same integral, but with $s \to -s$. Hence, the integral is $\sin^{1/2-s}(\theta/2)\cos^{1/2+s}(\theta/2)$ multiplied by something manifestly regular.

Thus, $\bar{A}_\xi$ is regular across the sphere if and only if $s = \pm\frac{1}{2}$. A separated variable ansatz for the remaining variable, φ_o, is therefore only possible if

$$\varphi_o = A_\varphi(\theta)\mathrm{e}^{\pm\mathrm{i}\chi/2}. \tag{5.170}$$

$\bar{\eth}\varphi_o - \mathrm{i}k\sqrt{2}\bar{\xi}_o = 0$ then reduces to a first order, linear ODE for $A_\varphi(\theta)$, which can immediately be solved using an integrating factor. In particular,

$$A_\varphi = \mathrm{i}k\sqrt{2}\exp\left(-\int \frac{1}{\overline{m}^2}(\bar{\beta} + \mathrm{i}s\overline{m}^3)\mathrm{d}\theta\right)\int \exp\left(\int \frac{1}{\overline{m}^2}(\bar{\beta} + \mathrm{i}s\overline{m}^3)\mathrm{d}\theta\right)\frac{1}{\overline{m}^2}\bar{A}_\xi\,\mathrm{d}\theta \tag{5.171}$$

$$\text{and } \exp\left(-\int \frac{1}{\overline{m}^2}(\bar{\beta} + \mathrm{i}s\overline{m}^3)\mathrm{d}\theta\right) \propto \frac{1}{\sin^{1/2+s}(\theta/2)\cos^{1/2-s}(\theta/2)}. \tag{5.172}$$

The bounds of the integral involving $\bar{A}_\xi$ must then be chosen so that the singularities in $\frac{1}{\sin(\theta/2)}$ and $\frac{1}{\cos(\theta/2)}$ respectively are canceled out. In summary, I have found two solutions to $\overline{m}^a\nabla_a\Phi = 0$ so far. They have

$$\varphi^1_\iota = \varphi^2_\iota = \xi^1_\iota = \xi^2_\iota = 0, \tag{5.173}$$

$$\bar{\xi}^1_o = \mathrm{e}^{\mathrm{i}\chi/2}\bar{A}^1_\xi(\theta),\ \bar{\xi}^2_o = \mathrm{e}^{-\mathrm{i}\chi/2}\bar{A}^2_\xi(\theta), \tag{5.174}$$

$$\bar{A}^1_\xi = \exp\left(\int \frac{1}{\overline{m}^2}(\alpha + \mathrm{i}\overline{m}^3/2)\mathrm{d}\theta\right), \tag{5.175}$$

$$\bar{A}^2_\xi = \exp\left(\int \frac{1}{\overline{m}^2}(\alpha - \mathrm{i}\overline{m}^3/2)\mathrm{d}\theta\right), \tag{5.176}$$

$$\begin{aligned}\varphi^1_o &= \mathrm{i}k\sqrt{2}\exp\left(-\int \frac{1}{\overline{m}^2}\left(\bar{\beta} + \frac{\mathrm{i}\overline{m}^3}{2}\right)\mathrm{d}\theta\right)\\ &\times \int_0^\theta \exp\left(\int \frac{1}{\overline{m}^2(\theta')}\left(\bar{\beta}(\theta') + \frac{\mathrm{i}\overline{m}^3(\theta')}{2}\right)\mathrm{d}\theta'\right)\frac{1}{\overline{m}^2(\theta')}\bar{A}^1_\xi(\theta')\,\mathrm{d}\theta'\,\mathrm{e}^{\mathrm{i}\chi/2}\ \text{ and}\end{aligned} \tag{5.177}$$

$$\begin{aligned}\varphi^2_o &= -\mathrm{i}k\sqrt{2}\exp\left(-\int \frac{1}{\overline{m}^2}\left(\bar{\beta} - \frac{\mathrm{i}\overline{m}^3}{2}\right)\mathrm{d}\theta\right)\\ &\times \int_\theta^\pi \exp\left(\int \frac{1}{\overline{m}^2(\theta')}\left(\bar{\beta}(\theta') - \frac{\mathrm{i}\overline{m}^3(\theta')}{2}\right)\mathrm{d}\theta'\right)\frac{1}{\overline{m}^2(\theta')}\bar{A}^2_\xi(\theta')\,\mathrm{d}\theta'\,\mathrm{e}^{-\mathrm{i}\chi/2}.\end{aligned} \tag{5.178}$$

Serendipitously, it turns out that

$$\exp\left(\int \frac{1}{\overline{m}^2}\left(\bar{\beta} + \frac{\mathrm{i}\overline{m}^3}{2}\right)\mathrm{d}\theta\right)\frac{\bar{A}^1_\xi(\theta)}{\overline{m}^2(\theta)},\ \exp\left(\int \frac{1}{\overline{m}^2}\left(\bar{\beta} - \frac{\mathrm{i}\overline{m}^3}{2}\right)\mathrm{d}\theta\right)\frac{\bar{A}^2_\xi(\theta)}{\overline{m}^2(\theta)} \propto \sin(\theta), \tag{5.179}$$

with the proportionality constant determined by the integration constant in the indefinite integral. Hence, with the appropriate choice of integration constants,

$$\varphi^1_o = 2\mathrm{i}k\sqrt{2}\exp\left(-\int \frac{1}{\overline{m}^2}\left(\bar{\beta} + \frac{\mathrm{i}\overline{m}^3}{2}\right)\mathrm{d}\theta\right)\sin^2(\theta/2)\mathrm{e}^{\mathrm{i}\chi/2}\ \text{ and} \tag{5.180}$$

$$\varphi^2_o = 2\mathrm{i}k\sqrt{2}\exp\left(-\int \frac{1}{\overline{m}^2}\left(\bar{\beta} - \frac{\mathrm{i}\overline{m}^3}{2}\right)\mathrm{d}\theta\right)\cos^2(\theta/2)\mathrm{e}^{-\mathrm{i}\chi/2}. \tag{5.181}$$

Moreover, the removal of the $\frac{1}{\sin(\theta/2)}$ and $\frac{1}{\cos(\theta/2)}$ singularities is now manifest.

Next, suppose that φ_ι was not initially zero. Proceeding completely analogously, this time the separated variable solution is

$$\varphi_\iota = B_\varphi(\theta)\mathrm{e}^{\mathrm{i}s\chi}\ \text{ for }\ B_\varphi = \exp\left(\int \frac{1}{\overline{m}^2}(\bar{\beta} - \mathrm{i}s\overline{m}^3)\mathrm{d}\theta\right), \tag{5.182}$$

$$\exp\left(\int \frac{1}{\overline{m}^2}(\bar{\beta} - \mathrm{i}s\overline{m}^3)\mathrm{d}\theta\right) \propto \sin^{1/2-s}(\theta/2)\cos^{1/2+s}(\theta/2) \tag{5.183}$$

and the proportionality function being manifestly regular. Therefore, φ_ι is regular if and only if $s=\pm\frac{1}{2}$. With this solution, I can separate variables and integrate the $\bar{\eth}\bar{\xi}_\iota$ equation to get

$$\bar{\xi}_\iota = \mathrm{e}^{\mathrm{i}s\chi}\bar{B}_\xi(\theta) \;\text{ for} \tag{5.184}$$

$$\bar{B}_\xi(\theta) = \mathrm{i}k\sqrt{2}\exp\left(\int \frac{1}{\overline{m}^2}\left(\alpha - \mathrm{i}s\overline{m}^3\right)\mathrm{d}\theta\right) \times \int \exp\left(-\int \frac{1}{\overline{m}^2(\theta')}\left(\alpha(\theta') - \mathrm{i}s\overline{m}^3(\theta')\right)\mathrm{d}\theta'\right)\frac{1}{\overline{m}^2(\theta')}B_\varphi(\theta')\,\mathrm{d}\theta'. \tag{5.185}$$

Again, the bounds of the $\mathrm{d}\theta'$ integral must be chosen to be $\int_0^\theta$ or $-\int_\theta^\pi$ to cancel the $\frac{1}{\sin(\theta/2)}$ or $\frac{1}{\cos(\theta/2)}$ singularity respectively coming from the leading factor. Again, it turns out the integrand of the $\mathrm{d}\theta'$ integral is just $\sin(\theta)$, up to a proportionality constant determined by the integration constant in the indefinite integral.

Having determined $\bar{B}_\xi(\theta)$, one could then continue analogously to earlier to determine $\bar{\xi}_o$ and φ_o. In summary, I find the following two additional solutions to $\overline{m}^a\nabla_a\Phi = 0$.

$$\varphi_o^3 = A_\varphi^3(\theta)\mathrm{e}^{\mathrm{i}\chi/2},\ \varphi_\iota^3 = B_\varphi^3(\theta)\mathrm{e}^{\mathrm{i}\chi/2},\ \bar{\xi}_o^3 = \bar{A}_\varphi^3(\theta)\mathrm{e}^{\mathrm{i}\chi/2},\ \bar{\xi}_\iota^3 = \bar{B}_\xi^3(\theta)\mathrm{e}^{\mathrm{i}\chi/2} \;\text{ and} \tag{5.186}$$

$$\varphi_o^4 = A_\varphi^4(\theta)\mathrm{e}^{-\mathrm{i}\chi/2},\ \varphi_\iota^4 = B_\varphi^4(\theta)\mathrm{e}^{-\mathrm{i}\chi/2},\ \bar{\xi}_o^4 = \bar{A}_\varphi^4(\theta)\mathrm{e}^{-\mathrm{i}\chi/2},\ \bar{\xi}_\iota^4 = \bar{B}_\xi^4(\theta)\mathrm{e}^{-\mathrm{i}\chi/2}, \;\text{ where} \tag{5.187}$$

$$B_\varphi^3(\theta) = \exp\left(\int \frac{1}{\overline{m}^2}\left(\bar{\beta} - \frac{\mathrm{i}\overline{m}^3}{2}\right)\mathrm{d}\theta\right), \tag{5.188}$$

$$\bar{B}_\xi^3(\theta) = 2\mathrm{i}k\sqrt{2}\exp\left(\int \frac{1}{\overline{m}^2}\left(\alpha - \frac{\mathrm{i}\overline{m}^3}{2}\right)\mathrm{d}\theta\right)\sin^2(\theta/2), \tag{5.189}$$

$$\bar{A}_\xi^3(\theta) = -\exp\left(-\int \frac{1}{\overline{m}^2}\left(\alpha + \frac{\mathrm{i}\overline{m}^3}{2}\right)\mathrm{d}\theta\right) \times \int_0^\theta \exp\left(\int \frac{1}{\overline{m}^2(\theta')}\left(\alpha(\theta') + \frac{\mathrm{i}\overline{m}^3(\theta')}{2}\right)\mathrm{d}\theta'\right)\frac{\lambda(\theta')}{\overline{m}^2(\theta')}\bar{B}_\xi^3(\theta')\mathrm{d}\theta', \tag{5.190}$$

$$A_\varphi^3(\theta) = \exp\left(-\int \frac{1}{\overline{m}^2}\left(\bar{\beta} + \frac{\mathrm{i}\overline{m}^3}{2}\right)\mathrm{d}\theta\right) \times \int_0^\theta \exp\left(\int \frac{1}{\overline{m}^2(\theta')}\left(\bar{\beta}(\theta') + \frac{\mathrm{i}\overline{m}^3(\theta')}{2}\right)\mathrm{d}\theta'\right)\frac{\mathrm{i}k\sqrt{2}\bar{A}_\xi^3(\theta') - \mu(\theta')B_\varphi^3(\theta')}{\overline{m}^2(\theta')}\mathrm{d}\theta', \tag{5.191}$$

$$B_\varphi^4(\theta) = \exp\left(\int \frac{1}{\overline{m}^2}\left(\bar{\beta} + \frac{\mathrm{i}\overline{m}^3}{2}\right)\mathrm{d}\theta\right), \tag{5.192}$$

