Positive energy theorems in asymptotically (locally) AdS spacetimes revisited

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Abstract

This work considers positive energy theorems in asymptotically, locally AdS spacetimes. Particular attention is given to spacetimes where conformal infinity has compact, Einstein cross-sections admitting Killing or parallel spinors; a positive energy theorem is derived for such spacetimes in terms of geometric data intrinsic to the cross-section. This is followed by the first complete proofs of the BPS inequalities in (the bosonic sectors of) 4D and 5D minimal, gauged supergravity, including with magnetic fields. The BPS inequalities are proven for asymptotically AdS spacetimes, but also generalised to the aforementioned class of asymptotically, locally AdS spacetimes. I wrote these notes in the process of producing [1]. The presentation here is much more pedagogical and written in a much more informal (but more opinionated) style. Some of the material has been superseded by [1] and there were only ever very limited checks of these notes.

Contents

1 Introduction

The positive energy theorem stands as one of the most treasured and significant results in mathematical general relativity - originally proved by Schoen & Yau based on minimal surface methods [2] and soon after by Witten [3] based on spinor techniques. Witten's method suggested a number of extensions, including allowing a negative cosmological constant - the focus of the present work. The first positive energy theorems for asymptotically AdS spacetimes [4, 5] followed soon after Witten's original work and were based on the Abbott-Deser definition of energy & asymptotics [6].

However, in the age of holography, a more natural choice of asymptotics is one based on a Fefferman-Graham expansion [7, 8, 9]. In particular, the Einstein equation is solved order by order from a timelike conformal boundary and the geometry of the boundary itself is arbitrary; the case of a static $\mathbb{R} \times S^2$ boundary reduces to the asymptotically (globally) AdS case. Rigorous definitions of energy were given in the latter context by [10, 11, 12] and corresponding positive energy theorems were subsequently proven[1].

Having understood the "global" case, the next logical extension is the "local" case. The example of a toroidal boundary was considered in [13] and a more general analysis was performed in [14]. One of the main aims of this work is to built upon the latter. I will adopt a few conceptual differences though. Most saliently, I will not follow the holographic renormalisation [9] approach pursued by [14]. Instead, energy will be defined using the background subtraction and Hamiltonian methods of [15, 12, 11]. Furthermore, Killing spinors will play a crucial role in the analysis. To this end, I develop a general formula for imaginary Killing spinors on time-symmetric metrics with cross-sections admitting either parallel or real Killing spinors. This formula allows a derivation of a positive energy theorem based on data intrinsic to the cross-section. The theorem decomposes the "Witten-Nester" energy [16] of [14] into further "conserved quantities" built from symmetries of the boundary geometry.

Given the deep connections between Witten's method and supergravity [17], another natural extension is to try prove BPS inequalities for (the bosonic sectors of) supergravity theories. This was realised very soon after Witten's original work to prove global mass-charge inequalities in asymptotically flat spacetimes in four and five dimensions [18, 19]. While some results already exist along these lines [20, 21, 22] in the context of asymptotically AdS spacetimes - i.e. in gauged supergravity theories - the magnetic field is set to zero in [20] and a non-gauge-covariant connection is used in [21, 22], thereby leading to some unnatural assumptions - which in fact never hold in electrovacuum - and different results to the present work when incorporating magnetic fields. I aim to build upon the literature by providing a more complete treatment of magnetic fields in the study of classical energy-charge inequalities with negative cosmological constant.

[1] Note that in the former two references, the asymptotics considered are Riemannian, not Lorentzian, and should be viewed as asymptotics for an initial data slice.

I begin in section 2, by deriving a definition of energy based on the techniques in [23] and [15]. However, I will not consider questions of geometric invariance á la [24], [10] or [12]. My main positive energy theorem - theorem 3.19 - follows in section 3. I use a Witten-style spinorial proof [3] and relevant spin assumptions will be stated as they arise. My presentation relies heavily on work in [25], [26] and [27] - especially the analysis of the Dirac operator and the use of modified, more general spin connections. In section 4, I'll apply my main result - theorem 3.19 - to various examples. To illustrate the effects of the boundary geometry, as a running theme I will compare asymptotically AdS spaces - with $\mathbb{R} \times S^{n-2}$ boundary - to spaces with $\mathbb{R}\times\mathbb{T}^{n-2}$ boundary. In section 4.2, based on the analysis in [13], I will also give a more complete analysis of the "Witten-Nester energy" than [28] or [14] and explain its relationship with the energy I define in section 2. In section 4.3, I consider general static boundary metrics with a parallel or Killing spinor on the cross-section and derive a positive energy theorem which is much more manifestly dependant only on the boundary data. The theorem is illustrated with some more exotic boundary geometries such as $\mathbb{R} \times L(p,1)$. Section 5 I will consider the minimal, gauged supergravities. The main results are theorems 3.19, 4.8, 4.3, 4.13, 4.14, 4.17, 5.5, 5.4, 5.8, 5.7 and 5.9.

Finally, readers are highly encouraged to familiarise themselves with my notational conventions - as listed in appendix A. I will use a litany of different types of indices[2]. Furthermore, only a very naive person would assume two people have common spinor/gamma matrix conventions.

2 Hamiltonian formulation

Definition 2.1 ($(n-1)+1$ split)**.** *The metric, g, is said to be written in an $(n-1)+1$ split if and only if*

$$g = -N^2 \mathrm{d}t \otimes \mathrm{d}t + h_{ij}(\mathrm{d}x^i + N^i \mathrm{d}t) \otimes (\mathrm{d}x^j + N^j \mathrm{d}t) \tag{1}$$

for some h_{ij}, N^i, N and coordinates, (t, x^i).

It is well known that this split admits a Hamiltonian formulation by the ADM formalism [29] - see also textbook treatments in [30] or [31]. I'll recount the story briefly. N and N^i turn out to be auxiliary fields and one finds the conjugate momentum to h_{ij} is

$$p_{ij} = \sqrt{h}(K_{ij} - K h_{ij}), \tag{2}$$

where $h = \det(h_{ij})$ and K_{ij} is the extrinsic curvature of Σ_t, namely[3]

$$K_{ij} = \frac{1}{2N}\left(\partial_t h_{ij} - D^{(h)}_i N_j - D^{(h)}_j N_i\right), \tag{3}$$

where $D^{(h)}$ refers to the Levi-Civita connection of h_{ij}.
Then, up to boundary terms, one finds the Hamiltonian (arising from the Einstein-Hilbert Lagrangian with cosmological constant[4]) is

$$H = \frac{1}{16\pi}\int_{\Sigma_t}\left(N\left(\frac{1}{h}p^{ij}p_{ij} - \frac{1}{(n-2)h}p^2 - R^{(h)} + 2\Lambda\right) - 2N^i D^{(h)j}\left(\frac{1}{\sqrt{h}}p_{ij}\right)\right)\mathrm{d}V. \tag{4}$$

[2]You have been warned.

[3]i, j, $\cdots$ indices are raised and lower by the h metric in the ADM formalism.

[4]The matter Lagrangian doesn't need to be considered in this process of defining gravitational energy.

I will take the same perspective as [23] and [15] in advocating energy to most simply be defined as the value of the Hamiltonian. However, by the constraint equations, the H in equation 4 is zero when the Einstein equation holds, suggesting the energy is always zero. The resolution is that the boundary terms do matter. As explained in [23], these boundary terms are essential to have a well defined variational principle. As shown in [23, 15, 30, 31], upon a variation to the metric, when the equations of motion hold, the Hamiltonian changes as

$$\begin{aligned}16\pi\delta H = &-\int_{\partial_\infty\Sigma_t} Nl^i\left(D^{(h)j}\delta h_{ij} - D^{(h)}_i(h^{jk}\delta h_{jk})\right)\mathrm{d}A - 2\int_{\partial_\infty\Sigma_t} l^i N^j\delta\left(K_{ij} - Kh_{ij}\right)\mathrm{d}A\\ &+\int_{\partial_\infty\Sigma_t} l^i\left(D^{(h)j}(N)\delta h_{ij} - h^{jk}\delta h_{jk}D^{(h)}_i N\right)\mathrm{d}A\\ &+\int_{\partial_\infty\Sigma_t} N^i\frac{1}{\sqrt{h}}\delta(h_{jk})\left(l_i p^{jk} - p_{il}l^l h^{jk} + 2p_i{}^k l^j\right)\mathrm{d}A,\end{aligned} \tag{5}$$

where $\partial_\infty\Sigma_t$ denotes the "boundary" at infinity of a constant t surface, Σ_t, and l^i is the (outward pointing) normal to $\partial_\infty\Sigma_t$. The first and thrid integrals come from $R^{(h)}$'s variation, the second integral comes from $\delta(p_{ij}/\sqrt{h})$ and the fourth integral comes from the variation of the Christoffel symbols when $D^{(h)j}$ acts on $p_{ij}/\sqrt{h}$.
One then defines the true Hamiltonian, H' say, to be $H + E$, where E is some quantity such that that $\delta E = -\delta H$. Hence, $\delta H' = 0$ when the equations of motion hold and the energy - the on-shell value of H' - is just E.
To make further progress, one needs to choose asymptotics, so the integrals in δH can be evaluated more precisely. In this work, I'm interested in asymptotically locally AdS spaces.

Definition 2.2 (Asymptotically locally AdS)**.** *A spacetime, (M, g), is said to be asymptotically locally AdS if and only if only if $\exists$ coordinates, (z, x^m), in an open neighbourhood of the "boundary" at infinity*[5] *such that $\{z = 0\}$ is the "boundary" itself and g admits a Fefferman-Graham expansion*[6] *[7],*

$$g = \frac{1}{z^2}\mathrm{d}z\otimes\mathrm{d}z + \frac{1}{z^2}\left(f_{(0)mn} + zf_{(1)mn} + z^2 f_{(2)mn} + \cdots\right)\mathrm{d}x^m\otimes\mathrm{d}x^n \tag{6}$$

for some $f_{(k)mn}$ that do not depend on z. By defining $r = -\ln(z)$, the "boundary" becomes $\{r = \infty\}$ and

$$g = \mathrm{d}r\otimes\mathrm{d}r + \mathrm{e}^{2r}\left(f_{(0)mn} + \mathrm{e}^{-r}f_{(1)mn} + \mathrm{e}^{-2r}f_{(2)mn} + \cdots\right)\mathrm{d}x^m\otimes\mathrm{d}x^n. \tag{7}$$

The series, $f_{(0)mn} + \mathrm{e}^{-r}f_{(1)mn} + \mathrm{e}^{-2r}f_{(2)mn} + \cdots$, will be denoted f_{mn} (when summed).

The first $n-2$ terms of f_{mn} are uniquely determined by the curvature of $f_{(0)mn}$ [7], i.e. specifying $f_{(0)}$ specifies f up to $O\left(\mathrm{e}^{-(n-2)r}\right)$.

Definition 2.3 (Asymptotically AdS)**.** *A spacetime, (M, g), is said to be asymptotically AdS if and only if it is asymptotically locally AdS and*

$$\begin{aligned}&f_{mn}\,\mathrm{d}x^m\otimes\mathrm{d}x^n\\ &= -\left(1 + \frac{1}{4}\mathrm{e}^{-2r}\right)^2\mathrm{d}t\otimes\mathrm{d}t + \left(1 - \frac{1}{4}\mathrm{e}^{-2r}\right)^2 g_{S^{n-2}} + \mathrm{e}^{-(n-1)r}f_{(n-1)mn}\,\mathrm{d}x^m\otimes\mathrm{d}x^n + \cdots.\end{aligned} \tag{8}$$

[5] First of all, such a notion of "boundary" at infinity should exist on (M, g).

[6] This expansion implicitly sets the "AdS length scale," to 1. Equivalently, one would choose units such that the cosmological constant is $\Lambda = -\frac{1}{2}(n-1)(n-2)$. The length scales can always be restored on dimensional grounds.

Equivalently,

$$g = \mathrm{d}r \otimes \mathrm{d}r + \mathrm{e}^{2r}\left(-\left(1+\frac{1}{4}\mathrm{e}^{-2r}\right)^2 \mathrm{d}t \otimes \mathrm{d}t + \left(1-\frac{1}{4}\mathrm{e}^{-2r}\right)^2 g_{S^{n-2}} + O\big(\mathrm{e}^{-(n-1)r}\big)\right) \tag{9}$$

$$= g_{\mathrm{AdS}} + \mathrm{e}^{2r}(\mathrm{e}^{-(n-1)r} f_{(n-1)mn} + \cdots)\,\mathrm{d}x^m \otimes \mathrm{d}x^n. \tag{10}$$

Note that when $n \leq 5$, there is an annoying subtlety that the "background metric," g_{AdS}, goes to an order in e^{-r} at least as high as the "leading correction" term, $\mathrm{e}^{-(n-1)r} f_{(n-1)mn}\,\mathrm{d}x^m \otimes \mathrm{d}x^n$. In such cases, especially in definition 3.15 later, I will always also include these fixed higher order terms in the background metric.

Having established asymptotics, I can now calculate the boundary integrals in equation 5. Since $f_{(1)}, f_{(2)}, \cdots, f_{(n-2)}$ are determined by $f_{(0)}$ and the Fefferman-Graham expansion always includes an exact $\mathrm{d}r \otimes \mathrm{d}r$ factor, I should let

$$\delta g = \mathrm{e}^{2r}\left(\mathrm{e}^{-(n-1)r}\delta f_{(n-1)mn} + O(\mathrm{e}^{-nr})\right)\mathrm{d}x^m \otimes \mathrm{d}x^n \tag{11}$$

in following the "background subtraction" method of [23] and [15]. Again, for $n \leq 5$, asymptotically AdS spactimes, the higher order terms of $-\left(1+\frac{1}{4}\mathrm{e}^{-2r}\right)^2 \mathrm{d}t \otimes \mathrm{d}t + \left(1-\frac{1}{4}\mathrm{e}^{-2r}\right)^2 g_{S^{n-2}}$ are included in the background metric and not considered in δg.

Theorem 2.4. *For variations given by equation 11,*

$$\delta H = -\delta\Bigg(\frac{n-1}{16\pi}\int_{\partial_\infty \Sigma_t}\left(\left(\tilde{f}^{mn}_{(0)} + f^{00}_{(0)}\tilde{f}^{mp}_{(0)}\tilde{f}^{nq}_{(0)} f_{(0)0p} f_{(0)0q}\right) f_{(n-1)mn} - f^{00}_{(0)}\tilde{f}^{mn}_{(0)} f_{(0)0n} f_{(n-1)0m}\right) \times \sqrt{\iota^* f_{(0)}/f^{00}_{(0)}}\,\mathrm{d}^{n-2}x\Bigg) \tag{12}$$

where $\mathrm{e}^{-2r}\tilde{f}^{mn}_{(0)} = \mathrm{e}^{-2r}(f^{mn}_{(0)} + n^m_{(0)} n^n_{(0)})$ *is the induced (inverse) metric on constant t and r surfaces and $\iota^* f_{(0)}$ is the pullback of $f_{(0)}$ to constant t surfaces*[7].

Proof. As I'm using Fefferman-Graham coordinates, $l^i = \delta^{i1}$. Hence, equation 5 becomes

$$\begin{aligned}16\pi\delta H = &-\int_{\partial_\infty \Sigma_t} N\left(D^{(h)i}\delta h_{1i} - D^{(h)}_1(h^{ij}\delta h_{ij})\right)\mathrm{d}A - 2\int_{\partial_\infty \Sigma_t} N^i \delta\left(K_{1i} - K h_{1i}\right)\mathrm{d}A \\ &+ \int_{\partial_\infty \Sigma_t}\left(D^{(h)i}(N)\delta h_{1i} - h^{ij}\delta h_{ij} D^{(h)}_1 N\right)\mathrm{d}A \\ &+ \int_{\partial_\infty \Sigma_t}\frac{1}{\sqrt{h}}\left(\delta(h_{ij})N_1 p^{ij} - \delta(h_{ij})N^k p_{k1} h^{ij} + 2\delta(h_{i1})N^j p_j{}^i\right)\mathrm{d}A\end{aligned} \tag{13}$$

Furthermore, comparing equations 1 and 7, it immediately follows that

$$N_i = \mathrm{e}^{2r}\delta^\alpha{}_i f_{0\alpha}, \quad -N^2 + N^i N_i = \mathrm{e}^{2r} f_{00} \text{ and} \tag{14}$$

$$h_{ij} = \delta_{i1}\delta_{j1} + \mathrm{e}^{2r}\delta^\alpha{}_i\delta^\beta{}_j f_{\alpha\beta} \equiv \begin{bmatrix}1 & 0 \\ 0 & \mathrm{e}^{2r} f_{\alpha\beta}\end{bmatrix}. \tag{15}$$

Since $N_1 = 0$, it follows that $\int_{\partial_\infty \Sigma_t}\frac{1}{\sqrt{h}}\delta(h_{ij})N_1 p^{ij}\mathrm{d}A = 0$.
Because f is artificially split in this way, I will denote the inverse of the $(n-2)\times(n-2)$ matrix, $f_{\alpha\beta}$, as $j^{\alpha\beta}$.

$$\therefore h^{ij} \equiv \begin{bmatrix}1 & 0 \\ 0 & \mathrm{e}^{-2r} j^{\alpha\beta}\end{bmatrix}. \tag{16}$$

[7] i.e. $\sqrt{-\iota^* f_{(0)}}$ is the square root of the determinant of $(n-2)\times(n-2)$ matrix that is $f_{(0)}$ restricted to constant t surfaces.

Note that $j^{\alpha\beta}$ is not the (α, β) component of f^{mn}; the two are related by

$$f^{mn} \equiv \frac{1}{f_{00} - j^{\theta\phi} f_{0\theta} f_{0\phi}} \begin{bmatrix} 1 & -j^{\beta\gamma} f_{0\gamma} \\ -j^{\alpha\gamma} f_{0\gamma} & j^{\alpha\beta} f_{00} + (j^{\alpha\gamma} j^{\beta\delta} - j^{\alpha\beta} j^{\gamma\delta}) f_{0\gamma} f_{0\delta} \end{bmatrix} \text{ because} \tag{17}$$

$$\frac{1}{f_{00} - j^{\theta\phi} f_{0\theta} f_{0\phi}} \begin{bmatrix} 1 & -j^{\gamma\delta} f_{0\delta} \\ -j^{\alpha\delta} f_{0\delta} & j^{\alpha\gamma} f_{00} + (j^{\alpha\delta} j^{\gamma\epsilon} - j^{\alpha\gamma} j^{\delta\epsilon}) f_{0\delta} f_{0\epsilon} \end{bmatrix} \begin{bmatrix} f_{00} & f_{0\beta} \\ f_{0\gamma} & f_{\gamma\beta} \end{bmatrix} \tag{18}$$

$$= \frac{1}{f_{00} - j^{\theta\phi} f_{0\theta} f_{0\phi}} \begin{bmatrix} f_{00} - j^{\gamma\delta} f_{0\delta} f_{0\gamma} & f_{0\beta} - j^{\gamma\delta} f_{0\delta} f_{\gamma\beta} \\ 0 & -j^{\alpha\delta} f_{0\delta} f_{0\beta} + j^{\alpha\gamma} f_{00} f_{\gamma\beta} + (j^{\alpha\delta} j^{\gamma\epsilon} - j^{\alpha\gamma} j^{\delta\epsilon}) f_{0\delta} f_{0\epsilon} f_{\gamma\beta} \end{bmatrix} \tag{19}$$

$$= \frac{1}{f_{00} - j^{\theta\phi} f_{0\theta} f_{0\phi}} \begin{bmatrix} f_{00} - j^{\gamma\delta} f_{0\delta} f_{0\gamma} & f_{0\beta} - \delta^{\delta}{}_{\beta} f_{0\delta} \\ 0 & -j^{\alpha\delta} f_{0\delta} f_{0\beta} + \delta^{\alpha}{}_{\beta} f_{00} + (j^{\alpha\delta} \delta^{\epsilon}{}_{\beta} - \delta^{\alpha}{}_{\beta} j^{\delta\epsilon}) f_{0\delta} f_{0\epsilon} \end{bmatrix} \tag{20}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & \delta^{\alpha}{}_{\beta} \end{bmatrix}. \tag{21}$$

Anyway, with this definition,

$$N^i = h^{ij} N_j = \mathrm{e}^{-2r} \delta^i{}_\alpha \delta^j{}_\beta j^{\alpha\beta} \mathrm{e}^{2r} \delta^\gamma{}_j f_{0\gamma} = \delta^i{}_\alpha j^{\alpha\beta} f_{0\beta}. \tag{22}$$

$$\therefore N^i N_i = \delta^i{}_\alpha j^{\alpha\beta} f_{0\beta} \mathrm{e}^{2r} \delta^\gamma{}_i f_{0\gamma} = \mathrm{e}^{2r} j^{\alpha\beta} f_{0\alpha} f_{0\beta}. \tag{23}$$

$$\therefore N = \sqrt{N^i N_i - \mathrm{e}^{2r} f_{00}} = \mathrm{e}^r \sqrt{j^{\alpha\beta} f_{0\alpha} f_{0\beta} - f_{00}} = \mathrm{e}^r \sqrt{-\frac{1}{f^{00}}}. \tag{24}$$

Similarly, in the $(n-1)+1$ split, h_{ij} is just the "space part" of g, so

$$\delta h_{ij} = \delta g_{ij} \tag{25}$$

$$= \delta_i{}^m \delta_j{}^n \mathrm{e}^{-(n-3)r} \delta f_{(n-1)mn} + O(\mathrm{e}^{-(n-2)r}) \text{ by equation 11} \tag{26}$$

$$= \delta_i{}^\alpha \delta_j{}^\beta \mathrm{e}^{-(n-3)r} \delta f_{(n-1)\alpha\beta} + O(\mathrm{e}^{-(n-2)r}). \tag{27}$$

An immediate corollary is that $\delta h_{1i} = 0$; this is effectively just stating that the $\mathrm{d}r \otimes \mathrm{d}r$ part of the Fefferman-Graham expansion is unchanged.
$\therefore \int_{\partial_\infty \Sigma_t} D^{(h)i}(N) \delta h_{1i} \mathrm{d}A$ and $\int_{\partial_\infty \Sigma_t} \frac{1}{\sqrt{h}} 2\delta(h_{i1}) N^j p_j{}^i \mathrm{d}A$ are both zero in equation 13.
Next, I'll likewise calculate all the other terms in integrands of equation 13. It will suffice to calculate them to leading order, as will become apparent later.

$$h^{ij} \delta h_{ij} D_1^{(h)} N = \mathrm{e}^{-(n-1)r} j^{\alpha\beta} \delta f_{(n-1)\alpha\beta} \partial_r(N) \tag{28}$$

$$= \mathrm{e}^{-(n-1)r} j^{\alpha\beta} \delta f_{(n-1)\alpha\beta} \partial_r \left(\mathrm{e}^r \sqrt{j^{\alpha\beta} f_{0\alpha} f_{0\beta} - f_{00}} \right) \text{ by equation 24} \tag{29}$$

$$= \mathrm{e}^{-(n-1)r} N j^{\alpha\beta} \delta f_{(n-1)\alpha\beta}. \tag{30}$$

$$D_1^{(h)} (h^{ij} \delta h_{ij}) = \partial_r \left(h^{ij} \delta_i{}^\alpha \delta_j{}^\beta \mathrm{e}^{-(n-3)r} \delta f_{(n-1)\alpha\beta} \right) \tag{31}$$

$$= \partial_r \left(\mathrm{e}^{-(n-1)r} j^{\alpha\beta} \delta f_{(n-1)\alpha\beta} \right) \tag{32}$$

$$= -(n-1) \mathrm{e}^{-(n-1)r} j^{\alpha\beta} \delta f_{(n-1)\alpha\beta}. \tag{33}$$

$$D^{(h)i} \delta h_{1i} = h^{ij} D_j^{(h)} \delta h_{1i} \tag{34}$$

$$= h^{ij} \left(\partial_j \delta h_{1i} - \Gamma^{(h)k}{}_{1j} \delta h_{ki} - \Gamma^{(h)k}{}_{ij} \delta h_{1k} \right) \tag{35}$$

$$= 0 - h^{ij} \Gamma^{(h)k}{}_{1j} \delta h_{ki} - 0 \tag{36}$$

$$= -\mathrm{e}^{-2r} j^{\alpha\gamma} \Gamma^{(h)\beta}{}_{1\gamma} \mathrm{e}^{-(n-3)r} \delta f_{(n-1)\alpha\beta} \tag{37}$$

The Christoffel symbol simplifies as

$$\Gamma^{(h)\beta}{}_{1\gamma} = \frac{1}{2}h^{\beta i}\left(\partial_r h_{\gamma i} + \partial_\gamma h_{i1} - \partial_i h_{1\gamma}\right) \tag{38}$$
$$= \frac{1}{2}h^{\beta\delta}\left(\partial_r h_{\gamma\delta} + \partial_\gamma h_{\delta 1} - 0\right) \tag{39}$$
$$= \frac{1}{2}\mathrm{e}^{-2r} j^{\beta\delta}\partial_r(\mathrm{e}^{2r} f_{\gamma\delta}) + 0 \tag{40}$$
$$= \delta^\beta{}_\gamma - \frac{1}{2}\mathrm{e}^{-r} j^{\beta\delta}_{(0)} f_{(1)\gamma\delta} + O(\mathrm{e}^{-2r}). \tag{41}$$
$$\therefore D^{(h)i}\delta h_{1i} = -\mathrm{e}^{-(n-1)r} j^{\alpha\beta}\delta f_{(n-1)\alpha\beta} \text{ to leading order.} \tag{42}$$

$$N^i\delta(Kh_{1i}) = \delta^i{}_\alpha j^{\alpha\beta} f_{0\beta}\delta(Kh_{1i}) = j^{\alpha\beta} f_{0\beta}\delta(Kh_{1\alpha}) = 0 \tag{43}$$

$$\frac{1}{\sqrt{h}}\delta(h_{ij})N^k p_{k1}h^{ij} = (K_{k1} - Kh_{k1})\delta^k{}_\alpha j^{\alpha\beta} f_{0\beta}\mathrm{e}^{-2r} j^{\gamma\delta}\mathrm{e}^{-(n-3)r}\delta f_{(n-1)\gamma\delta} \tag{44}$$
$$= \mathrm{e}^{-(n-1)r}(K_{\alpha 1} - 0) j^{\alpha\beta} f_{0\beta} j^{\gamma\delta}\delta f_{(n-1)\gamma\delta}. \tag{45}$$

For this expression, start by calculating $K_{\alpha 1}$.

$$K_{\alpha 1} = \frac{1}{2N}(\partial_t h_{\alpha 1} - D^{(h)}_\alpha N_1 - D^{(h)}_1 N_\alpha) \tag{46}$$
$$= 0 - \frac{1}{2N}(D^{(h)}_\alpha N_1 + D^{(h)}_1 N_\alpha) \tag{47}$$
$$= -\frac{1}{2N}\left(\partial_\alpha N_1 - \Gamma^{(h)i}{}_{1\alpha}N_i + \partial_r N_\alpha - \Gamma^{(h)i}{}_{\alpha 1}N_i\right) \tag{48}$$
$$= -\frac{1}{2N}\left(0 - \Gamma^{(h)\beta}{}_{1\alpha}N_\beta + \partial_r N_\alpha - \Gamma^{(h)\beta}{}_{\alpha 1}N_\beta\right) \tag{49}$$
$$= -\frac{1}{2N}\left(-2N_\alpha + \mathrm{e}^{-r} j^{\beta\gamma}_{(0)} f_{(1)\gamma\alpha}N_\beta + O(\mathrm{e}^{-2r}) + \partial_r N_\alpha\right) \text{ by equation 41.} \tag{50}$$
$$-\partial_r N_\alpha + 2N_\alpha = -\partial_r(\mathrm{e}^{2r} f_{(0)0\alpha} + \mathrm{e}^r f_{(1)0\alpha} + \cdots) + 2(\mathrm{e}^{2r} f_{(0)0\alpha} + \mathrm{e}^r f_{(1)0\alpha} + \cdots) \tag{51}$$
$$= \mathrm{e}^r f_{(1)0\alpha} + 2f_{(2)0\alpha} + 3\mathrm{e}^{-r} f_{(3)0\alpha} + \cdots. \tag{52}$$
$$\therefore K_{\alpha 1} = \frac{1}{2N}\mathrm{e}^r\left(f_{(1)0\alpha} - j^{\beta\gamma}_{(0)} f_{(1)\gamma\alpha} f_{(0)0\beta} + O(\mathrm{e}^{-r})\right) \tag{53}$$
$$= \frac{1}{2}\sqrt{-f^{00}}\left(f_{(1)0\alpha} - j^{\beta\gamma}_{(0)} f_{(1)\gamma\alpha} f_{(0)0\beta} + O(\mathrm{e}^{-r})\right). \tag{54}$$

Thus, to leading order,

$$\frac{1}{\sqrt{h}}\delta(h_{ij})N^k p_{k1}h^{ij} = \frac{1}{2}\mathrm{e}^{-(n-1)r}\sqrt{-f^{00}} j^{\alpha\beta} f_{0\beta} j^{\gamma\delta}\delta f_{(n-1)\gamma\delta}\left(f_{(1)0\alpha} + j^{\theta\phi}_{(0)} f_{(1)\theta\alpha} f_{(0)0\phi}\right), \tag{55}$$

which integrates to zero because the measure, $\mathrm{d}A$, is only $O(\mathrm{e}^{(n-2)r})$.
Finally, there's

$$N^i\delta K_{1i} = j^{\alpha\beta} f_{0\beta}\delta K_{1\alpha} \tag{56}$$
$$= j^{\alpha\beta} f_{0\beta}\delta\left(\frac{1}{2N}\left(-\partial_r N_\alpha + 2\Gamma^{(h)\gamma}{}_{1\alpha}N_\gamma\right)\right) \text{ from equation 49} \tag{57}$$
$$= j^{\alpha\beta} f_{0\beta}\delta\left(\frac{1}{2N}\left(-\partial_r N_\alpha + 2N_\alpha + j^{\delta\gamma}\partial_r(f_{\gamma\alpha})N_\delta\right)\right) \text{ by equation 40} \tag{58}$$

To leading order

$$\begin{aligned}
&\delta\left(\frac{1}{2N}\right)\left(-\partial_r N_\alpha + 2N_\alpha + j^{\beta\gamma}\partial_r(f_{\gamma\alpha})N_\beta\right)\\
&= \mathrm{e}^r\left(f_{(1)0\alpha} - j^{\beta\gamma}_{(0)} f_{(1)\gamma\alpha} f_{(0)0\beta}\right)\left(-\frac{1}{2N^2}\delta(N)\right) \text{ by equation 52} && (59)\\
&= O(\mathrm{e}^r \mathrm{e}^{-2r}\mathrm{e}^r\mathrm{e}^{-(n-1)r}) && (60)\\
&= O(\mathrm{e}^{-(n-1)r}), && (61)
\end{aligned}$$

which integrates to zero because the measure, $\mathrm{d}A$, is only $O(\mathrm{e}^{(n-2)r})$. Hence, to leading order,

$$N^i\delta K_{1i} = \frac{j^{\alpha\beta}f_{0\beta}}{2N}\delta\left(-\partial_r N_\alpha + 2N_\alpha + j^{\gamma\delta}\partial_r(f_{\delta\alpha})N_\gamma\right). \quad (62)$$

$$\begin{aligned}
\delta\left(-\partial_r N_\alpha + 2N_\alpha\right) &= (n-1)\mathrm{e}^{-(n-3)}\delta f_{(n-1)0\alpha} \text{ by equation 52.} && (63)\\
\delta\left(j^{\beta\gamma}\partial_r(f_{\gamma\alpha})N_\beta\right) &= \delta\left(\mathrm{e}^{2r} j^{\beta\gamma}\left(-\mathrm{e}^{-r}f_{(1)\gamma\alpha} - 2\mathrm{e}^{-2r}f_{(2)\gamma\alpha} - \cdots\right) f_{0\beta}\right) && (64)\\
&= -(n-1)\mathrm{e}^{-(n-3)r} j^{\beta\gamma}_{(0)} f_{(0)0\beta}\delta f_{(n-1)\gamma\alpha} \text{ to leading order} && (65)\\
\therefore N^i\delta K_{1i} &= \frac{n-1}{2N}\mathrm{e}^{-(n-3)r} j^{\alpha\beta} f_{0\beta}\left(\delta f_{(n-1)0\alpha} - j^{\gamma\delta}f_{0\gamma}\delta f_{(n-1)\delta\alpha}\right). && (66)
\end{aligned}$$

Substituting all these results back into equation 13, I get,

$$\begin{aligned}
&16\pi\delta H\\
&= \int_{\partial_\infty\Sigma_t}\Bigg(N\mathrm{e}^{-(n-1)r}j^{\alpha\beta}\delta f_{(n-1)\alpha\beta} - N(n-1)\mathrm{e}^{-(n-1)r}j^{\alpha\beta}\delta f_{(n-1)\alpha\beta}\\
&\quad - \frac{n-1}{N}\mathrm{e}^{-(n-3)r}j^{\alpha\beta}f_{0\beta}\left(\delta f_{(n-1)0\alpha} - j^{\gamma\delta}f_{0\gamma}\delta f_{(n-1)\delta\alpha}\right) - \mathrm{e}^{-(n-1)r}Nj^{\alpha\beta}\delta f_{(n-1)\alpha\beta}\Bigg)\mathrm{d}A. && (67)\\
&\therefore -\frac{16\pi}{n-1}\delta H\\
&= \int_{\partial_\infty\Sigma_t}\left(N\mathrm{e}^{-(n-1)r}j^{\alpha\beta}\delta f_{(n-1)\alpha\beta} + \frac{1}{N}\mathrm{e}^{-(n-3)r}j^{\alpha\beta}f_{0\beta}\left(\delta f_{(n-1)0\alpha} - j^{\gamma\delta}f_{0\gamma}\delta f_{(n-1)\delta\alpha}\right)\right)\mathrm{d}A && (68)\\
&= \int_{\partial_\infty\Sigma_t} N\mathrm{e}^{-(n-1)r}\left(j^{\alpha\beta}\delta f_{(n-1)\alpha\beta} + \frac{1}{N^2}\mathrm{e}^{2r}j^{\alpha\beta}f_{0\beta}\left(\delta f_{(n-1)0\alpha} - j^{\gamma\delta}f_{0\gamma}\delta f_{(n-1)\delta\alpha}\right)\right)\mathrm{d}A && (69)\\
&= \int_{\partial_\infty\Sigma_t}\sqrt{-\frac{1}{f^{00}}}\mathrm{e}^{-(n-2)r}\left(j^{\alpha\beta}\delta f_{(n-1)\alpha\beta} - f^{00}j^{\alpha\beta}f_{0\beta}\left(\delta f_{(n-1)0\alpha} - j^{\gamma\delta}f_{0\gamma}\delta f_{(n-1)\delta\alpha}\right)\right)\\
&\quad\times \mathrm{e}^{(n-2)r}\sqrt{-\iota^* f}\,\mathrm{d}^{n-2}x && (70)\\
&= \int_{\partial_\infty\Sigma_t}\left(j^{\alpha\beta}\delta f_{(n-1)\alpha\beta} - f^{00}j^{\alpha\beta}f_{0\beta}\left(\delta f_{(n-1)0\alpha} - j^{\gamma\delta}f_{0\gamma}\delta f_{(n-1)\delta\alpha}\right)\right)\sqrt{\frac{\iota^* f}{f^{00}}}\mathrm{d}^{n-2}x. && (71)
\end{aligned}$$

My earlier assertion that it suffices to go to leading order is now apparent. Anything higher than leading order for f, j etc. would integrate to zero in this expression, due to the $r \to \infty$ limit. Then, also noting that $f_{(0)}$ is unaffected by the variation, the result is

$$\begin{aligned}
-\frac{16\pi}{n-1}\delta H = \delta\Bigg(&\int_{\partial_\infty\Sigma_t}\left(\left(j^{\alpha\beta}_{(0)} + f^{00}_{(0)}j^{\alpha\gamma}_{(0)}j^{\beta\delta}_{(0)}f_{(0)0\gamma}f_{(0)0\delta}\right)f_{(n-1)\alpha\beta} - f^{00}_{(0)}j^{\alpha\beta}_{(0)}f_{(0)0\beta}f_{(n-1)0\alpha}\right)\\
&\times\sqrt{\iota^* f_{(0)}/f^{00}_{(0)}}\,\mathrm{d}^{n-2}x\Bigg). \quad (72)
\end{aligned}$$

To see the final claimed result, it suffices to show $\tilde{f}^{mn}_{(0)} = \delta^m{}_\alpha \delta^n{}_\beta j^{\alpha\beta}_{(0)}$.
To see this, first note that $n_{(0)m} \equiv -\mathrm{e}^{-r} N \mathrm{d}t$ (as can be seen from equation 1). Then,

$$n^m_{(0)} \equiv \frac{1}{f_{00} - j^{\theta\phi} f_{0\theta} f_{0\phi}} \begin{bmatrix} 1 & -j^{\beta\gamma} f_{0\gamma} \\ -j^{\alpha\gamma} f_{0\gamma} & j^{\alpha\beta} f_{00} + (j^{\alpha\gamma} j^{\beta\delta} - j^{\alpha\beta} j^{\gamma\delta}) f_{0\gamma} f_{0\delta} \end{bmatrix} \begin{bmatrix} -N\mathrm{e}^{-r} \\ 0 \end{bmatrix} \Bigg|_{r=0} \tag{73}$$

$$= -\frac{N\mathrm{e}^{-r}}{f_{00} - j^{\theta\phi} f_{0\theta} f_{0\phi}} \begin{bmatrix} 1 \\ -j^{\alpha\beta} f_{0\beta} \end{bmatrix} \Bigg|_{r=0} \tag{74}$$

$$= \sqrt{-f^{00}_{(0)}} \begin{bmatrix} 1 \\ -j^{\alpha\beta}_{(0)} f_{(0)0\beta} \end{bmatrix} \tag{75}$$

and finally

$$\tilde{f}^{mn}_{(0)} = f^{mn}_{(0)} + n^m_{(0)} n^n_{(0)} \tag{76}$$

$$\equiv -f^{00}_{(0)} \begin{bmatrix} -1 & j^{\beta\gamma}_{(0)} f_{(0)0\gamma} \\ j^{\alpha\gamma}_{(0)} f_{(0)0\gamma} & -j^{\alpha\beta}_{(0)} f_{(0)00} - (j^{\alpha\gamma}_{(0)} j^{\beta\delta}_{(0)} - j^{\alpha\beta}_{(0)} j^{\gamma\delta}_{(0)}) f_{(0)0\gamma} f_{(0)0\delta} \end{bmatrix}$$
$$- f^{00}_{(0)} \begin{bmatrix} 1 & -j^{\beta\gamma}_{(0)} f_{(0)0\gamma} \\ -j^{\alpha\gamma}_{(0)} f_{(0)0\gamma} & j^{\alpha\gamma}_{(0)} j^{\beta\delta}_{(0)} f_{(0)0\delta} f_{(0)0\gamma} \end{bmatrix} \tag{77}$$

$$= -f^{00}_{(0)} \begin{bmatrix} 0 & 0 \\ 0 & j^{\alpha\beta}_{(0)} (-f_{(0)00} + j^{\gamma\delta}_{(0)} f_{(0)\gamma} f_{(0)\delta}) \end{bmatrix} \tag{78}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & j^{\alpha\beta}_{(0)} \end{bmatrix}. \tag{79}$$

Putting this result into equation 72 completes the proof. □

I should interpret δH as $-\delta E$, as discussed earlier. Thus, I immediately generate the definition of energy I'll be using in this work.

Definition 2.5 (Energy)**.** *The energy is defined to be*

$$E = \frac{n-1}{16\pi} \int_{\partial_\infty \Sigma_t} \left(\left(\tilde{f}^{mn}_{(0)} + f^{00}_{(0)} \tilde{f}^{mp}_{(0)} \tilde{f}^{nq}_{(0)} f_{(0)0p} f_{(0)0q} \right) f_{(n-1)mn} - f^{00}_{(0)} \tilde{f}^{mn}_{(0)} f_{(0)0n} f_{(n-1)0m} \right)$$
$$\times \sqrt{\iota^* f_{(0)} / f^{00}_{(0)}} \mathrm{d}^{n-2} x, \tag{80}$$

where $\mathrm{e}^{-2r} \tilde{f}^{mn}_{(0)} = \mathrm{e}^{-2r}(f^{mn}_{(0)} + n^m_{(0)} n^n_{(0)})$ *is the induced (inverse) metric on constant t and r surfaces and $\iota^* f_{(0)}$ is the pullback of $f_{(0)}$ to constant t surfaces*[8]*.*

Corollary 2.5.1. *For asymptotically Kottler metrics*[9]*,*

$$E = \frac{n-1}{16\pi} \int_{\partial_\infty \Sigma_t} \tilde{f}^{mn}_{(0)} f_{(n-1)mn} \sqrt{\iota^* f_{(0)}} \mathrm{d}^{n-2} x. \tag{81}$$

Proof. The Kottler metrics are

$$g = -(k + R^2) \mathrm{d}t \otimes \mathrm{d}t + \frac{\mathrm{d}R \otimes \mathrm{d}R}{k + R^2} + R^2 g^{(k)}, \tag{82}$$

where $k = 1, 0, -1$, $g^{(1)}$ is the metric on the unit $(n-2)-$sphere, $g^{(0)}$ is the metric on a unit $(n-2)-$torus and $g^{(-1)}$ is the metric on a compact identification of $(n-2)-$dimensional hyperbolic space.
In particular, these metrics have $f_{(0)mn} \mathrm{d}x^m \otimes \mathrm{d}x^n = -\mathrm{d}t \otimes \mathrm{d}t + g^{(k)}$. □

[8]i.e. $\sqrt{-\iota^* f_{(0)}}$ is the square root of the determinant of $(n-2) \times (n-2)$ matrix that is $f_{(0)}$ restricted to constant t surfaces.

[9]See equation 82 for what I mean by a Kottler metric.

This corollary means definition 2.5 is a very natural, Lorentzian analogue of the Wang energy for asymptotically Poincaré-Einstein Riemannian manifolds, defined in [10, 32].

Also note that in the case of vacuum spacetimes, the Fefferman-Graham expansion requires $f^{mn}_{(0)} f_{(n-1)mn} = 0$ as a result of the Hamiltonian constraint on constant r hypersurfaces [33]. Hence, $\tilde{f}^{mn}_{(0)} f_{(n-1)mn} = n^m_{(0)} n^n_{(0)} f_{(n-1)mn}$. If $f_{(n-1)mn}$ is viewed as an energy-momentum tensor, then $n^m_{(0)} n^n_{(0)} f_{(n-1)mn}$ would indeed be what one naturally associates with energy density. It turns out the "true" energy momentum tensor one requires for AdS/CFT applications is actually $f_{(n-1)mn}$ with corrections from the "conformal anomaly" [33, 34], but that will not be relevant for the present analysis.
Furthermore, I will not assume $f^{mn}_{(0)} f_{(n-1)mn} = 0$ because I will not assume the spacetime is vacuum. Since the Hamiltonian constraint is changed by the presence of non-zero T_{ab}, it may be that $f^{mn}_{(0)} f_{(n-1)mn}$ is also adjusted depending on T_{ab}'s decay rate.

3 Positive energy theorem

I will follow the Witten-style spinorial proof of the positive energy theorem [3]. Naturally, this will rely on (M, g) actually admitting spinors. To keep Lorentz invariance manifest and to avoid introducing extrinsic curvature terms, I will adopt Nester's formulation [16] of Witten's argument. The techniques of my proof are adapted from those developed in [27, 26, 25] and [14] for the asymptotically flat and the asymptotically locally AdS cases respectively. I will only consider complete spacetimes for simplicity. However, (marginally) outer trapped surface (inner) boundaries[10] can also be very naturally be accommodated into the analysis using the techniques of [4].

3.1 Elements of analysis

This subsection is devoted to sketching a proof that a certain modified Dirac operator admits a Green's function. Readers willing to take this fact for granted - as is often done to varying extent in the physics literature [19, 18, 21] - may assimilate the opening definitions and skip ahead to subsection 3.2. Historically, establishing the Green's function has been attempted via different approaches and with varying levels of rigour - from the highly technical operator analysis methods of [35, 24], to the weighted Poincaré inequality methods of [27, 26, 25] to the more heuristic method of the original [3]. In my sketch below, I will attempt to strike something of a compromise.

Definition 3.1 ($\mathbb{M}$ and A_μ)**.** *Define the matrix, $\mathbb{M}$, by*

$$\mathbb{M} = 4\pi T^{0\mu}\gamma_0\gamma_\mu + \gamma^{IJ} D_I A_J + \mathrm{i}\alpha(n-2)(\gamma^I A_I + A^\dagger_I \gamma^I) - A^\dagger_I \gamma^{IJ} A_J, \tag{83}$$

where A_μ is some unspecified matrix. Assume the following conditions holds.

- $\gamma^{IJ} A_J$ *is hermitian.*
- $||A_I||_0 = O(\mathrm{e}^{-(n-1)r})$ *near $\partial_\infty \Sigma_t$, where $||\cdot||_0$ denotes the operator norm, i.e. the biggest (by absolute value) eigenvalue of the matrix.*
- $\mathbb{M}$ *is non-negative definite.*
- $||\mathbb{M}||_0$ *decays quicker than $O(\mathrm{e}^{-(n-1)r})$ near $\partial_\infty \Sigma_t$.*

[10] These boundaries are typically interpreted as proxies for black holes.

- $\alpha = \frac{1}{2}$.[11]
- $\exists \tilde{A}_\mu$ *such that* $A_I^\dagger \gamma^I = -\gamma^I \tilde{A}_I$ *and the first two conditions above continue to hold if* A_μ *is replaced by* $\tilde{A}_\mu$ *and* α *is replaced by* $-\alpha$.

Definition 3.2 (Modified connection)**.** *When acting on any spinor,* ψ*, define the modified connection,* ∇*, by*

$$\nabla_\mu \psi = D_\mu \psi + \mathrm{i}\alpha\gamma_\mu \psi + A_\mu \psi \quad \text{and} \tag{84}$$

$$\nabla_\mu \bar{\psi} = D_\mu \bar{\psi} - \mathrm{i}\alpha\bar{\psi}\gamma_\mu + \bar{\psi}\gamma^0 A_\mu^\dagger \gamma^0 = (\nabla_\mu \psi)^\dagger \gamma^0. \tag{85}$$

Definition 3.3 (n^μ and Σ_t)**.** *In any material that follows, whenever there is a timelile coordinate,* t*, whose level sets are spacelilke hypersurface, denoted* Σ_t*, I will choose a vielbein so that* n^a*, the future directed, unit normal to* Σ_t*, is* $e^0 \equiv n^\mu = \delta^{\mu 0}$*.*

Definition 3.4 ($\langle \cdot, \cdot \rangle_{C_c^\infty}$)**.** *Define an inner product on* C_c^∞ *by*

$$\langle \psi, \chi \rangle_{C_c^\infty} = \int_{\Sigma_t} \left((\nabla_I \psi)^\dagger \nabla^I \chi + \psi^\dagger \mathbb{M} \chi \right) \mathrm{d}V. \tag{86}$$

Proof. It has to be checked this really is a well-defined inner product.
$\langle \cdot, \cdot \rangle_{C_c^\infty}$ is manifestly conjugate symmetric and linear in the second argument. Since $\mathbb{M}$ is assumed to be non-negative definite and $I, J, \ldots$ are raised and lowered by δ, it is also immediate that $\langle \psi, \psi \rangle_{C_c^\infty} \geq 0$. The only non-trivial part[12] is checking that $\langle \psi, \psi \rangle_{C_c^\infty} = 0$ only occurs for $\psi = 0$. I'll do this using a technique from [35].
Suppose $\langle \psi, \psi \rangle_{C_c^\infty} = 0$.
$\therefore \nabla_I \psi = 0$, or equivalently $D_I \psi = -\mathrm{i}\alpha\gamma_I \psi - A_I \psi$, by equation 86.
It will help to re-write the derivative in terms of the Levi-Civita connection of h, say $D^{(h)}$.
$D_I \psi = e_I{}^{\mu'} \partial_{\mu'} \psi - \frac{1}{4}\omega_{\mu\nu I}\gamma^{\mu\nu}\psi = e_I{}^0 \partial_t \psi + e_I{}^i \partial_i \psi - \frac{1}{2}\omega_{0JI}\gamma^0\gamma^J\psi - \frac{1}{4}\omega_{JKI}\gamma^{JK}\psi$.
From equation 1, one immediately sees that $e^0 = -N\mathrm{d}t$ and $e^I = e_i^{(h)I}(\mathrm{d}x^i + N^i \mathrm{d}t)$, where $e_i^{(h)I}\mathrm{d}x^i$ is a vielbein for h. Then, $e_0 = \frac{1}{N}(\partial_t - N^i\partial_i)$ and $e_I = e_I^{(h)i}\partial_i = e_I^{(h)}$ because

$$\begin{bmatrix} N & N^k e_k^{(h)I} \\ 0 & e_i^{(h)I} \end{bmatrix} \begin{bmatrix} \frac{1}{N} & -\frac{1}{N}N^j \\ 0 & e_I^{(h)j} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \delta^j{}_i \end{bmatrix}. \tag{87}$$

$\therefore D_I \psi = e_I^{(h)i}\partial_i\psi - \frac{1}{2}\omega_{0JI}\gamma^0\gamma^J\psi - \frac{1}{4}\omega_{JKI}\gamma^{JK}\psi$.
Since $e_I = e_I^{(h)}$ has no ∂_t in it,

$$\omega_{JKI} = \frac{1}{2}(g(e_I, [e_J, e_K]) - g(e_J, [e_K, e_I]) + g(e_K, [e_J, e_I])) \tag{88}$$

$$= \frac{1}{2}(h(e_I^{(h)}, [e_J^{(h)}, e_K^{(h)}]) - h(e_J^{(h)}, [e_K^{(h)}, e_I^{(h)}]) + h(e_K^{(h)}, [e_J^{(h)}, e_I^{(h)}])) \tag{89}$$

$$= \omega_{JKI}^{(h)}. \tag{90}$$

$\therefore D_I \psi = e_I^{(h)i}\partial_i\psi - \frac{1}{2}\omega_{0JI}\gamma^0\gamma^J\psi - \frac{1}{4}\omega_{JKI}^{(h)}\gamma^{JK}\psi = D_I^{(h)}\psi - \frac{1}{2}\omega_{0JI}\gamma^0\gamma^J\psi$.
$\therefore \nabla_I \psi = 0$ is equivalent to $D_I^{(h)}\psi = \frac{1}{2}\omega_{0JI}\gamma_J\gamma_0\psi - \mathrm{i}\alpha\gamma_I\psi - A_I\psi$.

[11] It seems bizarre to carry around α instead of just setting its value to 1/2 throughout. I do this to explicitly follow the effects of the cosmological constant; $\Lambda = 0$ would require $\alpha = 0$. Furthermore, $\alpha = 1/2$ only works in the length scale convention I adopted in definition 2.2.

[12] This would be trivial too if $\mathbb{M}$ were positive definite, but I am only assuming non-negative definiteness.

The matrix multiplying ψ on the RHS doesn't really matter, so I'll just denote it as $\mathcal{A}_I$. $\psi^\dagger\psi$ is a scalar on a Riemannian manifold. Let $||\psi||_S^2 = \psi^\dagger\psi$. Then, I get

$$\begin{aligned}
\left|\partial_i(\ln\left(||\psi||_S^2\right))\right| &= \frac{1}{||\psi||_S^2}\left|\partial_i(\psi^\dagger\psi)\right| && (91)\\
&= \frac{1}{||\psi||_S^2}\left|D_i^{(h)}(\psi^\dagger\psi)\right| && (92)\\
&\leq \frac{1}{||\psi||_S^2}\left(\left|D_i^{(h)}(\psi)^\dagger\psi\right| + \left|\psi^\dagger D_i^{(h)}(\psi)\right|\right) && (93)\\
&\leq \frac{2||\psi||_S||D_i^{(h)}\psi||_S}{||\psi||_S^2} \quad \text{by the Cauchy} - \text{Schwartz inequality} && (94)\\
&= \frac{2\left|\left|e_i^{(h)I}\mathcal{A}_I\psi\right|\right|_S}{||\psi||_S} && (95)\\
&\leq 2\left|\left|e_i^{(h)I}\mathcal{A}_I\right|\right|_0. && (96)
\end{aligned}$$

Unpacking the absolute value, this is equivalent to

$$-2\left|\left|e_i^{(h)I}\mathcal{A}_I\right|\right|_0 \leq \partial_i(\ln\left(||\psi||_S^2\right)) \leq 2\left|\left|e_i^{(h)I}\mathcal{A}_I\right|\right|_0. \tag{97}$$

Let $K = \overline{\text{supp}(\psi)}$. K is compact as $\psi \in C_c^\infty$.
$\therefore$ By the extreme value theorem, $\exists$ a point, $x_1 \in \Sigma_t$, where $||\psi||_S$ is maximised.
Likewise, there also exists a point in K where $||e_i^{(h)I}\mathcal{A}_I||_0$ is maximised.
Let $C_i = \max_{x\in K\cap\Sigma_t}(||e_i^{(h)I}\mathcal{A}_I||_0)$.
Let x_0 be a point on $\partial K \cap \Sigma_t$, where $\psi = 0$.
Choose a curve, s, between x_1 and x_0, with finite length, $l(s)$. The length is determined by the Riemannian metric on Σ_t.

$$\begin{aligned}
&\therefore -2\int_{x_0}^{x_1}\left|\left|e_i^{(h)I}\mathcal{A}_I\right|\right|_0 \mathrm{d}s^i \leq \int_{x_0}^{x_1}\partial_i(\ln\left(||\psi||_S^2\right))\mathrm{d}s^i \leq 2\int_{x_0}^{x_1}\left|\left|e_i^{(h)I}\mathcal{A}_I\right|\right|_0 \mathrm{d}s^i. && (98)\\
&\therefore -2l(s)\sqrt{C_iC_jh^{ij}} \leq \ln\left(||\psi||_S^2(x_1)\right) - \ln\left(||\psi||_S^2(x_0)\right) \leq 2l(s)\sqrt{C_iC_jh^{ij}}. && (99)\\
&\therefore ||\psi||_S^2(x_0)\mathrm{e}^{-2l(s)\sqrt{C_iC_jh^{ij}}} \leq ||\psi||_S^2(x_1) \leq ||\psi||_S^2(x_0)\mathrm{e}^{2l(s)\sqrt{C_iC_jh^{ij}}}. && (100)
\end{aligned}$$

Since ψ goes to zero as one approaches x_0, both extremes of the inequality are just zero.
$\therefore ||\psi||_S^2(x_1) = 0$.
But $||\psi||_S^2(x_1)$ is maximised as x_1, so it must be that $||\psi||_S^2 = 0$ everywhere. But $||\cdot||_S$ is positive definite, so this just implies that $\psi = 0$. $\square$

Definition 3.5 (G). *Define a linear operator,* $G : C_c^\infty \to L^2$, *by* $G : \psi \mapsto \gamma^I\nabla_I\psi$.

The modified Dirac operator, G, will be the main subject of this subsection. Note that ψ being compactly supported means $G(\psi)$ is definitely in L^2.

Lemma 3.6. *If* n^a *is a future directed, unit normal to a spacelike surface,* Σ_t, *then for any antisymmetric tensor,* M^{ab},

$$n_aD_bM^{ba} = \tilde{D}_b(n_aM^{ba}), \tag{101}$$

where $\tilde{D}$ *is the induced covariant derivative on* Σ_t.

Proof. Let $H_{\mu\nu}$ be the induced metric on Σ_t, i.e. $H_{ab} = g_{ab} + n_a n_b$.
Observe that $n_b M^{ba}$ is invariant under projection, i.e. because of M^{ab}'s antisymmetry, $H^a{}_c n_b M^{cb} = \delta^a{}_c n_b M^{cb} + n^a n_c n_b M^{bc} = n_b M^{ab}$.
$\therefore$ The induced covariant derivative acts as

$$\begin{aligned}
\tilde{D}_b(n_a M^{ba}) &= H^c{}_b H^b{}_d D_c(n_a M^{da}) && (102)\\
&= H^c{}_b D_c(n_a M^{ba}) && (103)\\
&= H^c{}_b D_c(n_a) M^{ba} + H^c{}_b n_a D_c M^{ba} && (104)\\
&= K_{ba} M^{ba} + \delta^c{}_b n_a D_c M^{ba} + n^c n_b n_a D_c M^{ba} \text{ where } K_{ab} = \text{extrinsic curvature} && (105)\\
&= n_a D_b M^{ba} \text{ by } M^{ba}\text{'s antisymmetry,} && (106)
\end{aligned}$$

which is the claimed result. □

Lemma 3.7. *For any $\psi, \chi \in C_c^\infty$,*

$$\langle \psi, \chi \rangle_{C_c^\infty} = \int_{\Sigma_t} (\gamma^I \nabla_I \psi)^\dagger \gamma^J \nabla_J(\chi)\, \mathrm{d}V = \langle G(\psi), G(\chi) \rangle_{L^2}. \tag{107}$$

Proof. Because Σ_t is assumed to be non-compact and the elements of C_c^∞ are compactly supported, I can freely integrate by parts without worrying about boundary terms.
But first, observe that because $n_\mu \equiv -\delta_{\mu 0}$ in my choice of vielbein, the integrand is

$$\begin{aligned}
(\gamma^I \nabla_I \psi)^\dagger \gamma^J \nabla_J \chi &= -\nabla_I(\psi)^\dagger \gamma^I \gamma^J \nabla_J \chi && (108)\\
&= -\nabla_I(\psi)^\dagger (\gamma^{(I}\gamma^{J)} + \gamma^{IJ}) \nabla_J \chi && (109)\\
&= \nabla_I(\psi)^\dagger \nabla^I \chi - \nabla_I(\psi)^\dagger \gamma^{IJ} \nabla_J \chi && (110)\\
&= \nabla_I(\psi)^\dagger \nabla^I \chi - \nabla_I(\bar{\psi}) \gamma^{0IJ} \nabla_J \chi && (111)\\
&= \nabla_I(\psi)^\dagger \nabla^I \chi - \nabla_\nu(\bar{\psi}) \gamma^{0\nu\rho} \nabla_\rho \chi && (112)\\
&= \nabla_I(\psi)^\dagger \nabla^I \chi + n_\mu \nabla_\nu(\bar{\psi}) \gamma^{\mu\nu\rho} \nabla_\rho \chi. && (113)
\end{aligned}$$

Then, the integral is

$$\begin{aligned}
\langle G(\psi), G(\chi) \rangle_{L^2} &= \int_{\Sigma_t} (\gamma^I \nabla_I \psi)^\dagger \gamma^J \nabla_J(\chi) \mathrm{d}V && (114)\\
&= \int_{\Sigma_t} \left(\nabla_I(\psi)^\dagger \nabla^I \chi + n_\mu \nabla_\nu(\bar{\psi}) \gamma^{\mu\nu\rho} \nabla_\rho \chi\right) \mathrm{d}V && (115)\\
&= \int_{\Sigma_t} \big(\nabla_I(\psi)^\dagger \nabla^I \chi + n_\mu D_\nu(\bar{\psi}) \gamma^{\mu\nu\rho} \nabla_\rho \chi - \mathrm{i}\alpha n_\mu \bar{\psi} \gamma_\nu \gamma^{\mu\nu\rho} \nabla_\rho \chi \\
&\qquad + n_\mu \bar{\psi} \gamma^0 A_\nu^\dagger \gamma^0 \gamma^{\mu\nu\rho} \nabla_\rho \chi\big) \mathrm{d}V. && (116)
\end{aligned}$$

By lemma 3.6, Stokes' theorem and compact support, I can re-write the second term as

$$\begin{aligned}
\int_{\Sigma_t} n_\mu D_\nu(\bar{\psi}) \gamma^{\mu\nu\rho} \nabla_\rho(\chi) \mathrm{d}V &= \int_{\Sigma_t} n_\mu D_\nu(\bar{\psi} \gamma^{\mu\nu\rho} \nabla_\rho \chi) \mathrm{d}V - \int_{\Sigma_t} n_\mu \bar{\psi} \gamma^{\mu\nu\rho} D_\nu(\nabla_\rho \chi) \mathrm{d}V && (117)\\
&= \int_{\Sigma_t} \tilde{D}_\nu(n_\mu \bar{\psi} \gamma^{\mu\nu\rho} \nabla_\rho \chi) \mathrm{d}V - \int_{\Sigma_t} n_\mu \bar{\psi} \gamma^{\mu\nu\rho} D_\nu(\nabla_\rho \chi) \mathrm{d}V && (118)\\
&= -\int_{\Sigma_t} n_\mu \bar{\psi} \gamma^{\mu\nu\rho} D_\nu(\nabla_\rho \chi) \mathrm{d}V. && (119)
\end{aligned}$$

Substituting back, I get

$$\begin{aligned}\langle G(\psi), G(\chi)\rangle_{L^2} &= \int_{\Sigma_t} \big(\nabla_I(\psi)^\dagger \nabla^I \chi - n_\mu \bar{\psi}\gamma^{\mu\nu\rho} D_\nu(\nabla_\rho \chi) - \mathrm{i}\alpha n_\mu \bar{\psi}\gamma_\nu \gamma^{\mu\nu\rho}\nabla_\rho\chi \\ &\quad + n_\mu \bar{\psi}\gamma^0 A_\nu^\dagger \gamma^0 \gamma^{\mu\nu\rho}\nabla_\rho\chi\big)\mathrm{d}V \qquad (120)\\ &= \int_{\Sigma_t} \big(\nabla_I(\psi)^\dagger \nabla^I \chi - n_\mu \bar{\psi}\gamma^{\mu\nu\rho} D_\nu D_\rho \chi - \mathrm{i}\alpha n_\mu \bar{\psi}\gamma^{\mu\nu\rho}\gamma_\rho D_\nu \chi - n_\mu\bar{\psi}\gamma^{\mu\nu\rho} D_\nu(A_\rho)\chi \\ &\quad - n_\mu \bar{\psi}\gamma^{\mu\nu\rho} A_\rho D_\nu \chi - \mathrm{i}\alpha n_\mu \bar{\psi}\gamma_\nu\gamma^{\mu\nu\rho}\nabla_\rho\chi + n_\mu\bar{\psi}\gamma^0 A_\nu^\dagger\gamma^0\gamma^{\mu\nu\rho}\nabla_\rho\chi\big)\mathrm{d}V. \qquad (121)\end{aligned}$$

Now I just have to simplify this term by term.

$$\begin{aligned}\gamma^{\mu\nu\rho} D_\nu D_\rho \chi &= \frac{1}{2}\gamma^{\mu\nu\rho}[D_\nu, D_\rho]\chi \text{ by antisymmetry} \qquad (122)\\ &= -\frac{1}{8} R^{\lambda\sigma}{}_{\nu\rho}\gamma^{\mu\nu\rho}\gamma_{\lambda\sigma}\chi \qquad (123)\\ &= -\frac{1}{8} R^{\lambda\sigma}{}_{\nu\rho}\left(\gamma^{\mu\nu\rho}{}_{\lambda\sigma} - 6\gamma^{[\mu\nu}{}_{[\sigma}\delta^{\rho]}{}_{\lambda]} + 6\gamma^{[\mu}\delta^{\nu}{}_{[\sigma}\delta^{\rho]}{}_{\lambda]}\right)\chi \qquad (124)\\ &= \frac{1}{8} R^{\lambda\sigma}{}_{\nu\rho}\left(6\gamma^{[\mu\nu}{}_{[\sigma}\delta^{\rho]}{}_{\lambda]} - 6\gamma^{[\mu}\delta^{\nu}{}_{[\sigma}\delta^{\rho]}{}_{\lambda]}\right)\chi \text{ by the Bianchi identity} \qquad (125)\\ &= \frac{3}{4} R^{\lambda\sigma}{}_{\nu\rho}\left(\gamma^{[\mu\nu}{}_{\sigma}\delta^{\rho]}{}_{\lambda} - \gamma^{[\mu}\delta^{\nu}{}_{\sigma}\delta^{\rho]}{}_{\lambda}\right)\chi \text{ by antisymmetry} \qquad (126)\\ &= \frac{1}{4} R^{\lambda\sigma}{}_{\nu\rho}\left(\gamma^{\mu\nu}{}_{\sigma}\delta^{\rho}{}_{\lambda} + \gamma^{\nu\rho}{}_{\sigma}\delta^{\mu}{}_{\lambda} + \gamma^{\rho\mu}{}_{\sigma}\delta^{\nu}{}_{\lambda}\right)\chi \\ &\quad - \frac{1}{4} R^{\lambda\sigma}{}_{\nu\rho}\left(\gamma^{\mu}\delta^{\nu}{}_{\sigma}\delta^{\rho}{}_{\lambda} + \gamma^{\nu}\delta^{\rho}{}_{\sigma}\delta^{\mu}{}_{\lambda} + \gamma^{\rho}\delta^{\mu}{}_{\sigma}\delta^{\nu}{}_{\lambda}\right)\chi \qquad (127)\\ &= \frac{1}{4}\left(-R_{\sigma\nu}\gamma^{\mu\nu\sigma} + R^{\mu}{}_{\sigma\nu\rho}\gamma^{\nu\rho\sigma} + R_{\sigma\rho}\gamma^{\rho\mu\sigma} + R\gamma^\mu - R^{\mu\nu}\gamma_\nu - R^{\mu\rho}\gamma_\rho\right)\chi \qquad (128)\\ &= \frac{1}{4}(0+0+0+R\gamma^\mu - 2R^{\mu\nu}\gamma_\nu)\chi \text{ by Bianchi identity and } R_{\mu\nu} = R_{\nu\mu} \qquad (129)\\ &= -\frac{1}{2}\left(R^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}R\right)\gamma_\nu\chi. \qquad (130)\end{aligned}$$

$$\gamma^{\mu\nu\rho}\gamma_\rho D_\nu\chi = -(n-2)\gamma^{\mu\nu}D_\nu\chi. \qquad (131)$$

$$\begin{aligned}\gamma_\nu\gamma^{\mu\nu\rho}\nabla_\rho\chi &= (n-2)\gamma^{\mu\nu}\nabla_\nu\chi \qquad (132)\\ &= (n-2)\gamma^{\mu\nu}D_\nu\chi + \mathrm{i}\alpha(n-2)\gamma^{\mu\nu}\gamma_\nu\chi + (n-2)\gamma^{\mu\nu}A_\nu\chi \qquad (133)\\ &= (n-2)\gamma^{\mu\nu}D_\nu\chi - \mathrm{i}\alpha(n-1)(n-2)\gamma^\mu\chi + (n-2)\gamma^{\mu\nu}A_\nu\chi. \qquad (134)\end{aligned}$$

$$\begin{aligned}\gamma^{\mu\nu\rho}\nabla_\rho\chi &= \gamma^{\mu\nu\rho}D_\rho\chi + \mathrm{i}\alpha\gamma^{\mu\nu\rho}\gamma_\rho\chi + \gamma^{\mu\nu\rho}A_\rho\chi \qquad (135)\\ &= \gamma^{\mu\nu\rho}D_\rho\chi - \mathrm{i}\alpha(n-2)\gamma^{\mu\nu}\chi + \gamma^{\mu\nu\rho}A_\rho\chi. \qquad (136)\end{aligned}$$

Substituting these expressions back into equation 119, I get

$$\begin{aligned}\langle G(\psi), G(\chi)\rangle_{L^2} = \int_{\Sigma_t} \bigg(&\nabla_I(\psi)^\dagger\nabla^I\chi + \frac{1}{2}n_\mu\overline{\psi}\left(R^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}R\right)\gamma_\nu\chi + \mathrm{i}\alpha(n-2)n_\mu\overline{\psi}\gamma^{\mu\nu}D_\nu\chi \\ &- n_\mu\overline{\psi}\gamma^{\mu\nu\rho}D_\nu(A_\rho)\chi - n_\mu\overline{\psi}\gamma^{\mu\nu\rho}A_\rho D_\nu\chi - \mathrm{i}\alpha(n-2)n_\mu\overline{\psi}\gamma^{\mu\nu}D_\nu\chi \\ &- \alpha^2(n-1)(n-2)n_\mu\overline{\psi}\gamma^\mu\chi - \mathrm{i}\alpha(n-2)n_\mu\overline{\psi}\gamma^{\mu\nu}A_\nu\chi \\ &+ n_\mu\overline{\psi}\gamma^0A_\nu^\dagger\gamma^0\gamma^{\mu\nu\rho}D_\rho\chi - \mathrm{i}\alpha(n-2)n_\mu\overline{\psi}\gamma^0A_\nu^\dagger\gamma^0\gamma^{\mu\nu}\chi \\ &+ n_\mu\overline{\psi}\gamma^0A_\nu^\dagger\gamma^0\gamma^{\mu\nu\rho}A_\rho\chi\bigg)\mathrm{d}V \end{aligned} \tag{137}$$

$$\begin{aligned} = \int_{\Sigma_t}\bigg(&n_\mu\overline{\psi}\bigg(\bigg(\frac{1}{2}\left(R^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}R\right)\gamma_\nu - \gamma^{\mu\nu\rho}D_\nu(A_\rho) - \alpha^2(n-1)(n-2)\gamma^\mu \\ &- \mathrm{i}\alpha(n-2)\gamma^{\mu\nu}A_\nu - \mathrm{i}\alpha(n-2)\gamma^0A_\nu^\dagger\gamma^0\gamma^{\mu\nu} + \gamma^0A_\nu^\dagger\gamma^0\gamma^{\mu\nu\rho}A_\rho\bigg)\chi \\ &+ \left(\mathrm{i}\alpha(n-2)\gamma^{\mu\nu} - \gamma^{\mu\nu\rho}A_\rho - \mathrm{i}\alpha(n-2)\gamma^{\mu\nu} - \gamma^0A_\rho^\dagger\gamma^0\gamma^{\mu\nu\rho}\right)D_\nu\chi\bigg) \\ &+ (\nabla_I\psi)^\dagger\nabla^I\chi\bigg)\mathrm{d}V. \end{aligned} \tag{138}$$

In the unit conventions I'm working, $\alpha = 1/2$ and $\Lambda = -\frac{1}{2}(n-1)(n-2)$, so $\alpha^2(n-1)(n-2)$ is just $-\frac{1}{2}\Lambda$.

$\therefore$ Applying the Einstein equation[13] to 138 implies

$$\begin{aligned}\langle G(\psi), G(\chi)\rangle_{L^2} = \int_{\Sigma_t}\bigg(&n_\mu\overline{\psi}\bigg(\big(4\pi T^{\mu\nu}\gamma_\nu - \gamma^{\mu\nu\rho}D_\nu(A_\rho) - \mathrm{i}\alpha(n-2)\gamma^{\mu\nu}A_\nu \\ &- \mathrm{i}\alpha(n-2)\gamma^0A_\nu^\dagger\gamma^0\gamma^{\mu\nu} + \gamma^0A_\nu^\dagger\gamma^0\gamma^{\mu\nu\rho}A_\rho\big)\chi \\ &+ \big(\gamma^{\mu\nu\rho}A_\rho - \gamma^0A_\rho^\dagger\gamma^0\gamma^{\mu\nu\rho}\big)D_\nu\chi\bigg) + (\nabla_I\psi)^\dagger\nabla^I\chi\bigg)\mathrm{d}V. \end{aligned} \tag{139}$$

I've chosen a vielbein where $n_\mu = -\delta_{\mu 0}$, so this last equation simplifies to

$$\begin{aligned}\langle G(\psi), G(\chi)\rangle_{L^2} = \int_{\Sigma_t}\bigg(&\psi^\dagger\bigg(\big(-4\pi T^{0\mu}\gamma^0\gamma_\mu + \gamma^{IJ}D_I(A_J) + \mathrm{i}\alpha(n-2)\gamma^IA_I \\ &+ \mathrm{i}\alpha(n-2)A_I^\dagger\gamma^I - A_I^\dagger\gamma^{IJ}A_J\big)\chi \\ &+ \big(\gamma^{IJ}A_J + A_J^\dagger\gamma^{IJ}\big)D_J\chi\bigg) + (\nabla_I\psi)^\dagger\nabla^I\chi\bigg)\mathrm{d}V. \end{aligned} \tag{140}$$

Then, from definition 3.1 I immediately get

$$\langle G(\psi), G(\chi)\rangle_{L^2} = \int_{\Sigma_t}\left((\nabla_I\psi)^\dagger\nabla^I\chi + \psi^\dagger\mathbb{M}\chi\right)\mathrm{d}V \tag{141}$$

and the RHS is exactly what I defined to be $\langle\psi,\chi\rangle_{C_c^\infty}$. □

Definition 3.8 ($\mathcal{H}$)**.** *Define $\mathcal{H}$ to be the (metric space) completion of C_c^∞ under the metric corresponding to $\langle\cdot,\cdot\rangle_{C_c^\infty}$.*

Lemma 3.9. *G extends to a continuous (i.e. bounded) linear operator from $\mathcal{H}$ to L^2 such that $\langle\psi,\chi\rangle_{\mathcal{H}} = \langle G(\psi), G(\chi)\rangle_{L^2}$.*

[13] This is one of only two places where the Einstein equation is used in this work.

Proof. G is already defined for $\psi \in C_c^\infty$. The points in $\mathcal{H}\backslash C_c^\infty$ are equivalence classes of Cauchy sequences.
Let $\{\psi_a\}_{a=0}^\infty$ be a Cauchy sequence in C_c^∞ with limit in $\mathcal{H}\backslash C_c^\infty$.
Observe that by lemma 3.7, $||G(\psi_a) - G(\psi_b)||_{L^2} = ||G(\psi_a - \psi_b)||_{L^2} = ||\psi_a - \psi_b||_{C_c^\infty}$.
$\therefore \{G(\psi_a)\}_{a=0}^\infty$ is a Cauchy sequence in L^2.
$\therefore$ Since L^2 is complete, $\exists \lim_{a\to\infty} G(\psi_a) \in L^2$.
Extend the definition of G to $\mathcal{H}\backslash C_c^\infty$ by defining $G(\lim_{a\to\infty}\psi_a) = \lim_{a\to\infty} G(\psi_a)$.
This definition is independent of my original choice of Cauchy sequence, $\{\psi_a\}_{a=0}^\infty$, because if I'd chosen a different Cauchy sequence with the same "limit," $\{\chi_a\}_{a=0}^\infty$, then $\{G(\psi_a), G(\chi_b)\}$ would be a Cauchy sequence in L^2 by a similar computation to above. Hence, they would have the same limit in L^2.
Next, observe that this definition implies lemma 3.7 extends to $\mathcal{H}$. In particular, suppose $\psi = \lim_{a\to\infty}\psi_a$ and $\chi = \lim_{a\to\infty}\chi_a$ for Cauchy sequences[14], $\{\psi_a\}_{a=0}^\infty, \{\chi_a\}_{a=0}^\infty \in C_c^\infty$. Then,

$$\begin{aligned}
\langle\psi,\chi\rangle_{\mathcal{H}} &= \lim_{a\to\infty}\lim_{b\to\infty}\langle\psi_a,\chi_b\rangle_{C_c^\infty} \text{ by the definition of } \langle\cdot,\cdot\rangle_{\mathcal{H}} && (142)\\
&= \lim_{a\to\infty}\lim_{b\to\infty}\langle G(\psi_a), G(\chi_b)\rangle_{L^2} \text{ by lemma 3.7} && (143)\\
&= \left\langle \lim_{a\to\infty} G(\psi_a), \lim_{b\to\infty} G(\chi_b)\right\rangle_{L^2} \text{ by } \langle\cdot,\cdot\rangle'_{L^2}s \text{ continuity} && (144)\\
&= \langle G(\psi), G(\chi)\rangle_{L^2} \text{ by } G's \text{ definition.} && (145)
\end{aligned}$$

As an immediate consequence, I get

$$||G(\psi)||_{L^2} = ||\psi||_{\mathcal{H}}, \tag{146}$$

which implies that G is a continuous/bounded linear operator. □

Theorem 3.10. *G is a continuous, linear isomorphism between $\mathcal{H}$ and L^2.*

Proof. Continuity and linearity are already given by lemma 3.9.
Next suppose $G(\psi) = 0$. Then, by equation 146, $||\psi||_{\mathcal{H}} = 0$ and thus $\psi = 0$.
$\therefore G$ is injective.
Sadly, surjectivity is far harder to prove.
Let θ be an arbitrary element of L^2.
Define $F_\theta : \mathcal{H} \to \mathbb{C}$ by

$$F_\theta(\psi) = \langle\theta, G(\psi)\rangle_{L^2}. \tag{147}$$

F_θ is manifestly linear. It is also continuous/bounded because the Cauchy-Schwarz inequality and lemma 3.9 imply $|F_\theta(\psi)| = |\langle\theta, G(\psi)\rangle_{L^2}| \leq ||\theta||_{L^2}||G(\psi)||_{L^2} = ||\theta||_{L^2}||\psi||_{\mathcal{H}}$.
$\therefore$ By the Riesz representation theorem, $\exists\varphi \in \mathcal{H}$ such that $F_\theta(\psi) = \langle\varphi,\psi\rangle_{\mathcal{H}}$.
$\therefore F_\theta(\psi) = \langle G(\varphi), G(\psi)\rangle_{L^2}$ by lemma 3.9.
By equation 147, if follows that

$$\langle\Phi, G(\psi)\rangle_{L^2} = 0 \;\forall\psi \in \mathcal{H}, \text{ where } \Phi = \theta - G(\varphi). \tag{148}$$

Let $G^\dagger$ be the formal adjoint to G. Then, equation 148 can equivalently be formally written as

$$0 = \int_{\Sigma_t} \psi^\dagger G^\dagger(\Phi)\mathrm{d}V. \tag{149}$$

[14] Strictly speaking, ψ and χ are equivalence classes of Cauchy sequences, but I'm going to abuse notation by denoting them as if they were ordinary spinors themselves.

Since ψ is an arbitrary element of $\mathcal{H}$ - in particular it can be chosen to be supported in an arbitrarily small neighbourhood of any point of Σ_t - equation 149 implies that Φ is a weak solution to $G^\dagger(\Phi) = 0$.
$G^\dagger$ can be defined by formally integrating by parts[15]. Explicitly,

$$0 = \int_{\Sigma_t} \left(\gamma^I \nabla_I(\psi)\right)^\dagger \Phi \,\mathrm{d}V \tag{150}$$
$$= -\int_{\Sigma_t} \nabla_I(\psi)^\dagger \gamma^I \Phi \,\mathrm{d}V \tag{151}$$
$$= -\int_{\Sigma_t} \nabla_I(\bar{\psi})\gamma^{0I} \Phi \,\mathrm{d}V \tag{152}$$
$$= \int_{\Sigma_t} n_\mu \nabla_\nu(\bar{\psi})\gamma^{\mu\nu} \Phi \,\mathrm{d}V \tag{153}$$
$$= \int_{\Sigma_t} n_\mu D_\nu(\bar{\psi})\gamma^{\mu\nu} \Phi \,\mathrm{d}V + \int_{\Sigma_t} n_\mu \bar{\psi} \left(-\mathrm{i}\alpha\gamma_\nu + \gamma^0 A_\nu^\dagger \gamma^0\right) \gamma^{\mu\nu}\Phi \,\mathrm{d}V. \tag{154}$$

$\gamma^{\mu\nu}\Phi$ is antisymmetric, so lemma 3.6 applies, at least formally.

$$\therefore 0 = \int_{\Sigma_t} n_\mu \bar{\psi} \left(-\gamma^{\mu\nu} D_\nu(\Phi) - \mathrm{i}\alpha(n-1)\gamma^\mu \Phi + \gamma^0 A_\nu^\dagger \gamma^0 \gamma^{\mu\nu}\Phi\right) \mathrm{d}V \tag{155}$$
$$= \int_{\Sigma_t} \psi^\dagger \left(\gamma^I D_I(\Phi) + \mathrm{i}\alpha(n-1)\Phi - A_I^\dagger \gamma^I \Phi\right) \mathrm{d}V. \tag{156}$$
$$\therefore G^\dagger \Phi = \gamma^I D_I(\Phi) + \mathrm{i}\alpha(n-1)\Phi - A_I^\dagger \gamma^I \Phi. \tag{157}$$

In definition 3.1, I've assumed[16] $\exists \tilde{A}_\mu$ such that $A_I^\dagger \gamma^I = -\gamma^I \tilde{A}_I$, $\tilde{A}_I$ decays at the same rate as A_μ and $\gamma^{IJ}\tilde{A}_J$ is hermitian.
$\therefore$ Analogously to the previous steps, I can define an $\widetilde{\mathbb{M}}$ (with $A_\mu \to \tilde{A}_\mu$ and $\alpha \to -\alpha$) and a connection, $\widetilde{\nabla}_\mu = D_\mu - \mathrm{i}\alpha\gamma_\mu + \tilde{A}_\mu$, to get

$$G^\dagger \Phi = \gamma^I \widetilde{\nabla}_I \Phi = \widetilde{G}(\Phi) \text{ and} \tag{158}$$
$$\langle \widetilde{G}(\psi), \widetilde{G}(\chi)\rangle_{L^2} = \int_{\Sigma_t} ((\widetilde{\nabla}_I \psi)^\dagger \widetilde{\nabla}^I \chi + \psi^\dagger \widetilde{\mathbb{M}} \chi) \mathrm{d}V \tag{159}$$

for $\psi, \chi \in C_c^\infty$.
First, suppose $\widetilde{\mathbb{M}}$ is positive definite, where I can provide a much more self-contained proof.
Currently, equation 159 is a purely formal expression based on the weak solution property above. However, based on elliptic regularity arguments, one can show $\Phi \in H^1_{\mathrm{loc}}$ (I will defer to theorems 8.8, 7.3 an 6.4 of [26] for the details).
Having established this regularity for Φ, some more concrete manipulations can be made.
For that, define a function, a_m, as follows.
Let $d(\cdot,\cdot) : \Sigma_t \times \Sigma_t \to \mathbb{R}$ be the metric function (in the sense of a metric space, not a Riemannian metric) induced on Σ_t by g.
Let $a : \mathbb{R} \to \mathbb{R}$ be any smooth function such that $a(x) = 1$ for $x \in (0, 1)$ and $a(x) = 0$ for $x \in (2, \infty)$.
Let q be an arbitrary point of Σ_t and define $a_m : \Sigma_t \to \mathbb{R}$ by $a_m : p \mapsto a(d(q, p)/m)$.
From Σ_t's assumed completeness and the Hopf-Rinow theorem, $a_m \in C_c^\infty$ and hence $a_m\psi \in \mathcal{H}$.

[15] The integration by parts is only formal because $\Phi \in L^2$ may not be continuously differentiable a priori.
[16] This is the only place in this work where this assumption will be used.

Then, by equation 148,

$$0 = \langle G(a_m\psi), \Phi\rangle_{L^2} \tag{160}$$
$$= \int_{\Sigma_t} a_m\psi^\dagger G^\dagger(\Phi)\mathrm{d}V \tag{161}$$
$$= \int_{\Sigma_t} \psi^\dagger \left(\widetilde{G}(a_m\Phi) - \gamma^I D_I(a_m)\Phi\right)\mathrm{d}V. \tag{162}$$

Since ψ is an arbitrary element of $\mathcal{H} \supset C_c^\infty$, this can only be true if $\widetilde{G}(a_m\Phi) = \gamma^I D_I(a_m)\Phi$.
For now, I'm assuming $\widetilde{\mathbb{M}}$ is positive definite, so I can form a Hilbert space, $\widetilde{\mathcal{H}}$, in the same way as $\mathcal{H}$. Then,

$$||a_m\Phi - a_n\Phi||_{\widetilde{\mathcal{H}}} = ||\widetilde{G}(a_m\Phi) - \widetilde{G}(a_n\Phi)||_{L^2} \text{ by equation 159} \tag{163}$$
$$= ||\gamma^I D_I(a_m - a_n)\Phi||_{L^2} \tag{164}$$
$$\to 0 \text{ as } n, m \to \infty \tag{165}$$

because $\Phi \in L^2$ and the derivative, $D_I(a_m - a_n)$, is (by construction) only non-zero in some "annulus" whose "inner radius" closer and closer to $\partial_\infty\Sigma_t$ as $n, m \to \infty$.
$\therefore \{a_m\Phi\}_{m=1}^\infty$ is a Cauchy sequence in $\widetilde{H}$.
The limit, $\lim_{m\to\infty} a_m\Phi \in \widetilde{\mathcal{H}}$, must be Φ itself[17].
Finally, I get

$$||\Phi||_{\widetilde{\mathcal{H}}} = \lim_{m\to\infty} ||a_m\Phi||_{\widetilde{\mathcal{H}}} \tag{166}$$
$$= \lim_{m\to\infty} ||\widetilde{G}(a_m\Phi)||_{L^2} \tag{167}$$
$$= \lim_{m\to\infty} ||\gamma^I D_I(a_m)\Phi||_{L^2} \tag{168}$$
$$= 0 \text{ by the same reasoning as equation 165.} \tag{169}$$

$\therefore \Phi = 0$ and thus $\theta = G(\varphi)$.
Since θ was arbitrary, G must be surjective.
It remains to consider the case when $\widetilde{\mathbb{M}}$ is not positive-definite[18].
The proof is a variation of some black magic from [35].
For this approach to the proof, let θ be an arbitrary element of L^2.
From equation 159, if $\psi \in C_c^\infty$, then

$$||\widetilde{G}(\psi)||_{L^2}^2 = \int_{\Sigma_t} ((\widetilde{\nabla}_I\psi)^\dagger\widetilde{\nabla}^I\psi + \psi^\dagger\widetilde{\mathbb{M}}\psi)\mathrm{d}V. \tag{170}$$

By construction,

$$||G(\psi)||_{L^2}^2 = \int_{\Sigma_t} ((\nabla_I\psi)^\dagger\nabla^I\psi + \psi^\dagger\mathbb{M}\psi)\mathrm{d}V \tag{171}$$

is finite $\forall\psi \in \mathcal{H}$.
$\widetilde{\nabla}_I\psi = \nabla_I\psi - 2\mathrm{i}\alpha\gamma_I\psi + (\widetilde{A}_I - A_I)\psi$, i.e. the difference in the connections is only $\alpha \to -\alpha$ and $A_\mu \to \widetilde{A}_\mu$.

[17]Note that in all these Sobolev type spaces, functions are only defined up to a re-definition on sets of measure zero.

[18]The connections in section 5 fall in this category. This is true even in the analogous calculation for asymptotically flat spacetimes. As explained in appendix A of [36], this issue was completely ignored by [18, 19] and dealt with incorrectly by [26]. As far as I know, this work is the first to try fix this problem.

Let ϕ be an arbitrary element of C_c^∞.
Let $C_c^\infty(r_0)$ be the set of compactly supported smooth functions whose support is within $\Sigma_t \backslash \{r \geq 3r_0\}$.
Choose r_0 large enough so that $\phi \in C_c^\infty(r_0)$ and so that the Fefferman-Graham coordinates are valid (otherwise $\{r \geq 3r_0\}$ would not be a meaningful set).
$C_c^\infty(r_0)$ is a subspace of $\mathcal{H}$ by inspection. Let $\mathcal{H}(r_0)$ be the (metric space) completion of $C_c^\infty(r_0)$ under the inner product of equation 86 (the same inner product as $\mathcal{H}$).
$\therefore \mathcal{H}(r_0)$ is a closed, Hilbert space subspace of $\mathcal{H}$.
Now I can define a functional, $S_\phi : \mathcal{H}(r_0) \to \mathbb{C}$ by

$$S_\phi(\psi) = \frac{1}{2}||\widetilde{G}(\psi)||^2_{L^2} - \langle \psi, \phi \rangle_{L^2}. \tag{172}$$

The effective "cut-off" at $r = 3r_0$, ∇_I & $\widetilde{\nabla}_I$ differing only by $\alpha \to -\alpha$ & $A_I \to \widetilde{A}_I$ and the assumptions on $\widetilde{A}_I$ in definition 3.1 ensure that S_ϕ is finite.
Since $\widetilde{G} = G^\dagger$ and $(G^\dagger)^\dagger = G$, the variational equation for minimising S_ϕ is $G(\widetilde{G}(\psi)) = \phi$.
The main technical tool applied by [35] is theorem 9.5 of [37], which states that for any finite, weakly lower semicontinuous functional, $f(x)$, defined on a reflexive Banach space, E, if

$$\lim_{R\to\infty} \sup_{||x||_E = R} f(x) \to \infty, \tag{173}$$

then $f(x)$ has a minimum point. In particular, $\exists$ a weak solution to the variational equation.
By elliptic regularity, it's then lifted to a strong solution with the same regularity as ϕ.
In my case, $E = \mathcal{H}(r_0)$ and $f = S_\phi$.
S is finite by construction and every Hilbert space is reflexive. S is strictly convex by inspection and then theorem 8.10 of [37] implies weak lower semicontinuity.
$\therefore$ Only the limit superior property remains to check.
For that, construct a spinor, $\psi \in C_c^\infty$, as follows. Let

$$\psi = \begin{cases} \mathrm{e}^r \psi_0 & \text{for } r_0 < r < 2r_0 \\ 0 & \text{for } r < r_0 - \epsilon \text{ or } r > 2r_0 + \epsilon \\ C^\infty \text{ interpolation} & \text{for all other } r \end{cases} \tag{174}$$

for a constant spinor, ψ_0. Choose r_0 to be sufficiently large that this ψ is supported deep in the asymptotic end[19]. Also, choose ϵ to be sufficiently small for the argument below.
Then, since ψ depends only on r by construction and it's natural to separate out $e^1 = \mathrm{d}r$ in Fefferman-Graham coordinates, for $r_0 < r < 2r_0$ I get

$$D_I \psi = \mathrm{e}^r \delta_{I1} \psi_0 - \frac{1}{4}\mathrm{e}^r \omega_{\mu\nu I}\gamma^{\mu\nu}\psi_0. \tag{175}$$

$$\therefore G(\psi) = \mathrm{e}^r \left(\gamma^1 - \mathrm{i}\alpha(n-1)I - \frac{1}{4}\omega_{\mu\nu I}\gamma^I\gamma^{\mu\nu} + \gamma^I A_I \right)\psi_0 \text{ and similarly} \tag{176}$$

$$\widetilde{G}(\psi) = \mathrm{e}^r \left(\gamma^1 + \mathrm{i}\alpha(n-1)I - \frac{1}{4}\omega_{\mu\nu I}\gamma^I\gamma^{\mu\nu} + \gamma^I \widetilde{A}_I \right)\psi_0. \tag{177}$$

Because of the decay that I've assumed for A_I and $\widetilde{A}_I$ in definition 3.1, $\gamma^I A_I$ and $\gamma^I \widetilde{A}_I$ are completely dominated by the other terms in the $G(\psi)$ and $\widetilde{G}(\psi)$ expressions.
Choose ψ_0 so that $(\gamma^1 \pm \mathrm{i}\alpha(n-1)I)\psi_0 \neq 0$ and then let $\psi_0^\dagger \psi_0 \to \infty$.

[19] As a corollary, ψ_0 is now well defined; "constant" is a frame dependent concept for spinors, but there is a natural frame deep in the asymptotic end which covers the end.

$\therefore ||G(\psi)||_{L^2} = ||\psi||_{\mathcal{H}(r_0)} \to \infty$, but $||\widetilde{G}(\psi)||_{L^2}$ will also go off to infinity.
Applying the Cauchy-Schwartz inequality pointwise,

$$S_\phi(\psi) \geq \frac{1}{2}||\widetilde{G}(\psi)||^2_{L^2} - \int_{\mathrm{supp}(\phi)} \sqrt{\psi^\dagger\psi}\sqrt{\phi^\dagger\phi}\,\mathrm{d}V. \tag{178}$$

The 1st term is quadratic in ψ but the 2nd term is effectively linear in ψ, so $S(\psi) \to \infty$ too[20].
$\therefore$ The limit superior condition is satisfied.
$\therefore \exists \psi_\phi \in \mathcal{H}(r_0)$ such that $G(\widetilde{G}(\psi_\phi)) = \phi$. In fact, by ϕ's compact support and elliptic regularity, $\psi_\phi \in C_c^\infty$.
Furthermore, by lemma 3.7 $\widetilde{G}(\psi_\phi) \in \mathcal{H}$ because

$$||\widetilde{G}(\psi_\phi)||_{\mathcal{H}} = ||G(\widetilde{G}(\psi_\phi))||_{L^2} = ||\phi||_{L^2} < \infty. \tag{179}$$

Denote $\widetilde{G}(\psi_\phi)$ as Ψ_ϕ.
In summary, I have shown that $\forall \phi \in C_c^\infty$, $\exists \Psi_\phi \in \mathcal{H}$ such that $G(\Psi_\phi) = \phi$.
Since C_c^∞ is dense in L^2, $\exists$ a Cauchy sequence, $\{\theta_m\}_{m=0}^\infty \subseteq C_c^\infty$, which converges to θ in L^2.
Given $\{\theta_m\}_{m=0}^\infty$, construct the corresponding sequence, $\{\Psi_{\theta_m}\}_{m=0}^\infty \in \mathcal{H}$.
$\{\Psi_{\theta_m}\}_{m=0}^\infty \in \mathcal{H}$ is a Cauchy sequence because

$$||\Psi_{\theta_m} - \Psi_{\theta_n}||_{\mathcal{H}} = ||G(\Psi_{\theta_m} - \Psi_{\theta_n})||_{L^2} = ||\theta_m - \theta_n||_{L^2} \to 0. \tag{180}$$

Let $\Psi = \lim_{m\to\infty} \Psi_{\theta_m} \in \mathcal{H}$. By theorem 3.9, G is bounded/continuous. Thus,

$$||G(\Psi) - \theta||_{L^2} = \left|\left| \lim_{m\to\infty} G(\Psi_{\theta_m}) - \theta \right|\right|_{L^2} = \left|\left| \lim_{m\to\infty} \theta_m - \theta \right|\right|_{L^2} = 0. \tag{181}$$

$\therefore G(\Psi) = \theta$.
Since θ is an arbitrary element of L^2, it follows that G is surjective. □

3.2 Main theorem

The main result of this work is theorem 3.19, but I'll still need a few more definitions and lemmas to set it up.

Definition 3.11 ($Q(\varepsilon)$)**.** *For a spinor, ε, define $Q(\varepsilon)$ by*

$$Q(\varepsilon) = \int_{\Sigma_t} n_\mu D_\nu(E^{\nu\mu})\mathrm{d}V, \quad \text{where} \tag{182}$$

$$E^{\mu\nu} = \bar{\varepsilon}\gamma^{\mu\nu\rho}\nabla_\rho\varepsilon + \text{c.c} = \bar{\varepsilon}\gamma^{\mu\nu\rho}\nabla_\rho\varepsilon - \nabla_\rho(\bar{\varepsilon})\gamma^{\mu\nu\rho}\varepsilon \tag{183}$$

and n^μ is a future directed unit normal to constant t surfaces, Σ_t.

Like Q, $E^{\mu\nu}$ also depends on ε. But, I'll suppress that dependence in situations where there is no ambiguity.