$$\bar{B}_\xi^4(\theta) = 2\mathrm{i}k\sqrt{2}\exp\left(\int \frac{1}{\overline{m}^2}\left(\alpha + \frac{\mathrm{i}\overline{m}^3}{2}\right)\mathrm{d}\theta\right)\cos^2(\theta/2), \tag{5.193}$$

$$\bar{A}_\xi^4(\theta) = \exp\left(-\int \frac{1}{\overline{m}^2}\left(\alpha - \frac{\mathrm{i}\overline{m}^3}{2}\right)\mathrm{d}\theta\right) \times \int_\theta^\pi \exp\left(\int \frac{1}{\overline{m}^2(\theta')}\left(\alpha(\theta') - \frac{\mathrm{i}\overline{m}^3(\theta')}{2}\right)\mathrm{d}\theta'\right)\frac{\lambda(\theta')}{\overline{m}^2(\theta')}\bar{B}_\xi^4(\theta')\mathrm{d}\theta' \;\text{ and} \tag{5.194}$$

$$A_\varphi^4(\theta) = -\exp\left(-\int \frac{1}{\overline{m}^2}\left(\bar{\beta} - \frac{\mathrm{i}\overline{m}^3}{2}\right)\mathrm{d}\theta\right) \times \int_\theta^\pi \exp\left(\int \frac{1}{\overline{m}^2(\theta')}\left(\bar{\beta}(\theta') - \frac{\mathrm{i}\overline{m}^3(\theta')}{2}\right)\mathrm{d}\theta'\right)\frac{\mathrm{i}k\sqrt{2}\bar{A}_\xi^4(\theta') - \mu(\theta')B_\varphi^4(\theta')}{\overline{m}^2(\theta')}\mathrm{d}\theta'. \tag{5.195}$$

If S_{v,r_0} is generic, then these four solutions form a basis for the solution space of $\overline{m}^a\nabla_a\Phi = 0$. It remains only to calculate the quasilocal mass using definitions 5.9, 5.2 and 5.4. Since $\rho = \varphi_\iota^1 = \xi_\iota^1 = \varphi_\iota^2 = \xi_\iota^2 = 0$ and all the χ integrals are trivial (namely $\int_0^{2\pi}\mathrm{e}^{\mathrm{i}s\chi}\mathrm{d}\chi = 2\pi\delta_{s,0}$), it can be seen that the only non-zero components in Q^{AB} are

$$Q^{33} = 8\pi \int_0^\pi \left(\mu(|B^3_\xi|^2 - |B^3_\varphi|^2) + \mathrm{i}k\sqrt{2}(B^3_\xi A^3_\varphi - \bar{A}^3_\varphi \bar{B}^3_\xi - A^3_\xi B^3_\varphi + \bar{B}^3_\varphi \bar{A}^3_\xi)\right) \sqrt{\beta}\,\mathrm{d}\theta, \quad (5.196)$$

$$Q^{44} = 8\pi \int_0^\pi \left(\mu(|B^4_\xi|^2 - |B^4_\varphi|^2) + \mathrm{i}k\sqrt{2}(B^4_\xi A^4_\varphi - \bar{A}^4_\varphi \bar{B}^4_\xi - A^4_\xi B^4_\varphi + \bar{B}^4_\varphi \bar{A}^4_\xi)\right) \sqrt{\beta}\,\mathrm{d}\theta, \quad (5.197)$$

$$Q^{13} = \overline{Q^{31}} = -8\pi\mathrm{i}k\sqrt{2}\int_0^\pi \left(\bar{A}^1_\varphi \bar{B}^3_\xi + A^1_\xi B^3_\varphi\right) \sqrt{\beta}\,\mathrm{d}\theta \text{ and} \quad (5.198)$$

$$Q^{24} = \overline{Q^{42}} = -8\pi\mathrm{i}k\sqrt{2}\int_0^\pi \left(\bar{A}^2_\varphi \bar{B}^4_\xi + A^2_\xi B^4_\varphi\right) \sqrt{\beta}\,\mathrm{d}\theta, \text{ where} \quad (5.199)$$

$$\sqrt{\beta} = \sqrt{\det(\beta_{\alpha\beta})} = \frac{(r_0^2 + a^2)\sin(\theta)}{\mathcal{Z}}. \quad (5.200)$$

Meanwhile, since T^{AB} is constant, it can be evaluated by setting $\theta = \chi = 0$. Then, since $\varphi^1_\iota = \varphi^2_\iota = \varphi^4_\iota = \varphi^1_o = \xi^1_\iota = \xi^2_\iota = \xi^3_\iota = \xi^2_o = \xi^3_o = 0$ at $\theta = 0$ by construction, it follows that the only non-zero components are

$$T^{14} = -T^{41} = -\sqrt{2}\bar{A}^1_\xi(0)\bar{B}^4_\xi(0), \quad (5.201)$$

$$T^{23} = -T^{32} = \sqrt{2}A^2_\varphi(0)B^3_\varphi(0) \text{ and} \quad (5.202)$$

$$T^{34} = -T^{43} = -\sqrt{2}B^3_\varphi(0)A^4_\varphi(0). \quad (5.203)$$

Now, all that remains is to evaluate the quasilocal mass,

$$m(S_{v,r_0})^2 = -\frac{1}{256\pi^2}\mathrm{tr}(QT^{-1}\overline{Q}\overline{T}^{-1}), \quad (5.204)$$

by numerically integrating the relevant quantities. The result is the pink graph in figure 5.1.

The most striking feature of the graph is that $m(S_{v,r_0})^2 < 0$ for large values of (r_0, a). This is not a contradiction of previous results because $\theta_l = 0$ on the horizon. Therefore, the technical condition, $\theta_l\theta_n < \frac{2\Lambda}{3}$, is not purely a limitation of proof technique, but instead a non-trivial physical restriction. It's not clear whether the restriction is sharp though.

When $\Lambda = 0$, the new quasilocal mass reduces to the Dougan-Mason definition, which only requires $\theta_n \leq 0$ to be non-negative. The Λ dependent terms that turn the Kerr metric into the Kerr-AdS metric are proportional to Λr^2, Λa^2 or Λra, in terms of dimensionless quantities. Thus, whenever $r_0^2, a^2, r_0 a \ll \frac{1}{\Lambda}$, the quasilocal mass, $m(S_{v,r_0})$, will be only a small correction to the Dougan-Mason quasilocal mass of the Kerr horizon. This is why $m(S_{v,r_0})^2$ only becomes negative in figure 5.1 for large values of horizon radius or rotation parameter. Furthermore, even for large r_0, if $\Lambda a^2 \ll 1$, the quasilocal mass remains positive because the metric is only slightly perturbed from spherical symmetry, where positivity is guaranteed by section 5.2.1.

It should also be mentioned that $m^2 < 0$ is not unprecedented in AdS. In particular, from sections 5.3 and 5.4, $m(S)$ appears to be a Casimir type mass and it's known from the representation theory of $\mathfrak{o}(3,2)$ that m^2 can appear negative [13, 47]. However, unlike [13, 47], figure 5.1 strongly suggests that $m(S^2_{v,r_0})^2$ is unbounded below.

More broadly, it's unclear what the "right" answer should be for the quasilocal mass of the horizon. I have provided a few alternatives in figure 5.1. The simplest is just the mass parameter, M, in green. The next simplest is the Hawking quasilocal mass of the horizon,

$$m_H(S_{v,r_0}) = \sqrt{\frac{A(S_{v,r_0})}{16\pi}}\left(1 - \frac{\Lambda A(S_{v,r_0})}{12\pi} + \frac{1}{2\pi}\int_{S_{v,r_0}} \rho\mu\,\mathrm{d}A\right) \quad (5.205)$$

$$= \sqrt{\frac{r_0^2 + a^2}{4\mathcal{Z}}}\left(1 - \frac{\Lambda(r_0^2 + a^2)}{3\mathcal{Z}}\right), \quad (5.206)$$

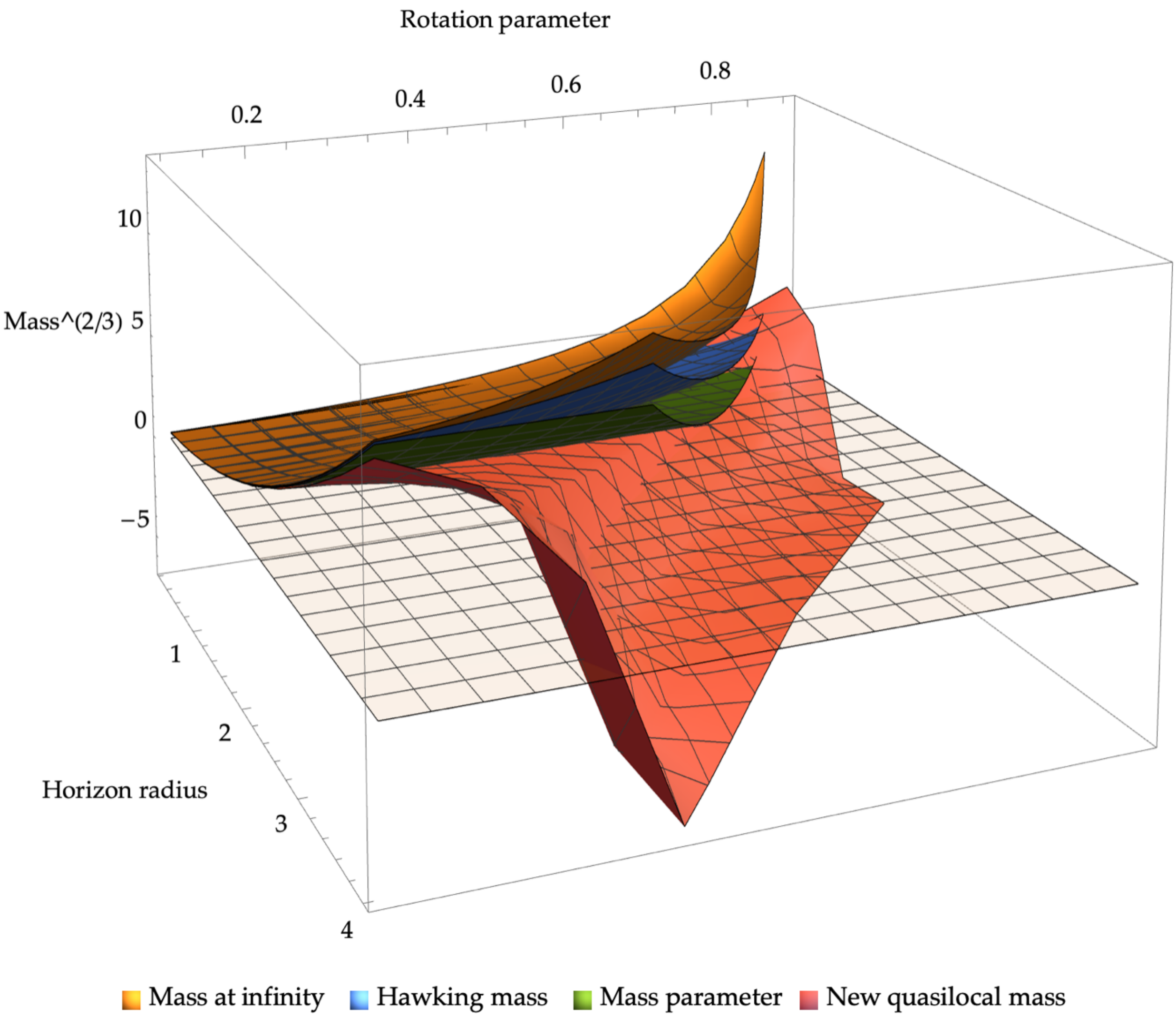

Figure 5.1: The new quasilocal mass defined in this chapter is evaluated on the horizon of Kerr-AdS for various values of the horizon radius, r_0, and rotation parameter, a. Both parameters are in units where $\Lambda = -3$. Points where r_0 and a are simultaneously large are not plotted for numerical stability reasons. For comparison, the mass at infinity, m_∞, the Hawking mass, $m_H(S_{v,r_0})$ and the mass parameter, M, are also plotted. To avoid huge numbers on the vertical axis, instead of a quasilocal mass, m_{QLM}, the vertical axis instead plots $(m_{QLM}^2)^{1/3}$. This allows $m(S_{v,r_0})^2$ to be negative for the new quasilocal mass. The pink mesh is created by interpolating between the $m(S_{v,r_0})^{2/3}$ values for 118 evenly chosen points in the (r_0, a) domain.

in blue. Finally, I've also plotted m_∞, the mass measured at infinity, in orange. From [54], the global energy and angular momentum of the Kerr-AdS spacetime should be

$$E = \frac{M}{\mathcal{Z}^2} \text{ and } J = \frac{Ma}{\mathcal{Z}^2}. \tag{5.207}$$

Therefore, based on the discussion in section 5.3, the mass measured at infinity (in $\Lambda = -3$

units) is[9]

$$m_\infty = \sqrt{E^2 + J^2} = \frac{M}{\mathcal{Z}^2}\sqrt{1 + a^2}. \tag{5.208}$$

It's clear from figure 5.1 that the new quasilocal mass behaves very differently to all three alternatives considered.

[9]The E and J appearing in section 5.3 are the energy and angular momentum defined using Fefferman-Graham expansions in chapter 4. In principle, these could be different to the energy and angular momentum defined using the prescription in [54]. However, from [78], the energy and angular momentum I derived for the equal angular momentum, 5D Myers-Perry black hole in section 4.3 agrees with the 5D discussion in [54]. For this reason, I am assuming my definition will also agree with [54] in 4D. As this is all purely for a qualitative comparison, it doesn't really matter if there is in fact a discrepancy.

Chapter 6

Outlook

Deep in the human unconsciousness is a pervasive need for a logical universe that makes sense. But the real universe is always one step beyond logic.
- Muad'Dib' in *Dune* by Frank Herbert

In this thesis, I have applied spinor methods to study energy in general relativity. The main results have been new positive energy theorems & BPS inequalities for asymptotically, locally AdS spacetimes and the development of a new quasilocal mass for generic surfaces in spacetimes with negative cosmological constant. Some avenues of further research are immediately apparent at this juncture.

The over-arching constraint in this thesis is the use of spinors. As explained in chapter 2, not every manifold admits a spin structure and a given manifold may admit multiple spin structures. As I've seen in section 4.3.5, at times this can be a serious limitation on the positive energy theorems derived. There is hence a pressing need in this field for further development of non-spinorial methods in studying energy in asymptotically, locally AdS spacetimes.

Most of the natural research directions suggested by this thesis are at the quasilocal level though. In terms of physical properties, one scenario I didn't study in chapter 5 is the small-sphere limit, i.e. property VI in section 2.3.2. This limit has been studied in the $\Lambda = 0$ case for both the Dougan-Mason and Penrose quasilocal masses [37, 73]. It would be interesting to study this limit for the quasilocal mass I defined in chapter 5 and see if it too leads to a kind of Casimir mass like the large sphere and linearised gravity limits.

Even for the definition of quasilocal mass itself, some improvements could be made. Two different definitions of generic were given in chapter 5 and it may be interesting to study further how the two definitions relate. It would be particularly desirable to find an example with toroidal S where the quasilocal mass construction can be carried out in full, unlike the examples in section 5.2.2. Then, perhaps a more concrete conclusion can be made about whether either definition is generic in practice or physically relevant for higher genus surfaces.

Likewise, there is the assumption of $\theta_l \theta_n < -8k^2$. As I found in section 5.5, this assumption is not purely technical, but instead has some physical meaning, albeit meaning that isn't fully understood. At a heuristic level, it can be understood as the surface being slightly more curved than the ground state, AdS, which has $\theta_l \theta_n = -8k^2$ on every S^2_r. But, this doesn't really mean anything concretely. Given the spherical symmetry calculation worked in section 5.2.1 even when this assumption failed, it needs to be further investigated how sharp this assumption is.

Another possible extension would be to consider spacetimes with $\Lambda > 0$ instead. Not only is the $\Lambda > 0$ case potentially most relevant to the real world, it is arguably also a pressing need for mathematical general relativity. Many familiar properties of conformal infinity break down when $\Lambda > 0$ [3] and this precludes defining anything directly analogous to the ADM [2] or Wang [121] masses. Nonetheless, a number of energy-momentum definitions have been devised in this context - see [114] for a review. Particularly relevant to this thesis are extensions based on spinor methods [71, 113]. Ultimately though, these successes still have to work around the global challenges imposed by $\Lambda > 0$ - e.g. compact Cauchy surfaces, spacelike $\mathcal{I}^+$ or cosmological horizons. It may be that quasilocal mass is a viable alternative for avoiding these issues. In fact, an analogue of Penrose's quasilocal mass can be defined for asymptotically de Sitter spacetimes, albeit it no longer retains some key properties, such as positivity [113, 115]. In the next section, based on the work in chapter 5, I'll collect some calculations that may go into a spinorial definition of quasilocal mass for spacetimes with positive cosmological constant.

6.1 Towards quasilocal mass for spacetimes with positive cosmological constant

The foundational principle of Witten's method is the Lichnerowicz identity. With $\Lambda > 0$, the modified connection should now be

$$\nabla_a \Psi = D_a \Psi + k\gamma_a \Psi, \tag{6.1}$$

where $k = \sqrt{\Lambda/12}$ now. In terms of the modified connection, the Witten-Nester two-form is formally identical, namely

$$E^{ab}(\Psi) = \overline{\Psi}\gamma^{abc}\nabla_c \Psi - \nabla_c(\overline{\Psi})\gamma^{abc}\Psi. \tag{6.2}$$

Indeed, these were the same connection and two-form used in [113, 71]. A curious fact, whose implications are not fully known to me, is that

$$\begin{aligned} E^{ab}(\Psi) &= \overline{\Psi}\gamma^{abc}\nabla_c \Psi - \nabla_c(\overline{\Psi})\gamma^{abc}\Psi &(6.3)\\ &= \overline{\Psi}\gamma^{abc}D_c \Psi + k\overline{\Psi}\gamma^{abc}\gamma_c\Psi - D_c(\overline{\Psi})\gamma^{abc}\Psi - k\overline{\Psi}\gamma_c\gamma^{abc}\Psi &(6.4)\\ &= \overline{\Psi}\gamma^{abc}D_c \Psi - D_c(\overline{\Psi})\gamma^{abc}\Psi, &(6.5)\end{aligned}$$

i.e. the Witten-Nester two-form is the same as it would be if there was no cosmological constant at all. Following the proof of theorem 3.2, this time the Lichnerowicz identity says

$$\begin{aligned} P_a D_b E^{ba}(\Psi) = 2(&\nabla_I(\Psi)^\dagger\nabla^I\Psi - (\gamma^I\nabla_I\Psi)^\dagger\gamma^J\nabla_J\Psi + 4\pi T_{0a}\overline{\Psi}\gamma^a\Psi \\ &+ 2k\Psi^\dagger\gamma^I\nabla_I\Psi + 2k(\gamma^I\nabla_I\Psi)^\dagger\Psi), \end{aligned} \tag{6.6}$$

This differs from the $\Lambda \leq 0$ case in that there are now cross-terms between Ψ and $\nabla_I\Psi$. Nonetheless, if $\gamma^I\nabla_I\Psi = 0$, then $P_a D_b E^{ba}(\Psi) \geq 0$.

If I follow the same idea as chapter 5 to extend the Dougan-Mason definition, then it makes sense to start with a spinor, $\Phi = (\varphi_\alpha, \bar{\xi}^{\dot{\alpha}})^T$, which solves $\overline{m}^a\nabla_a\Phi = 0$ on S. Following the same steps as lemma 5.1,

$$\begin{aligned} \overline{m}^a\nabla_a\Phi = 0 \iff 0 &= \bar{\eth}\varphi_o + \mu\varphi_\iota - k\sqrt{2}\,\bar{\xi}_o, &(6.7)\\ 0 &= \bar{\eth}\bar{\xi}_\iota - \rho\bar{\xi}_o - k\sqrt{2}\,\varphi_\iota, &(6.8)\\ 0 &= \bar{\eth}\varphi_\iota - \bar{\sigma}\varphi_o \text{ and} &(6.9)\\ 0 &= \bar{\eth}\bar{\xi}_o + \lambda\bar{\xi}_\iota &(6.10)\end{aligned}$$

on S. Then, the arguments in the $\Lambda < 0$ case regarding the number of solutions to this system of equations is unchanged. Likewise,

$$T^{AB} = \varepsilon^{\alpha\beta}\varphi^A_\alpha\varphi^B_\beta - \varepsilon^{\dot{\alpha}\dot{\beta}}\bar{\xi}^A_{\dot{\alpha}}\bar{\xi}^B_{\dot{\beta}} = \sqrt{2}\left(\varphi^A_o\varphi^B_\iota - \varphi^B_o\varphi^A_\iota + \bar{\xi}^B_o\bar{\xi}^A_\iota - \bar{\xi}^A_o\bar{\xi}^B_\iota\right) \tag{6.11}$$

remains constant on S by the $\overline{m}^a\nabla_a\Phi = 0$ equations and Liouville's theorem. Hence, a natural candidate for quasilocal mass is once again

$$m(S)^2 = -\frac{1}{256\pi^2}\mathrm{tr}(QT^{-1}\overline{Q}\,\overline{T}^{-1}), \tag{6.12}$$

like in the $\Lambda \leq 0$ case. Note that in light of equations 6.5, 6.7 & 6.8 and lemma 3.4, this time

$$\begin{aligned} Q(\Phi) &= 4\int_S \left(\varphi_\iota\eth\overline{\varphi}_o + \overline{\varphi}_\iota\bar{\eth}\varphi_o - \overline{\xi}_o\eth\xi_\iota - \xi_o\bar{\eth}\overline{\xi}_\iota + \rho|\varphi_o|^2 + \mu|\varphi_\iota|^2 + \rho|\xi_o|^2 + \mu|\xi_\iota|^2\right)\mathrm{d}A \quad (6.13) \\ &= 4\int_S \Big(\varphi_\iota(-\mu\overline{\varphi}_\iota + \sqrt{2}k\xi_o) + \overline{\varphi}_\iota(-\mu\varphi_\iota + \sqrt{2}k\overline{\xi}_o) - \overline{\xi}_o(\rho\xi_o + \sqrt{2}k\overline{\varphi}_\iota) \\ &\qquad - \xi_o(\rho\overline{\xi}_o + \sqrt{2}k\varphi_\iota) + \rho|\varphi_o|^2 + \mu|\varphi_\iota|^2 + \rho|\xi_o|^2 + \mu|\xi_\iota|^2\Big)\mathrm{d}A \quad (6.14) \\ &= 4\int_S \left(\rho|\varphi_o|^2 - \mu|\varphi_\iota|^2 - \rho|\xi_o|^2 + \mu|\xi_\iota|^2\right)\mathrm{d}A. \quad (6.15) \end{aligned}$$