Lemma 3.12. *$Q(\varepsilon)$ is conserved $\forall\varepsilon$.*

Proof. Consider two values of t, say t_1 and t_2. Then,

$$Q(\varepsilon)|_{t_2} - Q(\varepsilon)|_{t_1} = \int_{\Sigma_{t_2}} n_\mu D_\nu(E^{\nu\mu})\mathrm{d}V - \int_{\Sigma_{t_1}} n_\mu D_\nu(E^{\nu\mu})\mathrm{d}V \tag{184}$$

$$= \int_{t_1}^{t_2}\int_{\Sigma_t} D_\mu(D_\nu E^{\nu\mu})\mathrm{d}V\mathrm{d}t \;\text{ by Stokes' theorem.} \tag{185}$$

[20] I could even choose r_0 big enough so that the second term is just zero.

The integrand is however zero because

$$D_\mu(D_\nu E^{\nu\mu}) = \frac{1}{2}[D_\mu, D_\nu]E^{\nu\mu} \text{ as } E^{\nu\mu} \text{ is antisymmetric} \tag{186}$$
$$= \frac{1}{2}(R^\nu{}_{\rho\mu\nu}E^{\rho\mu} + R^\mu{}_{\rho\mu\nu}E^{\nu\rho}) \tag{187}$$
$$= -R_{\mu\nu}E^{\mu\nu} \tag{188}$$
$$= 0 \text{ as } R_{\mu\nu} = R_{\nu\mu} \text{ but } E^{\mu\nu} = -E^{\nu\mu}. \tag{189}$$

Hence the value of $Q(\varepsilon)$ does not depend on t. □

Lemma 3.13. *Choose a vielbein where* $n^\mu = \delta^{\mu 0} \equiv e^0$ *and* $e^1 = \mathrm{d}r$. *Then,*

$$Q(\varepsilon) = \int_{\partial_\infty \Sigma_t} E^{01}\mathrm{d}A \text{ and} \tag{190}$$
$$E^{01} = \varepsilon^\dagger\gamma^1\gamma^A D_A\varepsilon + D_A(\varepsilon)^\dagger\gamma^A\gamma^1\varepsilon - 2\mathrm{i}\alpha(n-2)\varepsilon^\dagger\gamma^1\varepsilon + \varepsilon^\dagger\gamma^1\gamma^A A_A\varepsilon + \varepsilon^\dagger A_A^\dagger\gamma^A\gamma^1\varepsilon. \tag{191}$$

Proof. Let l_a denote the normal to constant r surfaces. Then,

$$Q(\varepsilon) = \int_{\Sigma_t} n_\mu D_\nu(E^{\nu\mu})\mathrm{d}V \tag{192}$$
$$= \int_{\Sigma_t} \tilde{D}_\nu(n_\mu E^{\nu\mu})\mathrm{d}V \text{ by lemma 3.6} \tag{193}$$
$$= \int_{\partial_\infty \Sigma_t} l_\nu n_\mu E^{\nu\mu}\mathrm{d}A \text{ by Stokes' theorem} \tag{194}$$
$$= -\int_{\partial_\infty \Sigma_t} E^{10}\mathrm{d}A \text{ by my vielbein choice} \tag{195}$$
$$= \int_{\partial_\infty \Sigma_t} E^{01}\mathrm{d}A, \tag{196}$$

which proves the first half of the lemma. Meanwhile, from equation 183,

$$E^{01} = \bar{\varepsilon}\gamma^{01\mu}\nabla_\mu\varepsilon - \nabla_\mu(\bar{\varepsilon})\gamma^{01\mu}\varepsilon \tag{197}$$
$$= \bar{\varepsilon}\gamma^0\gamma^1\gamma^A\nabla_A\varepsilon - \nabla_A(\bar{\varepsilon})\gamma^0\gamma^1\gamma^A\varepsilon \tag{198}$$
$$= \varepsilon^\dagger\gamma^1\gamma^A\nabla_A\varepsilon - \nabla_A(\varepsilon)^\dagger\gamma^1\gamma^A\varepsilon \tag{199}$$
$$\begin{aligned}= {} & \varepsilon^\dagger\gamma^1\gamma^A D_A\varepsilon + \mathrm{i}\alpha\varepsilon^\dagger\gamma^1\gamma^A\gamma_A\varepsilon + \varepsilon^\dagger\gamma^1\gamma^A A_A\varepsilon \\ & - D_A(\varepsilon)^\dagger\gamma^1\gamma^A\varepsilon - \mathrm{i}\alpha\varepsilon^\dagger\gamma_A\gamma^1\gamma^A\varepsilon - \varepsilon^\dagger A_A^\dagger\gamma^1\gamma^A\varepsilon\end{aligned} \tag{200}$$
$$= \varepsilon^\dagger\gamma^1\gamma^A D_A\varepsilon + D_A(\varepsilon)^\dagger\gamma^A\gamma^1\varepsilon - 2\mathrm{i}\alpha(n-2)\varepsilon^\dagger\gamma^1\varepsilon + \varepsilon^\dagger\gamma^1\gamma^A A_A\varepsilon + \varepsilon^\dagger A_A^\dagger\gamma^A\gamma^1\varepsilon, \tag{201}$$

which proves the second half of the lemma. □

Lemma 3.13 evaluated the boundary expression for $Q(\varepsilon)$ when applying lemma 3.6. In the next lemma, I'll find the bulk expression for the same quantity.

Lemma 3.14. *Assuming the Einstein equation is satisfied,*

$$Q(\varepsilon) = 2\int_{\Sigma_t} \left((\nabla_I\varepsilon)^\dagger\nabla^I\varepsilon - (\gamma^I\nabla_I\varepsilon)^\dagger\gamma^J\nabla_J\varepsilon + \varepsilon^\dagger\mathbb{M}\varepsilon\right)\mathrm{d}V. \tag{202}$$

Proof. In accordance with equations 182 and 183, I'll begin by expanding $D_\nu E^{\nu\mu}$.

$$
\begin{aligned}
D_\nu E^{\nu\mu} &= D_\nu \left(\bar{\varepsilon}\gamma^{\nu\mu\rho}\nabla_\rho\varepsilon - \nabla_\rho(\bar{\varepsilon})\gamma^{\nu\mu\rho}\varepsilon\right) && (203)\\
&= D_\nu(\bar{\varepsilon})\gamma^{\nu\mu\rho}\nabla_\rho\varepsilon + \bar{\varepsilon}\gamma^{\nu\mu\rho}D_\nu(\nabla_\rho\varepsilon) - D_\nu(\nabla_\rho\bar{\varepsilon})\gamma^{\nu\mu\rho}\varepsilon - \nabla_\rho(\bar{\varepsilon})\gamma^{\nu\mu\rho}D_\nu\varepsilon && (204)\\
&= \nabla_\nu(\bar{\varepsilon})\gamma^{\nu\mu\rho}\nabla_\rho\varepsilon + \mathrm{i}\alpha\bar{\varepsilon}\gamma_\nu\gamma^{\nu\mu\rho}\nabla_\rho\varepsilon - \bar{\varepsilon}\gamma^0 A_\nu^\dagger\gamma^0\gamma^{\nu\mu\rho}\nabla_\rho\varepsilon + \bar{\varepsilon}\gamma^{\nu\mu\rho}D_\nu(\nabla_\rho\varepsilon)\\
&\quad - D_\nu(\nabla_\rho\bar{\varepsilon})\gamma^{\nu\mu\rho}\varepsilon - \nabla_\rho(\bar{\varepsilon})\gamma^{\nu\mu\rho}\nabla_\nu\varepsilon + \mathrm{i}\alpha\nabla_\rho(\bar{\varepsilon})\gamma^{\nu\mu\rho}\gamma_\nu\varepsilon + \nabla_\rho(\bar{\varepsilon})\gamma^{\nu\mu\rho}A_\nu\varepsilon && (205)\\
&= 2\nabla_\nu(\bar{\varepsilon})\gamma^{\nu\mu\rho}\nabla_\rho\varepsilon - \mathrm{i}\alpha(n-2)\bar{\varepsilon}\gamma^{\mu\nu}\nabla_\nu\varepsilon - \bar{\varepsilon}\gamma^0 A_\nu^\dagger\gamma^0\gamma^{\nu\mu\rho}\nabla_\rho\varepsilon + \bar{\varepsilon}\gamma^{\nu\mu\rho}D_\nu(\nabla_\rho\varepsilon)\\
&\quad - D_\nu(\nabla_\rho\bar{\varepsilon})\gamma^{\nu\mu\rho}\varepsilon - \mathrm{i}\alpha(n-2)\nabla_\nu(\bar{\varepsilon})\gamma^{\mu\nu}\varepsilon + \nabla_\rho(\bar{\varepsilon})\gamma^{\nu\mu\rho}A_\nu\varepsilon && (206)\\
&= 2\nabla_\nu(\bar{\varepsilon})\gamma^{\nu\mu\rho}\nabla_\rho\varepsilon - \mathrm{i}\alpha(n-2)\bar{\varepsilon}\gamma^{\mu\nu}D_\nu\varepsilon + \alpha^2(n-2)\bar{\varepsilon}\gamma^{\mu\nu}\gamma_\nu\varepsilon - \mathrm{i}\alpha(n-2)\bar{\varepsilon}\gamma^{\mu\nu}A_\nu\varepsilon\\
&\quad - \bar{\varepsilon}\gamma^0 A_\nu^\dagger\gamma^0\gamma^{\nu\mu\rho}D_\rho\varepsilon - \mathrm{i}\alpha\bar{\varepsilon}\gamma^0 A_\nu^\dagger\gamma^0\gamma^{\nu\mu\rho}\gamma_\rho\varepsilon - \bar{\varepsilon}\gamma^0 A_\nu^\dagger\gamma^0\gamma^{\nu\mu\rho}A_\rho\varepsilon + \bar{\varepsilon}\gamma^{\nu\mu\rho}D_\nu D_\rho\varepsilon\\
&\quad + \mathrm{i}\alpha\bar{\varepsilon}\gamma^{\nu\mu\rho}\gamma_\rho D_\nu\varepsilon + \bar{\varepsilon}\gamma^{\nu\mu\rho}D_\nu(A_\rho)\varepsilon + \bar{\varepsilon}\gamma^{\nu\mu\rho}A_\rho D_\nu\varepsilon - D_\nu D_\rho(\bar{\varepsilon})\gamma^{\nu\mu\rho}\varepsilon\\
&\quad + \mathrm{i}\alpha D_\nu(\bar{\varepsilon})\gamma_\rho\gamma^{\nu\mu\rho}\varepsilon - D_\nu(\bar{\varepsilon})\gamma^0 A_\rho^\dagger\gamma^0\gamma^{\nu\mu\rho}\varepsilon - \bar{\varepsilon}\gamma^0 D_\nu(A_\rho^\dagger)\gamma^0\gamma^{\nu\mu\rho}\varepsilon\\
&\quad - \mathrm{i}\alpha(n-2)D_\nu(\bar{\varepsilon})\gamma^{\mu\nu}\varepsilon - \alpha^2(n-2)\bar{\varepsilon}\gamma_\nu\gamma^{\mu\nu}\varepsilon - \mathrm{i}\alpha(n-2)\bar{\varepsilon}\gamma^0 A_\nu^\dagger\gamma^0\gamma^{\mu\nu}\varepsilon\\
&\quad + D_\rho(\bar{\varepsilon})\gamma^{\nu\mu\rho}A_\nu\varepsilon - \mathrm{i}\alpha\bar{\varepsilon}\gamma_\rho\gamma^{\nu\mu\rho}A_\nu\varepsilon + \bar{\varepsilon}\gamma^0 A_\rho^\dagger\gamma^0\gamma^{\nu\mu\rho}A_\nu\varepsilon. && (207)
\end{aligned}
$$

Some of these terms can be simplified, as follows.

$$
\begin{aligned}
\alpha^2(n-2)\bar{\varepsilon}\gamma^{\mu\nu}\gamma_\nu\varepsilon &= -\alpha^2(n-1)(n-2)\bar{\varepsilon}\gamma^\mu\varepsilon. && (208)\\
-\mathrm{i}\alpha\bar{\varepsilon}\gamma^0 A_\nu^\dagger\gamma^0\gamma^{\nu\mu\rho}\gamma_\rho\varepsilon &= \mathrm{i}\alpha(n-2)\bar{\varepsilon}\gamma^0 A_\nu^\dagger\gamma^0\gamma^{\nu\mu}\varepsilon. && (209)\\
\bar{\varepsilon}\gamma^{\nu\mu\rho}D_\nu D_\rho\varepsilon &= \frac{1}{2}\left(R^{\mu\nu} - \frac{1}{2}R\eta^{\mu\nu}\right)\bar{\varepsilon}\gamma_\nu\varepsilon \text{ by the same steps as equation 130.} && (210)\\
\mathrm{i}\alpha\bar{\varepsilon}\gamma^{\nu\mu\rho}\gamma_\rho D_\nu\varepsilon &= -\mathrm{i}\alpha(n-2)\bar{\varepsilon}\gamma^{\nu\mu}D_\nu\varepsilon. && (211)\\
-D_\nu D_\rho(\bar{\varepsilon})\gamma^{\nu\mu\rho}\varepsilon &= \frac{1}{2}\left(R^{\mu\nu} - \frac{1}{2}R\eta^{\mu\nu}\right)\bar{\varepsilon}\gamma_\nu\varepsilon \text{ by taking equation 210's conjugate.} && (212)\\
\mathrm{i}\alpha D_\nu(\bar{\varepsilon})\gamma_\rho\gamma^{\nu\mu\rho}\varepsilon &= -\mathrm{i}\alpha(n-2)D_\nu(\bar{\varepsilon})\gamma^{\nu\mu}\varepsilon. && (213)\\
-\alpha^2(n-2)\bar{\varepsilon}\gamma_\nu\gamma^{\mu\nu}\varepsilon &= -\alpha^2(n-1)(n-2)\bar{\varepsilon}\gamma^\mu\varepsilon. && (214)\\
-\mathrm{i}\alpha\bar{\varepsilon}\gamma_\rho\gamma^{\nu\mu\rho}A_\nu\varepsilon &= \mathrm{i}\alpha(n-2)\bar{\varepsilon}\gamma^{\nu\mu}A_\nu\varepsilon. && (215)
\end{aligned}
$$

Substituting these back up,

$$
\begin{aligned}
D_\nu E^{\nu\mu} &= 2\nabla_\nu(\bar{\varepsilon})\gamma^{\nu\mu\rho}\nabla_\rho\varepsilon - \mathrm{i}\alpha(n-2)\bar{\varepsilon}\gamma^{\mu\nu}D_\nu\varepsilon - \alpha^2(n-1)(n-2)\bar{\varepsilon}\gamma^\mu\varepsilon - \mathrm{i}\alpha(n-2)\bar{\varepsilon}\gamma^{\mu\nu}A_\nu\varepsilon\\
&\quad - \bar{\varepsilon}\gamma^0 A_\nu^\dagger\gamma^0\gamma^{\nu\mu\rho}D_\rho\varepsilon + \mathrm{i}\alpha(n-2)\bar{\varepsilon}\gamma^0 A_\nu^\dagger\gamma^0\gamma^{\nu\mu}\varepsilon - \bar{\varepsilon}\gamma^0 A_\nu^\dagger\gamma^0\gamma^{\nu\mu\rho}A_\rho\varepsilon\\
&\quad + \frac{1}{2}\left(R^{\mu\nu} - \frac{1}{2}R\eta^{\mu\nu}\right)\bar{\varepsilon}\gamma_\nu\varepsilon - \mathrm{i}\alpha(n-2)\bar{\varepsilon}\gamma^{\nu\mu}D_\nu\varepsilon + \bar{\varepsilon}\gamma^{\nu\mu\rho}D_\nu(A_\rho)\varepsilon + \bar{\varepsilon}\gamma^{\nu\mu\rho}A_\rho D_\nu\varepsilon\\
&\quad + \frac{1}{2}\left(R^{\mu\nu} - \frac{1}{2}R\eta^{\mu\nu}\right)\bar{\varepsilon}\gamma_\nu\varepsilon - \mathrm{i}\alpha(n-2)D_\nu(\bar{\varepsilon})\gamma^{\nu\mu}\varepsilon - D_\nu(\bar{\varepsilon})\gamma^0 A_\rho^\dagger\gamma^0\gamma^{\nu\mu\rho}\varepsilon\\
&\quad - \bar{\varepsilon}\gamma^0 D_\nu(A_\rho^\dagger)\gamma^0\gamma^{\nu\mu\rho}\varepsilon - \mathrm{i}\alpha(n-2)D_\nu(\bar{\varepsilon})\gamma^{\mu\nu}\varepsilon - \alpha^2(n-1)(n-2)\bar{\varepsilon}\gamma^\mu\varepsilon\\
&\quad - \mathrm{i}\alpha(n-2)\bar{\varepsilon}\gamma^0 A_\nu^\dagger\gamma^0\gamma^{\mu\nu}\varepsilon + D_\rho(\bar{\varepsilon})\gamma^{\nu\mu\rho}A_\nu\varepsilon + \mathrm{i}\alpha(n-2)\bar{\varepsilon}\gamma^{\nu\mu}A_\nu\varepsilon\\
&\quad + \bar{\varepsilon}\gamma^0 A_\rho^\dagger\gamma^0\gamma^{\nu\mu\rho}A_\nu\varepsilon && (216)
\end{aligned}
$$

In the unit conventions I'm working, $\alpha = 1/2$ and $\Lambda = -\frac{1}{2}(n-1)(n-2)$, so $\alpha^2(n-1)(n-2)$

is just $-\frac{1}{2}\Lambda$. Using that in conjunction with the Einstein equation[21], I get

$$\begin{aligned}D_\nu E^{\nu\mu} &= \bar{\varepsilon}(8\pi T^{\mu\nu}\gamma_\nu - \mathrm{i}\alpha(n-2)\gamma^{\mu\nu}A_\nu + \mathrm{i}\alpha(n-2)\gamma^0 A_\nu^\dagger\gamma^0\gamma^{\nu\mu} - \gamma^0 A_\nu^\dagger\gamma^0\gamma^{\nu\mu\rho}A_\rho \\ &\quad + \gamma^{\nu\mu\rho}D_\nu A_\rho - \gamma^0 D_\nu(A_\rho^\dagger)\gamma^0\gamma^{\nu\mu\rho} - \mathrm{i}\alpha(n-2)\gamma^0 A_\nu^\dagger\gamma^0\gamma^{\mu\nu} + \mathrm{i}\alpha(n-2)\gamma^{\nu\mu}A_\nu \\ &\quad + \gamma^0 A_\rho^\dagger\gamma^0\gamma^{\nu\mu\rho}A_\nu)\varepsilon + 2\nabla_\nu(\bar{\varepsilon})\gamma^{\nu\mu\rho}\nabla_\rho\varepsilon \\ &\quad + \bar{\varepsilon}(-\mathrm{i}\alpha(n-2)\gamma^{\mu\nu} - \gamma^0 A_\rho^\dagger\gamma^0\gamma^{\rho\mu\nu} - \mathrm{i}\alpha(n-2)\gamma^{\nu\mu} + \gamma^{\nu\mu\rho}A_\rho)D_\nu\varepsilon \\ &\quad + D_\nu(\bar{\varepsilon})(-\mathrm{i}\alpha(n-2)\gamma^{\nu\mu} - \gamma^0 A_\rho^\dagger\gamma^0\gamma^{\nu\mu\rho} - \mathrm{i}\alpha(n-2)\gamma^{\mu\nu} + \gamma^{\rho\mu\nu}A_\rho)\varepsilon \qquad (217) \\ &= \bar{\varepsilon}(8\pi T^{\mu\nu}\gamma_\nu - 2\mathrm{i}\alpha(n-2)\gamma^{\mu\nu}A_\nu - 2\mathrm{i}\alpha(n-2)\gamma^0 A_\nu^\dagger\gamma^0\gamma^{\mu\nu} - 2\gamma^0 A_\nu^\dagger\gamma^0\gamma^{\nu\mu\rho}A_\rho \\ &\quad + \gamma^{\nu\mu\rho}D_\nu A_\rho - \gamma^0 D_\nu(A_\rho^\dagger)\gamma^0\gamma^{\nu\mu\rho})\varepsilon + 2\nabla_\nu(\bar{\varepsilon})\gamma^{\nu\mu\rho}\nabla_\rho\varepsilon \\ &\quad + \bar{\varepsilon}(\gamma^{\nu\mu\rho}A_\rho - \gamma^0 A_\rho^\dagger\gamma^0\gamma^{\rho\mu\nu})D_\nu\varepsilon + D_\nu(\bar{\varepsilon})(\gamma^{\rho\mu\nu}A_\rho - \gamma^0 A_\rho^\dagger\gamma^0\gamma^{\nu\mu\rho})\varepsilon. \qquad (218)\end{aligned}$$

I'm working in a vielbein where $n_\mu \equiv -\delta_{\mu 0}$. Hence,

$$\begin{aligned}n_\mu D_\nu E^{\nu\mu} &= -\bar{\varepsilon}(8\pi T^{0\nu}\gamma_\nu - 2\mathrm{i}\alpha(n-2)\gamma^{0\nu}A_\nu - 2\mathrm{i}\alpha(n-2)\gamma^0 A_\nu^\dagger\gamma^0\gamma^{0\nu} - 2\gamma^0 A_\nu^\dagger\gamma^0\gamma^{\nu 0\rho}A_\rho \\ &\quad + \gamma^{\nu 0\rho}D_\nu A_\rho - \gamma^0 D_\nu(A_\rho^\dagger)\gamma^0\gamma^{\nu 0\rho})\varepsilon - 2\nabla_\nu(\bar{\varepsilon})\gamma^{\nu 0\rho}\nabla_\rho\varepsilon \\ &\quad - \bar{\varepsilon}(\gamma^{\nu 0\rho}A_\rho - \gamma^0 A_\rho^\dagger\gamma^0\gamma^{\rho 0\nu})D_\nu\varepsilon - D_\nu(\bar{\varepsilon})(\gamma^{\rho 0\nu}A_\rho - \gamma^0 A_\rho^\dagger\gamma^0\gamma^{\nu 0\rho})\varepsilon \qquad (219) \\ &= \varepsilon^\dagger(8\pi T^{0\mu}\gamma_0\gamma_\mu + 2\mathrm{i}\alpha(n-2)\gamma^I A_I + 2\mathrm{i}\alpha(n-2)A_I^\dagger\gamma^I - 2A_I^\dagger\gamma^{IJ}A_J \\ &\quad + \gamma^{IJ}D_I A_J - D_I(A_J^\dagger)\gamma^{IJ})\varepsilon + 2\nabla_I(\varepsilon)^\dagger\gamma^{IJ}\nabla_J\varepsilon \\ &\quad + \varepsilon^\dagger(\gamma^{IJ}A_J - A_J^\dagger\gamma^{JI})D_I\varepsilon - D_I(\varepsilon)^\dagger(\gamma^{IJ}A_J - A_J^\dagger\gamma^{JI})\varepsilon. \qquad (220)\end{aligned}$$

Then, by the definition of $\mathbb{M}$ and the $\gamma^{IJ}A_J = (\gamma^{IJ}A_J)^\dagger$ assumption - see definition 3.1 - this expression reduces to

$$\begin{aligned}n_\mu D_\nu E^{\nu\mu} &= \varepsilon^\dagger(8\pi T^{0\mu}\gamma_0\gamma_\mu + 2\mathrm{i}\alpha(n-2)\gamma^I A_I + 2\mathrm{i}\alpha(n-2)A_I^\dagger\gamma^I - 2A_I^\dagger\gamma^{IJ}A_J \\ &\quad + \gamma^{IJ}D_I A_J - D_I(\gamma^{JI}A_J))\varepsilon + 2\nabla_I(\varepsilon)^\dagger\gamma^{IJ}\nabla_J\varepsilon + 0 - 0 \qquad (221) \\ &= \varepsilon^\dagger(8\pi T^{0\mu}\gamma_0\gamma_\mu + 2\mathrm{i}\alpha(n-2)\gamma^I A_I + 2\mathrm{i}\alpha(n-2)A_I^\dagger\gamma^I - 2A_I^\dagger\gamma^{IJ}A_J + 2\gamma^{IJ}D_I A_J)\varepsilon \\ &\quad + 2\nabla_I(\varepsilon)^\dagger\gamma^{IJ}\nabla_J\varepsilon \qquad (222) \\ &= 2\varepsilon^\dagger\mathbb{M}\varepsilon + 2\nabla_I(\varepsilon)^\dagger\gamma^{IJ}\nabla_J\varepsilon. \qquad (223)\end{aligned}$$

The second term can be re-written as

$$\nabla_I(\varepsilon)^\dagger\gamma^{IJ}\nabla_J\varepsilon = \nabla_I(\varepsilon)^\dagger(\gamma^I\gamma^J + \delta^{IJ}I)\nabla_J\varepsilon = -(\gamma^I\nabla_I\varepsilon)^\dagger\gamma^J\nabla_J\varepsilon + \nabla_I(\varepsilon)^\dagger\nabla^I\varepsilon. \qquad (224)$$

$$\therefore n_\mu D_\nu E^{\nu\mu} = 2(\varepsilon^\dagger\mathbb{M}\varepsilon - (\gamma^I\nabla_I\varepsilon)^\dagger\gamma^J\nabla_J\varepsilon + \nabla_I(\varepsilon)^\dagger\nabla^I\varepsilon), \qquad (225)$$

which is exactly the claimed integrand. □

Definition 3.15 (Background Killing spinor). *Let ε_k denote a Killing spinor of the background metric. In particular, ε_k is defined to satisfy*

$$\overline{D}_\mu\varepsilon_k + \mathrm{i}\alpha\gamma_\mu\varepsilon_k = 0, \qquad (226)$$

where $\overline{D}_\mu$ is the Levi-Civita connection of the background metric[22]*,*

$$\bar{g} = \mathrm{d}r\otimes\mathrm{d}r + \mathrm{e}^{2r}\left(f_{(0)mn} + \mathrm{e}^{-r}f_{(1)mn} + \mathrm{e}^{-2r}f_{(2)mn} + \cdots \mathrm{e}^{-(n-2)r}f_{(n-2)mn}\right)\mathrm{d}x^m\otimes\mathrm{d}x^n. \qquad (227)$$

Similarly, denote the vielbeins[23] *associated to $\bar{g}$ as $\bar{e}^\mu$ and $\bar{e}_\mu$.*

[21] This is one of only two places the Einstein equation is used in this work.

[22] As explained after equation 8, in asymptotically AdS spaces there is a subtlety with the powers of e^{-r} when $n \leq 5$. In these cases, I will always take $\bar{g}$ to include the higher order terms in $-\left(1+\frac{1}{4}\mathrm{e}^{-2r}\right)^2\mathrm{d}t\otimes\mathrm{d}t + \left(1-\frac{1}{4}\mathrm{e}^{-2r}\right)^2 g_{S^{n-2}}$.

[23] When the meaning is clear or the distinction is unimportant, I use the word "vielbein" to refer to both the vielbein and the inverse vielbein.

Not every choice of $f_{(0)mn}$ will lead to a background metric that admits a non-zero solution to equation 226. However, the Witten-style proof - as far as I know - can only be applied to background metrics that do admit a non-zero ε_k. I won't attempt to classify such backgrounds - interested readers may consult [38, 39] and references therein for this problem - however some general remarks will be made in sections 4 as I consider various possibilities.

There is also a more subtle issue with background Killing spinors. ε_k may only be defined in an open neighbourhood of the "boundary" at infinity or equation 226 may only have a solution in such a region. This in itself is not a problem because equation 226 will only really be required in an open neighbourhood of infinity, say $\overline{M}$, and ε_k can be extended to a spinor on all of Σ_t by multiplying it with a smooth function that's 1 near infinity but falls to zero within $\overline{M}$. The problem is that $(\overline{M}, \bar{g})$ may admit multiple spin structures and the spin structure which admits a non-zero solution, ε_k, may not be compatible with the spin structure on (M, g). This is exactly the issue behind the AdS soliton [40, 41], which I'll discuss again in section 4. But in short, like others working on similar problems [13], my proof will only work when the spin structure admitting a non-zero ε_k on $\overline{M}$ is compatible with a spin structure on M.

Lemma 3.16. *If $\bar{e}^{(\bar{f})m}_M \partial_m$ is a vielbein for $\bar{f}$, then*

$$e_M = \mathrm{e}^{-r} \bar{e}^{(\bar{f})m}_M \left(\partial_m - \frac{1}{2} \mathrm{e}^{-(n-1)r} f_{(n-1)mp} \bar{f}^{pn} \partial_n + O(\mathrm{e}^{-nr}) \right), \tag{228}$$

together with ∂_r, forms a vielbein for g.

Proof. The candidate vielbein satisfies

$$\begin{aligned} g(e_M, e_N) &= \mathrm{e}^{-2r} \bar{e}^{(\bar{f})m}_M \bar{e}^{(\bar{f})n}_N g\Big(\partial_m - \frac{1}{2} \mathrm{e}^{-(n-1)r} f_{(n-1)mp} \bar{f}^{pq} \partial_q + O(\mathrm{e}^{-nr}), \\ &\qquad \partial_n - \frac{1}{2} \mathrm{e}^{-(n-1)r} f_{(n-1)nr} \bar{f}^{rs} \partial_s + O(\mathrm{e}^{-nr}) \Big) \qquad (229) \\ &= \mathrm{e}^{-2r} \bar{e}^{(\bar{f})m}_M \bar{e}^{(\bar{f})n}_N \Big(g(\partial_m, \partial_n) - \frac{1}{2} \mathrm{e}^{-(n-1)r} f_{(n-1)mp} \bar{f}^{pq} g(\partial_q, \partial_n) \\ &\qquad - \frac{1}{2} \mathrm{e}^{-(n-1)r} f_{(n-1)nr} \bar{f}^{rs} g(\partial_m, \partial_s) + O(\mathrm{e}^{-nr}) O(g) \Big) \qquad (230) \\ &= \bar{e}^{(\bar{f})m}_M \bar{e}^{(\bar{f})n}_N \Big(f_{mn} - \frac{1}{2} \mathrm{e}^{-(n-1)r} f_{(n-1)mp} \bar{f}^{pq} f_{qn} \\ &\qquad - \frac{1}{2} \mathrm{e}^{-(n-1)r} f_{(n-1)nr} \bar{f}^{rs} f_{ms} + O(\mathrm{e}^{-nr}) \Big) \qquad (231) \\ &= \bar{e}^{(\bar{f})m}_M \bar{e}^{(\bar{f})n}_N \Big(f_{mn} - \frac{1}{2} \mathrm{e}^{-(n-1)r} f_{(n-1)mp} \bar{f}^{pq} \bar{f}_{qn} \\ &\qquad - \frac{1}{2} \mathrm{e}^{-(n-1)r} f_{(n-1)nr} \bar{f}^{rs} \bar{f}_{ms} + O(\mathrm{e}^{-nr}) \Big) \qquad (232) \\ &= \bar{e}^{(\bar{f})m}_M \bar{e}^{(\bar{f})n}_N \Big(f_{mn} - \frac{1}{2} \mathrm{e}^{-(n-1)r} f_{(n-1)mn} - \frac{1}{2} \mathrm{e}^{-(n-1)r} f_{(n-1)nm} + O(\mathrm{e}^{-nr}) \Big) \qquad (233) \\ &= \bar{e}^{(\bar{f})m}_M \bar{e}^{(\bar{f})n}_N (\bar{f}_{mn} + O(\mathrm{e}^{-nr})) \qquad (234) \\ &= \eta_{MN} + O(\mathrm{e}^{-nr}), \qquad (235) \end{aligned}$$

which is all that's required because I'm leaving the $O(\mathrm{e}^{-nr})$ part undetermined. $\square$

Lemma 3.17. *If ε_k is $O(\mathrm{e}^{r/2})$ near $\partial_\infty \Sigma_t$, then $\nabla_I \varepsilon_k \in L^2$.*

Proof. First note that to be in L^2, an object must decay faster than $O(\mathrm{e}^{-(n-2)r/2})$ because the integration measure over Σ_t is $O(\mathrm{e}^{(n-2)r})$.
Next, recall that given a vielbein, $e_\mu{}^{\mu'}\partial_{\mu'}$, the spin connection coefficients are defined as

$$\omega_{\nu\rho\mu} = \frac{1}{2}\left(g(e_\mu, [e_\nu, e_\rho]) - g(e_\nu, [e_\rho, e_\mu]) + g(e_\rho, [e_\nu, e_\mu])\right). \tag{236}$$

In particular, when the one-form index is 1, corresponding to r,

$$\omega_{\mu\nu 1} = \frac{1}{2}\left(g(e_1, [e_\mu, e_\nu]) - g(e_\mu, [e_\nu, e_1]) + g(e_\nu, [e_\mu, e_1])\right) \tag{237}$$

$$= \frac{1}{2}\left(g(\partial_r, [e_\mu, e_\nu]) - g(e_\mu, [e_\nu, \partial_r]) + g(e_\nu, [e_\mu, \partial_r])\right). \tag{238}$$

Also, since ε_k is a background Killing spinor, from equation 226 and lemma 3.16, I get

$$D_M\varepsilon_k = e_M{}^m\partial_m\varepsilon_k - \frac{1}{4}\omega_{\mu\nu M}\gamma^{\mu\nu}\varepsilon_k \tag{239}$$

$$= (e_M{}^m - \bar{e}_M{}^m)\partial_m\varepsilon_k - \frac{1}{4}(\omega_{\mu\nu M} - \bar{\omega}_{\mu\nu M})\gamma^{\mu\nu}\varepsilon_k - \mathrm{i}\alpha\gamma_M\varepsilon_k \tag{240}$$

$$= \left(-\frac{1}{2}\mathrm{e}^{-nr}f_{(n-1)np}\bar{f}^{pm}\bar{e}_M^{(\bar{f})n} + O(\mathrm{e}^{-(n+1)r})\right)\partial_m\varepsilon_k - \frac{1}{4}(\omega_{\mu\nu M} - \bar{\omega}_{\mu\nu M})\gamma^{\mu\nu}\varepsilon_k - \mathrm{i}\alpha\gamma_M\varepsilon_k \quad \text{by lemma 3.16} \tag{241}$$

$$\text{and } D_1\varepsilon_k = \partial_r\varepsilon_k - \frac{1}{4}\omega_{\mu\nu 1}\gamma^{\mu\nu}\varepsilon_k \tag{242}$$

$$= -\frac{1}{4}(\omega_{\mu\nu 1} - \bar{\omega}_{\mu\nu 1})\gamma^{\mu\nu}\varepsilon_k - \mathrm{i}\alpha\gamma_1\varepsilon_k. \tag{243}$$

Therefore, with the modified connection,

$$\nabla_M\varepsilon_k = \left(-\frac{1}{2}\mathrm{e}^{-nr}f_{(n-1)np}\bar{f}^{pm}\bar{e}_M^{(\bar{f})n} + O(\mathrm{e}^{-(n+1)r})\right)\partial_m\varepsilon_k - \frac{1}{4}(\omega_{\mu\nu M} - \bar{\omega}_{\mu\nu M})\gamma^{\mu\nu}\varepsilon_k + A_M\varepsilon_k \quad \text{and} \tag{244}$$

$$\nabla_1\varepsilon_k = -\frac{1}{4}(\omega_{\mu\nu 1} - \bar{\omega}_{\mu\nu 1})\gamma^{\mu\nu}\varepsilon_k + A_1\varepsilon_k. \tag{245}$$

In definition 3.1 I'm assuming $||A_I||_0$ decays as $O(\mathrm{e}^{-(n-1)r})$ near $\partial_\infty\Sigma_t$, so $A_I\varepsilon_k = O(\mathrm{e}^{-(n-3/2)r})$.
Since I'm assuming $n \geq 4$, this is easily a faster decay than $O(\mathrm{e}^{-(n-2)r/2})$.
$\therefore A_I\varepsilon_k \in L^2$.
Partial derivatives don't change the order of exponentials, so I have

$$-\frac{1}{2}\mathrm{e}^{-nr}f_{(n-1)np}\bar{f}^{pm}\bar{e}_M^{(\bar{f})n}\partial_m\varepsilon_k = O(\mathrm{e}^{-(n-1/2)r}). \tag{246}$$

This is a quicker decay than $A_I\varepsilon_k$, so $-\frac{1}{2}\mathrm{e}^{-nr}f_{(n-1)mp}\bar{f}^{pn}\bar{e}_M^{(\bar{f})m}\partial_m\varepsilon_k \in L^2$ too.
For the terms with the connection coefficients, I'll have to split into different cases for μ and

ν. First consider $(\mu, \nu) = (N, P)$.

$$\begin{aligned}
\omega_{NPM} - \bar{\omega}_{NPM} &= \frac{1}{2}\left(g(e_M, [e_N, e_P]) - g(e_N, [e_P, e_M]) + g(e_P, [e_N, e_M])\right) \\
&\quad - \frac{1}{2}\left(\bar{g}(\bar{e}_M, [\bar{e}_N, \bar{e}_P]) - \bar{g}(\bar{e}_N, [\bar{e}_P, \bar{e}_M]) + \bar{g}(\bar{e}_P, [\bar{e}_N, \bar{e}_M])\right) \qquad (247) \\
&= \frac{1}{2}\Bigg(\left(\bar{g} + O(\mathrm{e}^{-(n-3)r})\right)\left(\bar{e}_M + O(\mathrm{e}^{-nr}), [\bar{e}_N + O(\mathrm{e}^{-nr}), \bar{e}_P + O(\mathrm{e}^{-nr})]\right) \\
&\quad - \left(\bar{g} + O(\mathrm{e}^{-(n-3)r})\right)\left(\bar{e}_N + O(\mathrm{e}^{-nr}), [\bar{e}_P + O(\mathrm{e}^{-nr}), \bar{e}_M + O(\mathrm{e}^{-nr})]\right) \\
&\quad + \left(\bar{g} + O(\mathrm{e}^{-(n-3)r})\right)\left(\bar{e}_P + O(\mathrm{e}^{-nr}), [\bar{e}_N + O(\mathrm{e}^{-nr}), \bar{e}_M + O(\mathrm{e}^{-nr})]\right)\Bigg) \\
&\quad - \frac{1}{2}\left(\bar{g}(\bar{e}_M, [\bar{e}_N, \bar{e}_P]) - \bar{g}(\bar{e}_N, [\bar{e}_P, \bar{e}_M]) + \bar{g}(\bar{e}_P, [\bar{e}_N, \bar{e}_M])\right) \qquad (248) \\
&= O(\mathrm{e}^{-nr}) \text{ since } g \text{ is } O(\mathrm{e}^{2r}) \text{ and } \bar{e}_M \text{ is } O(\mathrm{e}^{-r}). \qquad (249)
\end{aligned}$$

When the one-form index is 1, I can use $[e_M, e_N] \in \operatorname{span}(\{\partial_m\})$ and $\partial_r \perp \operatorname{span}(\{\partial_m\})$ to get rid of a term. Then, I similarly get

$$\begin{aligned}
\omega_{MN1} - \bar{\omega}_{MN1} &= \frac{1}{2}\left(g(\partial_r, [e_M, e_N]) - g(e_M, [e_N, \partial_r]) + g(e_N, [e_M, \partial_r])\right) \\
&\quad - \frac{1}{2}\left(\bar{g}(\partial_r, [\bar{e}_M, \bar{e}_N]) - \bar{g}(\bar{e}_M, [\bar{e}_N, \partial_r]) + \bar{g}(\bar{e}_N, [\bar{e}_M, \partial_r])\right) \qquad (250) \\
&= \frac{1}{2}\left(0 - g(e_M, [e_N, \partial_r]) + g(e_N, [e_M, \partial_r])\right) \\
&\quad - \frac{1}{2}\left(0 - \bar{g}(\bar{e}_M, [\bar{e}_N, \partial_r]) + \bar{g}(\bar{e}_N, [\bar{e}_M, \partial_r])\right) \qquad (251) \\
&= \frac{1}{2}\Bigg(\left(\bar{g} + O(\mathrm{e}^{-(n-3)r})\right)\left(\bar{e}_N + O(\mathrm{e}^{-nr}), [\bar{e}_M + O(\mathrm{e}^{-nr}), \partial_r]\right) \\
&\quad - \left(\bar{g} + O(\mathrm{e}^{-(n-3)r})\right)\left(\bar{e}_N + O(\mathrm{e}^{-nr}), [\bar{e}_M + O(\mathrm{e}^{-nr}), \partial_r]\right)\Bigg) \\
&\quad - \frac{1}{2}\left(\bar{g}(\bar{e}_N, [\bar{e}_M, \partial_r]) - \bar{g}(\bar{e}_M, [\bar{e}_N, \partial_r])\right) \qquad (252) \\
&= O(\mathrm{e}^{-(n-1)r}). \qquad (253)
\end{aligned}$$

The other case is when one of μ or ν is 1. By the antisymmety in these two indices, I can assume $\mu = 1$ and $\nu = N$. Then,

$$\begin{aligned}
\omega_{1NM} - \bar{\omega}_{1NM} &= \frac{1}{2}(g(e_M, [\partial_r, e_N]) - g(\partial_r, [e_N, e_M]) + g(e_N, [\partial_r, e_M])) \\
&\quad - \frac{1}{2}(\bar{g}(\bar{e}_M, [\partial_r, \bar{e}_N]) - \bar{g}(\partial_r, [\bar{e}_N, \bar{e}_M]) + \bar{g}(\bar{e}_N, [\partial_r, \bar{e}_M])) \qquad (254) \\
&= \frac{1}{2}(g(e_M, [\partial_r, e_N]) + g(e_N, [\partial_r, e_M]) - \bar{g}(\bar{e}_M, [\partial_r, \bar{e}_N]) - \bar{g}(\bar{e}_N, [\partial_r, \bar{e}_M])) \qquad (255) \\
&= O(\mathrm{e}^{-(n-1)r}) \text{ by the same logic as above.} \qquad (256)
\end{aligned}$$

The final case is

$$\begin{aligned}
\omega_{1M1} - \bar{\omega}_{1M1} &= \frac{1}{2}(g(\partial_r, [\partial_r, e_M]) - g(\partial_r, [e_M, \partial_r]) + g(e_M, [\partial_r, \partial_r])) \\
&\quad - \frac{1}{2}(\bar{g}(\partial_r, [\partial_r, \bar{e}_M]) - \bar{g}(\partial_r, [\bar{e}_M, \partial_r]) + \bar{g}(\bar{e}_M, [\partial_r, \partial_r])) \qquad (257) \\
&= \frac{1}{2}(0 - 0 + 0) - \frac{1}{2}(0 - 0 + 0) \qquad (258) \\
&= 0. \qquad (259)
\end{aligned}$$

In summary, the connection coefficient difference terms are at least $O(\mathrm{e}^{-(n-1)r})$ in their decay. When combined with the assumed $O(\mathrm{e}^{r/2})$ growth of ε_k, I get a $O(\mathrm{e}^{-(n-3/2)r})$ decay, which is again fast enough to get into L^2.
$\therefore$ I can conclude that $\nabla_I \varepsilon_k$ is a sum of terms that are in L^2. □

Corollary 3.17.1. $\gamma^I \nabla_I \varepsilon_k \in L^2$.

Proof. The gamma matrices are $O(1)$. □

Definition 3.18 (p_M)**.** *For future notational convenience, define*

$$p_M = e_M^{(f_{(0)})m} e_0^{(f_{(0)})n} f_{(n-1)mn} + \delta_{M0} f_{(0)}^{mn} f_{(n-1)mn} \tag{260}$$

$$= \delta_{M0} \tilde{f}_{(0)}^{mn} f_{(n-1)mn} + \delta^A{}_M e_A^{(f_{(0)})m} n_{(0)}^n f_{(n-1)mn}. \tag{261}$$

I've chosen the letter p for this quantity because it looks like a relativistic momentum vector if one views $f_{(n-1)mn}$ like an energy-momentum tensor. This is especially so given its 0th component is the integrand of equation 81. Furthermore, this is also qualitatively p_M's role in the positive energy theorem proven immediately below. Finally, note that since $f_{(0)}^{mn} f_{(n-1)mn}$ is 0 in vacuum [33], p_M is just $e_M^{(f_{(0)})m} n_{(0)}^n f_{(n-1)mn}$ in that case.

Theorem 3.19 (Positive energy theorem)**.** *If the Einstein equation holds and $\exists$ a non-zero ε_k with ε_k being $O(\mathrm{e}^{r/2})$ near $\partial_\infty \Sigma_t$, then $\exists \epsilon$ such that $\gamma^I \nabla_I \varepsilon = 0$ and*

$$Q(\varepsilon) = \frac{n-1}{2} \mathrm{e}^{-r} \int_{\partial_\infty \Sigma_t} p_M \bar{\varepsilon}_k \gamma^M \varepsilon_k \sqrt{\iota^* f_{(0)}} \, \mathrm{d}^{n-2}x$$
$$+ \mathrm{e}^{(n-2)r} \int_{\partial_\infty \Sigma_t} \varepsilon_k^\dagger \left(\gamma^1 \gamma^A A_A + A_A^\dagger \gamma^A \gamma^1 \right) \varepsilon_k \sqrt{\iota^* f_{(0)}} \mathrm{d}^{n-2}x \tag{262}$$

$$= 2 \int_{\Sigma_t} \left((\nabla_I \varepsilon)^\dagger \nabla^I \varepsilon + \varepsilon^\dagger \mathbb{M} \varepsilon \right) \mathrm{d}V \tag{263}$$

$$\geq 0. \tag{264}$$

Proof. By corollary 3.17.1 and theorem 3.10, $\exists \Psi \in \mathcal{H}$ such that $G(\Psi) = \gamma^I \nabla_I \varepsilon_k$.
Let $\varepsilon = \varepsilon_k - \Psi$, so that

$$\gamma^I \nabla_I \varepsilon = 0. \tag{265}$$

Let $\{\psi_a\}_{a=0}^\infty$ be a Cauchy sequence in C_c^∞ whose limit is Ψ.
Let $\varepsilon_a = \varepsilon_k - \psi_a$. Then, $\lim_{a\to\infty} \varepsilon_a = \varepsilon$.
By lemma 3.13,

$$Q(\varepsilon_a) = \int_{\partial_\infty \Sigma_t} E^{01}(\varepsilon_a) \mathrm{d}A. \tag{266}$$

However, since ψ_a is compactly supported,

$$Q(\varepsilon_a) = \int_{\partial_\infty \Sigma_t} E^{01}(\varepsilon_k) \mathrm{d}A, \tag{267}$$

which does not actually depend on a.

$$\therefore \lim_{a\to\infty} Q(\varepsilon_a) = \int_{\partial_\infty \Sigma_t} E^{01}(\varepsilon_k) \mathrm{d}A. \tag{268}$$

I'll evaluate the RHS before finding the limit on the LHS.
From equation 201,

$$E^{01}(\varepsilon) = \varepsilon^\dagger \gamma^1 \gamma^A D_A \varepsilon + D_A(\varepsilon)^\dagger \gamma^A \gamma^1 \varepsilon - 2\mathrm{i}\alpha(n-2)\varepsilon^\dagger \gamma^1 \varepsilon + \varepsilon^\dagger \gamma^1 \gamma^A A_A \varepsilon + \varepsilon^\dagger A_A^\dagger \gamma^A \gamma^1 \varepsilon. \quad (269)$$

Let $(\iota^\star f)_{mn}$ denote the pullback of f_{mn} to the constant t surface. Then, the measure, $\mathrm{d}A$, is

$$\mathrm{d}A = \sqrt{\det(\iota^\star(\mathrm{e}^{2r} f_{mn}))}\mathrm{d}x^2 \cdots \mathrm{d}x^{n-1} x \quad (270)$$

$$= \mathrm{e}^{(n-2)r} \sqrt{\iota^\star f_{(0)} + O(\mathrm{e}^{-r})}\, \mathrm{d}^{n-2} x \quad (271)$$

$$= \mathrm{e}^{(n-2)r} \sqrt{\iota^\star f_{(0)}}\, \mathrm{d}^{n-2} x \ \text{ to leading order.} \quad (272)$$

This $\mathrm{e}^{(n-2)r}$ growth and the $O(\mathrm{e}^{r/2})$ growth of ε_k means I only need to keep terms that decay as $O(\mathrm{e}^{-(n-1)r})$ or slower in the matrices in $E^{01}(\varepsilon_k)$.
A_A is assumed to decay as $O(\mathrm{e}^{-(n-1)r})$ in definition 3.1, so I just keep those terms as they are.
Consider the derivative terms next.

$$D_A \varepsilon_k = e_A{}^m \partial_m \varepsilon_k - \frac{1}{4}\omega_{\mu\nu A}\gamma^{\mu\nu}\varepsilon_k \quad (273)$$

$$= (e_A{}^m - \bar{e}_A{}^m)\partial_m \varepsilon_k - \frac{1}{4}(\omega_{\mu\nu A} - \bar{\omega}_{\mu\nu A})\gamma^{\mu\nu}\varepsilon_k - \mathrm{i}\alpha\gamma_A \varepsilon_k. \quad (274)$$

From equations 249 and 256, I only need to keep the connection coefficient difference terms when one of μ or ν is 1.
Likewise, from lemma 3.16, $e_A{}^m - \bar{e}_A{}^m$ is $O(\mathrm{e}^{-nr})$, so I can ignore that term too.

$$\therefore D_A \varepsilon_k \to -\frac{1}{2}(\omega_{1MA} - \bar{\omega}_{1MA})\gamma^1 \gamma^M \varepsilon_k - \mathrm{i}\alpha\gamma_A \varepsilon_k. \quad (275)$$

$$\therefore \gamma^A D_A \varepsilon_k \to -\frac{1}{2}(\omega_{1MA} - \bar{\omega}_{1MA})\gamma^A \gamma^1 \gamma^M \varepsilon_k - \mathrm{i}\alpha\gamma^A \gamma_A \varepsilon_k \quad (276)$$

$$= \frac{1}{2}(\omega_{1MA} - \bar{\omega}_{1MA})\gamma^1 \gamma^A \gamma^M \varepsilon_k + \mathrm{i}\alpha(n-2)\varepsilon_k. \quad (277)$$

$$\therefore \varepsilon_k^\dagger \gamma^1 \gamma^A D_A \varepsilon_k \to \frac{1}{2}(\omega_{1MA} - \bar{\omega}_{1MA})\varepsilon_k^\dagger \gamma^1 \gamma^1 \gamma^A \gamma^M \varepsilon_k + \mathrm{i}\alpha(n-2)\varepsilon_k^\dagger \gamma^1 \varepsilon_k \quad (278)$$

$$= -\frac{1}{2}(\omega_{1MA} - \bar{\omega}_{1MA})\varepsilon_k^\dagger \gamma^A \gamma^M \varepsilon_k + \mathrm{i}\alpha(n-2)\varepsilon_k^\dagger \gamma^1 \varepsilon_k. \quad (279)$$

From equation 255

$$\omega_{1MA} - \bar{\omega}_{1MA} = \frac{1}{2}(g(e_A, [\partial_r, e_M]) + g(e_M, [\partial_r, e_A]) - \bar{g}(\bar{e}_A, [\partial_r, \bar{e}_M]) - \bar{g}(\bar{e}_M, [\partial_r, \bar{e}_A])). \quad (280)$$

This is symmetric in A and M. Hence,

$$-\frac{1}{2}(\omega_{1MA} - \bar{\omega}_{1MA})\varepsilon_k^\dagger \gamma^A \gamma^M \varepsilon_k$$

$$= -\frac{1}{4}(g(e_A, [\partial_r, e_M]) + g(e_M, [\partial_r, e_A]) - \bar{g}(\bar{e}_A, [\partial_r, \bar{e}_M]) - \bar{g}(\bar{e}_M, [\partial_r, \bar{e}_A]))\varepsilon_k^\dagger \gamma^A \gamma^M \varepsilon_k \quad (281)$$

$$= -\frac{1}{4}(g(e_A, [\partial_r, e_B]) + g(e_B, [\partial_r, e_A]) - \bar{g}(\bar{e}_A, [\partial_r, \bar{e}_B]) - \bar{g}(\bar{e}_B, [\partial_r, \bar{e}_A]))\varepsilon_k^\dagger \gamma^A \gamma^B \varepsilon_k$$
$$- \frac{1}{4}(g(e_A, [\partial_r, e_0]) + g(e_0, [\partial_r, e_A]) - \bar{g}(\bar{e}_A, [\partial_r, \bar{e}_0]) - \bar{g}(\bar{e}_0, [\partial_r, \bar{e}_A]))\varepsilon_k^\dagger \gamma^A \gamma^0 \varepsilon_k \quad (282)$$

$$= \frac{1}{2}\delta^{AB}(g(e_A, [\partial_r, e_B]) - \bar{g}(\bar{e}_A, [\partial_r, \bar{e}_B]))\varepsilon_k^\dagger \varepsilon_k$$
$$- \frac{1}{4}(g(e_A, [\partial_r, e_0]) + g(e_0, [\partial_r, e_A]) - \bar{g}(\bar{e}_A, [\partial_r, \bar{e}_0]) - \bar{g}(\bar{e}_0, [\partial_r, \bar{e}_A]))\varepsilon_k^\dagger \gamma^A \gamma^0 \varepsilon_k. \quad (283)$$

To go further, I'll need more concrete expressions for $g(e_M, [\partial_r, e_N])$ and $\bar{g}(\bar{e}_M, [\partial_r, \bar{e}_N])$. Using lemma 3.16,

$$
\begin{aligned}
&g(e_M, [\partial_r, e_N]) \\
&= g\Bigg(\mathrm{e}^{-r}\bar{e}_M^{(\bar{f})m}\partial_m - \frac{1}{2}\mathrm{e}^{-nr}\bar{e}_M^{(\bar{f})m} f_{(n-1)mp}\bar{f}^{pn}\partial_n + O(\mathrm{e}^{-(n+1)r}), \\
&\quad - \mathrm{e}^{-r}\bar{e}_N^{(\bar{f})q}\partial_q + \mathrm{e}^{-r}\partial_r\left(\bar{e}_N^{(\bar{f})q}\right)\partial_q + \frac{n}{2}\mathrm{e}^{-nr}\bar{e}_N^{(\bar{f})q} f_{(n-1)qr}\bar{f}^{rs}\partial_s + O(\mathrm{e}^{-(n+1)r})\Bigg) && (284) \\
&= \mathrm{e}^{-2r}\bar{e}_M^{(\bar{f})m}\bar{e}_N^{(\bar{f})q}\Bigg(- g(\partial_m, \partial_q) + \frac{1}{2}\mathrm{e}^{-(n-1)r} f_{(n-1)mp}\bar{f}^{pn} g(\partial_n, \partial_q) \\
&\quad + \frac{n}{2}\mathrm{e}^{-(n-1)r} f_{(n-1)qr}\bar{f}^{rs} g(\partial_m, \partial_s) + O(\mathrm{e}^{-nr})O(g)\Bigg) \\
&\quad + \mathrm{e}^{-2r}\bar{e}_M^{(\bar{f})m}\partial_r\left(\bar{e}_N^{(\bar{f})q}\right)\left(g(\partial_m, \partial_q) - \frac{1}{2}\mathrm{e}^{-(n-1)r} f_{(n-1)mp}\bar{f}^{pn} g(\partial_n, \partial_q) + O(\mathrm{e}^{-nr})O(g)\right) && (285) \\
&= \bar{e}_M^{(\bar{f})m}\bar{e}_N^{(\bar{f})q}\Bigg(- \bar{f}_{mq} - \mathrm{e}^{-(n-1)r} f_{(n-1)mq} + \frac{1}{2}\mathrm{e}^{-(n-1)r} f_{(n-1)mp}\bar{f}^{pn}\bar{f}_{nq} \\
&\quad + \frac{n}{2}\mathrm{e}^{-(n-1)r} f_{(n-1)qr}\bar{f}^{rs}\bar{f}_{ms} + O(\mathrm{e}^{-nr})\Bigg) \\
&\quad + \bar{e}_M^{(\bar{f})m}\partial_r\left(\bar{e}_N^{(\bar{f})q}\right)\left(\bar{f}_{mq} + \mathrm{e}^{-(n-1)r} f_{(n-1)mq} - \frac{1}{2}\mathrm{e}^{-(n-1)r} f_{(n-1)mp}\bar{f}^{pn}\bar{f}_{nq} + O(\mathrm{e}^{-nr})\right) && (286) \\
&= -\eta_{MN} + \bar{e}_M^{(\bar{f})m}\partial_r\left(\bar{e}_N^{(\bar{f})q}\right)\bar{f}_{mq} + \frac{n-1}{2}\mathrm{e}^{-(n-1)r}\bar{e}_M^{(\bar{f})m}\bar{e}_N^{(\bar{f})n} f_{(n-1)mn} + O(\mathrm{e}^{-nr}). && (287)
\end{aligned}
$$

Likewise,

$$
\begin{aligned}
\bar{g}(\bar{e}_M, [\partial_r, \bar{e}_N]) &= \bar{g}\left(\mathrm{e}^{-r}\bar{e}_M^{(\bar{f})m}\partial_m, -\mathrm{e}^{-r}\bar{e}_N^{(\bar{f})n}\partial_n + \mathrm{e}^{-r}\partial_r\left(\bar{e}_N^{(\bar{f})n}\partial_n\right)\right) && (288) \\
&= -\bar{e}_M^{(\bar{f})m}\bar{e}_N^{(\bar{f})n}\bar{f}_{mn} + \bar{e}_M^{(\bar{f})m}\partial_r\left(\bar{e}_N^{(\bar{f})n}\right)\bar{f}_{mn} && (289) \\
&= -\eta_{MN} + \bar{e}_M^{(\bar{f})m}\partial_r\left(\bar{e}_N^{(\bar{f})n}\right)\bar{f}_{mn}. && (290)
\end{aligned}
$$

$$
\therefore g(e_M, [\partial_r, e_N]) - \bar{g}(\bar{e}_M, [\partial_r, \bar{e}_N]) = \frac{n-1}{2}\mathrm{e}^{-(n-1)r}\bar{e}_M^{(\bar{f})m}\bar{e}_N^{(\bar{f})n} f_{(n-1)mn} + O(\mathrm{e}^{-nr}) \tag{291}
$$

Substituting this back into equation 283, to leading order I get

$$
\begin{aligned}
&-\frac{1}{2}(\omega_{1MA} - \bar{\omega}_{1MA})\varepsilon_k^\dagger\gamma^A\gamma^M\varepsilon_k \\
&= \frac{1}{2}\delta^{AB}(g(e_A, [\partial_r, e_B]) - \bar{g}(\bar{e}_A, [\partial_r, \bar{e}_B]))\varepsilon_k^\dagger\varepsilon_k \\
&\quad - \frac{1}{4}(g(e_A, [\partial_r, e_0]) + g(e_0, [\partial_r, e_A]) - \bar{g}(\bar{e}_A, [\partial_r, \bar{e}_0]) - \bar{g}(\bar{e}_0, [\partial_r, \bar{e}_A]))\varepsilon_k^\dagger\gamma^A\gamma^0\varepsilon_k && (292) \\
&= \delta^{AB}\frac{n-1}{4}\mathrm{e}^{-(n-1)r}\bar{e}_A^{(\bar{f})m}\bar{e}_B^{(\bar{f})n} f_{(n-1)mn}\varepsilon_k^\dagger\varepsilon_k - \frac{n-1}{4}\mathrm{e}^{-(n-1)r}\bar{e}_A^{(\bar{f})m}\bar{e}_0^{(\bar{f})n} f_{(n-1)mn}\varepsilon_k^\dagger\gamma^A\gamma^0\varepsilon_k && (293) \\
&= \frac{n-1}{4}\mathrm{e}^{-(n-1)r}\eta^{MN}\bar{e}_M^{(\bar{f})m}\bar{e}_N^{(\bar{f})n} f_{(n-1)mn}\varepsilon_k^\dagger\varepsilon_k + \frac{n-1}{4}\mathrm{e}^{-(n-1)r}\bar{e}_0^{(\bar{f})m}\bar{e}_0^{(\bar{f})n} f_{(n-1)mn}\varepsilon_k^\dagger\varepsilon_k \\
&\quad + \frac{n-1}{4}\mathrm{e}^{-(n-1)r}\bar{e}_A^{(\bar{f})m}\bar{e}_0^{(\bar{f})n} f_{(n-1)mn}\varepsilon_k^\dagger\gamma^0\gamma^A\varepsilon_k && (294) \\
&= \frac{n-1}{4}\mathrm{e}^{-(n-1)r}\bar{f}^{mn} f_{(n-1)mn}\varepsilon_k^\dagger\varepsilon_k + \frac{n-1}{4}\mathrm{e}^{-(n-1)r}\bar{e}_0^{(\bar{f})m}\bar{e}_0^{(\bar{f})n} f_{(n-1)mn}\varepsilon_k^\dagger\varepsilon_k \\
&\quad + \frac{n-1}{4}\mathrm{e}^{-(n-1)r}\bar{e}_A^{(\bar{f})m}\bar{e}_0^{(\bar{f})n} f_{(n-1)mn}\varepsilon_k^\dagger\gamma^0\gamma^A\varepsilon_k && (295) \\
&= \frac{n-1}{4}\mathrm{e}^{-(n-1)r}\bar{f}^{mn} f_{(n-1)mn}\varepsilon_k^\dagger\varepsilon_k + \frac{n-1}{4}\mathrm{e}^{-(n-1)r}\bar{e}_M^{(\bar{f})m}\bar{e}_0^{(\bar{f})n} f_{(n-1)mn}\bar{\varepsilon}_k\gamma^M\varepsilon_k. && (296)
\end{aligned}
$$

The $\mathrm{e}^{-(n-1)r}$ factor and $\varepsilon_k = O(\mathrm{e}^{r/2})$ mean I only need everything else to $O(1)$; anything higher order will integrate to zero in equation 268.
$\therefore \bar{f}^{mn} f_{(n-1)mn} \to f_{(0)}^{mn} f_{(n-1)mn}$ and hence by definition 3.18,

$$-\frac{1}{2}(\omega_{1MA} - \bar{\omega}_{1MA})\varepsilon_k^\dagger \gamma^A \gamma^M \varepsilon_k = \frac{n-1}{4}\mathrm{e}^{-(n-1)r} p_M \bar{\varepsilon}_k \gamma^M \varepsilon_k + O(\mathrm{e}^{-(n-1)r}). \tag{297}$$

Substituting back into equation 279 then gives

$$\varepsilon_k^\dagger \gamma^1 \gamma^A D_A \varepsilon_k \to \frac{n-1}{4}\mathrm{e}^{-(n-1)r} p_M \bar{\varepsilon}_k \gamma^M \varepsilon_k + \mathrm{i}\alpha(n-2)\varepsilon_k^\dagger \gamma^1 \varepsilon_k. \tag{298}$$

$$\therefore D_A(\varepsilon_k)^\dagger \gamma^A \gamma^1 \varepsilon_k \to \frac{n-1}{4}\mathrm{e}^{-(n-1)r} p_M \bar{\varepsilon}_k \gamma^M \varepsilon_k + \mathrm{i}\alpha(n-2)\varepsilon_k^\dagger \gamma^1 \varepsilon_k \text{ too.} \tag{299}$$

Substituting these two expressions into equation 269 implies

$$E^{01}(\varepsilon_k) \to \frac{n-1}{2}\mathrm{e}^{-(n-1)r} p_M \bar{\varepsilon}_k \gamma^M \varepsilon_k + \varepsilon_k^\dagger \gamma^1 \gamma^A A_A \varepsilon_k + \varepsilon_k^\dagger A_A^\dagger \gamma^A \gamma^1 \varepsilon_k. \tag{300}$$

The lower order terms I've omitted integrate to zero under $\int_{\partial_\infty \Sigma_t} \mathrm{d}A$, so equation 268 becomes

$$\lim_{a\to\infty} Q(\varepsilon_a) = \int_{\partial_\infty \Sigma_t} \left(\frac{n-1}{2}\mathrm{e}^{-(n-1)r} p_M \bar{\varepsilon}_k \gamma^M \varepsilon_k + \varepsilon_k^\dagger \gamma^1 \gamma^A A_A \varepsilon_k + \varepsilon_k^\dagger A_A^\dagger \gamma^A \gamma^1 \varepsilon_k \right) \mathrm{d}A \tag{301}$$

$$\begin{aligned} &= \frac{n-1}{2}\mathrm{e}^{-r} \int_{\partial_\infty \Sigma_t} p_M \bar{\varepsilon}_k \gamma^M \varepsilon_k \sqrt{\iota^* f_{(0)}}\, \mathrm{d}^{n-2}x \\ &\quad + \mathrm{e}^{(n-2)r} \int_{\partial_\infty \Sigma_t} \varepsilon_k^\dagger \left(\gamma^1 \gamma^A A_A + A_A^\dagger \gamma^A \gamma^1 \right) \varepsilon_k \sqrt{\iota^* f_{(0)}} \mathrm{d}^{n-2}x. \end{aligned} \tag{302}$$

It's now time to evaluate the bulk expression for $\lim_{a\to\infty} Q(\varepsilon_a)$.
First note that by lemma 3.14,

$$\begin{aligned} Q(\varepsilon_k - \chi) = 2\int_{\Sigma_t} \Big(&\nabla_I(\varepsilon_k - \chi)^\dagger \nabla^I(\varepsilon_k - \chi) - (\gamma^I \nabla_I(\varepsilon_k - \chi))^\dagger \gamma^J \nabla_J(\varepsilon_k - \chi) \\ &+ (\varepsilon_k - \chi)^\dagger \mathbb{M}(\varepsilon_k - \chi)\Big)\mathrm{d}V. \end{aligned} \tag{303}$$

Hence, by equation 86, definition 3.8, definition 3.5, lemma 3.9 and corollary 3.17.1,

$$\begin{aligned} \frac{1}{2}(Q(\varepsilon) - Q(\varepsilon_a)) = &||\Psi||_{\mathcal{H}}^2 - ||\psi_a||_{\mathcal{H}}^2 - ||G(\Psi)||_{L^2}^2 + ||G(\psi_a)||_{L^2}^2 + \langle G(\Psi - \psi_a), \gamma^I \nabla_I \varepsilon_k \rangle_{L^2} \\ &+ \langle \gamma^I \nabla_I \varepsilon_k, G(\Psi - \psi_a)\rangle_{L^2} - \int_{\Sigma_t} (\nabla_I(\Psi - \psi_a))^\dagger \nabla^I(\varepsilon_k)\mathrm{d}V \\ &- \int_{\Sigma_t} \nabla^I(\varepsilon_k)^\dagger \nabla_I(\Psi - \psi_a)\mathrm{d}V - \int_{\Sigma_t} (\Psi - \psi_a)^\dagger \mathbb{M}\varepsilon_k \,\mathrm{d}V \\ &- \int_{\Sigma_t} \varepsilon_k^\dagger \mathbb{M}(\Psi - \psi_a)\,\mathrm{d}V. \end{aligned} \tag{304}$$

Inner products are continuous. By lemma 3.9, so is G.
$\therefore$ I immediately get

$$\begin{aligned} &\lim_{a\to\infty} ||\psi_a||_{\mathcal{H}}^2 = ||\Psi||_{\mathcal{H}}^2, \quad \lim_{a\to\infty} ||G(\psi_a)||_{L^2}^2 = ||G(\Psi)||_{L^2}^2, \quad \lim_{a\to\infty} \langle G(\Psi - \psi_a), \gamma^I \nabla_I \varepsilon_k \rangle_{L^2} = 0 \\ &\text{and} \quad \lim_{a\to\infty} \langle \gamma^I \nabla_I \varepsilon_k, G(\Psi - \psi_a)\rangle_{L^2} = 0. \end{aligned} \tag{305}$$

$$\begin{aligned} \therefore \lim_{a\to\infty} \frac{1}{2}(Q(\varepsilon) - Q(\varepsilon_a)) = \lim_{a\to\infty} \Big(&- \int_{\Sigma_t} (\nabla_I(\Psi - \psi_a))^\dagger \nabla^I(\varepsilon_k)\mathrm{d}V - \int_{\Sigma_t} \nabla^I(\varepsilon_k)^\dagger \nabla_I(\Psi - \psi_a)\mathrm{d}V \\ &- \int_{\Sigma_t} (\Psi - \psi_a)^\dagger \mathbb{M}\varepsilon_k \,\mathrm{d}V - \int_{\Sigma_t} \varepsilon_k^\dagger \mathbb{M}(\Psi - \psi_a)\,\mathrm{d}V \Big). \end{aligned} \tag{306}$$

Since the inner product on $\mathcal{H}$ is $\langle \psi, \chi \rangle_{\mathcal{H}} = \int_{\Sigma_t} \left((\nabla_I \psi)^\dagger \nabla^I \chi + \psi^\dagger \mathbb{M} \chi \right) \mathrm{d}V$ (with limits of Cauchy sequences taken appropriately when ψ or χ is in $\mathcal{H} \backslash C_c^\infty$) and $\mathbb{M}$ is assumed to be non-negative definite,

$$\int_{\Sigma_t} (\nabla_I \psi)^\dagger \nabla^I (\psi) \, \mathrm{d}V \leq ||\psi||_{\mathcal{H}}^2 < \infty. \tag{307}$$

$\therefore \nabla_I \psi \in L^2$ and $\psi \mapsto \nabla_I \psi$ is a continuous (i.e. bounded) linear operator.

$$\begin{aligned} \therefore \lim_{a \to \infty} \int_{\Sigma_t} (\nabla_I (\Psi - \psi_a))^\dagger \nabla^I (\varepsilon_k) \mathrm{d}V &= \lim_{a \to \infty} \langle \nabla_I (\Psi - \psi_a), \nabla^I \varepsilon_k \rangle_{L^2} &(308)\\ &= \left\langle \nabla_I \Big(\lim_{a \to \infty} (\Psi - \psi_a) \Big), \nabla^I \varepsilon_k \right\rangle_{L^2} &(309)\\ &= 0 &(310)\end{aligned}$$

and likewise for $\int_{\Sigma_t} \nabla^I (\varepsilon_k)^\dagger \nabla_I (\Psi - \psi_a) \mathrm{d}V$. That leaves

$$\lim_{a \to \infty} \frac{1}{2} (Q(\varepsilon) - Q(\varepsilon_a)) = \lim_{a \to \infty} \Bigg(- \int_{\Sigma_t} (\Psi - \psi_a)^\dagger \mathbb{M} \varepsilon_k \, \mathrm{d}V - \int_{\Sigma_t} \varepsilon_k^\dagger \mathbb{M} (\Psi - \psi_a) \, \mathrm{d}V \Bigg). \tag{311}$$

Because I'm assuming $\mathbb{M}$ is non-negative definite, $||\mathbb{M}||_0$ decays faster than $O(\mathrm{e}^{-(n-1)r})$ near $\partial_\infty \Sigma_t$ and ε_k grows at $O(\mathrm{e}^{r/2})$ near $\partial_\infty \Sigma_t$,

$$\left| \int_{\Sigma_t} \varepsilon_k^\dagger \mathbb{M} \varepsilon_k \, \mathrm{d}V \right| = \int_{\Sigma_t} \varepsilon_k^\dagger \mathbb{M} \varepsilon_k \, \mathrm{d}V \leq \int_{\Sigma_t} \varepsilon_k^\dagger \varepsilon_k ||\mathbb{M}||_0 \, \mathrm{d}V < \infty. \tag{312}$$

$\therefore \varepsilon_k \sqrt{||\mathbb{M}||_0} \in L^2$.
Likewise, $(\Psi - \psi_a) \sqrt{||\mathbb{M}||_0} \in L^2$ because

$$\int_{\Sigma_t} (\Psi - \psi_a)^\dagger (\Psi - \psi_a) ||\mathbb{M}||_0 \, \mathrm{d}V \leq \int_{\Sigma_t} (\Psi - \psi_a)^\dagger \mathbb{M} (\Psi - \psi_a) \mathrm{d}V \leq ||\Psi - \psi_a||_{\mathcal{H}}^2 < \infty. \tag{313}$$

Hence, effectively by the Cauchy-Schwartz inequality applied pointwise and the continuity of inner products,

$$\begin{aligned} \lim_{a \to \infty} \left| \int_{\Sigma_t} (\Psi - \psi_a)^\dagger \mathbb{M} \varepsilon_k \, \mathrm{d}V \right| &\leq \lim_{a \to \infty} ||(\Psi - \psi_a) \sqrt{||\mathbb{M}||_0}||_{L^2} ||\varepsilon_k \sqrt{||\mathbb{M}||_0}||_{L^2} &(314)\\ &= \left|\left| \lim_{a \to \infty} (\Psi - \psi_a) \sqrt{||\mathbb{M}||_0} \right|\right|_{L^2} ||\varepsilon_k \sqrt{||\mathbb{M}||_0}||_{L^2} &(315)\\ &= 0. &(316)\end{aligned}$$

$$\therefore \lim_{a \to \infty} \int_{\Sigma_t} (\Psi - \psi_a)^\dagger \mathbb{M} \varepsilon_k \, \mathrm{d}V = 0 \tag{317}$$

Analogously, $\int_{\Sigma_t} \varepsilon_k^\dagger \mathbb{M} (\Psi - \psi_a) \, \mathrm{d}V = 0$ too.
The net result is that

$$\begin{aligned} \lim_{a \to \infty} Q(\varepsilon_a) &= Q(\varepsilon) &(318)\\ &= 2 \int_{\Sigma_t} \left((\nabla_I \varepsilon)^\dagger \nabla^I \varepsilon - (\gamma^I \nabla_I \varepsilon)^\dagger \gamma^J \nabla_J \varepsilon + \varepsilon^\dagger \mathbb{M} \varepsilon \right) \mathrm{d}V \text{ by lemma 3.14} &(319)\\ &= 2 \int_{\Sigma_t} \left((\nabla_I \varepsilon)^\dagger \nabla^I \varepsilon + \varepsilon^\dagger \mathbb{M} \varepsilon \right) \mathrm{d}V \text{ by equation 265.} &(320)\end{aligned}$$

Substituting this into equation 302 completes the proof. □

Corollary 3.19.1.

$$e^{-r}\int_{\partial_\infty\Sigma_t} p_M\bar{\varepsilon}_k\gamma^M\varepsilon_k\sqrt{\iota^* f_{(0)}}\,\mathrm{d}^{n-2}x \tag{321}$$

is a conserved quantity.

Proof. Choose $A_\mu = 0$. Then, the result follows immediately from lemma 3.12. □

Corollary 3.19.2. *If equality holds in theorem 3.19, then*[24] $\exists$ *a non-zero spinor,* ε*, such that* $\nabla_I\varepsilon = 0$.

The example A_μs I'll be considering in the rest of this work are ones such that $\nabla_\mu\varepsilon$ would be the gravitino transformation in some theory of supergravity. Then, $\nabla_\mu\varepsilon = 0$ is the Killing spinor condition. The existence of a non-zero solution, ε, would imply (M, g) is a supersymmetric solution, i.e. some level of rigid supersymmetry is preserved by the spacetime. Corollary 3.19.2 almost implies that only a supersymmetric solution can achieve equality in theorem 3.19.

4 Examples with $A_\mu = 0$

Throughout this section, A_μ is set to zero.

Lemma 4.1. *The assumptions of definition 3.1 are satisfied if the energy-momentum tensor,* T_{ab}*, satisfies the dominant energy condition and* $T^{0\mu}$ *decays faster than* $O(e^{-(n-1)r})$ *near* $\partial_\infty\Sigma_t$.

Proof. Since $A_\mu = 0$ in this section, all the conditions about A_I in definition 3.1 are trivially satisfied (with $\tilde{A}_\mu = 0$ too).
Only the conditions about $\mathbb{M}$ remain.
Definition 3.1 implies $\mathbb{M} = 4\pi T^{0\mu}\gamma_0\gamma_\mu = 4\pi(T^{00}I + T^{0I}\gamma_0\gamma_I)$ when $A_\mu = 0$.
The eigenvalues of $T^{0I}\gamma_0\gamma_I$ are[25] $\pm\sqrt{T^{0I}T^0{}_I}$, so $\mathbb{M}$ being non-negative definite is equivalent to $T^{00} \geq \sqrt{T^{0I}T^0{}_I}$.
The dominant energy condition says $-T^a{}_bV^b$ is future directed and causal for any future directed, causal vector, V^a.
Choose $V^\mu = \delta^{\mu 0}$.
$\therefore -T^\mu{}_0 = T^{0\mu}$ is future directed and causal.
$\therefore T^{00} \geq 0$ and $0 \geq \eta_{\mu\nu}T^{0\mu}T^{0\nu} \iff (T^{00})^2 \geq T^{0I}T^0{}_I$, which is the condition found above for $\mathbb{M}$ to be non-negative definite.
Finally, since the gamma matrices are $O(1)$, the assumed condition on $T^{0\mu}$'s decay is exactly the condition in definition 3.1 about $||\mathbb{M}||_0$'s decay. □

Corollary 4.1.1. *Theorem 3.19 reduces to*

$$Q(\varepsilon) = \frac{n-1}{2}e^{-r}\int_{\partial_\infty\Sigma_t} p_M\bar{\varepsilon}_k\gamma^M\varepsilon_k\sqrt{\iota^* f_{(0)}}\,\mathrm{d}^{n-2}x \tag{322}$$

$$= 2\int_{\Sigma_t}\left((\nabla_I\varepsilon)^\dagger\nabla^I\varepsilon + 4\pi T^{0\mu}\varepsilon^\dagger\gamma_0\gamma_\mu\varepsilon\right)\mathrm{d}V \tag{323}$$

$$\geq 0 \tag{324}$$

[24]Note that lemma 3.12 implies the equality hold $\forall t$ if it holds for any one value of t.

[25]This can be seen by supposing $T^{0I}\gamma_0\gamma_I v = \lambda v$. Then, $\lambda^2 v = T^{0I}T^0{}_I v$ by the Clifford algebra. Both $\pm\sqrt{T^{0I}T^0{}_I}$ must be eigenvalues because if v is in one eigenspace, then $\gamma_0 v$ is in the other eigenspace.

The main task for the remainder of this section is to give physical meaning to the boundary term, $\frac{n-1}{2}\mathrm{e}^{-r}\int_{\partial_\infty \Sigma_t} p_M \bar{\varepsilon}_k \gamma^M \varepsilon_k \sqrt{\iota^* f_{(0)}}\,\mathrm{d}^{n-2}x$, for different boundary geometries, $f_{(0)}$. In [28], and to a lesser extent in [14], it's effectively argued that the entirety of $\frac{n-1}{2}\mathrm{e}^{-r}\int_{\partial_\infty \Sigma_t} p_M \bar{\varepsilon}_k \gamma^M \varepsilon_k \sqrt{\iota^* f_{(0)}}\,\mathrm{d}^{n-2}x$ should be interpreted as an energy. This interpretation is supported by the fact that ε_k being a background Killing spinor automatically makes $\bar{\varepsilon}_k \gamma^\mu \varepsilon_k$ a Killing vector for $\bar{g}$. However, as I'll show, often a bit more can be said and $\frac{n-1}{2}\mathrm{e}^{-r}\int_{\partial_\infty \Sigma_t} p_M \bar{\varepsilon}_k \gamma^M \varepsilon_k \sqrt{\iota^* f_{(0)}}\,\mathrm{d}^{n-2}x$ can be concretely connected to the energy I defined in section 2.

4.1 Toroidal boundary

Although asymptotically AdS spacetimes are more familiar, ε_k takes a simpler form in the case of the toroidal boundary, i.e. $\mathbb{R} \times \mathbb{T}^{n-2}$ with the metric,

$$f_{(0)} = -\mathrm{d}t \otimes \mathrm{d}t + \delta_{\alpha\beta}\,\mathrm{d}\theta^\alpha \otimes \mathrm{d}\theta^\beta = \eta_{mn}\mathrm{d}x^m \otimes \mathrm{d}x^n. \tag{325}$$

Hence, I will present applications in this class first. This choice of boundary metric is motivated by the Kottler metrics,

$$g = -(k + R^2)\mathrm{d}t \otimes \mathrm{d}t + \frac{\mathrm{d}R \otimes \mathrm{d}R}{k + R^2} + R^2 g^{(k)}, \tag{326}$$

where $k = 1, 0, -1$, $g^{(1)}$ is the metric on the unit $(n-2)-$sphere, $g^{(0)}$ is the metric on a unit $(n-2)-$torus and $g^{(-1)}$ is the metric on a compact identification of $(n-2)-$dimensional hyperbolic space, $\mathbb{H}^{n-2}$. The Kottler metrics are the simplest generalisation of AdS (the $k = 1$ case is AdS itself) and in Fefferman-Graham coordinates, they are [32]

$$g = \mathrm{d}r \otimes \mathrm{d}r + \mathrm{e}^{2r}\left(-\left(1 + \frac{k}{4}\mathrm{e}^{-2r}\right)^2 \mathrm{d}t \otimes \mathrm{d}t + \left(1 - \frac{k}{4}\mathrm{e}^{-2r}\right)^2 g^{(k)}\right). \tag{327}$$

Hence, this subsection studies the $k = 0$ case. $k = 1$, i.e. AdS, will be studied in section 4.2. No immediate progress can be made in the $k = -1$ case because compactifying $\mathbb{H}^{n-2}$ by identification is incompatible with retaining any of $\mathbb{H}^{n-2}$'s Killing spinors. Indeed, negative energy solutions are possible in spacetimes with compact hyperbolic cross-sections [42], albeit it isn't known whether the energy is unbounded below.

Lemma 4.2. *The most general Killing spinor for the $f_{(0)}$ in equation 325 is*

$$\varepsilon_k = \mathrm{e}^{r/2} P_1^- \varepsilon_0, \tag{328}$$

where $P_1^\pm = \frac{1}{2}(I \pm \mathrm{i}\gamma^1)$ and ε_0 is an arbitrary constant spinor.

Proof. As discussed above, the background metric is

$$\bar{g} = \mathrm{d}r \otimes \mathrm{d}r + \mathrm{e}^{2r}\eta_{mn}\mathrm{d}x^m \otimes \mathrm{d}x^n. \tag{329}$$

$\therefore$ The natural vielbein is $e^0 = \mathrm{e}^r\mathrm{d}t$, $e^1 = \mathrm{d}r$ and $e^A = \mathrm{e}^r \delta^A{}_\alpha \mathrm{d}\theta^\alpha$.
To solve the Killing spinor equation, I need to first find the spin connection coefficients. I'll do so by the structure equation, $\mathrm{d}e^\mu = -\omega^\mu{}_\nu \wedge e^\nu$.

$$\mathrm{d}e^0 = \mathrm{e}^r\mathrm{d}r \wedge \mathrm{d}t = e^1 \wedge e^0,\ \mathrm{d}e^1 = 0 \text{ and } \mathrm{d}e^A = \mathrm{e}^r \delta^A{}_\alpha \mathrm{d}r \wedge \mathrm{d}\theta^\alpha = e^1 \wedge e^A. \tag{330}$$

Hence, by inspection, the non-zero connection 1-forms are (up to antisymmetries) are

$$\omega_{01} = -e^0 \text{ and } \omega_{A1} = e^A \tag{331}$$
$$\iff \omega_{010} = -1 \text{ and } \omega_{A1A} = 1 \text{ (no sum).} \tag{332}$$

The background Killing spinor equation - equation 226 - says

$$0 = \partial_{\mu'}\varepsilon_k - \frac{1}{4}e_{\mu'}{}^{\mu}\omega_{\nu\rho\mu}\gamma^{\nu\rho}\varepsilon_k + \frac{\mathrm{i}}{2}e_{\mu'}{}^{\mu}\gamma_\mu\varepsilon_k \tag{333}$$

and it now reduces to

$$0 = \partial_t\varepsilon_k + \frac{\mathrm{e}^r}{2}\gamma^0\gamma^1\varepsilon_k - \frac{\mathrm{i}\mathrm{e}^r}{2}\gamma^0\varepsilon_k, \tag{334}$$
$$0 = \partial_r\varepsilon_k + \frac{\mathrm{i}}{2}\gamma^1\varepsilon_k \text{ and} \tag{335}$$
$$0 = \partial_\alpha\varepsilon_k - \frac{\mathrm{e}^r}{2}\delta_{A\alpha}\gamma^A\gamma^1\varepsilon_k + \frac{\mathrm{i}\mathrm{e}^r}{2}\delta_{A\alpha}\gamma^A\varepsilon_k. \tag{336}$$

Equation 335 immediately implies $\varepsilon_k = \mathrm{e}^{-\mathrm{i}\gamma^1 r/2}\tilde{\varepsilon}$, for some spinor, $\tilde{\varepsilon}$, that doesn't depend on r. Split $\tilde{\varepsilon}$ up into γ^1 eigenspaces, i.e. $\tilde{\varepsilon} = P_1^-\varepsilon_- + P_1^+\varepsilon_+$ for some $\varepsilon_\pm$ that also don't depend on r.
$\therefore \varepsilon_k = \mathrm{e}^{r/2}P_1^-\varepsilon_- + \mathrm{e}^{-r/2}P_1^+\varepsilon_+$.
Substituting this into equation 336 implies

$$\begin{aligned}0 &= \mathrm{e}^{r/2}\partial_\alpha P_1^-\varepsilon_- + \mathrm{e}^{-r/2}\partial_\alpha P_1^+\varepsilon_+ - \frac{\mathrm{i}\mathrm{e}^{3r/2}}{2}\delta_{A\alpha}\gamma^A P_1^-\varepsilon_- + \frac{\mathrm{i}\mathrm{e}^{r/2}}{2}\delta_{A\alpha}\gamma^A P_1^+\varepsilon_+ \\ &\quad + \frac{\mathrm{i}\mathrm{e}^{3r/2}}{2}\delta_{A\alpha}\gamma^A P_1^-\varepsilon_- + \frac{\mathrm{i}\mathrm{e}^{r/2}}{2}\delta_{A\alpha}\gamma^A P_1^+\varepsilon_+ &(337)\\ &= \mathrm{e}^{r/2}\partial_\alpha P_1^-\varepsilon_- + \mathrm{e}^{-r/2}\partial_\alpha P_1^+\varepsilon_+ + \mathrm{i}\mathrm{e}^{r/2}\delta_{A\alpha}\gamma^A P_1^+\varepsilon_+ &(338)\\ &= \mathrm{e}^{r/2}P_1^-(\partial_\alpha\varepsilon_- + \mathrm{i}\delta_{A\alpha}\gamma^A\varepsilon_+) + \mathrm{e}^{-r/2}P_1^+\partial_\alpha\varepsilon_+. &(339)\end{aligned}$$

Applying $P_1^\pm$ to this equation yields $\partial_\alpha P_1^+\varepsilon_+ = 0$ and $\partial_\alpha P_1^-\varepsilon_- = -\mathrm{i}\delta_{A\alpha}\gamma^A P_1^+\varepsilon_+$.
The first of these equations implies $P_1^+\varepsilon_+$ is independent of θ^α. Consequently, the 2nd equation integrates to $P_1^-\varepsilon_- = -\mathrm{i}\theta^\alpha\delta_{A\alpha}\gamma^A P_1^+\varepsilon_+ + P_1^-\varepsilon_0$, for some ε_0 independent of r and θ^α.
θ^α is an angle around a circle though; it is periodic. Spinors must be periodic or antiperiodic around a circle. $-\mathrm{i}\theta^\alpha\delta_{A\alpha}\gamma^A P_1^+\varepsilon_+$ is neither unless $P_1^+\varepsilon_+ = 0$.
$\therefore$ I'm left with $\varepsilon_k = \mathrm{e}^{r/2}P_1^-\varepsilon_0$, where ε_0 can only depend on t.
It remains to satisfy equation 334, which now says

$$0 = \mathrm{e}^{r/2}P_1^-\partial_t\varepsilon_0 + \frac{\mathrm{i}\mathrm{e}^{3r/2}}{2}\gamma^0 P_1^-\varepsilon_0 - \frac{\mathrm{i}\mathrm{e}^{3r/2}}{2}\gamma^0 P_1^-\varepsilon_0 = \mathrm{e}^{r/2}P_1^-\partial_t\varepsilon_0. \tag{340}$$

Hence, I'm left with $\varepsilon_k = \mathrm{e}^{r/2}P_1^-\varepsilon_0$ for just a constant spinor, ε_0. □

Corollary 4.2.1. *Theorem 3.19 only applies if (M, g) admits a spin structure where spinors are periodic in the torus' circle directions in an open neighbourhood of $\partial_\infty\Sigma_t$, say $\overline{M}$.*

A circle admits two spin structures - one where spinors are periodic and one where spinors are anti-periodic. In $\mathbb{T}^{n-2}$, this applies to each of the $n-2$ circles. Of all these different spin structures, equation 328 requires the one which is periodic in all $n-2$ circles. However, it's possible that spin structure, while fine on $(\overline{M}, \bar{g})$, does not extend to all of (M, g). Theorem 3.19 would therefore not apply in such a scenario. Indeed this is exactly the situation for the AdS soliton [40, 41] - see [43] for yet more exotic constructions. To make progress, I must henceforth restrict attention - just as the authors of [13] did - to manifolds where the spin structure required by equation 328 does extend to all of (M, g).

Theorem 4.3 (Toroidal positve energy theorem). *If the Einstein equation holds, T_{ab} satisfies the dominant energy condition, $T^{0\mu}$ decays faster than $O(\mathrm{e}^{-(n-1)r})$ near $\partial_\infty\Sigma_t$ and (M,g)'s spin structure is compatible with having periodic spinors near $\partial_\infty\Sigma_t$, then*

$$E \geq \sqrt{\mathbb{J}_A\mathbb{J}^A}, \tag{341}$$

where $\mathbb{J}_A = \frac{n-1}{16\pi}\int_{\partial_\infty\Sigma_t} p_A \mathrm{d}^{n-2}\theta$.

Proof. The proof is simply a matter of evaluating corollary 4.1.1 for the ε_k in equation 328. In particular, $\sqrt{\iota^\star f_{(0)}} = 1$ and

$$\begin{aligned} p_M\bar{\varepsilon}_k\gamma^M\varepsilon_k &= p_M\varepsilon_k^\dagger\gamma^0\gamma^M\varepsilon_k & (342)\\ &= \varepsilon_k^\dagger(p_0 I + p_A\gamma^0\gamma^A)\varepsilon_k & (343)\\ &= \mathrm{e}^r\varepsilon_0^\dagger P_1^-(p_0 I + p_A\gamma^0\gamma^A)P_1^-\varepsilon_0. & (344)\end{aligned}$$

The spinors and matrices in this equation are all constants, so can be freely moved in and out of integrals. Thus, corollary 4.1.1 says

$$\begin{aligned} 0 &\leq Q(\varepsilon) & (345)\\ &= \varepsilon_0^\dagger P_1^-\left(\frac{n-1}{2}\int_{\partial_\infty\Sigma_t} p_0\mathrm{d}^{n-2}\theta\, I + \frac{n-1}{2}\int_{\partial_\infty\Sigma_t} p_A\mathrm{d}^{n-2}\theta\gamma^0\gamma^A\right)P_1^-\varepsilon_0 & (346)\\ &= 8\pi\varepsilon_0^\dagger P_1^-\left(EI + \mathbb{J}_A\gamma^0\gamma^A\right)P_1^-\varepsilon_0 \text{ by equation 81.} & (347)\end{aligned}$$

The eigenvalues of the matrix, $\mathbb{J}_A\gamma^0\gamma^A$, are[26] $\pm\sqrt{\mathbb{J}_A\mathbb{J}^A}$, so $EI + \mathbb{J}_A\gamma^0\gamma^A$ has eigenvalues, $E \pm \sqrt{\mathbb{J}_A\mathbb{J}^A}$.
Choose ε_0 to be in intersection of $E-\sqrt{\mathbb{J}_A\mathbb{J}^A}$ eigenspace with the interection of the i eigenspace of γ^1 (so that $P_1^-\varepsilon_0 = \varepsilon_0$). Then, equation 347 can only hold if $E \geq \sqrt{\mathbb{J}_A\mathbb{J}^A}$. □

The quantity, $\frac{n-1}{16\pi}\int_{\partial_\infty\Sigma_t} p_A\mathrm{d}^{n-2}\theta$, has been suggestively denoted $\mathbb{J}_A$, hinting angular momentum. Indeed, a quantity analogous to $\mathbb{J}_A$ has been interpreted as an angular momentum vector in [13]. It is natural to make the same interpretation here because the boundary topology is $\mathbb{R}\times\mathbb{T}^{n-2}$; each component of $\mathbb{J}_A$ describes the rotation around one of the $n-2$ circles comprising $\mathbb{T}^{n-2}$. However, note that angular momentum will look quite different in section 4.2 because of the different boundary topology there.

4.2 Asymptotically *AdS*

In this subsection, I'll apply theorem 3.19 to the example of greatest physical interest, namely $A_\mu = 0$ and $f_{(0)} = -\mathrm{d}t\otimes\mathrm{d}t + g_{S^{n-2}}$. By definition 2.3, this corresponds to asymptotically AdS spacetimes; the background metric is

$$\bar{g} = g_{\mathrm{AdS}} = \mathrm{d}r\otimes\mathrm{d}r + \mathrm{e}^{2r}\left(-\left(1+\frac{1}{4}\mathrm{e}^{-2r}\right)^2\mathrm{d}t\otimes\mathrm{d}t + \left(1-\frac{1}{4}\mathrm{e}^{-2r}\right)^2 g_{S^{n-2}}\right) \tag{348}$$

The open neighbourhood of the "boundary" at infinity has only one spin structure now, so, unlike the toroidal case, the issues about compatibility raised in corollary 4.2.1 do not arise.

[26] The eigenvalues can be found by noting that if $\mathbb{J}_A\gamma^0\gamma^A v = \lambda v$, then $\lambda^2 v = \mathbb{J}_A\mathbb{J}_B\gamma^0\gamma^A\gamma^0\gamma^B v = \mathbb{J}_A\mathbb{J}^A v$ by the Clifford algebra. Both $\lambda = \sqrt{\mathbb{J}_A\mathbb{J}^A}$ and $\lambda = -\sqrt{\mathbb{J}_A\mathbb{J}^A}$ must occur because if v is in one eigenspace, then $\gamma^0 v$ is in the other eigenspace.

In Fefferman-Graham coordinates, AdS is given by equation 9. The Killing spinor - in the natural vielbien associated to those coordinates - is calculated in [14]. However, the Fefferman-Graham coordinates - especially when $g_{S^{n-2}}$ is written in the nested sines form of [14] - are very asymmetrical. The $\bar{\varepsilon}_k\gamma^M\varepsilon_k$ in theorem 3.19 will be practically impossible to calculate in this frame. Luckily for me, $\bar{\varepsilon}_k\gamma^\mu\varepsilon_k$ is a Lorentz vector. Hence, I can choose a more convenient frame, $e'_\mu = \Lambda^\nu{}_\mu(x)e_\nu$, calculate the Killing spinor, ε'_k, in the e'_μ frame and then determine $\bar{\varepsilon}_k\gamma^\mu\varepsilon_k$ by $\bar{\varepsilon}_k\gamma^\mu\varepsilon_k = \Lambda^\mu{}_\nu(x)\bar{\varepsilon}'_k\gamma^\nu\varepsilon'_k$. The most convenient e'_μ results from viewing AdS as $\mathbb{R}\times\mathbb{H}^{n-1}$ with the metric,

$$g_{\mathrm{AdS}} = -\left(\frac{1+\rho^2}{1-\rho^2}\right)^2 \mathrm{d}t\otimes\mathrm{d}t + \frac{4}{(1-\rho^2)^2}\delta_{IJ}\mathrm{d}x^I\otimes\mathrm{d}x^J, \tag{349}$$

where $\rho = \sqrt{x_I x^I}$ and x^I are Cartesian coordinates[27] in the unit disk (centred at the origin). $\mathbb{H}^{n-1}$ is thus being represented by the Poincaré disk/ball in these coordinates.