However, problems start to arise from here. First, imitating the positivity proof in the $\Lambda < 0$ case would suggest that I start with a Dirac spinor, $\Psi = (\psi_\alpha, \overline{\chi}^{\dot{\alpha}})^T$, such that $\gamma^I\nabla_I\Psi = 0$ on Σ and $\psi_o = \varphi_o$ & $\chi_\iota = \xi_\iota$ on S. Then, from equations 6.15 & 6.5 and lemma 3.4,

$$\begin{aligned} Q(\Psi) &= 4\int_S \left(\psi_\iota\eth\overline{\psi}_o + \overline{\psi}_\iota\bar{\eth}\psi_o - \overline{\chi}_o\eth\chi_\iota - \chi_o\bar{\eth}\overline{\chi}_\iota + \rho|\psi_o|^2 + \mu|\psi_\iota|^2 + \rho|\chi_o|^2 + \mu|\chi_\iota|^2\right)\mathrm{d}A \quad (6.16) \\ &= 4\int_S \Big(\psi_\iota(-\mu\overline{\varphi}_\iota + \sqrt{2}k\xi_o) + \overline{\psi}_\iota(-\mu\varphi_\iota + \sqrt{2}k\overline{\xi}_o) - \overline{\chi}_o(\rho\xi_o + \sqrt{2}k\overline{\varphi}_\iota) \\ &\qquad - \chi_o(\rho\overline{\xi}_o + \sqrt{2}k\varphi_\iota) + \rho|\varphi_o|^2 + \mu|\psi_\iota|^2 + \rho|\chi_o|^2 + \mu|\xi_\iota|^2\Big)\mathrm{d}A \quad (6.17) \\ &= Q(\Phi) + 4\int_S \left(\mu|\psi_\iota - \varphi_\iota|^2 + \rho|\chi_o - \xi_o|^2 + \sqrt{2}k(\psi_\iota\xi_o + \overline{\psi}_\iota\overline{\xi}_o - \overline{\chi}_o\overline{\varphi}_\iota - \chi_o\varphi_\iota)\right)\mathrm{d}A, \quad (6.18) \end{aligned}$$

from which I can't see any obvious positivity property for $Q(\Phi)$. Incidentally, loss of positivity is a feature of Tod's generalisation of the Penrose quasilocal mass to spacetimes with $\Lambda > 0$ [115], so perhaps it is not surprising if the present prescription were to also suffer from the same affliction.

When $\Lambda < 0$, McSharry and Reall [88] show that producing a positive geometric invariant is easier when choosing Φ to solve an equation from $m^a\nabla_a\Phi = 0$ and an equation from $\overline{m}^a\nabla_a\Phi = 0$. Briefly suppose once again that $\Lambda < 0$. Then,

$$\begin{aligned} m^a\nabla_a\Phi = 0 \iff 0 &= \eth\varphi_\iota - \rho\varphi_o + \mathrm{i}k\sqrt{2}\,\overline{\xi}_\iota, & (6.19) \\ 0 &= \eth\overline{\xi}_o + \mu\overline{\xi}_\iota + \mathrm{i}k\sqrt{2}\,\varphi_\iota, & (6.20) \\ 0 &= \eth\varphi_o + \bar{\lambda}\varphi_\iota \text{ and} & (6.21) \\ 0 &= \eth\overline{\xi}_o - \sigma\overline{\xi}_o. & (6.22) \end{aligned}$$

Suppose Φ solves

$$\bar{\eth}\varphi_o + \mu\varphi_\iota - \mathrm{i}\sqrt{2}k\overline{\xi}_o = 0 \text{ and } \eth\overline{\xi}_o + \mu\overline{\xi}_\iota + \mathrm{i}\sqrt{2}k\varphi_o = 0 \tag{6.23}$$

on S, i.e. one equation from the $m^a\nabla_a\Phi = 0$ set and one equation from the $\overline{m}^a\nabla_a\Phi = 0$ set. Then, [88] show

$$Q(\Phi) = Q(\Psi) - 4\int_S \mu\left(|\psi_\iota - \varphi_\iota|^2 + |\chi_\iota - \xi_\iota|^2\right)\mathrm{d}A, \tag{6.24}$$

which implies $Q(\Phi) \geq 0$ whenever $\theta_n \leq 0$, which is a much weaker condition than I needed in theorem 5.3. McSharry and Reall [88] were interested in the third law of black hole mechanics, rather than quasilocal mass, which meant that wider applicability of a non-negativity result was more desirable to them than the other properties which I studied in chapter 5.

Nonetheless, if I am seeking a non-negative geometric invariant when $\Lambda > 0$, it's worth also trying their approach. Hence, suppose

$$\bar{\eth}\varphi_o + \mu\varphi_\iota - \sqrt{2}k\overline{\xi}_o = 0 \text{ and } \bar{\eth}\overline{\xi}_o + \mu\overline{\xi}_\iota + \sqrt{2}k\varphi_o = 0 \tag{6.25}$$

on S. Then,

$$Q(\Phi) = 4\int_S \Big(\rho|\varphi_o|^2 + \rho|\xi_o|^2 - \mu|\varphi_\iota|^2 - \mu|\xi_\iota|^2 + \sqrt{2}k(\varphi_\iota\xi_o - \varphi_o\xi_\iota + \overline{\varphi}_\iota\overline{\xi}_o - \overline{\varphi}_o\overline{\xi}_\iota)\Big)\mathrm{d}A. \tag{6.26}$$

Again, suppose $\gamma^I\nabla_I\Psi = 0$ on Σ and $\psi_o = \varphi_o$ & $\chi_\iota = \xi_\iota$ on S. Then, this time the result is

$$\begin{aligned} Q(\Phi) = Q(\Psi) - 4\int_S \big(&\mu|\psi_\iota - \varphi_\iota|^2 + \mu|\xi_\iota - \chi_\iota|^2 \\ &+ \sqrt{2}k(\xi_o(\psi_\iota - \varphi_\iota) + \varphi_o(\xi_\iota - \chi_\iota) + \overline{\xi}_o(\overline{\psi}_\iota - \overline{\varphi}_\iota) + \overline{\varphi}_o(\overline{\xi}_\iota - \overline{\chi}_\iota))\big)\mathrm{d}A, \end{aligned} \tag{6.27}$$

which again doesn't seem to me to have any obvious positivity property.

Continuing again with my $\overline{m}^a\nabla_a\Phi = 0$ prescription and equation 6.12, one might hope that although positivity might fail, the other desirable properties may still hold. Alas, I will show this is not the case.

The simplest property to study is the behaviour of the candidate quasilocal mass in spherical symmetry. Once again, I will use double null coordinates, where

$$g = -\Omega(u,v)^2(\mathrm{d}u \otimes \mathrm{d}v + \mathrm{d}v \otimes \mathrm{d}u) + r(u,v)^2 g_{S^2}, \tag{6.28}$$

$\partial_u r > 0$ and $\partial_v r < 0$. Once again, the natural NP tetrad for any S^2_r is

$$l = \frac{1}{\Omega}\frac{\partial}{\partial u},\ n = \frac{1}{\Omega}\frac{\partial}{\partial v} \text{ and } m = \frac{1}{r\sqrt{2}}\left(\frac{\partial}{\partial\theta} + \frac{\mathrm{i}}{\sin(\theta)}\frac{\partial}{\partial\phi}\right). \tag{6.29}$$

Then, following the same steps as section 5.2.1,

$$\sigma = \lambda = 0,\ \rho = -\frac{\partial_u r}{\Omega r},\ \mu = \frac{\partial_v r}{\Omega r},\ \beta = -\alpha = \frac{1}{2\sqrt{2}r}\cot(\theta), \tag{6.30}$$

$$\xi_o = \bar{c}_1\left({}_{1/2}Y_{1/2,-1/2}\right) + \bar{c}_2\left({}_{1/2}Y_{1/2,1/2}\right), \tag{6.31}$$

$$\varphi_\iota = c_3\left({}_{-1/2}Y_{1/2,-1/2}\right) + c_4\left({}_{-1/2}Y_{1/2,1/2}\right), \tag{6.32}$$

$$\varphi_o = -\left(\frac{\sqrt{2}}{\Omega}\partial_v(r)c_3 + 2krc_2\right)\left({}_{1/2}Y_{1/2,-1/2}\right) - \left(\frac{\sqrt{2}}{\Omega}\partial_v(r)c_4 - 2krc_1\right)\left({}_{1/2}Y_{1/2,1/2}\right), \tag{6.33}$$

$$\xi_\iota = \left(\frac{\sqrt{2}}{\Omega}\partial_u(r)\bar{c}_1 - 2kr\bar{c}_4\right)\left({}_{-1/2}Y_{1/2,-1/2}\right) + \left(\frac{\sqrt{2}}{\Omega}\partial_u(r)\bar{c}_2 + 2kr\bar{c}_3\right)\left({}_{-1/2}Y_{1/2,1/2}\right), \tag{6.34}$$

$$T^{AB} \equiv \frac{1}{\pi\Omega}\begin{bmatrix} 0 & -\partial_u(r) & -\sqrt{2}\Omega kr & 0 \\ \partial_u(r) & 0 & 0 & -\sqrt{2}\Omega kr \\ \sqrt{2}\Omega kr & 0 & 0 & -\partial_v(r) \\ 0 & \sqrt{2}\Omega kr & \partial_v(r) & 0 \end{bmatrix}, \tag{6.35}$$

$$Q^{AB} \equiv \frac{4r(2\partial_u(r)\partial_v(r) + \Omega^2(1-4k^2r^2))}{\Omega^3}\begin{bmatrix} \partial_u(r) & 0 & 0 & 0 \\ 0 & \partial_u(r) & 0 & 0 \\ 0 & 0 & -\partial_v(r) & 0 \\ 0 & 0 & 0 & -\partial_v(r) \end{bmatrix} \text{ and} \tag{6.36}$$

$$m(S^2_r)^2 = \left(1 + \frac{2\Omega^2k^2r^2}{\partial_u(r)\partial_v(r) - 2\Omega^2k^2r^2}\right)m_{\mathrm{MS}}(S^2_r)^2, \tag{6.37}$$

where $m_{\text{MS}}(S_r^2)$ is the Misner-Sharp mass for $\Lambda > 0$. Hence, there is never agreement with the Misner-Sharp mass.

While this result is not encouraging, it is not necessarily terminal for this definition of quasilocal mass. In the standard Schwarzschild spacetime (i.e. where $\Lambda = 0$), one recognises the parameter, M, as the mass by taking the Newtonian limit. However, when $\Lambda > 0$, there is typically no such weak, far field regime. Therefore, it is not necessarily clear whether the paramater, M, in the Schwarzschild-dS metric represents mass. Hence, disagreement with the Misner-Sharp mass (which yields M for every S_r^2 in Schwarzschild-dS) is not guaranteed to be a result of any deficiency in the candidate quasilocal mass definition.

Another test which may help navigate these rocky shores is the analogue of section 5.4, i.e. gravity linearised about dS due to some infinitesimal energy-momentum tensor, T_{ab}. dS has the full set of solutions to $\nabla_a \varepsilon_k = 0$. In particular, in cosmological coordinates (which are the simplest coordinates for dS, albeit they are not global coordinates) and $k = 1/2$ units,

$$g_{\text{dS}} = -\mathrm{d}t \otimes \mathrm{d}t + \mathrm{e}^{2t}(\mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y + \mathrm{d}z \otimes \mathrm{d}z) \tag{6.38}$$

leads to the natural frame,

$$e^0 = \mathrm{d}t, \;\; e^1 = \mathrm{e}^t \mathrm{d}x \;\; e^2 = \mathrm{e}^t \mathrm{d}y, \;\; e^3 = \mathrm{e}^t \mathrm{d}z \;\text{ and subsequently} \tag{6.39}$$

$$\varepsilon_k = \mathrm{e}^{t/2} P_0^+ \varepsilon_+ - \left(\mathrm{e}^{t/2} x_I \gamma^I - \mathrm{e}^{-t/2} I\right) P_0^- \varepsilon_- \tag{6.40}$$

for $P_0^\pm = \frac{1}{2}(I \pm \gamma^0)$ and constant spinors, $\varepsilon_\pm$. Note however that unlike $\Lambda \leq 0$, this time $\bar{\varepsilon}_k \gamma^a \varepsilon_k$ is only a conformal Killing vector. Let

$$g = g_{\text{dS}} + \eta h \tag{6.41}$$

for η infinitesimal. Assume T_{ab} is $O(\eta)$. Then, the arguments in the $\Lambda \leq 0$ case go through to imply $\Phi = \varepsilon_k + \eta \mathcal{Z}$. However, this time, when $Q(\Phi)$ is converted from an integral over S into an integral over Σ using the Lichnerowicz identity of equation 6.6,

$$\begin{aligned} Q(\Phi) = 2 \int_\Sigma \big(& \nabla_I(\Phi)^\dagger \nabla^I \Phi - (\gamma^I \nabla_I \Phi)^\dagger \gamma^J \nabla_J \Phi + 4\pi T_{0a} \overline{\Phi} \gamma^a \Phi \\ &+ \Phi^\dagger \gamma^I \nabla_I \Phi + (\gamma^I \nabla_I \Phi)^\dagger \Phi\big) \mathrm{d}V \end{aligned} \tag{6.42}$$

$$= 2 \int_\Sigma \left(4\pi T_{0a} \bar{\varepsilon}_k \gamma^a \varepsilon_k + \eta \varepsilon_k^\dagger \gamma^I \nabla_I \mathcal{Z} + \eta (\gamma^I \nabla_I \mathcal{Z})^\dagger \varepsilon_k + O(\eta^2)\right) \mathrm{d}V. \tag{6.43}$$

Hence, unlike equation 5.124, it appears that even calculating the lineared limit requires some knowledge about how Φ is extended inside Σ; the result is not constructed only out of T_{ab} and special vectors of the background like when $\Lambda \leq 0$. This issue can only be avoided if $\nabla_I \Phi$ itself was $O(\eta^2)$, instead of $O(\eta)$.

When Σ is non-compact - e.g. as in chapter 4 - an essential part of Witten's method is a spinor, ε, which solves $\gamma^I \nabla_I \varepsilon = 0$ on Σ. However, in the Dougan-Mason approach, a spinor, Ψ, solving $\gamma^I \nabla_I \Psi = 0$ is a mere tool in proving $Q(\Phi) \geq 0$, but serves no actual purpose in defining the quasilocal mass itself. Perhaps this is unsurprising. After all, in chapter 4, $Q(\varepsilon)$ could be evaluated on $\Sigma_{t,\infty}$ solely on the basis that $\varepsilon \to \varepsilon_k$, without any reference to $\gamma^I \nabla_I \varepsilon = 0$ on Σ. Alternatively, perhaps solutions of the Dirac equation should be more directly exploited in defining a spinorial quasilocal energy.

It's possible the linearised gravity issue and the positivity issue for quasilocal mass with $\Lambda > 0$ can both be solved by working with just Ψ (the spinor solving $\gamma^I \nabla_I \Psi = 0$ in Σ with appropriate boundary conditions on $\partial\Sigma = S$) instead of Φ (the spinor solving $\overline{m}^a \nabla_a \Phi = 0$ on S). Instead of a 4D space of solutions to $\overline{m}^a \nabla_a \Phi = 0$ on S, one would have to find an appropriate boundary condition that fixes two out of four components of Ψ on S (with the other two components being fixed by enforcing $\gamma^I \nabla_I \Psi = 0$ on Σ). Then, there would a 2D space of freedom. This boundary condition would have to be chosen such that the PDE problem is well-posed and such that there exists an analogue of T^{AB} (which could then be normalised as ε^{AB} since the matrix is now only 2×2). Then, perhaps one could proceed analogously to Dougan-Mason and even define a quasilocal energy-momentum vector.

It would be very interesting in future to pursue this and the other lines of research suggested in this chapter. For now, the quest to find a truly satisfactory definition of quasilocal mass remains unfulfilled.

Appendix A

Conventions

And may I say - not in a shy way
Oh no, oh no, not me
I did it my way
- From *My way*, as popularised by Frank Sinatra

My conventions are based largely off [17]; the main points are listed below.

The metric signature is $(-1, +1, \cdots, +1)$.

The following symbols are frequently used.

- M: The full spacetime
- n: The dimension of M
- g: The Lorentzian metric on M
- Σ: $(n-1)$D, spacelike hypersurface
- S: $(n-2)$D, spacelike surface
- $\mathcal{I}$: Conformal infinity
- Σ_t: A spacelike hypersurface with constant t coordinate in an open neighbourhood of $\mathcal{I}$
- $\Sigma_{t,\infty} = \Sigma_t \cap \mathcal{I}$
- Λ: A negative cosmological constant, often normalised to $-\frac{1}{2}(n-1)(n-2)$
- $k = \sqrt{-\frac{\Lambda}{2(n-1)(n-2)}}$
- $f_{(l)}$: The $O(\mathrm{e}^{2r}\mathrm{e}^{-lr})$ tensor in a Fefferman-Graham expansion
- C_c^∞: The space of smooth Dirac spinors on Σ subject to the conditions in definition 3.5.
- $\mathcal{H}$: The completion of C_c^∞ under the inner product in lemma 3.6 or lemma 3.7.
- $\overline{\Psi} = \Psi^\dagger \gamma^0$ for a Dirac spinor, Ψ
- I: The identity matrix or operator
- D_a: The Levi-Civita connection of g

- $\nabla_a \Psi = D_a \Psi + \mathrm{i}k\gamma_a \Psi + \mathcal{A}_a \Psi$ for a Dirac spinor, Ψ, and a spinor endomorphism-valued one-form, $\mathcal{A}_a$
- $\nabla_a \overline{\Psi} = D_a \overline{\Psi} - \mathrm{i}k\overline{\Psi}\gamma_a + \overline{\Psi}\gamma^0 \mathcal{A}_a^\dagger \gamma^0 = (\nabla_a \Psi)^\dagger \gamma^0$
- $\mathfrak{D}$: Dirac operator from $\mathcal{H} \to L^2$ defined by $\Psi \mapsto \gamma^I \nabla_I \Psi$
- $\{o_\alpha,\, \iota_\alpha\}$: A GHP spinor dyad
- $\delta = m^a D_a$ in the context of the NP formalism
- $\bar{\delta} = \overline{m}^a D_a$ in the context of the NP formalism
- $D = l^a D_a$ in the context of the NP formalism
- $\Delta = n^a D_a$ in the context of the NP formalism

Many different types of indices are used, as given below. Unless otherwise stated, they carry the following meaning.

- $a, b, c, \cdots$ are vielbein indices running $0, 1, \cdots, n-1$. However, in most equations it will be apparent that these could equally well denote abstract indices.
- $\mu, \nu, \rho, \cdots$ are coordinate indices running $0, 1, \cdots, n-1$.
- $I, J, K, \cdots$ are vielbein indices running $1, 2, \cdots, n-1$.
- Depending on context, $\alpha, \beta, \gamma, \cdots$ are either two-component spinor indices for the $(1/2, 0)$ representation (left-handed Weyl spinors) running 1, 2 or coordinate indices running $2, 3, \cdots, n-1$.
- $\dot{\alpha}, \dot{\beta}, \dot{\gamma}, \cdots$ are two-component spinor indices for the $(0, 1/2)$ representation (right-handed Weyl spinors) and run $\dot{1}$, $\dot{2}$.
- Depending on context, $A, B, C, \cdots$ are either vielbein indices running $2, 3, \cdots, n-1$ or indices for the linearly independent solutions to $\overline{m}^a \nabla_a \Phi = 0$.
- $m, n, p, q, \cdots$ are coordinate indices running $0, 2, 3, \cdots, n-1$.
- $M, N, P, Q, \cdots$ are vielbein indices running $0, 2, 3, \cdots, n-1$. However, in section 5.3 only, they run $0, 1, 2, 3, 4$ and index the embedding Cartesian coordinates when AdS is viewed as a surface in $\mathbb{R}^{3,2}$.

The Riemann tensor is defined such that $[D_a, D_b]V^c = R^c{}_{dab} V^d$.

Complex conjugation of an object - unless it is a Dirac spinor - will be denoted by a bar over the object, e.g. $\bar{z}$.

Levi-Civita symbols are normalised by $\varepsilon_{12} = -1$, $\varepsilon^{12} = 1$, $\varepsilon_{\dot{1}\dot{2}} = -1$, $\varepsilon^{\dot{1}\dot{2}} = 1$, $\varepsilon_{0123} = -1$ and $\varepsilon^{0123} = 1$. Then, $\varepsilon^{\alpha\gamma}\varepsilon_{\gamma\beta} = \delta^\alpha{}_\beta$ and likewise for the dotted indices.

Two-component spinors are raised and lowered from the left, i.e. $\psi_\alpha = \varepsilon_{\alpha\beta}\psi^\beta$ and $\psi^\alpha = \varepsilon^{\alpha\beta}\psi_\beta$.

The extended Pauli matrices are

$$(\sigma_a)_{\alpha\dot{\alpha}} \equiv (I, \sigma_1, \sigma_2, \sigma_3) \text{ and} \tag{A.1}$$

$$(\tilde{\sigma}_a)^{\dot{\alpha}\alpha} = \varepsilon^{\alpha\beta}\varepsilon^{\dot{\alpha}\dot{\beta}}(\sigma_a)_{\beta\dot{\beta}} \equiv (I, -\sigma_1, -\sigma_2, -\sigma_3), \tag{A.2}$$

with $\sigma_{1,2,3}$ being the standard Pauli matrices.

The conversion between vierbein indices and two-component spinor indices is by $V_{\alpha\dot{\alpha}} = (\sigma_a)_{\alpha\dot{\alpha}} V^a$ and $V_a = -\frac{1}{2}(\tilde{\sigma}_a)^{\dot{\alpha}\alpha} V_{\alpha\dot{\alpha}}$.

In 4D, Dirac spinors are decomposed into two-component spinors by $\Psi = (\psi_\alpha, \bar{\chi}^{\dot{\alpha}})^T$.

The gamma matrices are chosen so that $\gamma^a\gamma^b + \gamma^b\gamma^a = -2g^{ab}I$, $(\gamma^0)^\dagger = \gamma^0$ and $(\gamma^I)^\dagger = -\gamma^I$.

In 4D, the gamma matrices are in the Weyl representation, i.e.

$$\gamma_a = \begin{bmatrix} 0 & (\sigma_a)_{\alpha\dot{\alpha}} \\ (\tilde{\sigma}_a)^{\dot{\alpha}\alpha} & 0 \end{bmatrix}. \tag{A.3}$$

The Dirac conjuage is defined to be $\overline{\Psi} = \Psi^\dagger\gamma^0$. In 4D that becomes $\overline{\Psi} = (-\chi^\alpha, -\overline{\psi}_{\dot{\alpha}})$.

The antisymmetric product, $\gamma^{[a_1}\cdots\gamma^{a_n]}$, is denoted by $\gamma^{a_1\cdots a_n}$.