Lemma 4.4. *The area radius function, R, is $R = \frac{2\rho}{1-\rho^2}$ and the Fefferman-Graham coordinate is $r = \ln\left(R+\sqrt{1+R^2}\right) - \ln(2)$.*

Proof. Writing the $\delta_{IJ}\mathrm{d}x^I\otimes\mathrm{d}x^J$ in equation 349 in spherical coordinates,

$$g_{\mathrm{AdS}} = -\left(\frac{1+\rho^2}{1-\rho^2}\right)^2 \mathrm{d}t\otimes\mathrm{d}t + \frac{4}{(1-\rho^2)^2}\mathrm{d}\rho\otimes\mathrm{d}\rho + \frac{4\rho^2}{(1-\rho^2)^2}g_{S^{n-2}}. \tag{350}$$

By inspection, the area-radius function is $R = \frac{2\rho}{1-\rho^2}$.
In terms of R, g_{AdS} takes the standard form,

$$g_{\mathrm{AdS}} = -(1+R^2)\mathrm{d}t\otimes\mathrm{d}t + \frac{\mathrm{d}R\otimes\mathrm{d}R}{1+R^2} + R^2 g_{S^{n-2}}, \quad \text{because} \tag{351}$$

$$1+R^2 = 1+\frac{4\rho^2}{(1-\rho^2)^2} = \left(\frac{1+\rho^2}{1-\rho^2}\right)^2 \quad \text{and} \tag{352}$$

$$\frac{\mathrm{d}R\otimes\mathrm{d}R}{1+R^2} = \left(\frac{1-\rho^2}{1+\rho^2}\right)^2\left(\frac{2(1-\rho^2)-2\rho(-2\rho)}{(1-\rho^2)^2}\right)^2\mathrm{d}\rho\otimes\mathrm{d}\rho = \frac{4\mathrm{d}\rho\otimes\mathrm{d}\rho}{(1-\rho^2)^2}. \tag{353}$$

The natural way to find the Fefferman-Graham coordinate is thus to choose r to depend only on R and fix it so that $\mathrm{d}r\otimes\mathrm{d}r = \frac{\mathrm{d}R\otimes\mathrm{d}R}{1+R^2}$.

$$\therefore \frac{\mathrm{d}r}{\mathrm{d}R} = \pm\frac{1}{\sqrt{1+R^2}}. \tag{354}$$

The RHS can be integrated (e.g. by computer algebra software) to get

$$r = \pm\ln\left(R+\sqrt{1+R^2}\right) + c. \tag{355}$$

I need the boundary at infinity to be $r\to\infty$, so I must choose the + in $\pm$.
The choice of $c = -\ln(2)$ is just to ensure the $\mathrm{d}t\otimes\mathrm{d}t$ in $f_{(0)}$ has coefficient -1; for other choices it would be $-2\mathrm{e}^c$. □

Corollary 4.4.1. *The boundary at infinity, $r\to\infty$, corresponds to $\rho\to 1^-$.*

[27] In particular, the I in x^I is not a vielbein index. However, I will still lower that index by δ_{IJ}, just as if it were a vielbein index.

In the coordinates of equation 349, the natural choice of vielbein is

$$e'_0 = \frac{1-\rho^2}{1+\rho^2}\partial_t \text{ and } e'_I = \frac{1-\rho^2}{2}\partial_I. \tag{356}$$

Meanwhile the coordinates of equation 348 are naturally viewed in vielbein,

$$e_0 = \frac{\mathrm{e}^{-r}}{1+\frac{1}{4}\mathrm{e}^{-2r}}\partial_t, \;\; e_1 = \partial_r \text{ and } e_A = \frac{\mathrm{e}^{-r}}{1-\frac{1}{4}\mathrm{e}^{-2r}}e^{(s)\alpha}_A\partial_\alpha \tag{357}$$

where $e^{(s)A}_\alpha$ is a vielbein for $g_{S^{n-2}}$ and θ^α are local coordinates for S^{n-2}.

Lemma 4.5. *The vielbeins, e_μ and e'_μ, are related by*

$$e'_0 = e_0, \;\; \text{and } e'_I = \hat{x}_I e_1 + \rho\frac{\partial\theta^\alpha}{\partial x^I}e^{(s)A}_\alpha e_A, \tag{358}$$

where $\hat{x}^I$ are unit vectors, i.e. $x^I = \rho\hat{x}^I$. Hence, the local Lorentz transformation relating e_μ and e'_μ, i.e. $e'_\mu = \Lambda^\nu{}_\mu(x)e_\nu$, is given by

$$\Lambda^\mu{}_0 = \delta^\mu{}_0 \text{ and } \Lambda^\mu{}_I = \delta^\mu{}_1\hat{x}_I + \delta^\mu{}_A\rho\frac{\partial\theta^\alpha}{\partial x^I}e^{(s)A}_\alpha. \tag{359}$$

Proof. The proof is essentially just applying lemma 4.4.
$e'_0 = e_0$ immediately because all I'm doing is re-writing r in terms of ρ. For e'_I,

$$e'_I = \frac{1-\rho^2}{2}\partial_I \tag{360}$$

$$= \frac{1-\rho^2}{2}\frac{\partial r}{\partial x^I}\partial_r + \frac{1-\rho^2}{2}\frac{\partial\theta^\alpha}{\partial x^I}\partial_\alpha \tag{361}$$

$$= \frac{1-\rho^2}{2}\frac{\partial r}{\partial x^I}e_1 + \frac{1-\rho^2}{2}\frac{\partial\theta^\alpha}{\partial x^I}e^{(s)A}_\alpha e^{(s)}_A \tag{362}$$

$$= \frac{1-\rho^2}{2}\frac{\partial r}{\partial x^I}e_1 + \frac{1-\rho^2}{2}\frac{\partial\theta^\alpha}{\partial x^I}e^{(s)A}_\alpha R e_A. \tag{363}$$

For the 2nd term,

$$\frac{1-\rho^2}{2}R = \frac{1-\rho^2}{2}\frac{2\rho}{1-\rho^2} = \rho. \tag{364}$$

Meanwhile, for the 1st term,

$$\frac{1-\rho^2}{2}\frac{\partial r}{\partial x^I} = \frac{1-\rho^2}{2}\frac{\partial}{\partial x^I}\left(\ln\left(R+\sqrt{1+R^2}\right)-\ln(2)\right) \tag{365}$$

$$= \frac{1-\rho^2}{2}\frac{1}{R+\sqrt{1+R^2}}\left(1+\frac{R}{\sqrt{1+R^2}}\right)\frac{\partial R}{\partial x^I} \tag{366}$$

$$= \frac{1-\rho^2}{2}\frac{1}{\sqrt{1+R^2}}\frac{\partial}{\partial x^I}\left(\frac{2\rho}{1-\rho^2}\right) \tag{367}$$

$$= \frac{1-\rho^2}{2}\frac{1}{\sqrt{1+\frac{4\rho^2}{(1-\rho^2)^2}}}\frac{2(1-\rho^2)-2\rho(-2\rho)}{(1-\rho^2)^2}\frac{\partial\rho}{\partial x^I} \tag{368}$$

$$= \frac{1-\rho^2}{2}\frac{1-\rho^2}{1+\rho^2}\frac{2(1+\rho^2)}{(1-\rho^2)^2}\frac{x^I}{\rho} \tag{369}$$

$$= \hat{x}^I. \tag{370}$$

Substituting these expressions back into equation 363 gives the claimed result. □

Lemma 4.6. *In the frame of equation 356,*

$$\varepsilon_k' = \frac{1}{\sqrt{1-\rho^2}}\left(I - \mathrm{i}x_I\gamma^I\right)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0 \tag{371}$$

is a background Killing spinor, for any constant spinor, ε_0.

Proof. To check whether equation 226 holds, I first need to find the spin connection coefficients.

$$\mathrm{d}e^0 = \mathrm{d}\left(\frac{1+\rho^2}{1-\rho^2}\mathrm{d}t\right) \tag{372}$$

$$= \frac{2\rho(1-\rho^2)-(1+\rho^2)(-2\rho)}{(1-\rho^2)^2}\mathrm{d}\rho\wedge\mathrm{d}t \tag{373}$$

$$= \frac{4}{(1-\rho^2)^2}x_I\mathrm{d}x^I\wedge\mathrm{d}t \tag{374}$$

$$= \frac{2}{1+\rho^2}x_I e^I\wedge e^0. \tag{375}$$

$$\mathrm{d}e^I = \mathrm{d}\left(\frac{2}{1-\rho^2}\mathrm{d}x^I\right) = \frac{4\rho}{(1-\rho^2)^2}\mathrm{d}\rho\wedge\mathrm{d}x^I \tag{376}$$

$$= \frac{4\rho}{(1-\rho^2)^2}\frac{x_J}{\rho}\frac{1-\rho^2}{2}e^J\wedge\frac{1-\rho^2}{2}e^I \tag{377}$$

$$= x_J e^J\wedge e^I. \tag{378}$$

${\omega^\mu}_\nu\wedge e^\nu = -\mathrm{d}e^\mu$ by the structure equations, so[28] $\omega_{0I} = -\frac{2}{1+\rho^2}x_I e^0$ and $\omega_{IJ} = x_J e^I - x_I e^J$. $\therefore \omega_{0I0} = -\frac{2}{1+\rho^2}x_I$ and $\omega_{IJI} = x_J$ for $J\neq I$ and no sum on I.
Equation 226, for background Killing spinors, is

$$0 = {e_\mu}^{\mu'}\partial_{\mu'}\varepsilon_k - \frac{1}{4}\omega_{\nu\rho\mu}\gamma^{\nu\rho}\varepsilon_k + \frac{\mathrm{i}}{2}\gamma_\mu\varepsilon_k. \tag{379}$$

First try $\mu=0$. Then,

$$ {e_\mu}^{\mu'}\partial_{\mu'}\varepsilon_k - \frac{1}{4}\omega_{\nu\rho\mu}\gamma^{\nu\rho}\varepsilon_k + \frac{\mathrm{i}}{2}\gamma_\mu\varepsilon_k \tag{380}$$

$$= \frac{1-\rho^2}{1+\rho^2}\partial_t\varepsilon_k + \frac{1}{1+\rho^2}x_I\gamma^0\gamma^I\varepsilon_k - \frac{\mathrm{i}}{2}\gamma^0\varepsilon_k \tag{381}$$

$$= \frac{1-\rho^2}{1+\rho^2}\frac{1}{\sqrt{1-\rho^2}}\left(I-\mathrm{i}x_I\gamma^I\right)\frac{\mathrm{i}}{2}\gamma^0\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0 + \frac{1}{1+\rho^2}x_J\gamma^0\gamma^J\frac{1}{\sqrt{1-\rho^2}}\left(I-\mathrm{i}x_I\gamma^I\right)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0$$
$$ - \frac{\mathrm{i}}{2}\gamma^0\frac{1}{\sqrt{1-\rho^2}}\left(I-\mathrm{i}x_I\gamma^I\right)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0 \tag{382}$$

$$= \frac{1}{2(1+\rho^2)\sqrt{1-\rho^2}}\Big(\mathrm{i}(1-\rho^2)\gamma^0 - (1-\rho^2)x_I\gamma^0\gamma^I + 2x_I\gamma^0\gamma^I - 2\mathrm{i}x_Jx_I\gamma^0\gamma^J\gamma^I$$
$$ - \mathrm{i}(1+\rho^2)\gamma^0 - (1+\rho^2)x_I\gamma^0\gamma^I\Big)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0 \tag{383}$$

$$= \frac{1}{2(1+\rho^2)\sqrt{1-\rho^2}}\Big(\mathrm{i}(1-\rho^2)\gamma^0 - (1-\rho^2)x_I\gamma^0\gamma^I + 2x_I\gamma^0\gamma^I + 2\mathrm{i}\rho^2\gamma^0$$
$$ - \mathrm{i}(1+\rho^2)\gamma^0 - (1+\rho^2)x_I\gamma^0\gamma^I\Big)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0 \tag{384}$$

$$= 0, \quad \text{as required.} \tag{385}$$

[28] As indices are raised and lowered by δ, the matching of upstairs and downstairs index position need not be too strict. I also won't list zero components or those determined by antisymmetries.

Next, consider $\mu = I$. The derivative term is

$$\partial_I \varepsilon_k = \partial_I \left(\frac{1}{\sqrt{1-\rho^2}} \left(I - \mathrm{i}x_J\gamma^J\right) \mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0 \right) \tag{386}$$

$$= \left(- \frac{1}{2(1-\rho^2)^{3/2}}(-2\rho)\frac{x_I}{\rho}(I - \mathrm{i}x_J\gamma^J) - \frac{\mathrm{i}}{\sqrt{1-\rho^2}}\gamma_I \right) \mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0 \tag{387}$$

$$= \frac{1}{(1-\rho^2)^{3/2}} \left(x_I I - \mathrm{i}x_I x_J \gamma^J - \mathrm{i}(1-\rho^2)\gamma^I \right) \mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0. \tag{388}$$

Then, the actual expression to be checked is

$$e_\mu{}^{\mu'}\partial_{\mu'}\varepsilon_k - \frac{1}{4}\omega_{\nu\rho\mu}\gamma^{\nu\rho}\varepsilon_k + \frac{\mathrm{i}}{2}\gamma_\mu\varepsilon_k \tag{389}$$

$$= \frac{1-\rho^2}{2}\frac{1}{(1-\rho^2)^{3/2}} \left(x_I I - \mathrm{i}x_I x_J \gamma^J - \mathrm{i}(1-\rho^2)\gamma^I \right) \mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0$$
$$- \frac{1}{2}x_J\gamma^{IJ}\frac{1}{\sqrt{1-\rho^2}}\left(I - \mathrm{i}x_K\gamma^K\right)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0 + \frac{\mathrm{i}}{2}\gamma^I\frac{1}{\sqrt{1-\rho^2}}\left(I - \mathrm{i}x_J\gamma^J\right)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0 \tag{390}$$

$$= \frac{1}{2\sqrt{1-\rho^2}} \Big(x_I I - \mathrm{i}x_I x_J\gamma^J - \mathrm{i}(1-\rho^2)\gamma^I - x_J\gamma^{IJ} + \mathrm{i}x_J x_K \gamma^{IJ}\gamma^K$$
$$+ \mathrm{i}\gamma^I + x_J\gamma^I\gamma^J \Big) \mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0 \tag{391}$$

$$= \frac{1}{2\sqrt{1-\rho^2}} \Big(x_I I - \mathrm{i}x_I x_J\gamma^J + \mathrm{i}\rho^2\gamma^I - x_J(\gamma^I\gamma^J + \delta^{IJ}I)$$
$$+ \mathrm{i}x_J x_K\left(\gamma^{IJK} - \delta^{KJ}\gamma^I + \delta^{KI}\gamma^J\right) + x_J\gamma^I\gamma^J \Big) \mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0 \tag{392}$$

$$= 0 \ \text{ too.} \tag{393}$$

Hence, the postulated ε'_k is indeed a background Killing spinor in the frame of equation 356. □

Definition 4.7 (Momentum, momentum, momentum!)**.** *Define the linear momentum, angular momentum and centre of mass momentum as*

$$P_I = \frac{n-1}{16\pi}\int_{\partial_\infty\Sigma_t} \tilde{f}_{(0)}^{mn} f_{(n-1)mn}\hat{x}_I \mathrm{d}(g_{S^{n-2}}) = \frac{n-1}{16\pi}\int_{\partial_\infty\Sigma_t} p_0\hat{x}_I \mathrm{d}(g_{S^{n-2}}), \tag{394}$$

$$J_{IJ} = \frac{n-1}{16\pi}\int_{\partial_\infty\Sigma_t} f_{(n-1)0\alpha}\left(\hat{x}_I \frac{\partial\theta^\alpha}{\partial x^J}\bigg|_{\rho=1} - \hat{x}_J \frac{\partial\theta^\alpha}{\partial x^I}\bigg|_{\rho=1} \right) \mathrm{d}(g_{S^{n-2}}) \ \text{ and} \tag{395}$$

$$K_I = \frac{n-1}{16\pi}\int_{\partial_\infty\Sigma_t} f_{(n-1)0\alpha}\frac{\partial\theta^\alpha}{\partial x^J}\bigg|_{\rho=1} \left(\delta^J{}_I - \hat{x}^J\hat{x}_I\right)\mathrm{d}(g_{S^{n-2}}) \tag{396}$$

respectively. In these expressions, θ^α denote local coordinates on S^{n-2}, $\hat{x}^I$ denote unit vector Cartesian coordinates and $\rho = \sqrt{x_I x^I}$, i.e. $x^I = \rho\hat{x}^I$.

These definitions are based off the discussion in [13]. The exact form is motivated by the terms that appear in the next theorem. However, some heuristics can be discussed now. It was shown in [10] that the Riemannian analogue of (E, P_I) transforms as a Lorentz vector when one chooses a different conformal class representative for the boundary metric, $f_{(0)}$. Hence, P_I naturally behaves like linear momentum. Next, observe that the vector, $(\hat{x}_I\frac{\partial\theta^\alpha}{\partial x^J}|_{\rho=1} - \hat{x}_J\frac{\partial\theta^\alpha}{\partial x^A}|_{\rho=1})\partial_\alpha$ equals $\hat{x}_I\partial_J - \hat{x}_J\partial_I$, which is the generator of rotations. Hence, it's natural to expect what I've defined as J_{IJ} above to behave like angular momentum. I will do an example illustrating this in section 4.2.1. Likewise, $\frac{\partial\theta^\alpha}{\partial x^J}|_{\rho=1}\left(\delta^J{}_I - \hat{x}^J\hat{x}_I\right)\partial_\alpha = \left(\delta^J{}_I - \hat{x}^J\hat{x}_I\right)\partial_J$ can be seen a generator of boosts, suggesting the K_I above should be interpreted as a centre of mass momentum.

Theorem 4.8 (Asymptotically AdS positive energy theorem)**.** *If the Einstein equation holds, T_{ab} satisfies the dominant energy condition and $T^{0\mu}$ decays faster than $O(\mathrm{e}^{-(n-1)r})$ near $\partial_\infty\Sigma_t$, then*

$$EI - \mathrm{i}P_I\gamma^I + \frac{\mathrm{i}}{2}J_{IJ}\gamma^0\gamma^{IJ} + K_I\gamma^0\gamma^I \tag{397}$$

is a non-negative definite matrix.

Proof. The proof is mostly a matter of evaluating corollary 4.1.1 for asymptotically AdS spaces. As explained at the start of section 4.2, I'll find $\bar{\varepsilon}'_k\gamma^\mu\varepsilon'_k$ and use that to find $p_M\bar{\varepsilon}_k\gamma^M\varepsilon_k$. When $\mu = 0$,

$$\bar{\varepsilon}'_k\gamma^0\varepsilon'_k = \varepsilon'^{\dagger}_k\varepsilon'_k \tag{398}$$

$$= \frac{1}{1-\rho^2}\varepsilon_0^\dagger \mathrm{e}^{-\mathrm{i}\gamma^0 t/2}(I - \mathrm{i}x_I\gamma^I)(I - \mathrm{i}x_J\gamma^J)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0 \text{ by lemma 4.6} \tag{399}$$

$$= \frac{1}{1-\rho^2}\varepsilon_0^\dagger \mathrm{e}^{-\mathrm{i}\gamma^0 t/2}(I - 2\mathrm{i}x_I\gamma^I - x_Ix_J\gamma^I\gamma^J)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0 \tag{400}$$

$$= \frac{1}{1-\rho^2}\varepsilon_0^\dagger \mathrm{e}^{-\mathrm{i}\gamma^0 t/2}((1+\rho^2)I - 2\mathrm{i}x_I\gamma^I)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0. \tag{401}$$

I only need this expression near $\partial_\infty\Sigma_t$, where $r \to \infty$. By lemma 4.4, that means $\rho \to 1$ and $\frac{1}{1-\rho^2} = \frac{R}{2\rho} \to \frac{1}{2}\mathrm{e}^r$.

$$\therefore \bar{\varepsilon}'_k\gamma^0\varepsilon'_k \to \mathrm{e}^r\varepsilon_0^\dagger \mathrm{e}^{-\mathrm{i}\gamma^0 t/2}(I - \mathrm{i}\hat{x}_I\gamma^I)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0. \tag{402}$$

Hence, using lemma 4.5,

$$\frac{n-1}{2}\mathrm{e}^{-r}\int_{\partial_\infty\Sigma_t} p_0\bar{\varepsilon}_k\gamma^0\varepsilon_k\sqrt{\iota^* f_{(0)}}\mathrm{d}^{n-2}x$$

$$= \frac{n-1}{2}\mathrm{e}^{-r}\int_{\partial_\infty\Sigma_t} p_0\Lambda^0{}_\mu\bar{\varepsilon}'_k\gamma^\mu\varepsilon'_k\sqrt{\iota^* f_{(0)}}\mathrm{d}^{n-2}x \tag{403}$$

$$= \frac{n-1}{2}\mathrm{e}^{-r}\int_{\partial_\infty\Sigma_t} p_0\bar{\varepsilon}'_k\gamma^0\varepsilon'_k\sqrt{\iota^* f_{(0)}}\mathrm{d}^{n-2}x \tag{404}$$

$$= \frac{n-1}{2}\int_{\partial_\infty\Sigma_t} p_0\varepsilon_0^\dagger \mathrm{e}^{-\mathrm{i}\gamma^0 t/2}(I - \mathrm{i}\hat{x}_I\gamma^I)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0\sqrt{\iota^* f_{(0)}}\mathrm{d}^{n-2}x \tag{405}$$

$$= \varepsilon_0^\dagger \mathrm{e}^{-\mathrm{i}\gamma^0 t/2}\frac{n-1}{2}\left(\int_{\partial_\infty\Sigma_t} p_0\sqrt{\iota^* f_{(0)}}\mathrm{d}^{n-2}x - \mathrm{i}\int_{\partial_\infty\Sigma_t} p_0\hat{x}_I\sqrt{\iota^* f_{(0)}}\mathrm{d}^{n-2}x\gamma^I\right)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0 \tag{406}$$

$$= 8\pi\varepsilon_0^\dagger \mathrm{e}^{-\mathrm{i}\gamma^0 t/2}(EI - \mathrm{i}P_I\gamma^I)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0 \text{ by definition 4.7.} \tag{407}$$

Likewise, when $\mu = I$,

$$\bar{\varepsilon}'_k\gamma^I\varepsilon'_k = \varepsilon'^{\dagger}_k\gamma^0\gamma^I\varepsilon'_k \tag{408}$$

$$= \frac{1}{1-\rho^2}\varepsilon_0^\dagger \mathrm{e}^{-\mathrm{i}\gamma^0 t/2}(I - \mathrm{i}x_J\gamma^J)\gamma^0\gamma^I(I - \mathrm{i}x_K\gamma^K)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0 \tag{409}$$

$$= \frac{1}{1-\rho^2}\varepsilon_0^\dagger \mathrm{e}^{-\mathrm{i}\gamma^0 t/2}(\gamma^0 - \mathrm{i}x_J\gamma^J\gamma^0)(\gamma^I - \mathrm{i}x_K\gamma^I\gamma^K)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0 \tag{410}$$

$$= \frac{1}{1-\rho^2}\varepsilon_0^\dagger \mathrm{e}^{-\mathrm{i}\gamma^0 t/2}(\gamma^0\gamma^I - \mathrm{i}x_J\gamma^J\gamma^0\gamma^I - \mathrm{i}x_J\gamma^0\gamma^I\gamma^J - x_Jx_K\gamma^J\gamma^0\gamma^I\gamma^K)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0 \tag{411}$$

$$= \frac{1}{1-\rho^2}\varepsilon_0^\dagger \mathrm{e}^{-\mathrm{i}\gamma^0 t/2}(\gamma^0\gamma^I - 2\mathrm{i}x_J\gamma^0\gamma^{IJ} - 2x^Ix_J\gamma^0\gamma^J + \rho^2\gamma^0\gamma^I)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0 \tag{412}$$

$$= \frac{1}{1-\rho^2}\varepsilon_0^\dagger \mathrm{e}^{-\mathrm{i}\gamma^0 t/2}((1+\rho^2)\gamma^0\gamma^I - 2\mathrm{i}x_J\gamma^0\gamma^{IJ} - 2x^Ix_J\gamma^0\gamma^J)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0 \tag{413}$$

$$\to \mathrm{e}^r\varepsilon_0^\dagger \mathrm{e}^{-\mathrm{i}\gamma^0 t/2}(\gamma^0\gamma^I - \mathrm{i}\hat{x}_J\gamma^0\gamma^{IJ} - \hat{x}^I\hat{x}_J\gamma^0\gamma^J)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0. \tag{414}$$

For the integral, note that $p_A = e_A^{(f_{(0)})m} n^n_{(0)} f_{(n-1)mn}$. With the choice of $f_{(0)}$ here, $n^n_{(0)} = \delta^{n0}$ and $e_A^{(f_{(0)})m} = e_A^{(s)\alpha}\delta^m{}_\alpha$, meaning $p_A = e_A^{(s)\alpha} f_{(n-1)\alpha 0}$. Hence, using lemma 4.5 again,

$$\begin{aligned}
&\frac{n-1}{2}\mathrm{e}^{-r}\int_{\partial_\infty\Sigma_t} p_A \bar{\varepsilon}_k \gamma^A \varepsilon_k \sqrt{\iota^* f_{(0)}}\mathrm{d}^{n-2}x \\
&= \frac{n-1}{2}\mathrm{e}^{-r}\int_{\partial_\infty\Sigma_t} p_A \Lambda^A{}_\mu \bar{\varepsilon}'_k \gamma^\mu \varepsilon'_k \sqrt{\iota^* f_{(0)}}\mathrm{d}^{n-2}x && (415)\\
&= \frac{n-1}{2}\mathrm{e}^{-r}\int_{\partial_\infty\Sigma_t} e_A^{(s)\alpha} f_{(n-1)0\alpha} \left.\frac{\partial\theta^\beta}{\partial x^I}\right|_{\rho=1} e_\beta^{(s)A} \bar{\varepsilon}'_k \gamma^I \varepsilon'_k \sqrt{\iota^* f_{(0)}}\mathrm{d}^{n-2}x && (416)\\
&= \frac{n-1}{2}\int_{\partial_\infty\Sigma_t} f_{(n-1)0\alpha} \left.\frac{\partial\theta^\alpha}{\partial x^I}\right|_{\rho=1} \\
&\quad\times \varepsilon_0^\dagger \mathrm{e}^{-\mathrm{i}\gamma^0 t/2}(\gamma^0\gamma^I - \mathrm{i}\hat{x}_J\gamma^0\gamma^{IJ} - \hat{x}^I\hat{x}_J\gamma^0\gamma^J)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0 \sqrt{\iota^* f_{(0)}}\mathrm{d}^{n-2}x && (417)\\
&= \frac{n-1}{2}\varepsilon_0^\dagger \mathrm{e}^{-\mathrm{i}\gamma^0 t/2}\Bigg(\int_{\partial_\infty\Sigma_t} f_{(n-1)0\alpha}\left.\frac{\partial\theta^\alpha}{\partial x^I}\right|_{\rho=1}(\delta^I{}_J - \hat{x}^I\hat{x}_J)\sqrt{\iota^* f_{(0)}}\mathrm{d}^{n-2}x\gamma^0\gamma^J \\
&\quad - \mathrm{i}\int_{\partial_\infty\Sigma_t} f_{(n-1)0\alpha}\left.\frac{\partial\theta^\alpha}{\partial x^I}\right|_{\rho=1}\hat{x}_J\sqrt{\iota^* f_{(0)}}\mathrm{d}^{n-2}x\gamma^0\gamma^{IJ}\Bigg)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0 && (418)\\
&= 8\pi\varepsilon_0^\dagger \mathrm{e}^{-\mathrm{i}\gamma^0 t/2}\left(K_I\gamma^0\gamma^I + \frac{\mathrm{i}}{2}J_{IJ}\gamma^0\gamma^{IJ}\right)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0 \text{ by definition 4.7.} && (419)
\end{aligned}$$

The upshot of these calculations is that corollary 4.1.1 now says

$$0 \le Q(\varepsilon) = 8\pi\varepsilon_0^\dagger \mathrm{e}^{-\mathrm{i}\gamma^0 t/2}\left(EI - \mathrm{i}P_I\gamma^I + K_I\gamma^0\gamma^I + \frac{\mathrm{i}}{2}J_{IJ}\gamma^0\gamma^{IJ}\right)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0. \quad (420)$$

By lemma 3.12, $Q(\varepsilon)$ is conserved. Since ε_0 is an arbitrary constant spinor, it must then be that $\mathrm{e}^{-\mathrm{i}\gamma^0 t/2}(EI - \mathrm{i}P_I\gamma^I + K_I\gamma^0\gamma^I + \frac{\mathrm{i}}{2}J_{IJ}\gamma^0\gamma^{IJ})\mathrm{e}^{\mathrm{i}\gamma^0 t/2}$ is t-independent too.
The $\mathrm{e}^{-\mathrm{i}\gamma^0 t/2}$ and $\mathrm{e}^{\mathrm{i}\gamma^0 t/2}$ book-ending this expression are merely performing a unitary change of basis; they don't affect the eigenvalues of the hermitian matrix,
$EI - \mathrm{i}P_I\gamma^I + K_I\gamma^0\gamma^I + \frac{\mathrm{i}}{2}J_{IJ}\gamma^0\gamma^{IJ}$.
$\therefore$ Since ε_0 is arbitrary and $Q(\varepsilon) \ge 0$, it must be that $EI - \mathrm{i}P_I\gamma^I + K_I\gamma^0\gamma^I + \frac{\mathrm{i}}{2}J_{IJ}\gamma^0\gamma^{IJ}$ is non-negative definite. □

Corollary 4.8.1. *The energy is not unbounded below as a function of the other physical quantities. In particular,* $E \ge \max(\text{eigenvalues}(\mathrm{i}P_I\gamma^I - \frac{\mathrm{i}}{2}J_{IJ}\gamma^0\gamma^{IJ} - K_I\gamma^0\gamma^I))$.

Understanding the general case where $E = \max(\text{eigenvalues}(\mathrm{i}P_I\gamma^I - \frac{\mathrm{i}}{2}J_{IJ}\gamma^0\gamma^{IJ} - K_I\gamma^0\gamma^I))$ may be quite complicated - e.g. see [44] and references therein for the analogous problem in asymptotically flat spacetimes. However, the following special case - analysed for asymptotically flat spacetimes in [35] - is much more straightforward.

Corollary 4.8.2. *The only solution where E, P_I, J_{IJ} and K_I all vanish is AdS.*

Proof. Just as in the more general corollary 3.19.2, E, P_I, J_{IJ} and K_I all vanishing implies $\nabla_I\varepsilon = 0$. Furthermore, With the present assumptions, $\nabla_I\varepsilon$ is not just zero for some ε, but for the ε that results from any choice of ε_0 in ε_k.

First, $\nabla_I \varepsilon = 0$ implies the 'integrability condition,'

$$0 = [\nabla_I, \nabla_J]\varepsilon \tag{421}$$
$$= [D_I + \mathrm{i}\alpha\gamma_I, D_J + \mathrm{i}\alpha\gamma_J]\varepsilon \tag{422}$$
$$= [D_I, D_J]\varepsilon + \mathrm{i}\alpha\gamma_J D_I\varepsilon - \mathrm{i}\alpha\gamma_I D_J\varepsilon + \mathrm{i}\alpha\gamma_I\nabla_J\varepsilon - \mathrm{i}\alpha\gamma_J\nabla_I\varepsilon \tag{423}$$
$$= -\frac{1}{4}R_{\mu\nu IJ}\gamma^{\mu\nu}\varepsilon + \mathrm{i}\alpha\gamma_J(-\mathrm{i}\alpha\gamma_I\varepsilon) - \mathrm{i}\alpha\gamma_I(-\mathrm{i}\alpha\gamma_J\varepsilon) + 0 - 0 \tag{424}$$
$$= -\frac{1}{4}R_{\mu\nu IJ}\gamma^{\mu\nu}\varepsilon - 2\alpha^2\gamma_{IJ}\varepsilon \tag{425}$$
$$= -\frac{1}{4}(R_{\mu\nu IJ} + 2\eta_{\mu I}\eta_{\nu J})\gamma^{\mu\nu}\varepsilon. \tag{426}$$

Since ε_0 can be chosen arbitrarily, equation 426 holds for a basis of spinors near $\partial_\infty\Sigma_t$.
Suppose $\{\varepsilon_a\}_{a=0}^k$ is a set of spinors solving $\nabla_I\varepsilon_a = 0$ and linearly independent near $\partial_\infty\Sigma_t$. Let $\{c_a\}_{a=0}^k$ be constants in $\mathbb{C}$ and let $\psi = c_a\varepsilon_a$.
Suppose, for a contradiction, that the c_a are non-zero, but $\exists$ a point, p, where $\psi = 0$.
ε_a's linear independence near $\partial_\infty\Sigma_t \implies \psi \neq 0$ near $\partial_\infty\Sigma_t$. Furthermore, by construction, $\nabla_I\psi = 0$ everywhere.
Now, I can repeat the same argument I used between equations 91 and 100 - with $x_0 = p$ and x_1 being some point, q, near (but not on) $\partial_\infty\Sigma_t$ - to conclude that $\psi = 0$ at q.
This contradicts $\psi \neq 0$ near $\partial_\infty\Sigma_t$.
Hence, all the c_a must be zero to get $\psi = 0$ somewhere.
$\therefore$ Linear independence near $\partial_\infty\Sigma_t$ extends to linear independence on all of Σ_t.
$\therefore$ At any given point, ε could take an arbitrary value in equation 426.
$\therefore (R_{\mu\nu IJ} + 2\eta_{\mu I}\eta_{\nu J})\gamma^{\mu\nu} = 0$.
Since $\{\gamma^{\mu\nu}\}$ are also linearly independent, it must be that $R_{\mu\nu IJ} = -(\eta_{\mu I}\eta_{\nu J} - \eta_{\mu J}\eta_{\nu I})$.
It remains to be seen what happens for $R_{\mu\nu 0I}$.
$R_{JK0I} = R_{0IJK} = -(\eta_{0J}\eta_{IK} - \eta_{0K}\eta_{IJ}) = 0$.
That leaves $R_{0J0I} = -R_{00IJ} - R_{0IJ0} = R_{0I0J}$.
Since a basis of ε is allowed, theorem 4.8 also implies $T^{0\mu}\gamma_0\gamma_\mu = 0$. But, the eigenvalues of $T^{0\mu}\gamma_0\gamma_\mu = 0$ are $T^{00} \pm \sqrt{T^{0I}T^0{}_I}$, so it must be that T^{00} and T^{0I} are both zero, i.e. $T^{\mu 0} = 0$.
By the dominant energy condition, $-T^\mu{}_\nu V^\nu$ is future directed and causal whenever V^μ is future directed and causal.
Choose $V^\mu = \delta^{\mu 0} + \delta^{\mu I}$ for some value of I.
$\therefore -T^\mu{}_\nu V^\nu = -T^\mu{}_0 - T^\mu{}_I = 0 - \delta^{\mu J}T^{JI}$.
However, this can only be causal if $T^{IJ} = 0$.
$\therefore$ Ultimately, $T_{ab} = 0$.
$\therefore R_{\mu\nu} = \frac{2}{n-2}\Lambda\eta_{\mu\nu} = -(n-1)\eta_{\mu\nu}$.
$\therefore -(n-1)\delta_{IJ} = R^\mu{}_{I\mu J} = -R_{0I0J} + R^K{}_{IKJ} = -R_{0I0J} - (\delta^K{}_K\delta_{IJ} - \delta^K{}_J\delta_{KI}) = -R_{0I0J} - (n-2)\delta_{IJ}$.
$\therefore R_{0I0J} = \delta_{IJ}$.
$\therefore$ Putting all the components together, $R_{\mu\nu\rho\sigma} = -(\eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\sigma}\eta_{\nu\rho})$.
From [45], the only spacetime with $R_{\mu\nu\rho\sigma} = -(\eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\sigma}\eta_{\nu\rho})$ and the chosen $f_{(0)}$ is AdS. □

In general, the eigenvalues of $EI - \mathrm{i}P_I\gamma^I + \frac{\mathrm{i}}{2}J_{IJ}\gamma^0\gamma^{IJ} + K_I\gamma^0\gamma^I$ cannot be found analytically. However, progress can be made in specific cases. For example, if one assumes J_{IJ} and K_I are zero - as is effectively done in [20, 4], then the eigenvalues are $E \pm \sqrt{P_IP^I}$, meaning one must have $E \geq \sqrt{P_IP^I}$. There are further examples in specfic dimensions. For example, if $n = 4$ and $P_I = 0$, then one finds $E \geq \sqrt{\frac{1}{2}J_{IJ}J^{IJ} + K_IK^I + J_{IK}J_J{}^K K^I K^J}$. See [13] for many other permutations.

4.2.1 5D, equal angular momenta Myers-Perry solution example

The examples so far are still very abstract. It's best to calculate the various physical quantities for a concrete metric and illustrate the implications of theorem 3.19. A sufficiently simple, but non-trivial, example is the 5D, equal angular momenta Myers-Perry solution[29] (with cosmological constant). Following [46], this solution can be expressed as

$$\begin{aligned} g = &-S^2\mathrm{d}t\otimes\mathrm{d}t + f^2\mathrm{d}R\otimes\mathrm{d}R + \frac{1}{4}h^2(\mathrm{d}\psi+\cos(\theta)\mathrm{d}\phi-\Omega\mathrm{d}t)\otimes(\mathrm{d}\psi+\cos(\theta)\mathrm{d}\phi-\Omega\mathrm{d}t) \\ &+\frac{1}{4}R^2(\mathrm{d}\theta\otimes\mathrm{d}\theta+\sin^2(\theta)\mathrm{d}\phi\otimes\mathrm{d}\phi), \end{aligned} \tag{427}$$

$$\text{where } \frac{1}{f^2} = 1+R^2-\frac{2MZ}{R^2}+\frac{2Ma^2}{R^4},\quad h^2=R^2\left(1+\frac{2Ma^2}{R^4}\right),\quad \Omega=\frac{4Ma}{R^2h^2},$$

$$Z = 1-a^2,\quad S=\frac{R}{fh}\quad \text{and } M \text{ and } a \text{ are constants.} \tag{428}$$

In these coordinates, t is a "time coordinate" taking values in $\mathbb{R}$ and the remaining coordinates would parameterise $\mathbb{R}^4$ as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} R\cos(\theta/2)\cos((\psi+\phi)/2) \\ R\cos(\theta/2)\sin((\psi+\phi)/2) \\ R\sin(\theta/2)\cos((\psi-\phi)/2) \\ R\sin(\theta/2)\sin((\psi-\phi)/2) \end{bmatrix}. \tag{429}$$

This parameterisation implies $R\in(R_0,\infty)$, where R_0 is the radius of the event horizon, $\theta\in[0,\pi]$ and (ψ,ϕ) takes values in $\mathbb{R}^2$ such that (ψ,ϕ) lies within the square with vertices, $(0,0)$, $(4\pi,0)$, $(2\pi,-2\pi)$ and $(2\pi,2\pi)$. Furthermore, in these coordinates, the AdS metric is

$$\begin{aligned} g_{\mathrm{AdS}} = &-(1+R^2)\mathrm{d}t\otimes\mathrm{d}t + \frac{\mathrm{d}R\otimes\mathrm{d}R}{1+R^2} \\ &+\frac{1}{4}R^2\big((\mathrm{d}\psi+\cos(\theta)\mathrm{d}\phi)\otimes(\mathrm{d}\psi+\cos(\theta)\mathrm{d}\phi)+\mathrm{d}\theta\otimes\mathrm{d}\theta+\sin^2(\theta)\mathrm{d}\phi\otimes\mathrm{d}\phi\big). \end{aligned} \tag{430}$$

For my purposes, it will be more convenient to swap (ψ,ϕ) for (ϕ_1,ϕ_2), where

$$\phi_1=\frac{1}{2}(\psi+\phi) \text{ and } \phi_2=\frac{1}{2}(\psi-\phi). \tag{431}$$

Then, $\phi_1\in[0,2\pi]$, $\phi_2\in[0,2\pi]$,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} R\cos(\theta/2)\cos(\phi_1) \\ R\cos(\theta/2)\sin(\phi_1) \\ R\sin(\theta/2)\cos(\phi_2) \\ R\sin(\theta/2)\sin(\phi_2) \end{bmatrix} \tag{432}$$

$$\text{and } \mathrm{d}\psi+\cos(\theta)\mathrm{d}\phi = (1+\cos(\theta))\mathrm{d}\phi_1+(1-\cos(\theta))\mathrm{d}\phi_2. \tag{433}$$

The first step in exemplifying the results in sections 2 and 3 is writing equation 427 in Fefferman-Graham form for an asymptotically AdS space[30]. Since the $f^2\mathrm{d}R\otimes\mathrm{d}R$ in equation 427 depends

[29] This is a black hole solution, contrary to the assumption I made at the start of section 3.1. However, as mentioned then, the arguments can be adapted - as per [4] - to include (marginally) outer trapped surfaces. Hence, the Myers-Perry metrics are admissible for exemplar purposes.