Antisymmetric combinations of the Pauli matrices are sometimes written in terms of

$$(\sigma_{ab})_\alpha{}^\beta = -\frac{1}{4}((\sigma_a)_{\alpha\dot{\alpha}}(\tilde{\sigma}_b)^{\dot{\alpha}\beta} - (\sigma_b)_{\alpha\dot{\alpha}}(\tilde{\sigma}_a)^{\dot{\alpha}\beta}) \equiv \frac{1}{2}\begin{bmatrix} 0 & \sigma_1 & \sigma_2 & \sigma_3 \\ -\sigma_1 & 0 & \mathrm{i}\sigma_3 & -\mathrm{i}\sigma_2 \\ -\sigma_2 & -\mathrm{i}\sigma_3 & 0 & \mathrm{i}\sigma_1 \\ -\sigma_3 & \mathrm{i}\sigma_2 & -\mathrm{i}\sigma_1 & 0 \end{bmatrix} \tag{A.4}$$

$$\text{and}\ \ (\tilde{\sigma}_{ab})^{\dot{\alpha}}{}_{\dot{\beta}} = -\frac{1}{4}((\tilde{\sigma}_a)^{\dot{\alpha}\alpha}(\sigma_b)_{\alpha\dot{\beta}} - (\tilde{\sigma}_b)^{\dot{\alpha}\alpha}(\sigma_a)_{\alpha\dot{\beta}}) \equiv \frac{1}{2}\begin{bmatrix} 0 & -\sigma_1 & -\sigma_2 & -\sigma_3 \\ \sigma_1 & 0 & \mathrm{i}\sigma_3 & -\mathrm{i}\sigma_2 \\ \sigma_2 & -\mathrm{i}\sigma_3 & 0 & \mathrm{i}\sigma_1 \\ \sigma_3 & \mathrm{i}\sigma_2 & -\mathrm{i}\sigma_1 & 0 \end{bmatrix}. \tag{A.5}$$

The charge conjugation matrix is

$$C = \begin{bmatrix} \varepsilon_{\alpha\beta} & 0 \\ 0 & \varepsilon^{\dot{\alpha}\dot{\beta}} \end{bmatrix} \iff C^{-1} = \begin{bmatrix} \varepsilon^{\alpha\beta} & 0 \\ 0 & \varepsilon_{\dot{\alpha}\dot{\beta}} \end{bmatrix}. \tag{A.6}$$

The NP coefficients are defined as follows.

$$\kappa = -m^a D l_a \tag{A.7}$$
$$\tau = -m^a \Delta l_a \tag{A.8}$$
$$\sigma = -m^a \delta l_a \tag{A.9}$$
$$\rho = -m^a \bar{\delta} l_a \tag{A.10}$$
$$\pi = \overline{m}^a D n_a \tag{A.11}$$
$$\nu = \overline{m}^a \Delta n_a \tag{A.12}$$
$$\mu = \overline{m}^a \delta n_a \tag{A.13}$$
$$\lambda = \overline{m}^a \bar{\delta} n_a \tag{A.14}$$
$$2\varepsilon = \overline{m}^a D m_a - n^a D l_a \tag{A.15}$$
$$2\gamma = \overline{m}^a \Delta m_a - n^a \Delta l_a \tag{A.16}$$
$$2\beta = \overline{m}^a \delta m_a - n^a \delta l_a \tag{A.17}$$
$$2\alpha = \overline{m}^a \bar{\delta} m_a - n^a \bar{\delta} l_a \tag{A.18}$$

The spin-weighted spherical harmonics used in section 5.2 are

$$\begin{aligned}\left({}_{1/2}Y_{1/2,1/2}\right) &= \frac{\mathrm{i}}{\sqrt{2\pi}}\sin\left(\frac{\theta}{2}\right)\mathrm{e}^{\mathrm{i}\phi/2}, \quad \left({}_{1/2}Y_{1/2,-1/2}\right) = -\frac{\mathrm{i}}{\sqrt{2\pi}}\cos\left(\frac{\theta}{2}\right)\mathrm{e}^{-\mathrm{i}\phi/2},\\ \left({}_{-1/2}Y_{1/2,1/2}\right) &= \frac{\mathrm{i}}{\sqrt{2\pi}}\cos\left(\frac{\theta}{2}\right)\mathrm{e}^{\mathrm{i}\phi/2} \text{ and } \left({}_{-1/2}Y_{1/2,-1/2}\right) = \frac{\mathrm{i}}{\sqrt{2\pi}}\sin\left(\frac{\theta}{2}\right)\mathrm{e}^{-\mathrm{i}\phi/2}. \qquad \text{(A.19)}\end{aligned}$$

In some parts of chapter 4 it will be necessary to consider both Dirac spinors on an $(n-2)$-dimensional surface and Dirac spinors on the full spacetime. In these cases, the spacetime spinor space is viewed as a direct sum of the $(n-2)$-dimensional surface's spinor space with itself. Furthermore, the gamma matrices for the spacetime are then chosen as

$$\gamma^0 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \ \gamma^1 = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \text{ and } \gamma^A = \begin{bmatrix} 0 & \hat{\gamma}^A \\ \hat{\gamma}^A & 0 \end{bmatrix}, \qquad \text{(A.20)}$$

where $\{\hat{\gamma}^A\}$ are the gamma matrices (in any representation) of the $(n-2)$-dimensional surface. In this context, hats will be placed on all quantities intrinsic to the surface.

A.1 Comparison to Penrose-Rindler conventions

The monogrpahs of Penrose and Rindler [100, 101] (PR) are popular references for two-component spinors. However, their conventions differ significantly at times from mine; the key differences are listed below.

- I use a mostly plus metric while PR use a mostly minus metric.
- I use lowercase letters from the start of the Greek alphabet for two-component spinor indices while PR use uppercase Latin letters.
- For two-component spinors, my undotted indices correspond to their primed indices and my dotted indices correspond to their unprimed indices.
- I take two-component spinor indices to run over the values 1 and 2, whereas PR take them to run over 0 and 1.
- My spinor index conversion is $V_{\alpha\dot{\alpha}} = (\sigma_a)_{\alpha\dot{\alpha}}V^a$, while PR define $V_{A'A} = \frac{1}{\sqrt{2}}(\sigma_a)_{A'A}V^a$.
- The $\sqrt{2}$ difference when converting to spinor indices means $\iota^\alpha o_\alpha = \bar{\iota}^{\dot{\alpha}}\bar{o}_{\dot{\alpha}} = \sqrt{2}$ here while PR have $\iota^{A'}o_{A'} = \iota^A o_A = 1$.
- For any spinor, ψ_α, the ψ_o and ψ_ι in equation 2.60 would be $\frac{1}{\sqrt{2}}\psi_{1'}$ and $-\frac{1}{\sqrt{2}}\psi_{0'}$ respectively in their notation. Likewise, $\overline{\psi}_o$ and $\overline{\psi}_\iota$ are $\frac{1}{\sqrt{2}}\psi_1$ and $-\frac{1}{\sqrt{2}}\psi_0$ respectively.
- Dirac spinors are $\Psi = (\psi_\alpha,\, \bar{\chi}^{\dot{\alpha}})^T$ for me while PR would write $(\bar{\chi}^A,\, \psi_{A'})^T$, i.e. the left and right handed components are written in the opposite order.
- I raise and lower indices from the left, i.e. $\psi^\alpha = \varepsilon^{\alpha\beta}\psi_\beta$ and $\psi_\alpha = \varepsilon_{\alpha\beta}\psi^\beta$, while PR raise from the left, but lower from the right, i.e. $\psi^A = \varepsilon^{AB}\psi_B$ but $\psi_A = \psi^B\varepsilon_{BA}$. This difference means $\varepsilon^{12} = 1 \implies \varepsilon_{12} = -1$ for me. Furthermore, it means $\varepsilon^{\alpha\gamma}\varepsilon_{\gamma\beta} = \delta^\alpha{}_\beta$ for me while they have $\varepsilon^{AC}\varepsilon_{CB} = -\delta^A{}_B$.

Appendix B

Frequently used spinor identities

Part of the art of bowling spin is to make the batsman think something special is happening when it isn't.
- Shane Warne

In this appendix I seek to de-mystify the practical application of spin by collecting some identities I use liberally throughout this thesis.

I deploy the following gamma matrix identities regularly.

$$\gamma^a\gamma^b + \gamma^b\gamma^a = -2g^{ab}I \tag{B.1}$$

$$\gamma^{a_1\cdots a_k}\gamma_{a_k} = -(n-k-1)\gamma^{a_1\cdots a_{k-1}} \tag{B.2}$$

$$\gamma^{ab}\gamma_c = \gamma_c\gamma^{ab} - 2\delta^b{}_c\gamma^a + 2\delta^a{}_c\gamma^b \tag{B.3}$$

$$\gamma^{ab}\gamma_c = \gamma^{ab}{}_c - \delta^b{}_c\gamma^a + \delta^a{}_c\gamma^b \tag{B.4}$$

$$\gamma_a\gamma^{bc} = \gamma_a{}^{bc} - \delta^b{}_a\gamma^c + \delta^c{}_a\gamma^b \tag{B.5}$$

$$\gamma^{abc}\gamma_{de} = \gamma^{abc}{}_{de} - 6\gamma^{[ab}{}_{[e}\delta^{c]}{}_{d]} + 6\gamma^{[a}\delta^b{}_{[e}\delta^{c]}{}_{d]} \tag{B.6}$$

$$\gamma^{ab}\gamma_{cd} = \gamma^{ab}{}_{cd} + \delta^a{}_c\gamma^b\gamma_c - \delta^a{}_d\gamma^b\gamma_c - \delta^b{}_c\gamma^a\gamma_d + \delta^b{}_d\gamma^a\gamma_c + \delta^a{}_c\delta^b{}_dI - \delta^a{}_d\delta^b{}_cI \tag{B.7}$$

$$\gamma_a\gamma^{bcd} = \gamma_a{}^{bcd} - \delta^b{}_a\gamma^{cd} + \delta^c{}_a\gamma^{bd} - \delta^d{}_a\gamma^{bc} \tag{B.8}$$

The following two component spinor identities are indispensable.

$$V^{\alpha\dot\alpha}W_{\alpha\dot\alpha} = -2V^aW_a \tag{B.9}$$

$$\overline{(\psi_\alpha)} = \overline{\psi}_{\dot\alpha} \tag{B.10}$$

$$\psi_\alpha\chi^\alpha = -\psi^\alpha\chi_\alpha \tag{B.11}$$

$$\psi_\alpha\psi^\alpha = 0 \tag{B.12}$$

$$(\sigma_a)_{\alpha\dot\alpha}(\tilde\sigma_b)^{\dot\alpha\beta} + (\sigma_b)_{\alpha\dot\alpha}(\tilde\sigma_a)^{\dot\alpha\beta} = -2g_{ab}\delta_\alpha{}^\beta \tag{B.13}$$

$$(\tilde\sigma_a)^{\dot\alpha\alpha}(\sigma_b)_{\alpha\dot\beta} + (\tilde\sigma_b)^{\dot\alpha\alpha}(\sigma_a)_{\alpha\dot\beta} = -2g_{ab}\delta^{\dot\alpha}{}_{\dot\beta} \tag{B.14}$$

$$(\sigma_a)_{\alpha\dot\alpha}(\tilde\sigma_b)^{\dot\alpha\alpha} = -2g_{ab} \tag{B.15}$$

$$(\sigma^a)_{\alpha\dot\alpha}(\tilde\sigma_a)^{\dot\beta\beta} = -2\delta^\beta{}_\alpha\delta^{\dot\beta}{}_{\dot\alpha} \tag{B.16}$$

$$(\sigma_a)_{\alpha\dot\beta}(\tilde\sigma_b)^{\dot\beta\beta}(\sigma_c)_{\beta\dot\alpha} = g_{ca}(\sigma_b)_{\alpha\dot\alpha} - g_{bc}(\sigma_a)_{\alpha\dot\alpha} - g_{ab}(\sigma_c)_{\alpha\dot\alpha} + \mathrm{i}\varepsilon_{abcd}(\sigma^d)_{\alpha\dot\alpha} \tag{B.17}$$

$$(\tilde\sigma_a)^{\dot\alpha\beta}(\sigma_b)_{\beta\dot\beta}(\tilde\sigma_c)^{\dot\beta\alpha} = g_{ca}(\tilde\sigma_b)^{\dot\alpha\alpha} - g_{bc}(\tilde\sigma_a)^{\dot\alpha\alpha} - g_{ab}(\tilde\sigma_c)^{\dot\alpha\alpha} - \mathrm{i}\varepsilon_{abcd}(\tilde\sigma^d)^{\dot\alpha\alpha} \tag{B.18}$$

$$\varepsilon_{\alpha\beta}\varepsilon^{\gamma\delta} = -\left(\delta^\gamma{}_\alpha\delta^\delta{}_\beta - \delta^\delta{}_\alpha\delta^\gamma{}_\beta\right) \tag{B.19}$$