[30] Note that being able to do so is proof the metric is indeed asymptotically AdS.

only on R and $R \to \infty$ heuristically looks like the asymptotic end, it is natural to try $r \equiv r(R)$ as the Fefferman-Graham coordinate.

$$\therefore \mathrm{d}r \otimes \mathrm{d}r = f^2 \mathrm{d}R \otimes \mathrm{d}R \iff \frac{\mathrm{d}r}{\mathrm{d}R} = \pm f. \tag{434}$$

$$\therefore r = \pm \int \frac{1}{\sqrt{1 + R^2 - \frac{2MZ}{R^2} + \frac{2Ma^2}{R^4}}} \mathrm{d}R. \tag{435}$$

This integral cannot be done explicitly. However, it only needs to be done perturbatively to generate a Fefferman-Graham expansion. For AdS, the square root in the expression above would have just $1 + R^2$, so it makes sense to perturb around that.
Therefore, to leading order in perturbation (it will become apparent this is the extent of the necessary perturbation),

$$r = \pm \int \frac{1}{\sqrt{1+R^2}} \frac{1}{\sqrt{1 - \frac{2MZ}{R^2(1+R^2)} + \frac{2Ma^2}{R^4(1+R^2)}}} \mathrm{d}R \tag{436}$$

$$\to \pm \int \frac{1}{\sqrt{1+R^2}} \left(1 + \frac{MZ}{R^2(1+R^2)} - \frac{Ma^2}{R^4(1+R^2)}\right) \mathrm{d}R \tag{437}$$

$$\to \pm \int \left(\frac{1}{\sqrt{1+R^2}} + \frac{MZ}{R^5}\right) \mathrm{d}R \tag{438}$$

$$= \pm \left(\ln\left(R + \sqrt{1+R^2}\right) - \frac{MZ}{4R^4}\right) + C. \tag{439}$$

To get $r \to \infty$ as $R \to \infty$, I should choose the $+$ in $\pm$.

$$\therefore \mathrm{e}^r \to C(R + \sqrt{1+R^2})\mathrm{e}^{-MZ/4R^4} \tag{440}$$

$$\to C(R + \sqrt{1+R^2}) \left(1 - \frac{MZ}{4R^4}\right). \tag{441}$$

To match the AdS solution asymptotically, where $M = 0$, I should choose $C = \frac{1}{2}$.

$$\therefore \mathrm{e}^r \to \frac{1}{2}(R + \sqrt{1+R^2}) \left(1 - \frac{MZ}{4R^4}\right). \tag{442}$$

To write equation 427 in the form of equation 8, I'll need to calculate R^2 in terms of r (perturbatively). To leading order, the AdS calculation implies $R^2 = \mathrm{e}^{2r}\left(1 - \frac{1}{4}\mathrm{e}^{-2r}\right)^2$. To find the correction to this, I just have to track the leading order term containing an M factor. Hence,

$$\begin{aligned} \mathrm{e}^{2r}\left(1 - \frac{1}{4}\mathrm{e}^{-2r}\right)^2 &\to \frac{1}{4}(R + \sqrt{1+R^2})^2 \left(1 - \frac{MZ}{4R^4}\right)^2 \\ &\quad \times \left(1 - \frac{1}{4}\frac{4}{(R+\sqrt{1+R^2})^2}\left(1 + \frac{MZ}{4R^4}\right)^2\right)^2 \end{aligned} \tag{443}$$

$$\begin{aligned} &\to \frac{1}{4}(R + \sqrt{1+R^2})^2 \left(1 - \frac{MZ}{2R^4}\right) \\ &\quad \times \left(1 - \frac{1}{(R+\sqrt{1+R^2})^2}\left(1 + \frac{MZ}{2R^4}\right)\right)^2. \end{aligned} \tag{444}$$

The leading order term is R^2, from the AdS calculation. The leading term containing M comes in at $\frac{1}{R^2}$ from the first factor, but $\frac{1}{R^6}$ from the second factor. Thus, I just get

$$\mathrm{e}^{2r}\left(1 - \frac{1}{4}\mathrm{e}^{-2r}\right)^2 \to R^2 - \frac{MZ}{2R^2}. \tag{445}$$

Since $R = \mathrm{e}^r$ to leading order, I also immediately get

$$R^2 = \mathrm{e}^{2r}\left(\left(1-\frac{1}{4}\mathrm{e}^{-2r}\right)^2 + \frac{MZ}{2}\mathrm{e}^{-4r}\right) = \mathrm{e}^{2r}\left(\left(1-\frac{1}{4}\mathrm{e}^{-2r}\right)^2 + \frac{M(1-a^2)}{2}\mathrm{e}^{-4r}\right). \tag{446}$$

Next, consider equation 427 perturbatively.

$$\begin{aligned}
S^2 &= \frac{R^2}{f^2h^2} && (447)\\
&= R^2\left(1+R^2-\frac{2MZ}{R^2}+\frac{2Ma^2}{R^4}\right)\frac{1}{R^2(1+2Ma^2/R^4)} && (448)\\
&\to \left(1+R^2-\frac{2MZ}{R^2}\right)\left(1-\frac{2Ma^2}{R^4}\right) && (449)\\
&\to \left(1+R^2-\frac{2M(Z+a^2)}{R^2}\right) && (450)\\
&= \left(1+R^2-\frac{2M}{R^2}\right) && (451)\\
&\to \left(1+\mathrm{e}^{2r}\left(1-\frac{1}{4}\mathrm{e}^{-2r}\right)^2+\frac{M(1-a^2)}{2}\mathrm{e}^{-2r}-2M\mathrm{e}^{-2r}\right) && (452)\\
&\to \mathrm{e}^{2r}\left(\left(1+\frac{1}{4}\mathrm{e}^{-2r}\right)^2-\frac{M(a^2+3)}{2}\mathrm{e}^{-4r}\right). && (453)
\end{aligned}$$

$$\begin{aligned}
h^2 &= R^2\left(1+\frac{2Ma^2}{R^4}\right) && (454)\\
&\to \mathrm{e}^{2r}\left(\left(1-\frac{1}{4}\mathrm{e}^{-2r}\right)^2+\frac{M(1-a^2)}{2}\mathrm{e}^{-4r}\right)\left(1+2Ma^2\mathrm{e}^{-4r}\right) && (455)\\
&\to \mathrm{e}^{2r}\left(\left(1-\frac{1}{4}\mathrm{e}^{-2r}\right)^2+\left(2Ma^2+\frac{M(1-a^2)}{2}\right)\mathrm{e}^{-4r}\right) && (456)\\
&= \mathrm{e}^{2r}\left(\left(1-\frac{1}{4}\mathrm{e}^{-2r}\right)^2+\frac{M(1+3a^2)}{2}\mathrm{e}^{-4r}\right). && (457)
\end{aligned}$$

$$h^2\Omega = \frac{4Ma}{R^2} \to 4Ma\mathrm{e}^{-2r} \text{ to leading order.} \tag{458}$$

Substituting these expressions back into equation 427 yields

$$\begin{aligned}
g = \mathrm{d}r\otimes\mathrm{d}r + \mathrm{e}^{2r}\Bigg(&-\left(\left(1+\frac{1}{4}\mathrm{e}^{-2r}\right)^2-\frac{M(a^2+3)}{2}\mathrm{e}^{-4r}\right)\mathrm{d}t\otimes\mathrm{d}t\\
&+\frac{1}{4}\left(\left(1-\frac{1}{4}\mathrm{e}^{-2r}\right)^2+\frac{M(1+3a^2)}{2}\mathrm{e}^{-4r}\right)(\mathrm{d}\psi+\cos(\theta)\mathrm{d}\phi)\otimes(\mathrm{d}\psi+\cos(\theta)\mathrm{d}\phi)\\
&-Ma\mathrm{e}^{-4r}(\mathrm{d}t\otimes(\mathrm{d}\psi+\cos(\theta)\mathrm{d}\phi)+(\mathrm{d}\psi+\cos(\theta)\mathrm{d}\phi)\otimes\mathrm{d}t)\\
&+\frac{1}{4}\left(\left(1-\frac{1}{4}\mathrm{e}^{-2r}\right)^2+\frac{M(1-a^2)}{2}\mathrm{e}^{-4r}\right)(\mathrm{d}\theta\otimes\mathrm{d}\theta+\sin^2(\theta)\mathrm{d}\phi\otimes\phi)+O(\mathrm{e}^{-6r})\Bigg).
\end{aligned} \tag{459}$$

$\therefore$ The metric is indeed in the form of equation 8 and one can immediately read off

$$f_{(0)mn}\mathrm{d}x^m \otimes \mathrm{d}x^n = -\mathrm{d}t \otimes \mathrm{d}t + \frac{1}{4}(\mathrm{d}\psi + \cos(\theta)\mathrm{d}\phi) \otimes (\mathrm{d}\psi + \cos(\theta)\mathrm{d}\phi) + \frac{1}{4}(\mathrm{d}\theta \otimes \mathrm{d}\theta + \sin^2(\theta)\mathrm{d}\phi \otimes \phi) \quad \text{and} \tag{460}$$

$$\begin{aligned} f_{(4)mn}\mathrm{d}x^m \otimes \mathrm{d}x^n = {} & \frac{M(a^3+3)}{2}\mathrm{d}t \otimes \mathrm{d}t + \frac{M(1+3a^2)}{8}(\mathrm{d}\psi + \cos(\theta)\mathrm{d}\phi) \otimes (\mathrm{d}\psi + \cos(\theta)\mathrm{d}\phi) \\ & - Ma(\mathrm{d}t \otimes (\mathrm{d}\psi + \cos(\theta)\mathrm{d}\phi) + (\mathrm{d}\psi + \cos(\theta)\mathrm{d}\phi) \otimes \mathrm{d}t) \\ & + \frac{M(1-a^2)}{8}(\mathrm{d}\theta \otimes \mathrm{d}\theta + \sin^2(\theta)\mathrm{d}\phi \otimes \mathrm{d}\phi). \end{aligned} \tag{461}$$

The $f_{(0)}$ expression also implies $n^m_{(0)} \equiv \delta^{m0}$ and

$$f^{mn}_{(0)}\partial_m \otimes \partial_n = -\partial_t \otimes \partial_t + 4\partial_\psi \otimes \partial_\psi + 4\partial_\theta \otimes \partial_\theta + \frac{4}{\sin^2(\theta)}(-\cos(\theta)\partial_\psi + \partial_\phi) \otimes (-\cos(\theta)\partial_\psi + \partial_\phi), \tag{462}$$

the latter because

$$\begin{aligned} & \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & \cos(\theta)/4 \\ 0 & 0 & 1/4 & 0 \\ 0 & \cos(\theta)/4 & 0 & 1/4 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 4+4\cot^2(\theta) & 0 & -4\cos(\theta)/\sin^2(\theta) \\ 0 & 0 & 4 & 0 \\ 0 & -4\cos(\theta)/\sin^2(\theta) & 0 & 4/\sin^2(\theta) \end{bmatrix} \\ & = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1+\cot^2(\theta)-\cot^2(\theta) & 0 & -\cos(\theta)/\sin^2(\theta)+\cos(\theta)/\sin^2(\theta) \\ 0 & 0 & 1 & 0 \\ 0 & \cos(\theta)(1+\cos^2(\theta)/\sin^2(\theta)-1/\sin^2(\theta)) & 0 & -\cot^2(\theta)+1/\sin^2(\theta) \end{bmatrix} \end{aligned} \tag{463}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{464}$$

I can now finally calculate the physical quantites that appear in theorem 4.8. By equation 81,

$$E = \frac{4}{16\pi}\int_{S^3} \tilde{f}^{mn}_{(0)} f_{(4)mn}\mathrm{d}(g_{S^3}) \tag{465}$$

$$= \frac{1}{4\pi}\int_{S^3}\bigg(- f_{(4)00} + 4(1+\cot^2(\theta))f_{(4)22} - \frac{8\cos(\theta)}{\sin^2(\theta)}f_{(4)24} + 4f_{(4)33} + \frac{4}{\sin^2(\theta)}f_{(4)44} + f_{(4)00}\bigg)\mathrm{d}(g_{S^3}) \tag{466}$$

$$= \frac{1}{\pi}\int_{S^3}\bigg((1+\cot^2(\theta))\frac{M(1+3a^2)}{8} - \frac{2\cos(\theta)}{\sin^2(\theta)}\frac{M(1+3a^2)}{8}\cos(\theta) + \frac{M(1-a^2)}{8} + \frac{1}{\sin^2(\theta)}\bigg(\frac{M(1+3a^2)}{8}\cos^2(\theta) + \frac{M(1-a^2)}{8}\sin^2(\theta)\bigg)\bigg)\mathrm{d}(g_{S^3}) \tag{467}$$

$$= \frac{M(a^2+3)}{8\pi}\int_{S^3}\mathrm{d}(g_{S^3}) \tag{468}$$

$$= \frac{\pi M(a^2+3)}{4}, \tag{469}$$

which matches the result quoted in [46], but calculated via a different method[31].
Next, by definition 4.7,

$$P_I = \frac{4}{16\pi}\int_{S^3} \tilde{f}^{mn}_{(0)} f_{(4)mn}\hat{x}_I \mathrm{d}(g_{S^3}) \tag{470}$$

$$= \frac{M(a^2+3)}{8\pi}\int_{S^3} \hat{x}_I \mathrm{d}(g_{S^3}) \text{ by the same algebra as for } E \tag{471}$$

$$= 0, \tag{472}$$

which matches what one would intuitively expect for the Myers-Perry metrics.
When calculating K_I and J_{IJ}, the $\frac{\partial\theta^\alpha}{\partial x^I}$ terms in definition 4.7 are more easily calculated when using the (θ, ϕ_1, ϕ_2) coordinates on S^3, as opposed to the (ψ, θ, ϕ) coordinates used to calculate E and P_I.
For both K_I and J_{IJ}, I need to first calculate $f_{(4)0\alpha}\frac{\partial\theta^\alpha}{\partial x^I}|_{\rho=1}$.
From equation 461,

$$f_{(4)0\alpha}\mathrm{d}x^\alpha = -Ma(\mathrm{d}\psi + \cos(\theta)\mathrm{d}\phi) \tag{473}$$

$$= -Ma((1+\cos(\theta))\mathrm{d}\phi_1 + (1-\cos(\theta))\mathrm{d}\phi_2). \tag{474}$$

From equation 432, $\phi_1 = \tan^{-1}(x_2/x_1)$ and $\phi_2 = \tan^{-1}(x_4/x_3)$. Hence, on the unit 3-sphere,

$$\frac{\partial\phi_1}{\partial x_1} = \frac{1}{1+x_2^2/x_1^2}\left(-\frac{x_2}{x_1^2}\right) = -\frac{x_2}{x_1^2+x_2^2} = -\frac{\sin(\phi_1)}{\cos(\theta/2)}, \tag{475}$$

$$\frac{\partial\phi_1}{\partial x_2} = \frac{1}{1+x_2^2/x_1^2}\left(\frac{1}{x_1}\right) = \frac{x_1}{x_1^2+x_2^2} = \frac{\cos(\phi_1)}{\cos(\theta/2)} \tag{476}$$

$$\text{and } \frac{\partial\phi_1}{\partial x_3} = \frac{\partial\phi_1}{\partial x_4} = 0. \tag{477}$$

$$\text{Similarly, } \frac{\partial\phi_2}{\partial x_1} = \frac{\partial\phi_2}{\partial x_2} = 0,\ \frac{\partial\phi_2}{\partial x_3} = -\frac{\sin(\phi_2)}{\sin(\theta/2)} \text{ and } \frac{\partial\phi_2}{\partial x_4} = \frac{\cos(\phi_2)}{\sin(\theta/2)}. \tag{478}$$

Putting these expressions together with the $f_{(4)0\alpha}\mathrm{d}x^\alpha$ expression above,

$$f_{(4)0\alpha}\frac{\partial\theta^\alpha}{\partial x^I}$$

$$= -Ma\left((1+\cos(\theta))\frac{\partial\phi_1}{\partial x^I} + (1-\cos(\theta))\frac{\partial\phi_2}{\partial x^I}\right) \tag{479}$$

$$\equiv Ma\left[\tfrac{1+\cos(\theta)}{\cos(\theta/2)}\sin(\phi_1) \quad -\tfrac{1+\cos(\theta)}{\cos(\theta/2)}\cos(\phi_1) \quad \tfrac{1-\cos(\theta)}{\sin(\theta/2)}\sin(\phi_2) \quad -\tfrac{1-\cos(\theta)}{\sin(\theta/2)}\cos(\phi_2)\right]. \tag{480}$$

Since $\int_0^{2\pi}\sin(\phi_{1,2})\mathrm{d}\phi_{1,2} = 0$, I immediately get

$$\int_{S^3} f_{(4)0\alpha}\frac{\partial\theta^\alpha}{\partial x^I}\bigg|_{\rho=1}\mathrm{d}(g_{S^3}) = 0. \tag{481}$$

$$\therefore K_I = -\frac{1}{4\pi}\int_{S^3}\hat{x}^J\hat{x}_I f_{(4)0\alpha}\frac{\partial\theta^\alpha}{\partial x^J}\bigg|_{\rho=1}\mathrm{d}(g_{S^3}). \tag{482}$$

[31] Note that while the choice of the letter, M, for the constant, M, suggests it should be the mass/energy, this is not the case.

However, observe that the integrand contains

$$\hat{x}^J f_{(4)0\alpha} \frac{\partial \theta^\alpha}{\partial x^J}\bigg|_{\rho=1} \tag{483}$$

$$\equiv Ma \begin{bmatrix} \cos(\theta/2)\cos(\phi_1) & \cos(\theta/2)\sin(\phi_1) & \sin(\theta/2)\cos(\phi_2) & \sin(\theta/2)\sin(\phi_2) \end{bmatrix}$$

$$\times \begin{bmatrix} \frac{1+\cos(\theta)}{\cos(\theta/2)}\sin(\phi_1) \\ -\frac{1+\cos(\theta)}{\cos(\theta/2)}\cos(\phi_1) \\ \frac{1-\cos(\theta)}{\sin(\theta/2)}\sin(\phi_2) \\ -\frac{1-\cos(\theta)}{\sin(\theta/2)}\cos(\phi_2) \end{bmatrix} \tag{484}$$

$$= 0. \tag{485}$$

Hence $K_I = 0$ too, again matching what one would intuitively expect.
Finally, there's J_{IJ}. For that, I need

$$\hat{x}_I f_{(4)0\alpha} \frac{\partial \theta^\alpha}{\partial x^J}\bigg|_{\rho=1} \tag{486}$$

$$\equiv Ma \begin{bmatrix} \cos(\theta/2)\cos(\phi_1) \\ \cos(\theta/2)\sin(\phi_1) \\ \sin(\theta/2)\cos(\phi_2) \\ \sin(\theta/2)\sin(\phi_2) \end{bmatrix}$$

$$\times \begin{bmatrix} \frac{1+\cos(\theta)}{\cos(\theta/2)}\sin(\phi_1) & -\frac{1+\cos(\theta)}{\cos(\theta/2)}\cos(\phi_1) & \frac{1-\cos(\theta)}{\sin(\theta/2)}\sin(\phi_2) & -\frac{1-\cos(\theta)}{\sin(\theta/2)}\cos(\phi_2) \end{bmatrix} \tag{487}$$

$$= Ma \left[\begin{matrix} (1+\cos(\theta))\sin(\phi_1)\cos(\phi_1) & -(1+\cos(\theta))\cos^2(\phi_1) \\ (1+\cos(\theta))\sin^2(\phi_1) & -(1+\cos(\theta))\sin(\phi_1)\cos(\phi_1) \\ (1+\cos(\theta))\tan(\theta/2)\sin(\phi_1)\cos(\phi_2) & -(1+\cos(\theta))\tan(\theta/2)\cos(\phi_1)\cos(\phi_2) \\ (1+\cos(\theta))\tan(\theta/2)\sin(\phi_1)\sin(\phi_2) & -(1+\cos(\theta))\tan(\theta/2)\cos(\phi_1)\sin(\phi_2) \end{matrix}\right.$$

$$\left.\begin{matrix} (1-\cos(\theta))\cot(\theta/2)\cos(\phi_1)\sin(\phi_2) & -(1-\cos(\theta))\cot(\theta/2)\cos(\phi_1)\cos(\phi_2) \\ (1-\cos(\theta))\cot(\theta/2)\sin(\phi_1)\sin(\phi_2) & -(1-\cos(\theta))\cot(\theta/2)\sin(\phi_1)\cos(\phi_2) \\ (1-\cos(\theta))\cos(\phi_2)\sin(\phi_2) & -(1-\cos(\theta))\cos^2(\phi_2) \\ (1-\cos(\theta))\sin^2(\phi_2) & -(1-\cos(\theta))\sin(\phi_2)\cos(\phi_2) \end{matrix}\right]. \tag{488}$$

This appears as an integrand inside $\int_{S^3} \mathrm{d}(g_{S^3})$. In particular, the 2π range of ϕ_1 and ϕ_2 means those integrals can be done inspection, leaving

$$\int_0^{2\pi}\int_0^{2\pi} \hat{x}_I f_{(4)0\alpha} \frac{\partial \theta^\alpha}{\partial x^J}\bigg|_{\rho=1} \mathrm{d}\phi_1 \mathrm{d}\phi_2$$

$$\equiv 4\pi^2 Ma \begin{bmatrix} 0 & -(1+\cos(\theta))/2 & 0 & 0 \\ (1+\cos(\theta))/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -(1-\cos(\theta))/2 \\ 0 & 0 & (1-\cos(\theta))/2 & 0 \end{bmatrix}. \tag{489}$$

Finally, by definition 4.7,

$$J_{IJ} = \frac{1}{4\pi}\int_{S^3} f_{(4)0\alpha}\left(\hat{x}_I \frac{\partial \theta^\alpha}{\partial x^J}\bigg|_{\rho=1} - \hat{x}_J \frac{\partial \theta^\alpha}{\partial x^I}\bigg|_{\rho=1}\right) \mathrm{d}(g_{S^3}) \tag{490}$$

$$\equiv \pi Ma \int_0^\pi \begin{bmatrix} 0 & -(1+\cos(\theta)) & 0 & 0 \\ 1+\cos(\theta) & 0 & 0 & 0 \\ 0 & 0 & 0 & -(1-\cos(\theta)) \\ 0 & 0 & 1-\cos(\theta) & 0 \end{bmatrix} \frac{\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{2} \mathrm{d}\theta \tag{491}$$

Then, since $\int_0^\pi \frac{1}{2}\sin(\theta/2)\cos(\theta/2)\mathrm{d}\theta = \frac{1}{4}\int_0^\pi \sin(\theta)\mathrm{d}\theta = \frac{1}{2}$ and $\int_0^\pi \cos(\theta)\frac{1}{2}\sin(\theta/2)\cos(\theta/2)\mathrm{d}\theta = \frac{1}{8}\int_0^\pi \sin(2\theta)\mathrm{d}\theta = 0$, I get

$$J_{IJ} \equiv \frac{\pi Ma}{2}\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{492}$$

This result justifies interpreting the original metric - equation 427 - as containing two equal, independent angular momenta, $\pi Ma/2$. If one measures angular momenta with respect to $\frac{\partial}{\partial\psi}$ and $\frac{\partial}{\partial\phi}$ instead, then since $\frac{\partial}{\partial\psi} = \frac{1}{2}(\frac{\partial}{\partial\phi_1} + \frac{\partial}{\partial\phi_2})$ and $\frac{\partial}{\partial\phi} = \frac{1}{2}(\frac{\partial}{\partial\phi_1} - \frac{\partial}{\partial\phi_2})$, the angular momenta would be $\pi Ma/2$ and 0 respectively - matching the result in [46] up to a factor of two (which is presumably only a matter of conventions).
At last, I can consider theorem 4.8, which reduces to saying $EI + \mathrm{i}J_{12}\gamma^0\gamma^1\gamma^2 + \mathrm{i}J_{34}\gamma^0\gamma^3\gamma^4$ is non-negative definite.
Using computer algebra for example, one can check the eigenvalues of this matrix are
$E + J_{12} + J_{34}$, $E - J_{12} + J_{34}$, $E + J_{12} - J_{34}$ and $E - J_{12} - J_{34}$.
$\therefore$ Non-negative definiteness is equivalent to

$$E \geq |J_{12}| + |J_{34}|, \tag{493}$$

which one can recognise as a BPS bound of 5D gauged supergravity. In terms of the actual values I've calculated for E and J_{IJ}, equation 493 says

$$\frac{\pi M(a^2+3)}{4} \geq \pi Ma \iff (a-1)(a-3) \geq 0. \tag{494}$$

Therefore, supersymmetric limits are reached by taking $a \to 1^-$ or $a \to 3^+$. Unlike the charged, asymptotically flat case [18, 19], here the BPS bound does not coincide with the condition to have a regular event horizon[32]. Instead, the BPS bound can lead to singular horizons now. While perhaps strange, this is behaviour known to occur for supersymmetric limits of rotating black holes with $\Lambda < 0$ [47, 48].

4.3 General cross-sections

The Kottler metrics are

$$g = -(c+R^2)\mathrm{d}t \otimes \mathrm{d}t + \frac{\mathrm{d}R \otimes \mathrm{d}R}{c+R^2} + R^2 g^{(c)}, \tag{495}$$

where $c = 1, 0, -1$, $g^{(1)}$ is the metric on the unit $(n-2)-$sphere, $g^{(0)}$ is the metric on a unit $(n-2)-$torus and $g^{(-1)}$ is the metric on a compact identification of $(n-2)-$dimensional hyperbolic space. In the last two sections I have studied the round sphere and the torus. However, these metrics continue to satisfy the Einstein equation, $R_{ab} = -\frac{1}{2}(n-1)(n-2)g_{ab}$, as long as $g^{(c)}$ has Ricci tensor equal to $c(n-3)\delta_{AB}$. Re-writing the metric in Fefferman-Graham coordinates gives the following definition.

Definition 4.9 (Kottler with cross-section, H)**.** *A metric is defined to be Kottler with cross-section, H, if and only if*

$$g = \mathrm{d}r \otimes \mathrm{d}r + \mathrm{e}^{2r}\left(-\left(1+\frac{c}{4}\mathrm{e}^{-2r}\right)^2 \mathrm{d}t \otimes \mathrm{d}t + \left(1-\frac{c}{4}\mathrm{e}^{-2r}\right)^2 H\right) \tag{496}$$

and H is a Riemannian, Einstein metric on a compact manifold such that $R^{(H)}_{AB} = c(n-3)\delta_{AB}$ for $c = -1$, 0 or 1.

[32] This is especially manifest given $(a-1)(a-3)$ doesn't even depend on M.

The main objective of this section is to prove that if H is "symmetric" in some sense, then there is a positive energy theorem for spacetimes that are asymptotically Kottler with cross-section, H, i.e. asymptotically locally AdS spacetimes with $f_{(0)} = -\mathrm{d}t \otimes \mathrm{d}t + H$.

Lemma 4.10. *The non-zero connection 1-forms (up to antisymmetries) for equation 496 are*

$$\omega_{01} = -\frac{\mathrm{e}^r - \frac{c}{4}\mathrm{e}^{-r}}{\mathrm{e}^r + \frac{c}{4}\mathrm{e}^{-r}} e^0, \ \omega_{A1} = \frac{\mathrm{e}^r + \frac{c}{4}\mathrm{e}^{-r}}{\mathrm{e}^r - \frac{c}{4}\mathrm{e}^{-r}} e^A \text{ and } \omega_{AB} = \omega^{(H)}_{AB}, \tag{497}$$

$$\text{where } e^0 = \left(\mathrm{e}^r + \frac{c}{4}\mathrm{e}^{-r}\right)\mathrm{d}t, \ e^1 = \mathrm{d}r \text{ and } e^A = \left(\mathrm{e}^r - \frac{c}{4}\mathrm{e}^{-r}\right) e^{(H)A}. \tag{498}$$

Proof. For this choice of vielbein,

$$\mathrm{d}e^0 = \left(\mathrm{e}^r - \frac{c}{4}\mathrm{e}^{-r}\right)\mathrm{d}r \wedge \mathrm{d}t = \frac{\mathrm{e}^r - \frac{c}{4}\mathrm{e}^{-r}}{\mathrm{e}^r + \frac{c}{4}\mathrm{e}^{-r}} e^1 \wedge e^0, \tag{499}$$

$$\mathrm{d}e^1 = 0 \text{ and} \tag{500}$$

$$\mathrm{d}e^A = \left(\mathrm{e}^r + \frac{c}{4}\mathrm{e}^{-r}\right)\mathrm{d}r \wedge e^{(H)A} + \left(\mathrm{e}^r - \frac{c}{4}\mathrm{e}^{-r}\right)\mathrm{d}e^{(H)A} \tag{501}$$

$$= \frac{\mathrm{e}^r + \frac{c}{4}\mathrm{e}^{-r}}{\mathrm{e}^r - \frac{c}{4}\mathrm{e}^{-r}} e^1 \wedge e^A + \left(\mathrm{e}^r - \frac{c}{4}\mathrm{e}^{-r}\right)\mathrm{d}e^{(H)A}. \tag{502}$$

$\omega^{(H)}_{AB}$ satisfy $\mathrm{d}e^{(H)A} = -\omega^{(H)A}{}_B \wedge e^{(H)B}$ by definition.
Then, by inspection, the $\omega_{\mu\nu}$ claimed in the lemma likewise satisfy $\mathrm{d}e^\mu = -\omega^\mu{}_\nu \wedge e^\nu$.
Since connection coefficients are unique, the claimed coefficients must be correct. □

Lemma 4.11. *If $\nabla_\mu \varepsilon_k = 0$ for a Kottler metric with cross-section, H, then*

$$\varepsilon_k = \mathrm{e}^{r/2} P_1^- \varepsilon_- + \mathrm{e}^{-r/2} P_1^+ \varepsilon_+, \tag{503}$$

where $\varepsilon_\pm$ must solve $D^{(H)}_A \varepsilon_- = -\mathrm{i}\gamma_A \varepsilon_+$, $D^{(H)}_A \varepsilon_+ = \frac{\mathrm{i}c}{4}\gamma_A \varepsilon_-$, $\partial_t \varepsilon_- = \mathrm{i}\gamma^0 \varepsilon_+$, $\partial_t \varepsilon_+ = \frac{\mathrm{i}c}{4}\gamma^0 \varepsilon_-$ and $\partial_r \varepsilon_- = \partial_r \varepsilon_+ = 0$.

Proof. The equation to solve is

$$0 = \nabla_\mu \varepsilon_k = e_\mu{}^{\mu'} \partial_{\mu'} \varepsilon_k - \frac{1}{4}\omega_{\nu\rho\mu}\gamma^{\nu\rho}\varepsilon_k + \frac{\mathrm{i}}{2}\gamma_\mu \varepsilon_k. \tag{504}$$

Start with $\mu = 1$. Then, from lemma 4.10, equation 504 reduces to $0 = \partial_r \varepsilon_k + \frac{\mathrm{i}}{2}\gamma_1 \varepsilon_k$.
$\therefore \varepsilon_k = \mathrm{e}^{-\mathrm{i}\gamma^1 r/2}\varepsilon_r$ for some spinor, ε_r, that doesn't depend on r.
Split ε_r up into eigenspaces of γ^1, i.e. $\varepsilon_r = P_1^- \varepsilon_- + P_1^+ \varepsilon_+$.
$\therefore \varepsilon_k = \mathrm{e}^{-\mathrm{i}\gamma^1 r/2}(P_1^- \varepsilon_- + P_1^+ \varepsilon_+) = \mathrm{e}^{r/2} P_1^- \varepsilon_- + \mathrm{e}^{-r/2} P_1^+ \varepsilon_+$.
Next consider $\mu = 0$. Then, from lemma 4.10

$$\therefore 0 = \frac{1}{\mathrm{e}^r + \frac{c}{4}\mathrm{e}^{-r}}\partial_t \varepsilon_k + \frac{\mathrm{e}^r - \frac{c}{4}\mathrm{e}^{-r}}{2(\mathrm{e}^r + \frac{c}{4}\mathrm{e}^{-r})}\gamma^0\gamma^1 - \frac{\mathrm{i}}{2}\gamma^0 \varepsilon_k. \tag{505}$$

$$\begin{aligned}\therefore 0 = {} & \partial_t(\mathrm{e}^{r/2} P_1^- \varepsilon_- + \mathrm{e}^{-r/2} P_1^+ \varepsilon_+) + \frac{1}{2}\left(\mathrm{e}^r - \frac{c}{4}\mathrm{e}^{-r}\right)\gamma^0\gamma^1(\mathrm{e}^{r/2} P_1^- \varepsilon_- + \mathrm{e}^{-r/2} P_1^+ \varepsilon_+) \\ & - \frac{\mathrm{i}}{2}\left(\mathrm{e}^r + \frac{c}{4}\mathrm{e}^{-r}\right)\gamma^0(\mathrm{e}^{r/2} P_1^- \varepsilon_- + \mathrm{e}^{-r/2} P_1^+ \varepsilon_+)\end{aligned} \tag{506}$$

$$\begin{aligned} = {} & \mathrm{e}^{r/2} P_1^- \partial_t \varepsilon_- + \mathrm{e}^{-r/2} P_1^+ \partial_t \varepsilon_+ + \frac{\mathrm{i}}{2}\mathrm{e}^{3r/2}\gamma^0 P_1^- \varepsilon_- - \frac{\mathrm{i}c}{8}\mathrm{e}^{-r/2}\gamma^0 P_1^- \varepsilon_- - \frac{\mathrm{i}}{2}\mathrm{e}^{r/2}\gamma^0 P_1^+ \varepsilon_+ \\ & + \frac{\mathrm{i}c}{8}\mathrm{e}^{-3r/2}\gamma^0 P_1^+ \varepsilon_+ - \frac{\mathrm{i}}{2}\mathrm{e}^{3r/2}\gamma^0 P_1^- \varepsilon_- - \frac{\mathrm{i}c}{8}\mathrm{e}^{-r/2}\gamma^0 P_1^- \varepsilon_- - \frac{\mathrm{i}}{2}\mathrm{e}^{r/2}\gamma^0 P_1^+ \varepsilon_+ \\ & - \frac{\mathrm{i}c}{8}\mathrm{e}^{-3r/2}\gamma^0 P_1^+ \varepsilon_+\end{aligned} \tag{507}$$

$$= \mathrm{e}^{r/2} P_1^- \left(\partial_t \varepsilon_- - \mathrm{i}\gamma^0 \varepsilon_+\right) + \mathrm{e}^{-r/2} P_1^+ \left(\partial_t \varepsilon_+ - \frac{\mathrm{i}c}{4}\gamma^0 \varepsilon_-\right). \tag{508}$$

Since the two γ^1 eigenspaces have no non-trivial intersection, it follows that $\partial_t\varepsilon_- = \mathrm{i}\gamma^0\varepsilon_+$ and $\partial_t\varepsilon_+ = \frac{\mathrm{i}c}{4}\gamma^0\varepsilon_-$.
Finally, consider $\mu = A$. Before applying lemma 4.10 to equation 504, note that

$$\omega_{AB} = \omega^{(H)}_{AB} \implies \omega_{ABC}e^C = \omega^{(H)}_{ABC}e^{(H)C} \tag{509}$$

$$= \frac{1}{\mathrm{e}^r - \frac{c}{4}\mathrm{e}^{-r}}\omega^{(H)}_{ABC}e^C. \tag{510}$$

$$\therefore \omega_{ABC} = \frac{1}{\mathrm{e}^r - \frac{c}{4}\mathrm{e}^{-r}}\omega^{(H)}_{ABC}. \tag{511}$$

Hence, the Killing spinor equation says

$$0 = \frac{1}{\mathrm{e}^r - \frac{c}{4}\mathrm{e}^{-r}}e^{(H)\alpha}_A\partial_\alpha\varepsilon_k - \frac{1}{4(\mathrm{e}^r - \frac{c}{4}\mathrm{e}^{-r})}\omega^{(H)}_{BCA}\gamma^{BC}\varepsilon_k - \frac{\mathrm{e}^r + \frac{c}{4}\mathrm{e}^{-r}}{2(\mathrm{e}^r - \frac{c}{4}\mathrm{e}^{-r})}\gamma^A\gamma^1\varepsilon_k + \frac{\mathrm{i}}{2}\gamma_A\varepsilon_k. \tag{512}$$

$$\therefore 0 = D^{(H)}_A\varepsilon_k - \frac{1}{2}\left(\mathrm{e}^r + \frac{c}{4}\mathrm{e}^{-r}\right)\gamma^A\gamma^1\varepsilon_k + \frac{\mathrm{i}}{2}\left(\mathrm{e}^r - \frac{c}{4}\mathrm{e}^{-r}\right)\gamma_A\varepsilon_k \tag{513}$$

$$= D^{(H)}_A(\mathrm{e}^{r/2}P^-_1\varepsilon_- + \mathrm{e}^{-r/2}P^+_1\varepsilon_+) - \frac{1}{2}\left(\mathrm{e}^r + \frac{c}{4}\mathrm{e}^{-r}\right)\gamma^A\gamma^1(\mathrm{e}^{r/2}P^-_1\varepsilon_- + \mathrm{e}^{-r/2}P^+_1\varepsilon_+)$$
$$+ \frac{\mathrm{i}}{2}\left(\mathrm{e}^r - \frac{c}{4}\mathrm{e}^{-r}\right)\gamma_A(\mathrm{e}^{r/2}P^-_1\varepsilon_- + \mathrm{e}^{-r/2}P^+_1\varepsilon_+) \tag{514}$$

$$= \mathrm{e}^{r/2}P^-_1D^{(H)}_A\varepsilon_- + \mathrm{e}^{-r/2}P^+_1D^{(H)}_A\varepsilon_+ - \frac{\mathrm{i}}{2}\mathrm{e}^{3r/2}\gamma_AP^-_1\varepsilon_- - \frac{\mathrm{i}c}{8}\mathrm{e}^{-r/2}\gamma_AP^-_1\varepsilon_- + \frac{\mathrm{i}}{2}\mathrm{e}^{r/2}\gamma_AP^+_1\varepsilon_+$$
$$+ \frac{\mathrm{i}c}{8}\mathrm{e}^{-3r/2}\gamma_AP^+_1\varepsilon_+ + \frac{\mathrm{i}}{2}\mathrm{e}^{3r/2}\gamma_AP^-_1\varepsilon_- - \frac{\mathrm{i}c}{8}\mathrm{e}^{-r/2}\gamma_AP^-_1\varepsilon_- + \frac{\mathrm{i}}{2}\mathrm{e}^{r/2}\gamma_AP^+_1\varepsilon_+$$
$$- \frac{\mathrm{i}c}{8}\mathrm{e}^{-3r/2}\gamma_AP^+_1\varepsilon_+ \tag{515}$$

$$= \mathrm{e}^{r/2}P^-_1\left(D^{(H)}_A\varepsilon_- + \mathrm{i}\gamma_A\varepsilon_+\right) + \mathrm{e}^{-r/2}P^+_1\left(D^{(H)}_A\varepsilon_+ - \frac{\mathrm{i}c}{4}\gamma_A\varepsilon_-\right). \tag{516}$$

$\therefore D^{(H)}_A\varepsilon_- = -\mathrm{i}\gamma_A\varepsilon_+$ and $D^{(H)}_A\varepsilon_+ = \frac{\mathrm{i}c}{4}\gamma_A\varepsilon_-$. □

Theorem 4.12. *The most general solution to $\nabla_\mu\varepsilon_k = 0$ for a Kottler metric with cross-section, H, is*

$$\varepsilon_k = \begin{cases} \mathrm{e}^{r/2}P^-_1\varepsilon_H & \text{for } c = 0 \\ \mathrm{e}^{r/2}P^-_1\left(\mathrm{e}^{\mathrm{i}\gamma^0t/2} - \mathrm{i}\mathrm{e}^{-\mathrm{i}\gamma^0t/2}\right)\varepsilon_H + \frac{1}{2}\mathrm{e}^{-r/2}P^+_1\left(\mathrm{e}^{\mathrm{i}\gamma^0t/2} + \mathrm{i}\mathrm{e}^{-\mathrm{i}\gamma^0t/2}\right)\varepsilon_H & \text{for } c = 1 \\ 0 & \text{for } c = -1 \end{cases} \tag{517}$$

where ε_H solves $D^{(H)}_A\varepsilon_H = \frac{c}{2}\gamma_A\varepsilon_H$ and $\partial_t\varepsilon_H = 0$.

Proof. Start with $c = 0$. From lemma 4.11, I need $\partial_t\varepsilon_- = \mathrm{i}\gamma^0\varepsilon_+$, $\partial_t\varepsilon_+ = 0$, $D^{(H)}_A\varepsilon_- = -\mathrm{i}\gamma_A\varepsilon_+$ and $D^{(H)}_A\varepsilon_+ = 0$.
From the first two equations, it follows that $\varepsilon_- = \mathrm{i}t\gamma^0\varepsilon_+ + \varepsilon_H$ for some spinor, ε_H, that (like ε_+) doesn't depend on t.
$\therefore -\mathrm{i}\gamma_A\varepsilon_+ = D^{(H)}_A\varepsilon_- = \mathrm{i}t\gamma^0D^{(H)}_A\varepsilon_+ + D^{(H)}_A\varepsilon_H = 0 + D^{(H)}_A\varepsilon_H$.
$\therefore D^{(H)A}D^{(H)}_A\varepsilon_H = -\mathrm{i}\gamma^AD^{(H)}_A\varepsilon_+ = 0$.
Let $\Sigma_{t,r}$ be constant t and r surface. Then, by $\Sigma_{t,r}$'s assumed compactness,

$$0 = \int_{\Sigma_{t,r}}\varepsilon^\dagger_HD^{(H)A}D^{(H)}_A(\varepsilon_H)\mathrm{d}(H) \tag{518}$$

$$= -\int_{\Sigma_{t,r}}\left(D^{(H)A}\varepsilon_H\right)^\dagger D^{(H)}_A(\varepsilon_H)\mathrm{d}(H) \implies D^{(H)}_A\varepsilon_H = 0. \tag{519}$$

Since $-\mathrm{i}\gamma_A\varepsilon_+ = D_A^{(H)}\varepsilon_H$ from above, it follows that $\varepsilon_+ = 0$.
That leaves $\varepsilon_- = \varepsilon_H$ with ε_H solving $D_A^{(H)}\varepsilon_H = 0$.
It follows by inspection that all six conditions in lemma 4.11 are now satisfied - no further constraints are necessary.
Next, consider $c = 1$.
$\therefore$ I have to solve $\partial_t\varepsilon_- = \mathrm{i}\gamma^0\varepsilon_+$, $\partial_t\varepsilon_+ = \frac{\mathrm{i}}{4}\gamma^0\varepsilon_-$, $D_A^{(H)}\varepsilon_- = -\mathrm{i}\gamma_A\varepsilon_+$ and $D_A^{(H)}\varepsilon_+ = \frac{\mathrm{i}}{4}\gamma_A\varepsilon_-$.
Let

$$\psi = \varepsilon_- + 2\varepsilon_+ \text{ and } \varphi = \varepsilon_- - 2\varepsilon_+ \iff \varepsilon_- = \frac{1}{2}(\psi + \varphi) \text{ and } \varepsilon_+ = \frac{1}{4}(\psi - \varphi). \tag{520}$$

$\therefore \partial_t\psi = \frac{\mathrm{i}}{2}\gamma^0\psi$, $\partial_t\varphi = -\frac{\mathrm{i}}{2}\gamma^0\varphi$, $D_A^{(H)}\psi = \frac{\mathrm{i}}{2}\gamma_A\varphi$ and $D_A^{(H)}\varphi = -\frac{\mathrm{i}}{2}\gamma_A\psi$.
$\therefore\ \psi = 2\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\psi_t$ and $\varphi = 2\mathrm{e}^{-\mathrm{i}\gamma^0 t/2}\varphi_t$ for some spinors, ψ_t and φ_t, that don't depend on t.
Equivalently, $\varepsilon_- = \mathrm{e}^{\mathrm{i}\gamma^0 t/2}\psi_t + \mathrm{e}^{-\mathrm{i}\gamma^0 t/2}\varphi_t$ and $\varepsilon_+ = \frac{1}{2}(\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\psi_t - \mathrm{e}^{-\mathrm{i}\gamma^0 t/2}\varphi_t)$.
By construction, I can assume without loss of generality that $P_1^{\pm}\varepsilon_\pm = \varepsilon_\pm \iff \varepsilon_\pm = \pm\mathrm{i}\gamma^1\varepsilon_\pm$.
$\therefore \varepsilon_- = \mathrm{e}^{\mathrm{i}\gamma^0 t/2}\psi_t + \mathrm{e}^{-\mathrm{i}\gamma^0 t/2}\varphi_t = -\mathrm{i}\gamma^1(\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\psi_t + \mathrm{e}^{-\mathrm{i}\gamma^0 t/2}\varphi_t) = -\mathrm{i}\mathrm{e}^{-\mathrm{i}\gamma^0 t/2}\gamma^1\psi_t - \mathrm{i}\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\gamma^1\varphi_t$.
Setting $t = 0$ in the previous equation then implies

$$\psi_t + \varphi_t = -\mathrm{i}\gamma^1\psi_t - \mathrm{i}\gamma^1\varphi_t \tag{521}$$

Meanwhile, setting $t = \pi$ implies

$$\begin{aligned}
&(\cos(\pi/2)I + \mathrm{i}\sin(\pi/2)\gamma^0)\psi_t + (\cos(\pi/2)I - \mathrm{i}\sin(\pi/2)\gamma^0)\varphi_t \\
&= -\mathrm{i}(\cos(\pi/2)I - \mathrm{i}\sin(\pi/2)\gamma^0)\gamma^1\psi_t - \mathrm{i}(\cos(\pi/2)I + \mathrm{i}\sin(\pi/2)\gamma^0)\gamma^1\varphi_t && (522)\\
&\iff \mathrm{i}\gamma^0\psi_t - \mathrm{i}\gamma^0\varphi_t = -\gamma^0\gamma^1\psi_t + \gamma^0\gamma^1\varphi_t && (523)\\
&\iff \psi_t - \varphi_t = \mathrm{i}\gamma^1\psi_t - \mathrm{i}\gamma^1\varphi_t. && (524)
\end{aligned}$$

Putting the $t = 0, \pi$ equations together, it immediately follows that $\psi_t = -\mathrm{i}\gamma^1\varphi_t$ (and this relation solves both equations).

$$\therefore \varepsilon_- = \mathrm{e}^{\mathrm{i}\gamma^0 t/2}\psi_t + \mathrm{e}^{-\mathrm{i}\gamma^0 t/2}\varphi_t = (I - \mathrm{i}\gamma^1)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\psi_t \text{ and} \tag{525}$$

$$\varepsilon_+ = \frac{1}{2}(\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\psi_t - \mathrm{e}^{-\mathrm{i}\gamma^0 t/2}\varphi_t) = \frac{1}{2}(I + \mathrm{i}\gamma^1)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\psi_t. \tag{526}$$

Next, consider the $D_A^{(H)}$ constraints on ψ and φ. In terms of ψ_t and φ_t, they now imply

$$2\mathrm{e}^{\mathrm{i}\gamma^0 t/2}D_A^{(H)}\psi_t = \frac{\mathrm{i}}{2}\gamma_A 2\mathrm{e}^{-\mathrm{i}\gamma^0 t/2}\varphi_t \iff D_A^{(H)}\psi_t = \frac{\mathrm{i}}{2}\gamma_A\varphi_t = \frac{1}{2}\gamma_A\gamma^1\psi_t. \tag{527}$$

Let $\varepsilon_H = \frac{1}{2}(I + \gamma^1)\psi_t \iff \psi_t = (I - \gamma^1)\varepsilon_H$.
$\therefore D_A^{(H)}\varepsilon_H = \frac{1}{2}(I+\gamma^1)D_A^{(H)}\psi_t = \frac{1}{2}(I+\gamma^1)\frac{1}{2}\gamma_A\gamma^1\psi_t = \frac{1}{4}\gamma_A(I-\gamma^1)\gamma^1\psi_t = \frac{1}{4}\gamma_A(\gamma^1+I)\psi_t = \frac{1}{2}\gamma_A\varepsilon_H$.
Writing $\varepsilon_\pm$ in terms of ε_H, I get

$$\varepsilon_- = (I - \mathrm{i}\gamma^1)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}(I - \gamma^1)\varepsilon_H = (I - \mathrm{i}\gamma^1)(\mathrm{e}^{\mathrm{i}\gamma^0 t/2} - \mathrm{i}\mathrm{e}^{-\mathrm{i}\gamma^0 t/2})\varepsilon_H \text{ and} \tag{528}$$

$$\varepsilon_+ = \frac{1}{2}(I + \mathrm{i}\gamma^1)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}(I - \gamma^1)\varepsilon_H = \frac{1}{2}(I + \mathrm{i}\gamma^1)(\mathrm{e}^{\mathrm{i}\gamma^0 t/2} + \mathrm{i}\mathrm{e}^{-\mathrm{i}\gamma^0 t/2})\varepsilon_H, \tag{529}$$

which is the result claimed in the theorem.
For completeness, I'll check that the conditions in lemma 4.11 are indeed all satisfied. Take

$$\varepsilon_- = \left(\mathrm{e}^{\mathrm{i}\gamma^0 t/2} - \mathrm{i}\mathrm{e}^{-\mathrm{i}\gamma^0 t/2}\right)\varepsilon_H \text{ and } \varepsilon_+ = \frac{1}{2}\left(\mathrm{e}^{\mathrm{i}\gamma^0 t/2} + \mathrm{i}\mathrm{e}^{-\mathrm{i}\gamma^0 t/2}\right)\varepsilon_H. \tag{530}$$

The time derivative are

$$\therefore \partial_t\varepsilon_- = \left(\frac{1}{2}\mathrm{i}\gamma^0\mathrm{e}^{\mathrm{i}\gamma^0 t/2} - \frac{1}{2}\gamma^0\mathrm{e}^{-\mathrm{i}\gamma^0 t/2}\right)\varepsilon_H = \frac{\mathrm{i}}{2}\gamma^0\left(\mathrm{e}^{\mathrm{i}\gamma^0 t/2} + \mathrm{i}\mathrm{e}^{-\mathrm{i}\gamma^0 t/2}\right)\varepsilon_H = \mathrm{i}\gamma^0\varepsilon_+ \text{ and} \tag{531}$$

$$\partial_t\varepsilon_+ = \frac{1}{2}\left(\frac{1}{2}\mathrm{i}\gamma^0\mathrm{e}^{\mathrm{i}\gamma^0 t/2} + \frac{1}{2}\gamma^0\mathrm{e}^{-\mathrm{i}\gamma^0 t/2}\right)\varepsilon_H = \frac{\mathrm{i}}{4}\gamma^0\left(\mathrm{e}^{\mathrm{i}\gamma^0 t/2} - \mathrm{i}\mathrm{e}^{-\mathrm{i}\gamma^0 t/2}\right)\varepsilon_H = \frac{\mathrm{i}}{4}\gamma^0\varepsilon_-, \tag{532}$$

while the space derivatives are

$$D_A^{(H)}\varepsilon_- = \left(\mathrm{e}^{\mathrm{i}\gamma^0 t/2} - \mathrm{i}\mathrm{e}^{-\mathrm{i}\gamma^0 t/2}\right)D_A^{(h)}\varepsilon_H \tag{533}$$

$$= \left(\mathrm{e}^{\mathrm{i}\gamma^0 t/2} - \mathrm{i}\mathrm{e}^{-\mathrm{i}\gamma^0 t/2}\right)\frac{1}{2}\gamma_A\varepsilon_H \tag{534}$$

$$= \frac{1}{2}\gamma_A\left(\mathrm{e}^{-\mathrm{i}\gamma^0 t/2} - \mathrm{i}\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\right)\varepsilon_H \tag{535}$$

$$= -\mathrm{i}\gamma_A\varepsilon_+ \text{ and} \tag{536}$$

$$D_A^{(H)}\varepsilon_+ = \frac{1}{2}\left(\mathrm{e}^{\mathrm{i}\gamma^0 t/2} + \mathrm{i}\mathrm{e}^{-\mathrm{i}\gamma^0 t/2}\right)D_A^{(h)}\varepsilon_H \tag{537}$$

$$= \frac{1}{2}\left(\mathrm{e}^{\mathrm{i}\gamma^0 t/2} + \mathrm{i}\mathrm{e}^{-\mathrm{i}\gamma^0 t/2}\right)\frac{1}{2}\gamma_A\varepsilon_H \tag{538}$$

$$= \frac{1}{4}\gamma_A\left(\mathrm{e}^{-\mathrm{i}\gamma^0 t/2} + \mathrm{i}\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\right)\varepsilon_H \tag{539}$$

$$= \frac{\mathrm{i}}{4}\gamma_A\varepsilon_-, \tag{540}$$

which implies all six constraints in lemma 4.11 are satisfied.
Finally, consider $c = -1$.
The $D_A^{(H)}$ constraints are $D_A^{(H)}\varepsilon_- = -\mathrm{i}\gamma_A\varepsilon_+$ and $D_A^{(H)}\varepsilon_+ = -\frac{\mathrm{i}}{4}\gamma_A\varepsilon_-$.
$\therefore D^{(H)A}D_A^{(H)}\varepsilon_- = -\mathrm{i}\gamma^A D_A^{(H)}\varepsilon_+ = -\frac{1}{4}\gamma^A\gamma_A\varepsilon_- = \frac{n-2}{4}\varepsilon_-$.
Then, from $\Sigma_{t,r}$'s assumed compactness,

$$\int_{\Sigma_{t,r}}\varepsilon_-^\dagger\varepsilon_-\mathrm{d}(H) = \frac{4}{n-2}\int_{\Sigma_{t,r}}\varepsilon_-^\dagger D^{(H)A}D_A^{(H)}(\varepsilon_-)\mathrm{d}(H) \tag{541}$$

$$= -\frac{4}{n-2}\int_{\Sigma_{t,r}}(D^{(H)A}\varepsilon_-)^\dagger D_A^{(H)}(\varepsilon_-)\mathrm{d}(H). \tag{542}$$

The LHS is non-negative while the RHS is non-positive, meaning they both must be zero.
$\therefore \varepsilon_- = 0$
$\therefore \varepsilon_+ = 0$ too from $D_A^{(H)}\varepsilon_- = -\mathrm{i}\gamma_A\varepsilon_+$. □

The main upshot of this corollary is that given any static boundary, $f_{(0)} = -\mathrm{d}t\otimes\mathrm{d}t + H$, if H admits a parallel spinor or (real) Killing spinor, the full spacetime will admit a positive energy theorem[33]. Furthermore, the "boundary charge" can now be evaluated solely in terms of quantities defined on the boundary, $\partial_\infty\Sigma_t$.