Finally, the following NP coefficients are needed in terms of GHP variables.

$$\sqrt{2}\mu = \sqrt{2}\overline{m}^a \delta n_a = \bar{\iota}^{\dot{\alpha}} \delta \bar{\iota}_{\dot{\alpha}} \tag{B.20}$$

$$\sqrt{2}\rho = -\sqrt{2}m^a \bar{\delta} l_a = \bar{o}^{\dot{\alpha}} \bar{\delta} \bar{o}_{\dot{\alpha}} \tag{B.21}$$

$$\sqrt{2}\alpha = \frac{1}{\sqrt{2}} \left(\overline{m}^a \bar{\delta} m_a - n^a \bar{\delta} l_a \right) = \bar{\iota}^{\dot{\alpha}} \bar{\delta} \bar{o}_{\dot{\alpha}} \tag{B.22}$$

$$\sqrt{2}\beta = \frac{1}{\sqrt{2}} \left(\overline{m}^a \delta m_a - n^a \delta l_a \right) = \bar{\iota}^{\dot{\alpha}} \delta \bar{o}_{\dot{\alpha}} \tag{B.23}$$

$$\sqrt{2}\sigma = -\sqrt{2}m^a \delta l_a = \bar{o}^{\dot{\alpha}} \delta \bar{o}_{\dot{\alpha}} \tag{B.24}$$

$$\sqrt{2}\lambda = \sqrt{2}\overline{m}^a \bar{\delta} n_a = \bar{\iota}^{\dot{\alpha}} \bar{\delta} \bar{\iota}_{\dot{\alpha}} \tag{B.25}$$

Bibliography

[1] L.F. Abbott and S. Deser. Stability of gravity with a cosmological constant. *Nuclear Physics B*, 195(1):76–96, 1982.

[2] R. Arnowitt, S. Deser, and C.W. Misner. The Dynamics of General Relativity. In *Gravitation: An introduction to current research*, pages 227–264. Wiley, 1962.

[3] A. Ashtekar, B. Bonga, and A. Kesavan. Asymptotics with a positive cosmological constant: I. Basic framework. *Classical and Quantum Gravity*, 32(2):025004, 2014.

[4] A. Ashtekar and S. Das. Asymptotically anti-de Sitter spacetimes: conserved quantities. *Classical and Quantum Gravity*, 17(2):L17, 2000.

[5] C. Bär. Real Killing Spinors and Holonomy. *Communications in Mathematical Physics*, 154:509–521, 1993.

[6] R. Bartnik. The Mass of an Asymptotically Flat Manifold. *Communications on Pure and Applied Mathematics*, 39(5):661–693, 1986.

[7] R. Bartnik. New Definition of Quasilocal Mass. *Physical Review Letters*, 62(20):2346, 1989.

[8] R.A. Bartnik and P. T. Chrusciel. Boundary value problems for Dirac-type equations, with applications, 2003. arXiv[math.DG/0307278].

[9] R.A. Bartnik and P.T. Chrusciel. Boundary value problems for Dirac-type equations. *Journal für die reine und angewandte Mathematik*, 579:13–73, 2005.

[10] H. Baum. Complete Riemannian manifolds with imaginary Killing spinors. *Annals of Global Analysis and Geometry*, 7:205–226, 1989.

[11] G. Bergqvist. Quasilocal mass for event horizons. *Classical and Quantum Gravity*, 9(7):1753, 1992.

[12] H. Bondi, M.J. van der Burg, and A.K. Metzner. Gravitational waves in general relativity, VII. Waves from axi-symmetric isolated system. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences*, 269(1336):21–52, 1962.

[13] P. Breitenlohner and D.Z. Freedman. Stability in Gauged Extended Supergravity. *Annals of Physics*, 144(2):249–281, 1982.

[14] S. Brendle and P.K. Hung. Systolic inequalities and the Horowitz-Myers conjecture, 2024. arXiv[math.DG/2406.04283].

[15] J.D. Brown and J.W. York Jr. Quasilocal energy and conserved charges derived from the gravitational action. *Physical Review D*, 47(4):1407, 1993.

[16] J.D. Brown, S.R. Lau, and J.W. York. Canonical quasilocal energy and small spheres. *Physical Review D*, 59(6):064028, 1999.

[17] I.L. Buchbinder and S.M. Kuzenko. *Ideas and Methods of Supersymmetry and Supergravity or A Walk Through Superspace*. Institute of Physics, Bristol, 1998.

[18] A. Cabo-Bizet, D. Cassani, D. Martelli, and S. Murthy. Microscopic origin of the Bekenstein-Hawking entropy of supersymmetric AdS5 black holes. *Journal of High Energy Physics*, 2019(10):62, 2019.

[19] M.M. Caldarelli and D. Klemm. Supersymmetry of Anti-de Sitter Black Holes. *Nuclear Physics B*, 545(1-3):434–460, 1999.

[20] B. Carter. Hamilton-Jacobi and Schrodinger separable solutions of Einstein's equations. *Communications in Mathematical Physics*, 10(4):280–310, 1968.

[21] S. Chandrasekhar. *The Mathematical Theory of Black Holes*. Oxford University Press, 1983.

[22] P.N. Chen, M.T. Wang, and S.T. Yau. Evaluating Small Sphere Limit of the Wang–Yau Quasi-Local Energy. *Communications in Mathematical Physics*, 357(2):731–774, 2018.

[23] M.C. Cheng and K. Skenderis. Positivity of energy for asymptotically locally AdS spacetimes. *Journal of High Energy Physics*, 2005(08):107, 2005.

[24] Z.W. Chong, M. Cvetic, H. Lu, and C.N. Pope. General Non-Extremal Rotating Black Holes in Minimal Five-Dimensional Gauged Supergravity. *Physical Review Letters*, 95(16):161301, 2005.

[25] D. Christodoulou and S.T. Yau. Some remarks on the quasi-local mass. In *Mathematics and General Relativity (Santa Cruz, CA 1986)*, pages 9–14. American Mathematical Society, 1986.

[26] P. Chrusciel. Boundary conditions at spatial infinity: From a Hamiltonian point of view. In *Topological properties and global structure of space-time*, pages 49–59. Springer US, 1986.

[27] P.T. Chrusciel. Lectures on energy in general relativity, 2010. Accessed on May 2, 2023 from `https://homepage.univie.ac.at/piotr.chrusciel/teaching/Energy/Energy.pdf`.

[28] P.T. Chrusciel and E. Delay. On Asymptotically Locally Hyperbolic Metrics with Negative Mass. *Symmetry, Integrability and Geometry: Methods and Applications*, 19:5, 2023.

[29] P.T. Chrusciel and M. Herzlich. The mass of asymptotically hyperbolic Riemannian manifolds. *Pacific Journal of Mathematics*, 212(2):231–264, 2003.

[30] P.T. Chrusciel, D. Maerten, and P. Tod. Rigid upper bounds for the angular momentum and centre of mass of non-singular asymptotically anti-de Sitter space-times. *Journal of High Energy Physics*, 2006(11):084, 2006.

[31] P.T. Chrusciel and G. Nagy. The mass of spacelike hypersurfaces in asymptotically anti-de Sitter space-times. *Advances in Theoretical and Mathematical Physics*, 5(4):697–754, 2001.

[32] R. Clarkson and R.B. Mann. Eguchi–Hanson solitons in odd dimensions. *Classical and Quantum Gravity*, 23(5):1507, 2006.

[33] M. Cvetic, P. Gao, and J. Simon. Supersymmetric Kerr-Anti-de Sitter solutions. *Physical Review D*, 72(2):021701, 2005.

[34] M. Cvetic, G.W. Gibbons, H. Lu, and C.N. Pope. Rotating Black Holes in Gauged Supergravities; Thermodynamics, Supersymmetric Limits, Topological Solitons and Time Machines, 2005. arXiv[hep-th/0504080].

[35] A. Das and S. Okubo. *Lie Groups and Lie Algebras for Physicists*. World Scientific Publishing Co., Singapore, 2014.

[36] S. de Haro, K. Skenderis, and S.N. Solodukhin. Holographic Reconstruction of Spacetime and Renormalization in the AdS/CFT Correspondence. *Communications in Mathematical Physics*, 217:595–622, 2001.

[37] A.J. Dougan. Quasi-local mass for spheres. *Classical and Quantum Gravity*, 9(11):2461, 1992.

[38] A.J. Dougan and L.J. Mason. Quasilocal Mass Constructions with Positive Energy. *Physical Review Letters*, 67(16):2119, 1991.

[39] M.J. Duff, B.E. Nilsson, and C.N. Pope. Kaluza-Klein Supergravity. *Physics Reports*, 130(1-2):1–142, 1986.

[40] M. Dunajski and P. Tod. The Kijowski–Liu–Yau quasi-local mass of the Kerr black hole horizon. *Classical and Quantum Gravity*, 38(23):235001, 2021.

[41] T. Durgut and H.K. Kunduri. Supersymmetric asymptotically locally AdS5 gravitational solitons. *Annals of Physics*, 457:169435, 2023.

[42] C. Fefferman and C.R. Graham. Conformal invariants. In *Elie Cartan et les Mathematiques d'aujourd'hui*, page 95. Asterisque, 1985.

[43] J.M. Figueroa-O'Farrill. Lectures on spin geometry, 2010. Accessed on February 20, 2026, from `https://empg.maths.ed.ac.uk/Activities/Spin/`.

[44] G. Fournodavlos and J. Smulevici. On the initial boundary value problem for the Einstein vacuum equations in the maximal gauge, 2019. arXiv[math.AP/1912.07338].

[45] A. Franc. Spin structures and Killing spinors on lens spaces. *Journal of Geometry and Physics*, 4(3):277–287, 1987.

[46] D.Z. Freedman and A. Das. Gauge internal symmetry in extended supergravity. *Nuclear Physics B*, 120(2):221–230, 1977.

[47] C. Fronsdal. Elementary particles in a curved space. II. *Physical Review D*, 10(2):589, 1974.

[48] R. Geroch, A. Held, and R. Penrose. A space-time calculus based on pairs of null directions. *Journal of Mathematical Physics*, 14(7):874–881, 1973.

[49] G.W. Gibbons and S.W. Hawking. Action integrals and partition functions in quantum gravity. *Physical Review D*, 15(10):2752, 1977.

[50] G.W. Gibbons, S.W. Hawking, G.T. Horowitz, and M.J. Perry. Positive Mass Theorems for Black Holes. *Communications in Mathematical Physics*, 88:295–308, 1983.

[51] G.W. Gibbons and C.M. Hull. A Bogomolny bound for general relativity and solitons in N = 2 supergravity. *Physics Letters B*, 109(3):190–194, 1982.

[52] G.W. Gibbons, C.M. Hull, and N.P. Warner. The stability of gauged supergravity. *Nuclear Physics B*, 218(1):173–190, 1983.

[53] G.W. Gibbons, D. Kastor, L.A.J. London, P.K. Townsend, and J. Traschen. Supersymmetric Self Gravitating Solitons. *Nuclear Physics B*, 416(3):850–880, 1994.

[54] G.W. Gibbons, M.J. Perry, and C.N. Pope. The first law of thermodynamics for Kerr-anti-de Sitter black holes. *Classical and Quantum Gravity*, 22(9):1503–1526, 2005.

[55] J.B. Gutowski and H.S. Reall. Supersymmetric AdS5 black holes. *Journal of High Energy Physics*, 2004(2):6, 2004.

[56] M. Günaydin, G. Sierra, and P.K. Townsend. The geometry of N = 2 Maxwell-Einstein supergravity and Jordan algebras. *Nuclear Physics B*, 242(1):244–268, 1984.

[57] S.W. Hawking. Gravitational radiation in an expanding universe. *Journal of Mathematical Physics*, 9(4):598–604, 1968.

[58] M. Henneaux and C. Teitelboim. Asymptotically Anti-de Sitter Spaces. *Communications in Mathematical Physics*, 98:391–424, 1985.

[59] M. Herzlich. The positive mass theorem for black holes revisited. *Journal of Geometry and Physics*, 26(1-2):97–111, 1998.

[60] S. Hirsch, H.C. Jang, and Y.Zhang. Rigidity of Asymptotically Hyperboloidal Initial Data Sets with Vanishing Mass, 2024. arXiv[math.DG/2411.07357].

[61] S. Hirsch and Y.Zhang. Initial data sets with vanishing mass are contained in pp-wave spacetimes, 2024. arXiv[gr-qc/2403.15984].

[62] S. Hirsch and Y. Zhang. Causal character of imaginary Killing spinors and spinorial slicings, 2025. arXiv[gr-qc/2512.14569].

[63] S. Hollands, A. Ishibashi, and D. Marolf. Comparision between various notions of conserved charges in asymptotically AdS spacetimes. *Classical and Quantum Gravity*, 22(14):2881, 2005.

[64] G.T. Horowitz and R.C. Myers. AdS-CFT correspondence and a new positive energy conjecture for general relativity. *Physical Review D*, 59(2):026005, 1998.

[65] G.T. Horowitz and B.G. Schmidt. Note on gravitational energy. *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences*, 381(1780):215–224, 1982.

[66] G.T. Horowitz and A. Strominger. Witten's expression for gravitational energy. *Physical Review D*, 27(12):2793, 1983.

[67] K. Hristov, C. Toldo, and S. Vandoren. On BPS bounds in D = 4 N = 2 gauged supergravity. *Journal of High Energy Physics*, 2011(12):1–22, 2011.

[68] G. Huisken and T. Ilmanen. The inverse mean curvature flow and the Riemannian Penrose inequality. *Journal of Differential Geometry*, 59(3):353–437, 2001.

[69] C. Imbimbo, A. Schwimmer, S. Theisen, and S. Yankielowicz. Diffeomorphisms and holographic anomalies. *Classical and Quantum Gravity*, 17(5):1129, 2000.

[70] J. Jezierski, J. Kijowski, and P. Waluk. Gauge-invariant quadratic approximation of quasi-local mass and its relation with Hamiltonian for gravitational field. *Classical and Quantum Gravity*, 38(9):095006, 2021.

[71] D. Kastor and J. Traschen. A positive energy theorem for asymptotically de Sitter spacetimes. *Classical and Quantum Gravity*, 19(23):5901, 2002.

[72] R. Kelly. Penrose's quasi-local mass for asymptotically anti-de Sitter space-times, 1985. Twistor Newsletter 20. arXiv[gr-qc/1505.00214].

[73] R.M. Kelly, K.P. Tod, and N.M.J. Woodhouse. Quasi-local mass for small surfaces. *Classical and Quantum Gravity*, 3(6):1151, 1986.

[74] J. Kijowski. A Simple Derivation of Canonical Structure and Quasi-local Hamiltonians in General Relativity. *General Relativity and Gravitation*, 29(3):307–343, 1997.

[75] D. Klemm and W.A. Sabra. Charged Rotating Black Holes in 5d Einstein–Maxwell–(A)dS Gravity. *Physics Letters B*, 503(1-2):147–153, 2001.

[76] V.A. Kostelecky and M.J. Perry. Solitonic black holes in gauged N = 2 supergravity. *Physical Letters B*, 371(3-4):191–198, 1996.

[77] H.K. Kunduri and J. Lucietti. Notes on non-extremal, charged, rotating black holes in minimal D=5 gauged supergravity. *Nuclear Physics B*, 724(1-2):343–356, 2005.

[78] H.K. Kunduri, J. Lucietti, and H.S. Reall. Gravitational perturbations of higher dimensional rotating black holes: tensor perturbations. *Physical Review D*, 74(8):084021, 2006.

[79] F. Leitner. Imaginary Killing Spinors in Lorentzian Geometry. *Journal of Mathematical Physics*, 44(10):4795–4806, 2003.

[80] L.A.J. London. Arbitrary dimensional cosmological multi-black holes. *Nuclear Physics B*, 434(3):709–735, 1995.

[81] J. Lott. A spinorial quasilocal mass. *Journal of Mathematical Physics*, 64(2), 2023.

[82] H. Lu, C.N. Pope, and J. Rahmfeld. A construction of Killing Spinors on S^n. *Journal of Mathematical Physics*, 48(9):4518–4526, 1999.

[83] J. Lucietti and S.G. Ovchinnikov. Uniqueness of supersymmetric AdS5 black holes with SU(2) symmetry. *Classical and Quantum Gravity*, 38(19):195019, 2021.

[84] O. Madden and S.F. Ross. On uniqueness of charged Kerr–AdS black holes in five dimensions. *Classical and Quantum Gravity*, 22(3):515, 2005.

[85] C. Mantoulidis and R. Schoen. On the Bartnik mass of apparent horizons. *Classical and Quantum Gravity*, 32(20):205002, 2015.

[86] L.J. Mason and J. Frauendiener. The Sparling 3-form, Ashtekar variables and quasi-local mass. *London Mathematical Society Lecture Notes*, 156:189, 1990.

[87] S. McCormick. An Overview of Bartnik's Quasi-Local Mass. *Beijing Journal of Pure and Applied Mathematics*, 1(2):455–487, 2024.

[88] A.M. McSharry and H.S. Reall. On the third law of black hole mechanics for supersymmetric black holes. *Physical Review D*, 112(10):104009, 2025.

[89] C.W. Misner and D.H. Sharp. Relativistic Equations for Adiabatic, Spherically Symmetric Gravitational Collapse. *Physical Review*, 136(2B):B571, 1964.

[90] P. Mondal and S.T. Yau. Quasi-local masses in general relativity and their positivity: spinor approach, 2024. arXiv[math-ph/2401.13909].

[91] N. O Murchadha, L.B. Szabados, and K.P. Tod. Comment on "Positivity of Quasilocal Mass". *Physical Review Letters*, 92(25):259001, 2004.

[92] J.M. Nester. A new gravitational energy expression with a simple positivity proof. *Physics Letters A*, 83(6):241–242, 1981.

[93] E. Newman and R. Penrose. An Approach to Gravitational Radiation by a Method of Spin Coefficients. *Journal of Mathematical Physics*, 3(3):566–578, 1962.

[94] L. Nirenberg. The Weyl and Minkowski Problems in Differential Geometry in the Large. *Communications on Pure and Applied Mathematics*, 6(3):337–394, 1953.

[95] M. Nozawa and T. Shiromizu. Positive mass theorem in extended supergravities. *Nuclear Physics B*, 887:380–399, 2014.

[96] M. Obata. Certain conditions for a Riemannian manifold to be isometric with a sphere. *Journal of the Mathematical Society of Japan*, 14(3):333–340, 1962.

[97] I. Papadimitriou and K. Skenderis. Thermodynamics of asymptotically locally AdS spacetimes. *Journal of High Energy Physics*, 2005(8):004, 2005.

[98] T. Parker and C.H. Taubes. On Witten's Proof of the Positive Energy Theorem. *Communications in Mathematical Physics*, 84(2):223–238, 1982.

[99] R. Penrose. Quasi-local Mass and Angular Momentum in General Relativity. *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, 381(1780):53–63, 1982.

[100] R. Penrose and W. Rindler. *Spinors and space-time volume 1: two-spinor calculus and relativistic fields*. Cambridge University Press, 1984.

[101] R. Penrose and W. Rindler. *Spinors and space-time volume 2: spinor and twistor methods in space-time geometry*. Cambridge University Press, 1986.

[102] A.V.E. Pogorelov. Regularity of a convex surface with given Gaussian curvature. *Matematicheskii Sbornik*, 73(1):88–103, 1952.

[103] D. Pook-Kolb, B. Zhao, L. Andersson, B. Krishnan, and S.T. Yau. Properties of quasilocal mass in binary black hole mergers. *Physical Review D*, 108(12):124031, 2023.

[104] H.S. Reall. Part III black holes lecture notes, 2017. Accessed on March 4, 2024 from `https://www.damtp.cam.ac.uk/user/hsr1000/black_holes_lectures_2016.pdf`.

[105] H.S. Reall. Third law of black hole mechanics for supersymmetric black holes and a quasi-local mass-charge inequality. *Physical Review D*, 110(12):124059, 2024.

[106] T. Regge and C. Teitelboim. Role of Surface Integrals in the Hamiltonian Formulation of General Relativity. *Annals of Physics*, 88(1):286–318, 1974.

[107] R. Schoen and S.T. Yau. On the Proof of the Positive Mass Conjecture in General Relativity. *Communications in Mathematical Physics*, 65(1):45–76, 1979.

[108] R. Schoen and S.T. Yau. Proof of the Positive Mass Theorem. II. *Communications in Mathematical Physics*, 79(2):231–260, 1981.

[109] Y. Shi and L.F. Tam. Positive mass theorem and the boundary behaviours of compact manifolds with nonnegative scalar curvature. *Journal of Differential Geometry*, 62(1):79–125, 2002.

[110] K. Skenderis. Lecture notes on holographic renormalisation. *Classical and Quantum Gravity*, 19(22):5849, 2002.

[111] L.B. Szabados. Two-dimensional Sen connections and quasi-local energy-momentum. *Classical and Quantum Gravity*, 11(7):1847–1866, 1994.

[112] L.B. Szabados. Quasi-Local Energy-Momentum and Angular Momentum in General Relativity. *Living Reviews in Relativity*, 12(1):1–163, 2009.

[113] L.B. Szabados and P. Tod. A positive Bondi-type mass in asymptotically de Sitter spacetimes. *Classical and Quantum Gravity*, 32(20):205011, 2015.

[114] L.B. Szabados and P. Tod. A review of total energy–momenta in GR with a positive cosmological constant. *International Journal of Modern Physics D*, 28(01):1930003, 2019.

[115] P. Tod. Some geometry of de Sitter space, 2015. arXiv[gr-qc/1505.06123].

[116] R. Wald. *General Relativity.* The University of Chicago Press, 1984.

[117] M.T. Wang and S.T. Yau. A generalisation of Liu-Yau's quasi-local mass. *Communications in Analysis and Geometry*, 15(2):249–282, 2006.

[118] M.T. Wang and S.T. Yau. Isometric Embeddings into the Minkowski Space and New Quasi-Local Mass. *Communications in Mathematical Physics*, 288(3):919–942, 2009.

[119] M.Y. Wang. Parallel Spinors and Parallel Forms. *Annals of Global Analysis and Geometry*, 7(1):59–68, 1989.

[120] M.Y. Wang. On Non-Simply Connected Manifolds with Non-Trivial Parallel Spinors. *Annals of Global Analysis and Geometry*, 13(1):31–42, 1995.

[121] X. Wang. The mass of asymptotically hyperbolic manifolds. *Journal of Differential Geometry*, 57(2):273–299, 2001.

[122] Y. Wang, N. Xie, and X. Zhang. The positive energy theorem for asymptotically anti-de Sitter spacetimes. *Communications in Contemporary Mathematics*, 17(04):1550015, 2015.

[123] Y. Wang and X. Xu. Hyperbolic positive energy theorem with electromagnetic fields. *Classical and Quantum Gravity*, 32(2):025007, 2015.

[124] S. Weinberg. *The Quantum Theory of Fields, Volume 1: Foundations*. Cambridge University Press, 1995.

[125] E. Witten. A New Proof of the Positive Energy Theorem. *Communications in Mathematical Physics*, 80(3):381–402, 1981.

[126] X. Zhang. On a quasi-local mass. *Classical and Quantum Gravity*, 26(24):245018, 2009.