Theorem 4.13. *For spacetimes which are asymptotically Kottler with cross-section, H, if the Einstein equation and the dominant energy condition hold, then*

$$Q(\varepsilon) = (n-1)\int_{\partial_\infty\Sigma_t} p_M\varepsilon_H^\dagger\left(\cos(t/2)I - \sin(t/2)\gamma^0\right)\gamma^0\gamma^M P_1^- \times(\cos(t/2)I - \sin(t/2)\gamma^0)\varepsilon_H\,\mathrm{d}(H) \text{ when } c = 1 \tag{543}$$

$$\text{and } Q(\varepsilon) = \frac{n-1}{2}\int_{\partial_\infty\Sigma_t} p_M\varepsilon_H^\dagger\gamma^0\gamma^M P_1^-\varepsilon_H\,\mathrm{d}(H) \text{ when } c = 0. \tag{544}$$

[33]If the asymptotic region admits multiple spin structures, then the usual caveat about compatible spin structures still applies.

for ε_H solving $D^{(H)}_A \varepsilon_H = \frac{c}{2}\gamma_A \varepsilon_H$ and $\partial_t \varepsilon_H = \partial_r \varepsilon_H = 0$. In both cases,

$$Q(\varepsilon) = 2\int_{\Sigma_t} \left((\nabla_I \varepsilon)^\dagger \nabla^I \varepsilon + 4\pi T^{0\mu}\varepsilon^\dagger \gamma_0\gamma_\mu \varepsilon\right) \mathrm{d}V \geq 0. \tag{545}$$

Proof. The proof is simply a matter of substituting equation 517 into theorem 3.19.
As the boundary geometry is $f_{(0)} = -\mathrm{d}t \otimes \mathrm{d}t + H$, the integration measure, $\sqrt{\iota^\star f_{(0)}}\mathrm{d}^{n-2}x$ reduces to $\mathrm{d}(H)$.
For the rest of the integrand, consider $c = 0, 1$ separately. When $c = 0$,

$$\bar{\varepsilon}_k \gamma^M \varepsilon_k = \mathrm{e}^r \varepsilon_H^\dagger P_1^- \gamma^0 \gamma^M P_1^- \varepsilon_H = \mathrm{e}^r \varepsilon_H^\dagger \gamma^0 \gamma^M P_1^- \varepsilon_H \tag{546}$$

and hence the claimed result follows.
Next, consider $c = 1$. The e^{-r} factor in theorem 3.19 means it suffices to ignore any components of $\bar{\varepsilon}_k \gamma^M \varepsilon_k$ less than $O(\mathrm{e}^r)$.

$$\begin{aligned}
\therefore \bar{\varepsilon}_k \gamma^M \varepsilon_k &\to \mathrm{e}^r (P_1^- \varepsilon_-)^\dagger \gamma^0 \gamma^M P_1^- \varepsilon_- &(547)\\
&= \mathrm{e}^r \varepsilon_-^\dagger \gamma^0 \gamma^M P_1^- \varepsilon_- &(548)\\
&= \mathrm{e}^r \left(\left(\mathrm{e}^{\mathrm{i}\gamma^0 t/2} - \mathrm{i}\mathrm{e}^{-\mathrm{i}\gamma^0 t/2}\right)\varepsilon_H\right)^\dagger \gamma^0\gamma^M P_1^- \left(\mathrm{e}^{\mathrm{i}\gamma^0 t/2} - \mathrm{i}\mathrm{e}^{-\mathrm{i}\gamma^0 t/2}\right)\varepsilon_H &(549)\\
&= \mathrm{e}^r \varepsilon_H^\dagger \left(\mathrm{e}^{-\mathrm{i}\gamma^0 t/2} + \mathrm{i}\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\right)\gamma^0\gamma^M P_1^- \left(\mathrm{e}^{\mathrm{i}\gamma^0 t/2} - \mathrm{i}\mathrm{e}^{-\mathrm{i}\gamma^0 t/2}\right)\varepsilon_H &(550)\\
&= \mathrm{e}^r \varepsilon_H^\dagger \left(\cos(t/2)I - \mathrm{i}\sin(t/2)\gamma^0 + \mathrm{i}\cos(t/2)I - \sin(t/2)\gamma^0\right)\gamma^0\gamma^M P_1^- \\
&\quad \times \left(\cos(t/2)I + \mathrm{i}\sin(t/2)\gamma^0 - \mathrm{i}\cos(t/2)I - \sin(t/2)\gamma^0\right)\varepsilon_H &(551)\\
&= \mathrm{e}^r \varepsilon_H^\dagger (1+\mathrm{i})\left(\cos(t/2)I - \sin(t/2)\gamma^0\right)\gamma^0\gamma^M P_1^- \\
&\quad \times (1-\mathrm{i})\left(\cos(t/2)I - \sin(t/2)\gamma^0\right)\varepsilon_H &(552)\\
&= 2\mathrm{e}^r \varepsilon_H^\dagger \left(\cos(t/2)I - \sin(t/2)\gamma^0\right)\gamma^0\gamma^M P_1^- \left(\cos(t/2)I - \sin(t/2)\gamma^0\right)\varepsilon_H, &(553)
\end{aligned}$$

which gives the claimed result. □

Solutions to $D^{(H)}_A \varepsilon_H = 0$ and $D^{(H)}_A \varepsilon_H = \frac{1}{2}\gamma_A \varepsilon_H$ are well-studied problems for mathematicians. One subtlety in comparing with the maths literature is that $\{\gamma^A\}_{A=2}^{n-1}$ don't form an irreducible representation of the Clifford algebra; a (Riemannian) Clifford algebra with $n-2$ elements would have $2^{\lfloor (n-2)/2 \rfloor} \times 2^{\lfloor (n-2)/2 \rfloor}$ matrices, not $2^{\lfloor n/2 \rfloor} \times 2^{\lfloor n/2 \rfloor}$ matrices like $\{\gamma^A\}_{A=2}^{n-1} \subset \{\gamma^\mu\}_{\mu=0}^{n-1}$. The doubled size means there are effectively two irreducible representations summed in γ^A. This can be made much more concrete, as follows. Suppose $\{\hat{\gamma}^A\}_{A=2}^{n-1}$ form an irreducible representation of the Clifford algebra with $n-2$ elements. Then, γ^μ can be chosen to be

$$\gamma^0 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \ \gamma^1 = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \text{ and } \gamma^A = \begin{bmatrix} 0 & \hat{\gamma}^A \\ \hat{\gamma}^A & 0 \end{bmatrix}. \tag{554}$$

Likewise, split ε_H into two $2^{\lfloor n/2 \rfloor - 1}$ component blocks, i.e. $\varepsilon_H = [\psi, \varphi]^\mathrm{T}$. Then,

$$\begin{aligned}
D^{(H)}_A \varepsilon_H &= \left(e^{(H)\alpha}_A \partial_\alpha - \frac{1}{8}\omega^{(H)}_{BCA}\left(\begin{bmatrix} 0 & \hat{\gamma}^A \\ \hat{\gamma}^A & 0 \end{bmatrix}\begin{bmatrix} 0 & \hat{\gamma}^B \\ \hat{\gamma}^B & 0 \end{bmatrix} - \begin{bmatrix} 0 & \hat{\gamma}^B \\ \hat{\gamma}^B & 0 \end{bmatrix}\begin{bmatrix} 0 & \hat{\gamma}^A \\ \hat{\gamma}^A & 0 \end{bmatrix}\right)\right)\begin{bmatrix} \psi \\ \varphi \end{bmatrix} &(555)\\
&= \left(e^{(H)\alpha}_A \partial_\alpha - \frac{1}{4}\omega^{(H)}_{BCA}\begin{bmatrix} \hat{\gamma}^{AB} & 0 \\ 0 & \hat{\gamma}^{AB} \end{bmatrix}\right)\begin{bmatrix} \psi \\ \varphi \end{bmatrix} &(556)\\
&= \begin{bmatrix} \hat{D}^{(H)}_A \psi \\ \hat{D}^{(H)}_A \varphi \end{bmatrix}. &(557)
\end{aligned}$$

$\therefore$ The general solution to $D_A^{(H)}\varepsilon_H = 0$ on M is constructed from a pair of spinors on the cross-section, ψ and φ, each parallel with respect to $\hat{D}_A^{(H)}$. There are many compact Riemannian manifolds admitting parallel spinors - see [49] for a classification in the simply connected case and [50] for more general comments.
Similarly, since

$$\gamma_A\varepsilon_H = \begin{bmatrix} 0 & \hat{\gamma}_A \\ \hat{\gamma}_A & 0 \end{bmatrix} \begin{bmatrix} \psi \\ \varphi \end{bmatrix} = \begin{bmatrix} \hat{\gamma}_A\varphi \\ \hat{\gamma}_A\psi \end{bmatrix}, \tag{558}$$

$D_A^{(H)}\varepsilon = \frac{1}{2}\gamma_A\varepsilon_H \implies \hat{D}_A^{(H)}\psi = \frac{1}{2}\hat{\gamma}_A\varphi$ and $\hat{D}_A^{(H)}\varphi = \frac{1}{2}\hat{\gamma}_A\psi$.
Let $\hat{\varepsilon}_H^{(\pm)} = \psi \pm \varphi$.

$$\therefore \hat{D}_A^{(H)}\hat{\varepsilon}_H^{(\pm)} = \pm\frac{1}{2}\hat{\gamma}_A\hat{\varepsilon}_H^{(\pm)} \text{ and } \varepsilon_H = \frac{1}{2}\begin{bmatrix} \hat{\varepsilon}_H^{(+)} + \hat{\varepsilon}_H^{(-)} \\ \hat{\varepsilon}_H^{(+)} - \hat{\varepsilon}_H^{(-)} \end{bmatrix}. \tag{559}$$

$\therefore$ The general solution to $D_A^{(H)}\varepsilon_H = \frac{1}{2}\gamma_A\varepsilon_H$ on M is constructed from a pair of real Killing spinors on the cross-section, $\hat{\varepsilon}_H^{(\pm)}$, satisfying $\hat{D}_A^{(H)}\hat{\varepsilon}_H^{(\pm)} = \pm\frac{1}{2}\hat{\gamma}_A\hat{\varepsilon}_H^{(\pm)}$.
From [51], there are some strenuous constraints on solutions to $\hat{D}_A^{(H)}\hat{\varepsilon}_H^{(\pm)} = \pm\frac{1}{2}\hat{\gamma}_A\hat{\varepsilon}_H^{(\pm)}$. First of all, there is not necessarily any correspondence between solutions of $\hat{D}_A^{(H)}\hat{\varepsilon}_H^{(+)} = \frac{1}{2}\hat{\gamma}_A\hat{\varepsilon}_H^{(+)}$ and $\hat{D}_A^{(H)}\hat{\varepsilon}_H^{(-)} = -\frac{1}{2}\hat{\gamma}_A\hat{\varepsilon}_H^{(-)}$ (albeit swapping orientation swaps the meaning of $+$ and $-$). Moreover, for any even dimension, except $n-2=6$, the only metric, H, admitting a solution to either equations is the standard metric on S^{n-2}, which I've already considered in section 4.2 and will revisit in section 4.3.3. For odd dimensions, for simply connected cross-sections, other than the round sphere, one can also have Sasaki-Einstein spaces and Sasaki-3 spaces in general, besides some other specific examples when $n-2=7$. Furthermore, having both $\hat{\varepsilon}_H^{(+)}$ and $\hat{\varepsilon}_H^{(-)}$ non-zero is only possible for simply connected cross-sections when $n-2=1$ (mod) 4.

Having decomposed solutions of $D_A^{(H)} = \frac{c}{2}\gamma_A\varepsilon_H$ into spinors defined completely on the cross-section, it makes sense to re-write theorem 4.13 purely in terms of cross section data.

Theorem 4.14. *Suppose a spacetime is asymptotically Kottler with cross-section, H. Assume the Einstein equation and the dominant energy condition hold. Then, when $c=1$,*

$$Q(\varepsilon) = \frac{n-1}{4}\int_{\partial_\infty\Sigma_t}\left(\hat{\varepsilon}_H^{(+)\dagger} + \mathrm{i}e^{\mathrm{i}t}\hat{\varepsilon}_H^{(-)\dagger}\right)\left(p_0 I - \mathrm{i}p_A\hat{\gamma}^A\right)\left(\hat{\varepsilon}_H^{(+)} - \mathrm{i}e^{-\mathrm{i}t}\hat{\varepsilon}_H^{(-)}\right)\mathrm{d}(H). \tag{560}$$

for $\hat{\varepsilon}_H^{(\pm)}$ solving $\hat{D}_A^{(H)}\hat{\varepsilon}_H^{(\pm)} = \pm\frac{1}{2}\hat{\gamma}_A\hat{\varepsilon}_H^{(\pm)}$. Likewise, when $c=0$,

$$Q(\varepsilon) = \frac{n-1}{4}\int_{\partial_\infty\Sigma_t}\hat{\psi}^\dagger\left(p_0 I - \mathrm{i}p_A\hat{\gamma}^A\right)\hat{\psi}\,\mathrm{d}(H). \tag{561}$$

where $\hat{\psi}$ solves $\hat{D}_A^{(H)}\hat{\psi} = 0$. In both cases,

$$Q(\varepsilon) = 2\int_{\Sigma_t}\left((\nabla_I\varepsilon)^\dagger\nabla^I\varepsilon + 4\pi T^{0\mu}\varepsilon^\dagger\gamma_0\gamma_\mu\varepsilon\right)\mathrm{d}V \geq 0. \tag{562}$$

Proof. Start with $c=0$. From the discussion above, the most general solution to $D_A\varepsilon_H = 0$ is given by $\varepsilon_H = [\psi,\varphi]^{\mathrm{T}}$ where ψ and φ are both parallel with respect to $\hat{D}_A^{(H)}$.

Then, to apply theorem 4.13, I need to evaluate $p_M \varepsilon_H^\dagger \gamma^0 \gamma^M P_1^- \varepsilon_H$.

$$\varepsilon_H^\dagger \gamma^0 \gamma^0 P_1^- \varepsilon_H = \begin{bmatrix} \psi^\dagger & \varphi^\dagger \end{bmatrix}^{\mathrm{T}} \frac{1}{2} \begin{bmatrix} I & \mathrm{i}I \\ -\mathrm{i}I & I \end{bmatrix} \begin{bmatrix} \psi \\ \varphi \end{bmatrix} \tag{563}$$

$$= \frac{1}{2} \left(\psi^\dagger \psi + \mathrm{i}\psi^\dagger \varphi - \mathrm{i}\varphi^\dagger \psi + \varphi^\dagger \varphi \right). \tag{564}$$

$$\varepsilon_H^\dagger \gamma^0 \gamma^A P_1^- \varepsilon_H = \begin{bmatrix} \psi^\dagger & \varphi^\dagger \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} 0 & \hat\gamma^A \\ \hat\gamma^A & 0 \end{bmatrix} \frac{1}{2} \begin{bmatrix} I & \mathrm{i}I \\ -\mathrm{i}I & I \end{bmatrix} \begin{bmatrix} \psi \\ \varphi \end{bmatrix} \tag{565}$$

$$= \frac{1}{2} \begin{bmatrix} \psi^\dagger & \varphi^\dagger \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} 0 & \hat\gamma^A \\ -\hat\gamma^A & 0 \end{bmatrix} \begin{bmatrix} \psi + \mathrm{i}\varphi \\ -\mathrm{i}\psi + \varphi \end{bmatrix} \tag{566}$$

$$= \frac{1}{2} \begin{bmatrix} \psi^\dagger & \varphi^\dagger \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} -\mathrm{i}\hat\gamma^A \psi + \hat\gamma^A \varphi \\ -\hat\gamma^A \psi - \mathrm{i}\hat\gamma^A \varphi \end{bmatrix} \tag{567}$$

$$= \frac{1}{2} \left(-\mathrm{i}\psi^\dagger \hat\gamma^A \psi + \psi^\dagger \hat\gamma^A \varphi - \varphi^\dagger \hat\gamma^A \psi - \mathrm{i}\varphi^\dagger \hat\gamma^A \varphi \right). \tag{568}$$

Putting both parts together,

$$p_M \varepsilon_H^\dagger \gamma^0 \gamma^M P_1^- \varepsilon_H = \frac{1}{2} \left(\psi^\dagger - \mathrm{i}\varphi^\dagger \right) \left(p_0 I - \mathrm{i} p_A \hat\gamma^A \right) \left(\psi + \mathrm{i}\varphi \right). \tag{569}$$

Defining $\hat\psi = \psi + \mathrm{i}\varphi$ proves the claim for $c = 0$.
Next, let $c = 1$. This time, by the discussion earlier,

$$\varepsilon_H = \frac{1}{2} \begin{bmatrix} \hat\varepsilon_H^{(+)} + \hat\varepsilon_H^{(-)} \\ \hat\varepsilon_H^{(+)} - \hat\varepsilon_H^{(-)} \end{bmatrix} \tag{570}$$

for $\hat\varepsilon_H^{(\pm)}$ solving $\hat D_A^{(H)} \hat\varepsilon_H^{(\pm)} = \pm \frac{1}{2} \hat\gamma_A \hat\varepsilon_H^{(\pm)}$. By theorem 4.13 I need to evaluate

$$p_M \varepsilon_H^\dagger \left(\cos(t/2) I - \sin(t/2)\gamma^0 \right) \gamma^0 \gamma^M P_1^- (\cos(t/2) I - \sin(t/2)\gamma^0) \varepsilon_H. \tag{571}$$

When $M = 0$, I get

$$\begin{aligned}
&\varepsilon_H^\dagger \left(\cos(t/2) I - \sin(t/2)\gamma^0 \right) P_1^- (\cos(t/2) I - \sin(t/2)\gamma^0) \varepsilon_H \\
&= \frac{1}{8} \begin{bmatrix} \hat\varepsilon_H^{(+)\dagger} + \hat\varepsilon_H^{(-)\dagger} & \hat\varepsilon_H^{(+)\dagger} - \hat\varepsilon_H^{(-)\dagger} \end{bmatrix} \begin{bmatrix} (\cos(t/2) - \sin(t/2)) I & 0 \\ 0 & (\cos(t/2) + \sin(t/2)) I \end{bmatrix} \begin{bmatrix} I & \mathrm{i}I \\ -\mathrm{i}I & I \end{bmatrix} \\
&\quad \times \begin{bmatrix} (\cos(t/2) - \sin(t/2)) I & 0 \\ 0 & (\cos(t/2) + \sin(t/2)) I \end{bmatrix} \begin{bmatrix} \hat\varepsilon_H^{(+)} + \hat\varepsilon_H^{(-)} \\ \hat\varepsilon_H^{(+)} - \hat\varepsilon_H^{(-)} \end{bmatrix} && (572) \\
&= \frac{1}{8} \begin{bmatrix} \hat\varepsilon_H^{(+)\dagger} + \hat\varepsilon_H^{(-)\dagger} & \hat\varepsilon_H^{(+)\dagger} - \hat\varepsilon_H^{(-)\dagger} \end{bmatrix} \begin{bmatrix} (\cos(t/2) - \sin(t/2)) I & \mathrm{i}(\cos(t/2) - \sin(t/2)) I \\ -\mathrm{i}(\cos(t/2) + \sin(t/2)) I & (\cos(t/2) + \sin(t/2)) I \end{bmatrix} \\
&\quad \times \begin{bmatrix} (\cos(t/2) - \sin(t/2))(\hat\varepsilon_H^{(+)} + \hat\varepsilon_H^{(-)}) \\ (\cos(t/2) + \sin(t/2))(\hat\varepsilon_H^{(+)} - \hat\varepsilon_H^{(-)}) \end{bmatrix} && (573) \\
&= \frac{1}{8} \begin{bmatrix} \hat\varepsilon_H^{(+)\dagger} + \hat\varepsilon_H^{(-)\dagger} & \hat\varepsilon_H^{(+)\dagger} - \hat\varepsilon_H^{(-)\dagger} \end{bmatrix} \\
&\quad \times \begin{bmatrix} (\cos(t/2) - \sin(t/2))^2 (\hat\varepsilon_H^{(+)} + \hat\varepsilon_H^{(-)}) + \mathrm{i}(\cos^2(t/2) - \sin^2(t/2))(\hat\varepsilon_H^{(+)} - \hat\varepsilon_H^{(-)}) \\ -\mathrm{i}(\cos^2(t/2) - \sin^2(t/2))(\hat\varepsilon_H^{(+)} + \hat\varepsilon_H^{(-)}) + (\cos(t/2) + \sin(t/2))^2 (\hat\varepsilon_H^{(+)} - \hat\varepsilon_H^{(-)}) \end{bmatrix} && (574) \\
&= \frac{1}{8} \Big((\hat\varepsilon_H^{(+)\dagger} + \hat\varepsilon_H^{(-)\dagger})(1 - \sin(t))(\hat\varepsilon_H^{(+)} + \hat\varepsilon_H^{(-)}) + (\hat\varepsilon_H^{(+)\dagger} + \hat\varepsilon_H^{(-)\dagger})\mathrm{i}\cos(t)(\hat\varepsilon_H^{(+)} - \hat\varepsilon_H^{(-)}) \\
&\quad + (\hat\varepsilon_H^{(+)\dagger} - \hat\varepsilon_H^{(-)\dagger})(-\mathrm{i}\cos(t))(\hat\varepsilon_H^{(+)} + \hat\varepsilon_H^{(-)}) + (\hat\varepsilon_H^{(+)\dagger} - \hat\varepsilon_H^{(-)\dagger})(1 + \sin(t))(\hat\varepsilon_H^{(+)} - \hat\varepsilon_H^{(-)}) \Big) && (575) \\
&= \frac{1}{4} \left(\hat\varepsilon_H^{(+)\dagger} \hat\varepsilon_H^{(+)} - \mathrm{i}e^{-\mathrm{i}t} \hat\varepsilon_H^{(+)\dagger} \hat\varepsilon_H^{(-)} + \mathrm{i}e^{\mathrm{i}t} \hat\varepsilon_H^{(-)\dagger} \hat\varepsilon_H^{(+)} + \hat\varepsilon_H^{(-)\dagger} \hat\varepsilon_H^{(-)} \right). && (576)
\end{aligned}$$

Similarly, when $M = A$,

$$\begin{aligned}
&\varepsilon_H^\dagger \left(\cos(t/2)I - \sin(t/2)\gamma^0\right) \gamma^0\gamma^A P_1^-(\cos(t/2)I - \sin(t/2)\gamma^0)\varepsilon_H \\
&= \frac{1}{8}\begin{bmatrix}\hat{\varepsilon}_H^{(+)\dagger} + \hat{\varepsilon}_H^{(-)\dagger} & \hat{\varepsilon}_H^{(+)\dagger} - \hat{\varepsilon}_H^{(-)\dagger}\end{bmatrix} \begin{bmatrix}(\cos(t/2) - \sin(t/2))I & 0 \\ 0 & (\cos(t/2) + \sin(t/2))I\end{bmatrix} \begin{bmatrix}0 & \hat{\gamma}^A \\ -\hat{\gamma}^A & 0\end{bmatrix} \\
&\quad \times \begin{bmatrix}I & \mathrm{i}I \\ -\mathrm{i}I & I\end{bmatrix} \begin{bmatrix}(\cos(t/2) - \sin(t/2))I & 0 \\ 0 & (\cos(t/2) + \sin(t/2))I\end{bmatrix} \begin{bmatrix}\hat{\varepsilon}_H^{(+)} + \hat{\varepsilon}_H^{(-)} \\ \hat{\varepsilon}_H^{(+)} - \hat{\varepsilon}_H^{(-)}\end{bmatrix} && (577)\\
&= \frac{1}{8}\begin{bmatrix}\hat{\varepsilon}_H^{(+)\dagger} + \hat{\varepsilon}_H^{(-)\dagger} & \hat{\varepsilon}_H^{(+)\dagger} - \hat{\varepsilon}_H^{(-)\dagger}\end{bmatrix} \begin{bmatrix}0 & (\cos(t/2) - \sin(t/2))\hat{\gamma}^A \\ -(\cos(t/2) + \sin(t/2))\hat{\gamma}^A & 0\end{bmatrix} \\
&\quad \times \begin{bmatrix}(\cos(t/2) - \sin(t/2))I & \mathrm{i}(\cos(t/2) + \sin(t/2))I \\ -\mathrm{i}(\cos(t/2) - \sin(t/2))I & (\cos(t/2) + \sin(t/2))I\end{bmatrix} \begin{bmatrix}\hat{\varepsilon}_H^{(+)} + \hat{\varepsilon}_H^{(-)} \\ \hat{\varepsilon}_H^{(+)} - \hat{\varepsilon}_H^{(-)}\end{bmatrix} && (578)\\
&= \frac{1}{8}\begin{bmatrix}\hat{\varepsilon}_H^{(+)\dagger} + \hat{\varepsilon}_H^{(-)\dagger} & \hat{\varepsilon}_H^{(+)\dagger} - \hat{\varepsilon}_H^{(-)\dagger}\end{bmatrix} \begin{bmatrix}-\mathrm{i}(1 - \sin(t))\hat{\gamma}^A & \cos(t)\hat{\gamma}^A \\ -\cos(t)\hat{\gamma}^A & -\mathrm{i}(1 + \sin(t))\hat{\gamma}^A\end{bmatrix} \begin{bmatrix}\hat{\varepsilon}_H^{(+)} + \hat{\varepsilon}_H^{(-)} \\ \hat{\varepsilon}_H^{(+)} - \hat{\varepsilon}_H^{(-)}\end{bmatrix} && (579)\\
&= \frac{1}{4}\left(-\mathrm{i}\hat{\varepsilon}_H^{(+)\dagger}\hat{\gamma}^A\hat{\varepsilon}_H^{(+)} - \mathrm{e}^{-\mathrm{i}t}\hat{\varepsilon}_H^{(+)\dagger}\hat{\gamma}^A\hat{\varepsilon}_H^{(-)} + \mathrm{e}^{\mathrm{i}t}\hat{\varepsilon}_H^{(-)\dagger}\hat{\gamma}^A\hat{\varepsilon}_H^{(+)} - \mathrm{i}\hat{\varepsilon}_H^{(-)\dagger}\hat{\gamma}^A\hat{\varepsilon}_H^{(-)}\right). && (580)
\end{aligned}$$

Putting both parts together, I get

$$\begin{aligned}
&p_M\varepsilon_H^\dagger \left(\cos(t/2)I - \sin(t/2)\gamma^0\right) \gamma^0\gamma^M P_1^-(\cos(t/2)I - \sin(t/2)\gamma^0)\varepsilon_H \\
&= \left(\hat{\varepsilon}_H^{(+)\dagger} + \mathrm{i}\mathrm{e}^{\mathrm{i}t}\hat{\varepsilon}_H^{(-)\dagger}\right)\left(p_0 I - \mathrm{i}p_A\hat{\gamma}^A\right)\left(\hat{\varepsilon}_H^{(+)} - \mathrm{i}\mathrm{e}^{-\mathrm{i}t}\hat{\varepsilon}_H^{(-)}\right), \qquad (581)
\end{aligned}$$

which is the claimed integrand. □

When $c = 1$, the different spinor bilinears appearing in theorem 5.9 have some geometric interpretation.

Lemma 4.15. *$\hat{\varepsilon}_H^{(\pm)\dagger}\hat{\varepsilon}_H^{(\pm)}$ are constants and $\hat{k}^{(\pm)A} = -\mathrm{i}\hat{\varepsilon}_H^{(\pm)\dagger}\hat{\gamma}^A\hat{\varepsilon}_H^{(\pm)}$ are (real) Killing vectors for H. Define functions, $\hat{s}^{(\pm)} = \hat{\varepsilon}_H^{(\mp)\dagger}\hat{\varepsilon}_H^{(\pm)}$. Then, $\hat{\xi}^{(\pm)A} = \hat{\varepsilon}_H^{(\mp)\dagger}\hat{\gamma}^A\hat{\varepsilon}_H^{(\pm)}$ are (complex) conformal Killing vectors for H with $\hat{\xi}_A^{(\pm)} = \pm\hat{D}_A^{(H)}\hat{s}^{(\pm)}$ and $\hat{D}_A^{(H)}\hat{\xi}_B^{(\pm)} = -\delta_{AB}\hat{s}^{(\pm)}$.*

Proof. The proof is repeatedly applying the Killing spinor equation.

$$\begin{aligned}
\hat{D}_A^{(H)}\left(\hat{\varepsilon}_H^{(\pm)\dagger}\hat{\varepsilon}_H^{(\pm)}\right) &= \left(\pm\frac{1}{2}\hat{\gamma}_A\hat{\varepsilon}_H^{(\pm)}\right)^\dagger \hat{\varepsilon}_H^{(\pm)} + \hat{\varepsilon}_H^{(\pm)\dagger}\left(\pm\frac{1}{2}\hat{\gamma}_A\hat{\varepsilon}_H^{(\pm)}\right) && (582)\\
&= \mp\frac{1}{2}\hat{\varepsilon}_H^{(\pm)\dagger}\hat{\gamma}_A\hat{\varepsilon}_H^{(\pm)} + \pm\frac{1}{2}\hat{\varepsilon}_H^{(\pm)\dagger}\hat{\gamma}_A\hat{\varepsilon}_H^{(\pm)} && (583)\\
&= 0. && (584)
\end{aligned}$$

$\therefore \hat{\varepsilon}_H^{(\pm)\dagger}\hat{\varepsilon}_H^{(\pm)}$ are constants.

$$\hat{D}_A^{(H)}\hat{k}_B^{(\pm)} = -\mathrm{i}\left(\mp\frac{1}{2}\hat{\varepsilon}_H^{(\pm)\dagger}\hat{\gamma}_A\right)\hat{\gamma}_B\hat{\varepsilon}_H^{(\pm)} - \mathrm{i}\hat{\varepsilon}_H^{(\pm)\dagger}\hat{\gamma}_B\left(\pm\frac{1}{2}\hat{\gamma}_A\hat{\varepsilon}_H^{(\pm)}\right) = \pm\mathrm{i}\hat{\varepsilon}_H^{(\pm)\dagger}\hat{\gamma}_{AB}\hat{\varepsilon}_H^{(\pm)}. \qquad (585)$$

$\therefore$ The symmetric part of $\hat{D}_A^{(H)}\hat{k}_B^{(\pm)}$ is zero.
$\therefore \hat{k}_A^{(\pm)}$ are Killing vectors.

$$\hat{D}_A^{(H)}\hat{s}^{(\pm)} = \left(\pm\frac{1}{2}\hat{\varepsilon}_H^{(\mp)\dagger}\hat{\gamma}_A\right)\hat{\varepsilon}_H^{(\pm)} + \hat{\varepsilon}_H^{(\mp)\dagger}\left(\pm\frac{1}{2}\hat{\gamma}_A\hat{\varepsilon}_H^{(\pm)}\right) = \pm\hat{\xi}_A. \qquad (586)$$

$$\hat{D}^{(H)}_A\hat{\xi}^{(\pm)}_B = \left(\pm\frac{1}{2}\hat{\varepsilon}^{(\mp)\dagger}_H\hat{\gamma}_A\right)\hat{\gamma}_B\hat{\varepsilon}^{(\pm)}_H + \hat{\varepsilon}^{(\mp)\dagger}_H\hat{\gamma}_B\left(\pm\frac{1}{2}\hat{\gamma}_A\hat{\varepsilon}^{(\pm)}_H\right) = \mp\delta_{AB}\hat{\varepsilon}^{(\mp)\dagger}_H\hat{\varepsilon}^{(\pm)}_H = \mp\delta_{AB}\hat{s}^{(\pm)}. \quad (587)$$

$\therefore$ The symmetric and traceless part of $\hat{D}^{(H)}_A\hat{\xi}^{(\pm)}_B$ is zero.
$\therefore$ $\hat{\xi}^{(\pm)}$ is a conformal Killing vector. □

Corollary 4.15.1. *If H is not a round metric on a sphere, then $\hat{s}^{(\pm)} = 0$ and $\hat{\xi}^{(\pm)}_A = 0$.*

Proof. From the lemma,

$$\hat{D}^{(H)}_A\hat{D}^{(H)}_B\hat{s}^{(\pm)} = \hat{D}^{(H)}_A\left(\pm\hat{\xi}_B\right) = -\delta_{AB}\hat{s}^{(\pm)}. \quad (588)$$

From [52], the only compact, complete, Riemannian manifold admitting a solution to this equation is the round sphere. □

Note this corollary does not preclude having both $\hat{\varepsilon}^{(+)}_H$ and $\hat{\varepsilon}^{(-)}_H$ non-zero. However, it does mean $\hat{\varepsilon}^{(+)}_H$ and $\hat{\varepsilon}^{(-)}_H$ are orthogonal on anything aside from a round sphere cross-section.

Definition 4.16 ("Conserved quantities" on the cross-section)**.** *For a Killing vector, $\hat{k}$, of H, define a "conserved quantity" by*

$$Q_{\hat{k}} = \frac{n-1}{16\pi}\int_{\partial_\infty\Sigma_t} p_A\hat{k}^A\mathrm{d}(H) = \frac{n-1}{16\pi}\int_{\partial_\infty\Sigma_t} f_{(n-1)0\alpha}\hat{k}^\alpha\mathrm{d}(H). \quad (589)$$

Theorem 4.17. *Define $\hat{k}^{(\pm)A} = -\mathrm{i}\hat{\varepsilon}^{(\pm)\dagger}_H\hat{\gamma}^A\hat{\varepsilon}^{(\pm)}_H$. Without loss of generality, scale $\hat{\varepsilon}^{(\pm)}_H$ such that $\hat{\varepsilon}^{(\pm)\dagger}_H\hat{\varepsilon}^{(\pm)}_H = \hat{\delta}^{(\pm)}$, where $\hat{\delta}^{(\pm)} = 1$ if a non-trivial $\hat{\varepsilon}^{(\pm)}_H$ exists and $\hat{\delta}^{(\pm)} = 0$ otherwise. Then, if $c = 1$ and H is not a round metric on a sphere, theorem 4.14 can be re-written as*

$$Q(\varepsilon) = 4\pi\left(E(\hat{\delta}^{(+)} + \hat{\delta}^{(-)}) + Q_{\hat{k}^{(+)}} + Q_{\hat{k}^{(-)}}\right) \quad (590)$$

$$= 2\int_{\Sigma_t}\left((\nabla_I\varepsilon)^\dagger\nabla^I\varepsilon + 4\pi T^{0\mu}\varepsilon^\dagger\gamma_0\gamma_\mu\varepsilon\right)\mathrm{d}V \geq 0. \quad (591)$$

Proof. Direct application of theorem 4.14, definitions 3.18 & 4.16, equation 81, lemma 4.15 and corollary 4.15.1. □

I will illustrate some of these cross-section based theorems in the next two subsections by studying the lens space, $L(p,1)$, and a squashed S^7 - both some of the simplest deformations of the round sphere.

4.3.1 Squashed S^7

The simplest deformation that can be made to a sphere is squashing. However, squashed spheres typically don't admit Killing spinors. A rare exception is a particular squashed sphere,

$$H = \frac{9}{20}\left(\mathrm{d}a\otimes\mathrm{d}a + \frac{1}{4}\sin^2(a)b_x\otimes b_x + \frac{1}{20}(c_x+\cos(a)b_x)\otimes(c_x+\cos(a)b_x)\right), \quad (592)$$

$$b_x = \sigma_x - \Sigma_x,\ \ c_x = \sigma_x + \Sigma_x, \quad (593)$$

$$\sigma_1 = \cos(\psi)\mathrm{d}\theta + \sin(\psi)\sin(\theta)\mathrm{d}\phi,\ \ \sigma_2 = -\sin(\psi)\mathrm{d}\theta + \cos(\psi)\sin(\theta)\mathrm{d}\phi,$$

$$\sigma_3 = \mathrm{d}\psi + \cos(\theta)\mathrm{d}\phi \quad (594)$$

and Σ_x are defined identically to σ_x, but with (ψ,θ,ϕ) replaced by some analogous coordinates, (ψ',θ',ϕ'). The squashing comes from the factor of 1/20 in equation 592. If that 1/20 were

also 1/4, then H would be the usual round sphere.
From [53], H satisfies $R^{(H)}_{AB} = 6\delta_{AB}$ and admits exactly one linearly independent Killing spinor. Choose γ^μ as per equation 554. Then, solutions to $D^{(H)}_A \varepsilon_H = \frac{1}{2}\gamma_A \varepsilon_H$ are constructed as per equation 559 and theorem 4.14 is applicable.
From [51], the only solution to $\hat{D}_A \hat{\varepsilon}^{(-)}_H = -\frac{1}{2}\hat{\gamma}_A \hat{\varepsilon}^{(-)}_H$ is $\hat{\varepsilon}^{(-)}_H = 0$.
I will choose the same $\hat{\gamma}^A$ as [53]. Then, by [53], the only solution (up to constant scaling) to $\hat{D}_A \hat{\varepsilon}^{(+)}_H = \frac{1}{2}\hat{\gamma}_A \hat{\varepsilon}^{(+)}_H$ is

$$\hat{\varepsilon}^{(+)}_H = \begin{bmatrix} 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}. \tag{595}$$

Theorem 4.18. *For spacetimes asymptotically Kottler with the cross-section in equation 592, if the Einstein equation and dominant energy condition hold, then $E \geq 0$.*

Proof. The proof is simply applying theorem 4.14 with $\hat{\varepsilon}^{(-)}_H = 0$ and equation 595.
$\hat{\varepsilon}^{(-)}_H = 0$ means I only need to evaluate $\hat{\varepsilon}^{(+)\dagger}_H (p_0 I - \mathrm{i} p_A \hat{\gamma}^A)\hat{\varepsilon}^{(+)}_H$ for theorem 4.14.
$\hat{\varepsilon}^{(+)\dagger}_H \hat{\varepsilon}^{(+)}_H = 2$ by inspection and by computer algebra, one finds $\hat{\varepsilon}^{(+)\dagger}_H \hat{\gamma}^A \hat{\varepsilon}^{(+)}_H = 0$ for all A.
Thus, theorem 4.14 reduces to saying $0 \leq Q(\varepsilon) = 2\int_{\partial_\infty \Sigma_t} 2p_0 \mathrm{d}(H) = 8\pi E \iff E \geq 0$. □

4.3.2 Lens spaces, $L(p, 1)$

Definition 4.19 (Lens space, $L(p,1)$). *View S^3 as the level set,*

$$\{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}. \tag{596}$$

Then, the lens space, $L(p,1)$, is defined as the quotient of S^3 by the $\mathbb{Z}_p$ action,

$$(z_1, z_2) \to (z_1 \mathrm{e}^{2\pi\mathrm{i}/p}, z_2 \mathrm{e}^{2\pi\mathrm{i}/p}). \tag{597}$$

Lemma 4.20. *When $H = g_{S^3}$, the most general solution to $\hat{D}^{(H)}_A \hat{\varepsilon}^{(+)}_H = \frac{1}{2}\hat{\gamma}_A \hat{\varepsilon}^{(+)}_H$ is*

$$\hat{\varepsilon}^{(+)}_H = \mathrm{e}^{\theta\hat{\gamma}^3/4}\mathrm{e}^{\phi_1\hat{\gamma}^2/2}\mathrm{e}^{-\phi_2\hat{\gamma}^3\hat{\gamma}^4/2}\hat{\varepsilon}_0, \tag{598}$$

where $\hat{\varepsilon}_0$ is an arbitrary constant spinor, (θ, ϕ_1, ϕ_2) are defined by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \cos(\theta/2)\cos(\phi_1) \\ \cos(\theta/2)\sin(\phi_1) \\ \sin(\theta/2)\cos(\phi_2) \\ \sin(\theta/2)\sin(\phi_2) \end{bmatrix} \tag{599}$$

and the vielbein is[34]

$$e^2 = \cos(\theta/2)\mathrm{d}\phi_1,\ e^3 = \frac{1}{2}\mathrm{d}\theta \text{ and } e^4 = \sin(\theta/2)\mathrm{d}\phi_2 \tag{600}$$

$$\iff e_2 = \frac{1}{\cos(\theta/2)}\partial_{\phi_1},\ e_3 = 2\partial_\theta \text{ and } e_4 = \frac{1}{\sin(\theta/2)}\partial_{\phi_2}. \tag{601}$$

Furthermore, the most general solution to $\hat{D}^{(H)}_A \hat{\varepsilon}^{(-)}_H = -\frac{1}{2}\hat{\gamma}_A \hat{\varepsilon}^{(-)}_H$ is

$$\hat{\varepsilon}^{(-)}_H = \mathrm{e}^{-\theta\hat{\gamma}^3/4}\mathrm{e}^{-\phi_1\hat{\gamma}^2/2}\mathrm{e}^{-\phi_2\hat{\gamma}^3\hat{\gamma}^4/2}\hat{\varepsilon}_0. \tag{602}$$

[34]The ordering of e^A is chosen so that $\Lambda^I{}_J$ in lemma 4.5 has determinant 1, not -1.

Proof. To write the Killing spinor equation, I first need the connection coefficients[35]. For that,

$$\mathrm{d}e^3 = 0,\ \mathrm{d}e^2 = -\frac{1}{2}\sin(\theta/2)\mathrm{d}\theta\wedge\mathrm{d}\phi_1 = -\tan(\theta/2)e^3\wedge e^2$$
$$\text{and } \mathrm{d}e^4 = \frac{1}{2}\cos(\theta/2)\mathrm{d}\theta\wedge\mathrm{d}\phi_2 = \cot(\theta/2)e^3\wedge e^4. \tag{603}$$

From $\mathrm{d}e^A = -\omega^A{}_B\wedge e^B$, it follows that

$$\omega_{32} = \tan(\theta/2)e^2,\ \omega_{34} = -\cot(\theta/2)e^4 \text{ and } \omega_{24} = 0. \tag{604}$$

$\therefore 0 = e_A{}^\alpha\partial_\alpha\hat{\varepsilon}_H^{(+)} - \frac{1}{4}\omega_{BCA}\hat{\gamma}^{BC}\hat{\varepsilon}_H^{(+)} - \frac{1}{2}\hat{\gamma}_A\hat{\varepsilon}_H^{(+)}$ reduces to the three equations,

$$0 = 2\partial_\theta\hat{\varepsilon}_H^{(+)} - 0 - \frac{1}{2}\hat{\gamma}_3\hat{\varepsilon}_H^{(+)}, \tag{605}$$
$$0 = \frac{1}{\cos(\theta/2)}\partial_{\phi_1}\hat{\varepsilon}_H^{(+)} - \frac{1}{2}\tan(\theta/2)\hat{\gamma}^3\hat{\gamma}^2\hat{\varepsilon}_H^{(+)} - \frac{1}{2}\hat{\gamma}_2\hat{\varepsilon}_H^{(+)} \text{ and} \tag{606}$$
$$0 = \frac{1}{\sin(\theta/2)}\partial_{\phi_2}\hat{\varepsilon}_H^{(+)} + \frac{1}{2}\cot(\theta/2)\hat{\gamma}^3\hat{\gamma}^4\hat{\varepsilon}_H^{(+)} - \frac{1}{2}\hat{\gamma}_4\hat{\varepsilon}_H^{(+)}. \tag{607}$$
$$\therefore \partial_\theta\hat{\varepsilon}_H^{(+)} = \frac{1}{4}\hat{\gamma}^3\hat{\varepsilon}_H^{(+)}, \tag{608}$$
$$\partial_{\phi_1}\hat{\varepsilon}_H^{(+)} = \frac{1}{2}\sin(\theta/2)\hat{\gamma}^3\hat{\gamma}^2\hat{\varepsilon}_H^{(+)} + \frac{1}{2}\cos(\theta/2)\hat{\gamma}^2\hat{\varepsilon}_H^{(+)} \text{ and} \tag{609}$$
$$\partial_{\phi_2}\hat{\varepsilon}_H^{(+)} = -\frac{1}{2}\cos(\theta/2)\hat{\gamma}^3\hat{\gamma}^4\hat{\varepsilon}_H^{(+)} + \frac{1}{2}\sin(\theta/2)\hat{\gamma}^4\hat{\varepsilon}_H^{(+)}. \tag{610}$$

The first equation immediately integrates to give $\hat{\varepsilon}_H^{(+)} = \mathrm{e}^{\theta\hat{\gamma}^3/4}\hat{\varepsilon}_\theta$, for a spinor, $\hat{\varepsilon}_\theta$, that doesn't depend on θ.
Using $\mathrm{e}^{\theta\hat{\gamma}^3} = \cos(\theta)I + \sin(\theta)\hat{\gamma}^3$, the other two equations simplify as follows.

$$\partial_{\phi_1}\hat{\varepsilon}_\theta = \frac{1}{2}\sin(\theta/2)\mathrm{e}^{-\theta\hat{\gamma}^3/4}\hat{\gamma}^3\hat{\gamma}^2\mathrm{e}^{\theta\hat{\gamma}^3/4}\hat{\varepsilon}_\theta + \frac{1}{2}\cos(\theta/2)\mathrm{e}^{-\theta\hat{\gamma}^3/4}\hat{\gamma}^2\mathrm{e}^{\theta\hat{\gamma}^3/4}\hat{\varepsilon}_\theta \tag{611}$$
$$= \frac{1}{2}\sin(\theta/2)\mathrm{e}^{-\theta\hat{\gamma}^3/2}\hat{\gamma}^3\hat{\gamma}^2\hat{\varepsilon}_\theta + \frac{1}{2}\cos(\theta/2)\mathrm{e}^{-\theta\hat{\gamma}^3/2}\hat{\gamma}^2\hat{\varepsilon}_\theta \tag{612}$$
$$= \frac{1}{2}\sin(\theta/2)(\cos(\theta/2)I - \sin(\theta/2)\hat{\gamma}^3)\hat{\gamma}^3\hat{\gamma}^2\hat{\varepsilon}_\theta$$
$$+ \frac{1}{2}\cos(\theta/2)(\cos(\theta/2)I - \sin(\theta/2)\hat{\gamma}^3)\hat{\gamma}^2\hat{\varepsilon}_\theta \tag{613}$$
$$= \frac{1}{2}\hat{\gamma}^2\hat{\varepsilon}_\theta. \tag{614}$$

$$\partial_{\phi_2}\hat{\varepsilon}_\theta = -\frac{1}{2}\cos(\theta/2)\mathrm{e}^{-\theta\hat{\gamma}^3/4}\hat{\gamma}^3\hat{\gamma}^4\mathrm{e}^{\theta\hat{\gamma}^3/4}\hat{\varepsilon}_\theta + \frac{1}{2}\sin(\theta/2)\mathrm{e}^{-\theta\hat{\gamma}^3/4}\hat{\gamma}^4\mathrm{e}^{\theta\hat{\gamma}^3/4}\hat{\varepsilon}_\theta \tag{615}$$
$$= -\frac{1}{2}\cos(\theta/2)\mathrm{e}^{-\theta\hat{\gamma}^3/2}\hat{\gamma}^3\hat{\gamma}^4\hat{\varepsilon}_\theta + \frac{1}{2}\sin(\theta/2)\mathrm{e}^{-\theta\hat{\gamma}^3/2}\hat{\gamma}^4\hat{\varepsilon}_\theta \tag{616}$$
$$= -\frac{1}{2}\cos(\theta/2)(\cos(\theta/2)I - \sin(\theta/2)\hat{\gamma}^3)\hat{\gamma}^3\hat{\gamma}^4\hat{\varepsilon}_\theta$$
$$+ \frac{1}{2}\sin(\theta/2)(\cos(\theta/2)I - \sin(\theta/2)\hat{\gamma}^3)\hat{\gamma}^4\hat{\varepsilon}_\theta \tag{617}$$
$$= -\frac{1}{2}\hat{\gamma}^3\hat{\gamma}^4\hat{\varepsilon}_\theta. \tag{618}$$

[35]I will omit superscript (H)s in this lemma given everything is restricted to the cross-section.

Since $\hat{\gamma}^2$ and $\hat{\gamma}^3\hat{\gamma}^4$ commute, these two equations are simultaneously solved by

$$\hat{\varepsilon}_\theta = \mathrm{e}^{\phi_1\hat{\gamma}^2/2}\mathrm{e}^{-\phi_2\hat{\gamma}^3\hat{\gamma}^4/2}\hat{\varepsilon}_0, \tag{619}$$

for a constant spinor, $\hat{\varepsilon}_0$.
$\hat{\varepsilon}_H^{(-)}$ follows immediately because $\hat{D}_A^{(H)}\hat{\varepsilon}_H^{(-)} = -\frac{1}{2}\hat{\gamma}_A\hat{\varepsilon}_H^{(-)}$ is identical to $\hat{D}_A^{(H)}\hat{\varepsilon}_H^{(+)} = \frac{1}{2}\hat{\gamma}_A\hat{\varepsilon}_H^{(+)}$ except that $\hat{\gamma}_A$ is replaced by $-\hat{\gamma}_A$ everywhere. □

Corollary 4.20.1. *Choose $\hat{\gamma}^2 = \mathrm{i}\sigma_1$, $\hat{\gamma}^3 = \mathrm{i}\sigma_2$ and $\hat{\gamma}^4 = \mathrm{i}\sigma_3$. Then, the solutions to $\hat{D}_A^{(H)}\hat{\varepsilon}_H^{(\pm)} = \pm\frac{1}{2}\hat{\gamma}_A\hat{\varepsilon}_H^{(\pm)}$ for $L(p,1)$ cross-sections are given by $\hat{\varepsilon}_H^{(+)} = 0$ and*

$$\hat{\varepsilon}_H^{(-)} = \mathrm{e}^{-\mathrm{i}\theta\sigma_2/4}\mathrm{e}^{-\mathrm{i}(\phi_1-\phi_2)\sigma_1/2}\hat{\varepsilon}_0, \tag{620}$$

for an arbitrary constant spinor, $\hat{\varepsilon}_0$.

Proof. The metric is locally identical to S^3.
$\therefore$ Any solution will simply be a further restriction on the $\hat{\varepsilon}_H^{(\pm)}$ found in the main lemma. With the chosen gamma matrices, those solutions are

$$\hat{\varepsilon}_H^{(\pm)} = \mathrm{e}^{\pm\mathrm{i}\theta\sigma_2/4}\mathrm{e}^{\pm\mathrm{i}(\phi_1\pm\phi_2)\sigma_1/2}\hat{\varepsilon}_0^{(\pm)}. \tag{621}$$

For a Killing spinor of S^3 to remain a Killing spinor of $L(p,1)$, it must remain invariant under the $\mathbb{Z}_p$ action.

$$\therefore \hat{\varepsilon}_H^{(\pm)} \to \mathrm{e}^{\pm\mathrm{i}\theta\sigma_2/4}\mathrm{e}^{\pm\mathrm{i}((\phi_1+2\pi/p)\pm(\phi_2+2\pi/p))\sigma_1/2}\hat{\varepsilon}_0^{(\pm)}. \tag{622}$$

Choosing the $-$ in $\pm$ means the $2\pi/p$ factors immediately cancel and the spinor is left invariant.
$\therefore$ Every $\hat{\varepsilon}_H^{(-)}$ of S^3 is also a $\hat{\varepsilon}_H^{(-)}$ of $L(p,1)$.
Meanwhile, in the $+$ case,

$$\hat{\varepsilon}_H^{(+)} \to \mathrm{e}^{\mathrm{i}\theta\sigma_2/4}\mathrm{e}^{\mathrm{i}(\phi_1+\phi_2+4\pi/p)\sigma_1/2}\hat{\varepsilon}_0 = \mathrm{e}^{\mathrm{i}\theta\sigma_2/4}\mathrm{e}^{\mathrm{i}(\phi_1+\phi_2)\sigma_1/2}\mathrm{e}^{2\pi\mathrm{i}\sigma_1/p}\hat{\varepsilon}_0. \tag{623}$$

Since $\mathrm{e}^{\mathrm{i}\theta\sigma_2/4}\mathrm{e}^{\mathrm{i}(\phi_1+\phi_2)\sigma_1/2}$ is invertible, $\hat{\varepsilon}_H^{(+)}$ remains invariant if and only if $\mathrm{e}^{2\pi\mathrm{i}\sigma_1/p}\hat{\varepsilon}_0 = \hat{\varepsilon}_0$.
However, $\mathrm{e}^{2\pi\mathrm{i}\sigma_1/p} = \cos(2\pi/p)I + \mathrm{i}\sin(2\pi/p)\sigma_1$ has eigenvalues $\cos(2\pi/p) \pm \mathrm{i}\sin(2\pi/p)$, neither of which is 1.
$\therefore \mathrm{e}^{2\pi\mathrm{i}\sigma_1/p}\hat{\varepsilon}_0 = \hat{\varepsilon}_0$ has no solution.
$\therefore$ The only $\hat{\varepsilon}_H^{(+)}$ on $L(p,1)$ is 0. □

A more foundational issue not addressed in the corollary is the question of spin structures on $L(p,1)$. Luckily, it is shown in [54] that for odd p, there exists a unique spin structure, while for even p, there exist two inequivalent spin structures.

Lemma 4.21. *The Killing vectors on $L(p,1)$ are spanned by*

$$k_1 = \frac{\partial}{\partial\phi_1} + \frac{\partial}{\partial\phi_2} = 2\frac{\partial}{\partial\psi}, \tag{624}$$

$$k_2 = \frac{\partial}{\partial\phi_1} - \frac{\partial}{\partial\phi_2} = 2\frac{\partial}{\partial\phi}, \tag{625}$$

$$k_3 = \tan(\theta/2)\sin(\phi_1-\phi_2)\frac{\partial}{\partial\phi_1} + 2\cos(\phi_1-\phi_2)\frac{\partial}{\partial\theta} + \cot(\theta/2)\sin(\phi_1-\phi_2)\frac{\partial}{\partial\phi_2} \quad \text{and} \tag{626}$$

$$k_4 = \tan(\theta/2)\cos(\phi_1-\phi_2)\frac{\partial}{\partial\phi_1} - 2\sin(\phi_1-\phi_2)\frac{\partial}{\partial\theta} + \cot(\theta/2)\cos(\phi_1-\phi_2)\frac{\partial}{\partial\phi_2}. \tag{627}$$

Proof. The Killing vectors of a sphere are known to be spanned by

$$v_{IJ} = \left(\hat{x}_I \frac{\partial \theta^\alpha}{\partial x^J}\bigg|_{\rho=1} - \hat{x}_J \frac{\partial \theta^\alpha}{\partial x^I}\bigg|_{\rho=1} \right) \partial_\alpha. \tag{628}$$

From equations, 475 to 478,

$$\begin{aligned} &\frac{\partial \phi_1}{\partial x_1}\bigg|_{\rho=1} = -\frac{\sin(\phi_1)}{\cos(\theta/2)}, \ \frac{\partial \phi_1}{\partial x_2}\bigg|_{\rho=1} = \frac{\cos(\phi_1)}{\cos(\theta/2)}, \ \frac{\partial \phi_1}{\partial x_3}\bigg|_{\rho=1} = \frac{\partial \phi_1}{\partial x_4}\bigg|_{\rho=1} = 0, \\ &\frac{\partial \phi_2}{\partial x_1}\bigg|_{\rho=1} = \frac{\partial \phi_2}{\partial x_2}\bigg|_{\rho=1} = 0, \ \frac{\partial \phi_2}{\partial x_3}\bigg|_{\rho=1} = -\frac{\sin(\phi_2)}{\sin(\theta/2)} \text{ and } \frac{\partial \phi_2}{\partial x_4}\bigg|_{\rho=1} = \frac{\cos(\phi_2)}{\sin(\theta/2)}. \end{aligned} \tag{629}$$

For $\partial\theta/\partial x^I$,

$$\tan^2(\theta/2) = \frac{x_3^2 + x_4^2}{x_1^2 + x_2^2} \implies \frac{\partial}{\partial x^I}\left(\frac{x_3^2 + x_4^2}{x_1^2 + x_2^2}\right) = 2\tan(\theta/2)\frac{1}{\cos^2(\theta/2)}\frac{1}{2}\frac{\partial \theta}{\partial x^I}. \tag{630}$$

$$\therefore \frac{\partial \theta}{\partial x^I} = \frac{\cos^3(\theta/2)}{\sin(\theta/2)}\frac{\partial}{\partial x^I}\left(\frac{x_3^2 + x_4^2}{x_1^2 + x_2^2}\right). \tag{631}$$

$$\therefore \frac{\partial \theta}{\partial x^1}\bigg|_{\rho=1} = \frac{\cos^3(\theta/2)}{\sin(\theta/2)}\left(-\frac{x_3^2 + x_4^2}{(x_1^2 + x_2^2)^2}\right)2x_1 \tag{632}$$

$$= -\frac{\cos^3(\theta/2)}{\sin(\theta/2)}\frac{\sin^2(\theta/2)}{\cos^4(\theta/2)}2\cos(\theta/2)\cos(\phi_1) \tag{633}$$

$$= -2\sin(\theta/2)\cos(\phi_1), \tag{634}$$

$$\frac{\partial \theta}{\partial x^3}\bigg|_{\rho=1} = \frac{\cos^3(\theta/2)}{\sin(\theta/2)}\frac{2x_3}{x_1^2 + x_2^2} = \frac{\cos^3(\theta/2)}{\sin(\theta/2)}\frac{2\sin(\theta/2)\cos(\phi_2)}{\cos^2(\theta/2)} = 2\cos(\theta/2)\cos(\phi_2), \tag{635}$$

$$\text{and similarly } \frac{\partial \theta}{\partial x^2}\bigg|_{\rho=1} = -2\sin(\theta/2)\sin(\phi_1) \ \& \ \frac{\partial \theta}{\partial x^4}\bigg|_{\rho=1} = 2\cos(\theta/2)\sin(\phi_2). \tag{636}$$

Then, by computer algebra, one finds

$$v_{12} = \frac{\partial}{\partial \phi_1}, \tag{637}$$

$$v_{13} = 2\cos(\phi_1)\cos(\phi_2)\frac{\partial}{\partial \theta} + \tan(\theta/2)\sin(\phi_1)\cos(\phi_2)\frac{\partial}{\partial \phi_1} - \cot(\theta/2)\cos(\phi_1)\sin(\phi_2)\frac{\partial}{\partial \phi_2}, \tag{638}$$

$$v_{14} = 2\cos(\phi_1)\sin(\phi_2)\frac{\partial}{\partial \theta} + \tan(\theta/2)\sin(\phi_1)\sin(\phi_2)\frac{\partial}{\partial \phi_1} + \cot(\theta/2)\cos(\phi_1)\cos(\phi_2)\frac{\partial}{\partial \phi_2}, \tag{639}$$

$$v_{23} = 2\sin(\phi_1)\cos(\phi_2)\frac{\partial}{\partial \theta} - \tan(\theta/2)\cos(\phi_1)\cos(\phi_2)\frac{\partial}{\partial \phi_1} - \cot(\theta/2)\sin(\phi_1)\sin(\phi_2)\frac{\partial}{\partial \phi_2}, \tag{640}$$

$$v_{24} = 2\sin(\phi_1)\sin(\phi_2)\frac{\partial}{\partial \theta} - \tan(\theta/2)\cos(\phi_1)\sin(\phi_2)\frac{\partial}{\partial \phi_1} + \cot(\theta/2)\sin(\phi_1)\cos(\phi_2)\frac{\partial}{\partial \phi_2}, \tag{641}$$

$$v_{34} = \frac{\partial}{\partial \phi_2}. \tag{642}$$

Then, change to a new basis of Killing vectors by

$$k_1 = v_{12} + v_{34} = \frac{\partial}{\partial\phi_1} + \frac{\partial}{\partial\phi_2} = 2\frac{\partial}{\partial\psi}, \tag{643}$$

$$k_2 = v_{12} - v_{34} = \frac{\partial}{\partial\phi_1} - \frac{\partial}{\partial\phi_2} = 2\frac{\partial}{\partial\phi}, \tag{644}$$

$$k_3 = v_{24} + v_{13} \tag{645}$$

$$= \tan(\theta/2)\sin(\phi_1 - \phi_2)\frac{\partial}{\partial\phi_1} + 2\cos(\phi_1 - \phi_2)\frac{\partial}{\partial\theta} + \cot(\theta/2)\sin(\phi_1 - \phi_2)\frac{\partial}{\partial\phi_2}, \tag{646}$$

$$k_4 = v_{14} - v_{23} \tag{647}$$

$$= \tan(\theta/2)\cos(\phi_1 - \phi_2)\frac{\partial}{\partial\phi_1} - 2\sin(\phi_1 - \phi_2)\frac{\partial}{\partial\theta} + \cot(\theta/2)\cos(\phi_1 - \phi_2)\frac{\partial}{\partial\phi_2}, \tag{648}$$

$$k_5 = v_{13} - v_{24} \tag{649}$$

$$= \tan(\theta/2)\sin(\phi_1 + \phi_2)\frac{\partial}{\partial\phi_1} + 2\cos(\phi_1 + \phi_2)\frac{\partial}{\partial\theta} - \cot(\theta/2)\sin(\phi_1 + \phi_2)\frac{\partial}{\partial\phi_2} \text{ and} \tag{650}$$

$$k_6 = v_{14} + v_{23} \tag{651}$$

$$= -\tan(\theta/2)\cos(\phi_1 + \phi_2)\frac{\partial}{\partial\phi_1} + 2\sin(\phi_1 + \phi_2)\frac{\partial}{\partial\theta} + \cot(\theta/2)\cos(\phi_1 + \phi_2)\frac{\partial}{\partial\phi_2}. \tag{652}$$

The metric on $L(p,1)$ is locally identical to the metric on the S^3.
$\therefore$ The Killing vectors of $L(p,1)$ are a subspace of the Killing vectors of S^3.
In this case, k_1, k_2, k_3 and k_4 are manifestly well-defined on the lens space and any linear combination involving k_5 and k_6 is not well-defined on the lens space. □

Definition 4.22 (Angular momenta on $L(p,1)$)**.** *For each Killing vector, k_I, on $L(p,1)$, define a "conserved angular momentum,"*

$$J_I = \frac{1}{4\pi}\int_{\partial\Sigma_t} f_{(4)0\alpha}\, k_I^\alpha\, \mathrm{d}(L(p,1)). \tag{653}$$

Not that these J_Is are identical to the "conserved quantities" of definition 4.16.

Theorem 4.23 ($L(p,1)$ cross-section positive energy theorem)**.** *If the Einstein equation holds, T_{ab} satisfies the dominant energy condition and $T^{0\mu}$ decays faster than $O(\mathrm{e}^{-4r})$ near $\partial_\infty\Sigma_t$, then*

$$E \geq \sqrt{J_2^2 + J_3^2 + J_4^2}. \tag{654}$$

Note that J_1 does not appear in the theorem.

Proof. The proof is merely substituting the $\hat{\varepsilon}_H^{(\pm)}$ in corollary 4.20.1 into theorem 4.14. Since $\hat{\varepsilon}_H^{(+)} = 0$, the integrand is simply

$$\begin{aligned}
&p_0\hat{\varepsilon}_H^{(-)\dagger}\hat{\varepsilon}_H^{(-)} - \mathrm{i}p_A\hat{\varepsilon}_H^{(-)\dagger}\hat{\gamma}^A\hat{\varepsilon}_H^{(-)} \\
&= p_0\hat{\varepsilon}_0^\dagger \mathrm{e}^{\mathrm{i}(\phi_1-\phi_2)\sigma_1/2}\mathrm{e}^{\mathrm{i}\theta\sigma_2/4}\mathrm{e}^{-\mathrm{i}\theta\sigma_2/4}\mathrm{e}^{-\mathrm{i}(\phi_1-\phi_2)\sigma_1/2}\hat{\varepsilon}_0 + p_2\hat{\varepsilon}_0^\dagger \mathrm{e}^{\mathrm{i}(\phi_1-\phi_2)\sigma_1/2}\mathrm{e}^{\mathrm{i}\theta\sigma_2/4}\sigma_1\mathrm{e}^{-\mathrm{i}\theta\sigma_2/4}\mathrm{e}^{-\mathrm{i}(\phi_1-\phi_2)\sigma_1/2}\hat{\varepsilon}_0 \\
&\quad + p_3\hat{\varepsilon}_0^\dagger \mathrm{e}^{\mathrm{i}(\phi_1-\phi_2)\sigma_1/2}\mathrm{e}^{\mathrm{i}\theta\sigma_2/4}\sigma_2\mathrm{e}^{-\mathrm{i}\theta\sigma_2/4}\mathrm{e}^{-\mathrm{i}(\phi_1-\phi_2)\sigma_1/2}\hat{\varepsilon}_0 \\
&\quad + p_4\hat{\varepsilon}_0^\dagger \mathrm{e}^{\mathrm{i}(\phi_1-\phi_2)\sigma_1/2}\mathrm{e}^{\mathrm{i}\theta\sigma_2/4}\sigma_3\mathrm{e}^{-\mathrm{i}\theta\sigma_2/4}\mathrm{e}^{-\mathrm{i}(\phi_1-\phi_2)\sigma_1/2}\hat{\varepsilon}_0
\end{aligned} \tag{655}$$

$$\begin{aligned}
&= \hat{\varepsilon}_0^\dagger\big(p_0 I + (p_2\cos(\theta/2) - p_4\sin(\theta/2))\sigma_1 \\
&\quad + (p_2\sin(\theta/2)\sin(\phi_1 - \phi_2) + p_3\cos(\phi_1 - \phi_2) + p_4\cos(\theta/2)\sin(\phi_1 - \phi_2))\sigma_2 \\
&\quad + (p_2\sin(\theta/2)\cos(\phi_1 - \phi_2) - p_3\sin(\phi_1 - \phi_2) + p_4\cos(\theta/2)\cos(\phi_1 - \phi_2))\sigma_3\big)\hat{\varepsilon}_0,
\end{aligned} \tag{656}$$

using computer algebra.
From definition 3.18 and the vielbein I've chosen in lemma 4.20,

$$\tilde{f}_{(0)}^{mn} f_{(4)mn} = p_0, \quad f_{(4)03} = \frac{1}{2}p_3, \quad f_{(4)02} = \cos(\theta/2)p_2 \text{ and } f_{(4)04} = \sin(\theta/2)p_4. \tag{657}$$

Thus, equation 656 says

$$\begin{aligned}
&p_0\hat{\varepsilon}_H^{(-)\dagger}\hat{\varepsilon}_H^{(-)} - \mathrm{i}p_A\hat{\varepsilon}_H^{(-)\dagger}\hat{\gamma}^A\hat{\varepsilon}_H^{(-)} \\
&= \hat{\varepsilon}_0^\dagger\big((f_{(4)02}\tan(\theta/2)\sin(\phi_1-\phi_2) + 2f_{(4)03}\cos(\phi_1-\phi_2) + f_{(4)04}\cot(\theta/2)\sin(\phi_1-\phi_2))\sigma_2 \\
&\quad + (f_{(4)02}\tan(\theta/2)\cos(\phi_1-\phi_2) - 2f_{(4)03}\sin(\phi_1-\phi_2) + f_{(4)04}\cot(\theta/2)\cos(\phi_1-\phi_2))\sigma_3 \\
&\quad + \tilde{f}_{(0)}^{mn}f_{(4)mn}I + (f_{(4)02} - f_{(4)04})\sigma_1\big)\hat{\varepsilon}_0 \qquad (658) \\
&= \hat{\varepsilon}_0^\dagger\left(\tilde{f}_{(0)}^{mn}f_{(4)mn}I + f_{(4)0\alpha}k_2^\alpha\sigma_1 + f_{(4)0\alpha}k_3^\alpha\sigma_2 + f_{(4)0\alpha}k_4^\alpha\sigma_3\right)\hat{\varepsilon}_0 \text{ by lemma 4.21.} \qquad (659)
\end{aligned}$$

Then, from definition 4.22 and theorem 4.14,

$$\begin{aligned}
0 &\leq \int_{L(p,1)} \left(p_0\hat{\varepsilon}_H^{(-)\dagger}\hat{\varepsilon}_H^{(-)} - \mathrm{i}p_A\hat{\varepsilon}_H^{(-)\dagger}\hat{\gamma}^A\hat{\varepsilon}_H^{(-)}\right)\mathrm{d}(L(p,1)) \qquad (660) \\
&= 4\pi\hat{\varepsilon}_0^\dagger\left(EI + J_2\sigma_1 + J_3\sigma_2 + J_4\sigma_3\right)\hat{\varepsilon}_0. \qquad (661)
\end{aligned}$$

The eigenvalues of the matrix inbetween $\hat{\varepsilon}_0^\dagger$ and $\hat{\varepsilon}_0$ are

$$E \pm \sqrt{J_2^2 + J_3^2 + J_4^2} \tag{662}$$

and hence the theorem follows. □

4.3.3 The matrix reloaded - asymptotically AdS spaces once again

The simplest metric which can be written in the form of equation 496 is AdS itself. Hence, the results of section 4.2 should be reproducible using theorem 4.13. In particular, the matrix in equation 4.8 should appear via equation 4.13 as well. In this section, I'll prove this is indeed the case.

To deal with spheres, S^{n-2}, in arbitrary dimensions, the only practical coordinate system is the "nested spheres." In particular,

$$x^I = \rho\begin{bmatrix}\cos(\theta_2) \\ \sin(\theta_2)\cos(\theta_3) \\ \vdots \\ \sin(\theta_2)\cdots\sin(\theta_{n-2})\cos(\theta_{n-1}) \\ \sin(\theta_2)\cdots\sin(\theta_{n-2})\sin(\theta_{n-1})\end{bmatrix} \text{ and} \tag{663}$$

$$H = \rho^2(\mathrm{d}\theta_2\otimes\mathrm{d}\theta_2 + \sin^2(\theta_2)\mathrm{d}\theta_3\otimes\mathrm{d}\theta_3 + \cdots + \sin^2(\theta_2)\cdots\sin^2(\theta_{n-2})\mathrm{d}\theta_{n-1}\otimes\mathrm{d}\theta_{n-1}). \tag{664}$$

The natural vielbein to use on the unit sphere - $e^{(s)}$ as I've called it in section 4.2 - is thus

$$e^{(s)2} = \mathrm{d}\theta_2, \; e^{(s)3} = \sin(\theta_2)\mathrm{d}\theta_3, \; \cdots, \; e^{(s)n-1} = \sin(\theta_2)\cdots\sin(\theta_{n-2})\mathrm{d}\theta_{n-1}. \tag{665}$$

In this frame, the most general solution to $D_A^{(H)}\varepsilon_H = \frac{1}{2}\gamma_A\varepsilon_H$ on the unit sphere is [55]

$$\varepsilon_H = \mathrm{e}^{\theta_2\gamma^2/2}\mathrm{e}^{\theta_3\gamma^3\gamma^2/2}\cdots\mathrm{e}^{\theta_{n-1}\gamma^{n-1}\gamma^{n-2}/2}\varepsilon_0 \tag{666}$$

for a constant spinor, ε_0.

It is well known - and easily verifiable - that

$$(M_{IJ})_{KL} = \delta_{IK}\delta_{JL} - \delta_{IL}\delta_{JK} \tag{667}$$

satisfies the $\mathfrak{o}(\mathfrak{n}-\mathfrak{1})$ Lie algebra[36],

$$[M_{IJ}, M_{KL}] = \delta_{IL}M_{JK} - \delta_{IK}M_{JL} - \delta_{JL}M_{IK} + \delta_{JK}M_{IL}. \tag{668}$$

Likewise, it's also a standard result that this Lie algebra is also satisfied by

$$S_{IJ} = -\frac{1}{4}[\gamma_I, \gamma_J] = -\frac{1}{2}\gamma_{IJ}. \tag{669}$$

By definition, a spinor is an object that transforms as

$$\varepsilon \to S[\Lambda]\varepsilon = \mathrm{e}^{\frac{1}{2}\omega_{IJ}S^{IJ}}\varepsilon = \mathrm{e}^{-\frac{1}{4}\omega_{IJ}\gamma^{IJ}}\varepsilon \tag{670}$$

under a Lorentz transformation defined by $\Lambda = \mathrm{e}^{\frac{1}{2}\omega_{IJ}M^{IJ}}$.
The main objective of this subsection is to write the Λ from lemma 4.5 as a product of exponentials, $\mathrm{e}^{\frac{1}{2}\omega_{IJ}M^{IJ}}$, deduce the corresponding $S[\Lambda]$ and thereby prove the spinor in lemma 4.6 is equivalent to the spinor defined by theorem 4.12.

Lemma 4.24. *The Lorentz transformation, Λ, is given by*

$$\begin{bmatrix} \cos(\theta_2) & \sin(\theta_2)\cos(\theta_3) & \cdots & \sin(\theta_2)\cdots\sin(\theta_{n-2})\cos(\theta_{n-1}) & \sin(\theta_2)\cdots\sin(\theta_{n-2})\sin(\theta_{n-1}) \\ -\sin(\theta_2) & \cos(\theta_2)\cos(\theta_3) & \cdots & \cos(\theta_2)\sin(\theta_3)\cdots\sin(\theta_{n-2})\cos(\theta_{n-1}) & \cos(\theta_2)\sin(\theta_3)\cdots\sin(\theta_{n-1}) \\ 0 & -\sin(\theta_3) & \cdots & \cos(\theta_3)\sin(\theta_4)\cdots\sin(\theta_{n-2})\cos(\theta_{n-1}) & \cos(\theta_3)\sin(\theta_4)\cdots\sin(\theta_{n-1}) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -\sin(\theta_{n-1}) & \cos(\theta_{n-1}) \end{bmatrix} \tag{671}$$

$$= \mathrm{e}^{\theta_2 M_{12}} \cdots \mathrm{e}^{\theta_{n-1}M_{n-2,n-1}}. \tag{672}$$

Proof. From lemma 4.5,

$$\Lambda^I{}_J = \delta^I{}_1 \hat{x}_J + \delta^I{}_A \rho \frac{\partial\theta^\alpha}{\partial x^J} e^{(s)A}_\alpha. \tag{673}$$

$\therefore$ The first row Λ can be read-off from equation 663.
$\frac{\partial\theta^\alpha}{\partial x^J}$ can be calculated from

$$\frac{x^2_{\alpha-1}}{x^2_\alpha + \cdots + x^2_{n-1}} = \cot^2(\theta_\alpha). \tag{674}$$

$$\therefore \frac{\partial}{\partial x^I}\frac{x^2_{\alpha-1}}{x^2_\alpha + \cdots + x^2_{n-1}} = \frac{\partial}{\partial x^I}\cot^2(\theta_\alpha) = 2\cot(\theta_\alpha)\left(-\frac{1}{\sin^2(\theta_\alpha)}\right)\frac{\partial\theta_\alpha}{\partial x^I} = -\frac{2\cos(\theta_\alpha)}{\sin^3(\theta_\alpha)}\frac{\partial\theta_\alpha}{\partial x^I}. \tag{675}$$

$$\therefore \frac{\partial\theta_\alpha}{\partial x^I} = -\frac{\sin^3(\theta_\alpha)}{2\cos(\theta_\alpha)}\frac{\partial}{\partial x^I}\frac{x^2_{\alpha-1}}{x^2_\alpha + \cdots + x^2_{n-1}}. \tag{676}$$

[36] Really, I should consider $\mathfrak{o}(\mathfrak{1}, \mathfrak{n}-\mathfrak{1})$, but because the local Lorentz transformation of lemma 4.5 is only amongst the e^I, it suffices to consider $\mathfrak{o}(\mathfrak{n}-\mathfrak{1})$.

$\therefore \frac{\partial\theta_\alpha}{\partial x^I} = 0$ when $I < \alpha - 1$.
When $I = \alpha - 1$,

$$\frac{\partial\theta_\alpha}{\partial x^I} = -\frac{\sin^3(\theta_\alpha)}{2\cos(\theta_\alpha)}\frac{2x_{\alpha-1}}{x_\alpha^2+\cdots+x_{n-1}^2} \tag{677}$$
$$= -\frac{\sin^3(\theta_\alpha)}{\cos(\theta_\alpha)}\frac{\rho\sin(\theta_2)\cdots\sin(\theta_{\alpha-1})\cos(\theta_\alpha)}{\rho^2\sin^2(\theta_2)\cdots\sin^2(\theta_\alpha)} \tag{678}$$
$$= -\frac{1}{\rho\sin(\theta_2)\cdots\sin(\theta_{\alpha-1})}\sin(\theta_\alpha). \tag{679}$$

When $I \geq \alpha$ (taking $\cos(\theta_n)$ to mean 1 in one of the equations below),

$$\frac{\partial\theta_\alpha}{\partial x^I} = -\frac{\sin^3(\theta_\alpha)}{2\cos(\theta_\alpha)}\left(-\frac{x_{\alpha-1}^2}{(x_\alpha^2+\cdots+x_{n-1}^2)^2}\right)2x_I \tag{680}$$
$$= \frac{\sin^3(\theta_\alpha)}{\cos(\theta_\alpha)}\frac{\rho^2\sin^2(\theta_2)\cdots\sin^2(\theta_{\alpha-1})\cos^2(\theta_\alpha)}{\rho^4\sin^4(\theta_2)\cdots\sin^4(\theta_\alpha)}\rho\sin(\theta_2)\cdots\sin(\theta_I)\cos(\theta_{I+1}) \tag{681}$$
$$= \frac{1}{\rho\sin(\theta_2)\cdots\sin(\theta_{\alpha-1})}\cos(\theta_\alpha)\sin(\theta_{\alpha+1})\cdots\sin(\theta_I)\cos(\theta_{I+1}). \tag{682}$$

Since $e^{(s)A} = \delta^A{}_\alpha\sin(\theta_2)\cdots\sin(\theta_{\alpha-1})\mathrm{d}\theta^\alpha$, I get the matrix in equation 671.
The exponential product in equation 672 then follows from

$$\exp\left(\theta\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\right) = \begin{bmatrix}\cos(\theta) & \sin(\theta)\\ -\sin(\theta) & \cos(\theta)\end{bmatrix}, \tag{683}$$

$M_{I,I+1}$, $M_{I+1,I+2}$, $\cdots$ only acting non-trivially on rows & columns with index $\geq I$ and induction (on n). ☐

Corollary 4.24.1. *The ε_k from lemma 4.6 agrees with the ε_k from equations 517 and 666.*

Proof. By definition, $\Lambda = \mathrm{e}^{\theta_2 M_{12}}\cdots\mathrm{e}^{\theta_{n-1}M_{n-2,n-1}}$ means

$$S[\Lambda] = \mathrm{e}^{\theta_2\gamma^2\gamma^1/2}\cdots\mathrm{e}^{\theta_{n-1}\gamma^{n-1}\gamma^{n-2}/2}. \tag{684}$$

Hence, the constant spinor, ε_0, in lemma 4.6 goes to

$$S[\Lambda]\varepsilon_0 = \mathrm{e}^{\theta_2\gamma^2\gamma^1/2}\cdots\mathrm{e}^{\theta_{n-1}\gamma^{n-1}\gamma^{n-2}/2}\varepsilon_0 \tag{685}$$

upon the change of frame.
ε_0 is an arbitrary constant spinor, so I can redefine it as $\varepsilon_0 = \frac{1}{\sqrt{2}}(I-\gamma^1)\tilde{\varepsilon}_0$ for an arbitrary constant spinor, $\tilde{\varepsilon}_0$.
Since I and γ^1 both commute with γ^A, I can push $I-\gamma^1$ past all the matrix exponentials until $\mathrm{e}^{\theta_2\gamma^2\gamma^1/2}$. Then,

$$\mathrm{e}^{\theta_2\gamma^2\gamma^1/2}(I-\gamma^1) = (\cos(\theta_2/2)I - \sin(\theta_2/2)\gamma^2\gamma^1)(I-\gamma^1) \tag{686}$$
$$= (I-\gamma^1)\cos(\theta_2/2)I + \gamma^1\sin(\theta_2/2)\gamma^2 - \sin(\theta_2/2)\gamma^2 \tag{687}$$
$$= (I-\gamma^1)(\cos(\theta_2/2)I - \sin(\theta_2/2)\gamma^2) \tag{688}$$
$$= (I-\gamma^1)\mathrm{e}^{\theta_2\gamma^2/2}. \tag{689}$$
$$\therefore S[\Lambda]\varepsilon_0 = \frac{1}{\sqrt{2}}(I-\gamma^1)\mathrm{e}^{\theta_2\gamma^2/2}\mathrm{e}^{\theta_3\gamma^3\gamma^2/2}\cdots\mathrm{e}^{\theta_{n-1}\gamma^{n-1}\gamma^{n-2}/2}\tilde{\varepsilon}_0 \tag{690}$$
$$= \frac{1}{\sqrt{2}}(I-\gamma^1)\varepsilon_H \text{ by equation 666.} \tag{691}$$

e^0 and e_0 are unchanged, so the $\mathrm{e}^{\mathrm{i}\gamma^0 t/2}$ factor in lemma 4.6 is unchanged.
Next, consider the $x^I\gamma_I$ term. To view ε_k as a well-defined spinor, x^I should be a vector. Hence, upon the change of frame, x^I should go to $\Lambda^I{}_J x^J$.
From equations 671 and 663, it follows by inspection that $\Lambda^I{}_J x^J = \rho(1,0,\cdots,0)^{\mathrm{T}}$.
$\therefore x^I\gamma_I$ goes to $\rho\gamma^1$ upon the vielbien transformation.
Putting all the different pieces together, I get

$$\varepsilon_k \to \frac{1}{\sqrt{2(1-\rho^2)}}(I - \mathrm{i}\rho\gamma^1)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}(I-\gamma^1)\varepsilon_H. \tag{692}$$

From lemma 4.4,

$$\mathrm{e}^r = \frac{1}{2}\left(\frac{2\rho}{1-\rho^2} + \sqrt{1+\frac{4\rho^2}{(1-\rho^2)^2}}\right) = \frac{(1+\rho)^2}{2(1-\rho^2)}. \tag{693}$$

$$\therefore 0 = \rho^2(1+2\mathrm{e}^r) + 2\rho + 1 - 2\mathrm{e}^{\mathrm{r}}. \tag{694}$$

$$\therefore \rho = \frac{-2+\sqrt{4-4(1+2\mathrm{e}^r)(1-2\mathrm{e}^{\mathrm{r}})}}{2(1+2\mathrm{e}^r)} = \frac{-1+2\mathrm{r}^2}{1+2\mathrm{e}^r} = \frac{1-\frac{1}{2}\mathrm{e}^{-r}}{1+\frac{1}{2}\mathrm{e}^{-r}}. \tag{695}$$

$$\therefore 1-\rho^2 = \frac{2\mathrm{e}^{-r}}{\left(1+\frac{1}{2}\mathrm{e}^{-r}\right)^2} \quad\text{and}\quad \frac{\rho}{\sqrt{1-\rho^2}} = \frac{1}{\sqrt{2}}\mathrm{e}^{r/2}\left(1-\frac{1}{2}\mathrm{e}^{-r}\right). \tag{696}$$

Substituting these relations back into 692, I get

$$\varepsilon_k \to \frac{1}{2}\mathrm{e}^{r/2}\left(1+\frac{1}{2}\mathrm{e}^{-r}\right)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}(I-\gamma^1)\varepsilon_H - \frac{\mathrm{i}}{2}\mathrm{e}^{r/2}\left(1-\frac{1}{2}\mathrm{e}^{-r}\right)\gamma^1\mathrm{e}^{\mathrm{i}\gamma^0 t/2}(I-\gamma^1)\varepsilon_H \tag{697}$$

$$= \mathrm{e}^{r/2}P_1^-\mathrm{e}^{\mathrm{i}\gamma^0 t/2}(I-\gamma^1)\varepsilon_H + \frac{1}{2}\mathrm{e}^{-r/2}P_1^+\mathrm{e}^{\mathrm{i}\gamma^0 t/2}(I-\gamma^1)\varepsilon_H \tag{698}$$

$$= \mathrm{e}^{r/2}P_1^-\left(\mathrm{e}^{\mathrm{i}\gamma^0 t/2} - \mathrm{i}\mathrm{e}^{-\mathrm{i}\gamma^0 t/2}\right)\varepsilon_H + \frac{1}{2}\mathrm{e}^{-r/2}P_1^+\left(\mathrm{e}^{\mathrm{i}\gamma^0 t/2} + \mathrm{i}\mathrm{e}^{-\mathrm{i}\gamma^0 t/2}\right)\varepsilon_H, \tag{699}$$

which is exactly the result in theorem 4.12. □

5 BPS inequalities

In this section, I'll apply theorem 3.19 to various supergravity theories by appropriately choosing A_μ. In particular, A_μ will be chosen so that the resulting ∇_μ generates the local transformation of the gravitino field, ψ_μ, in the supergravity, i.e.

$$\delta\psi_\mu = \nabla_\mu\varepsilon, \tag{700}$$

for transformation parameter, ε.
In a supergravity context, positive energy theorems typically go by the name BPS inequalities and I will adopt this terminology.
The presence of $\Lambda < 0$ necessitates considering not just supergravity, but gauged supergravity. I will focus on $\mathcal{N}=2$ theories in four and five dimensions in this work.
Both theories contain Maxwell fields. For convenience, if not physical significance, I'll split the Maxwell field as follows.

Definition 5.1 (Electric and magnetic components)**.** *Given a Maxwell field, F_{ab}, the electric and magnetic components will be defined as $E_I = F_{I0}$ and F_{IJ} respectively. Likewise, any current density, j^a, will be split into electric charge and current as $j^\mu \equiv (\rho, j^I)$.*

Note that the split is with respect to the vielbein indices, not the coordinate indices.
It is also natural to consider electric charges in the presence of Maxwell fields.

Definition 5.2 (Electric charge)**.** *The electric charge, q_e, is defined to be*

$$q_e = \frac{1}{4\pi}\int_{\partial_\infty \Sigma_t} \star F. \tag{701}$$

Corollary 5.2.1. *When the metric is written in Fefferman-Graham form,*

$$q_e = \frac{1}{4\pi}\int_{\partial_\infty \Sigma_t} E_1 \mathrm{d}A. \tag{702}$$

Proof. Writing F out in a vielbein basis,

$$q_e = \frac{1}{4\pi}\int_{\partial_\infty \Sigma_t} \frac{1}{(n-2)!}(\star F)_{\mu_1\cdots\mu_{n-2}} e^{\mu_1}\wedge\cdots\wedge e^{\mu_{n-2}} \tag{703}$$

$$= \frac{1}{4\pi}\int_{\partial_\infty \Sigma_t} \frac{1}{2(n-2)!}\varepsilon_{\nu\rho\mu_1\cdots\mu_{n-2}} F^{\nu\rho} e^{\mu_1}\wedge\cdots\wedge e^{\mu_{n-2}}. \tag{704}$$

I've been working in a vielbein where $e^0 \perp \Sigma_t$ and $e^1 \perp \Sigma_r$. Hence, $\partial_\infty \Sigma_t \perp e^0$, e^1. Then,

$$q_e = \frac{1}{4\pi}\int_{\partial_\infty \Sigma_t} \frac{1}{2}\varepsilon_{\mu\nu 2\cdots n-1} F^{\mu\nu} e^2\wedge\cdots\wedge e^{n-1} \tag{705}$$

$$= \frac{1}{4\pi}\int_{\partial_\infty \Sigma_t} \varepsilon_{012\cdots n-1} F^{01} e^2\wedge\cdots\wedge e^{n-1} \tag{706}$$

$$= \frac{1}{4\pi}\int_{\partial_\infty \Sigma_t} F^{01} \mathrm{d}A \tag{707}$$

$$= \frac{1}{4\pi}\int_{\partial_\infty \Sigma_t} E_1 \mathrm{d}A, \tag{708}$$

as claimed. □

In 4D, it is also possible to have magnetic charge, an essentially topological effect due to the inability to define a global one-form, a_a, such that $F = \mathrm{d}a$. However, the choices of A_μ in the subsequent sections will explicitly involve a_μ as part of gauge covariant derivatives. On it's own, this may not be an issue. Changing from one patch to another imposes a transition function, $a \to a + \mathrm{d}\lambda$; likewise performing a phase transformation, $\varepsilon \to \mathrm{e}^{\mathrm{i}\lambda}\varepsilon$ keeps the result consistent. Unfortunately, such a (pointwise) phase transformation in inconsistent with my effective choice of ε_k boundary conditions. Hence, I must restrict to electromagnetic fields sourced by a global one-form, a_a. Furthermore, even if this problem could be surmounted, I would still encounter the more practical problem that F_{AB}'s decay rate for magnetically charged solutions is too slow for the requirements of section 3.1.

5.1 4D, $\mathcal{N} = 2$, gauged supergravity

The bosonic sector of 4D, $\mathcal{N} = 2$, gauged supergravity is described the action,

$$S = \frac{1}{16\pi}\int_M \left(R - 2\Lambda - F_{ab}F^{ab}\right) \mathrm{d}V(g) + S^{\text{other}}_{\text{matter}}, \tag{709}$$

where $F_{ab} = D_a a_b - D_b a_a$, for some locally defined gauge field, a_a. Note this is nothing but Einstein-Maxwell theory with a cosmological constant. Strictly speaking, $S^{\text{other}}_{\text{matter}}$ should be zero for the supergravity theory or for Einstein-Maxwell theory, but, for completeness, I've left open the possibility of having further matter fields. I will however assume $S^{\text{other}}_{\text{matter}}$ couples to the Maxwell field at most through a term of the form, $\int_M j^a a_a \mathrm{d}V(g)$, where a_a is a (local) gauge field.

The equations of motion in this theory are well known to be

$$R_{ab} - \frac{1}{2}R g_{ab} + \Lambda g_{ab} = 8\pi T_{ab} = 2\left(F_a{}^c F_{bc} - \frac{1}{4}g_{ab}F^{cd}F_{cd}\right) + 8\pi T^{\text{other}}_{ab}, \tag{710}$$

$$D_b F^{ba} = -4\pi j^a, \tag{711}$$

$$D_{[a}F_{bc]} = 0 \tag{712}$$

and whatever equations the fields in $S^{\text{other}}_{\text{matter}}$ solve.

It is known that in this supergravity theory, the gravitino, ψ_μ, transforms under local supersymmetry transformations as

$$\delta\psi_\mu = D_\mu\varepsilon - \frac{1}{4}F_{\nu\rho}\gamma^{\nu\rho}\gamma_\mu\varepsilon + \mathrm{i}a_\mu\varepsilon + \frac{\mathrm{i}}{2}\gamma_\mu\varepsilon, \tag{713}$$

for a given spinor parameter, ε. Hence, I will choose

$$A_\mu = -\frac{1}{4}F_{\nu\rho}\gamma^{\nu\rho}\gamma_\mu + \mathrm{i}a_\mu I = \frac{1}{2}E_I\gamma^0\gamma^I\gamma_\mu - \frac{1}{4}F_{IJ}\gamma^{IJ}\gamma_\mu + \mathrm{i}a_\mu I. \tag{714}$$

Theorem 5.3 (4D, $\mathcal{N} = 2$ supergravity BPS inequality)**.** *If the equations of motion hold, $T^{\text{other}}_{0\mu}$, ρ decay faster than $O(\mathrm{e}^{-(n-1)r})$ near $\partial_\infty\Sigma_t$ and $T^{\text{other}}_{00} \geq \sqrt{T^{\text{other}}_{0I}T^{\text{other}I}_{0} + \rho^2}$ and E_1, a_A & F_{23} decay fast enough for the integrals below to be convergent, then theorem 3.19 implies*

$$\begin{aligned} Q(\varepsilon) = &\frac{n-1}{2}\mathrm{e}^{-r}\int_{\partial_\infty\Sigma_t} p_M\bar{\varepsilon}_k\gamma^M\varepsilon_k\sqrt{\iota^* f_{(0)}}\,\mathrm{d}^{n-2}x \\ &- 2\int_{\partial_\infty\Sigma_t} E_1\bar{\varepsilon}_k\varepsilon_k \mathrm{d}A - 2\int_{\partial_\infty\Sigma_t} F_{23}\varepsilon_k^\dagger\gamma^1\gamma^2\gamma^3\varepsilon_k\mathrm{d}A + 2\mathrm{i}\int_{\partial_\infty\Sigma_t} a_A\varepsilon_k^\dagger\gamma^1\gamma^A\varepsilon_k\mathrm{d}A \end{aligned} \tag{715}$$

$$= 2\int_{\Sigma_t}\left((\nabla_I\varepsilon)^\dagger\nabla^I\varepsilon + 4\pi T^{0\mu}_{\text{other}}\varepsilon^\dagger\gamma_0\gamma_\mu\varepsilon - 4\pi\rho\varepsilon^\dagger\gamma^0\varepsilon\right)\mathrm{d}V \tag{716}$$

$$\geq 0 \tag{717}$$

Note that when $\rho = 0$, the $T^{\text{other}}_{0\mu}$ inequality is automatically satisfied if T^{other}_{ab} satisfies the dominant energy condition.

Proof. I'll begin by validating the assumptions of definition 3.1 for the connection chosen in equation 714. From equation 714,

$$A_I = \frac{1}{2}E_J\gamma^0\gamma^J\gamma_I - \frac{1}{4}F_{JK}\gamma^{JK}\gamma_I + \mathrm{i}a_I I. \tag{718}$$

First, I'll check

$$\gamma^{IJ}A_J = \frac{1}{2}E_K\gamma^{IJ}\gamma^0\gamma^K\gamma_J - \frac{1}{4}F_{KL}\gamma^{IJ}\gamma^{KL}\gamma_J + \mathrm{i}a_J\gamma^{IJ} \tag{719}$$

is hermitian. The 1st term simplifies as

$$\frac{1}{2}E_K\gamma^{IJ}\gamma^0\gamma^K\gamma_J = -\frac{1}{2}E_K\gamma^{IJ}\gamma^0\gamma_J\gamma^K - E_K\gamma^{IJ}\gamma^0\delta^K{}_J \tag{720}$$

$$= \frac{1}{2}E_K\gamma^{IJ}\gamma_J\gamma^0\gamma^K - E_J\gamma^{IJ}\gamma^0 \tag{721}$$

$$= -E_J\gamma^I\gamma^0\gamma^J - E_J\gamma^{IJ}\gamma^0 \tag{722}$$

$$= E_J\gamma^0(\gamma^I\gamma^J - \gamma^{IJ}) \tag{723}$$

$$= -E^I\gamma^0, \tag{724}$$

while the 2nd term simplifies as

$$-\frac{1}{4}F_{KL}\gamma^{IJ}\gamma^{KL}\gamma_J = -\frac{1}{4}F_{KL}\gamma^{IJ}(\gamma_J\gamma^{KL} - 2\delta^L{}_J\gamma^K + 2\delta^K{}_J\gamma^L) \tag{725}$$

$$= \frac{1}{2}F_{JK}\gamma^I\gamma^{JK} - F_{JK}\gamma^{IJ}\gamma^K \tag{726}$$

$$= \frac{1}{2}F_{JK}(\gamma^{IJK} - \delta^{IJ}\gamma^K + \delta^{IK}\gamma^J - 2\gamma^{IJK} + 2\delta^{KJ}\gamma^I - 2\delta^{KI}\gamma^J) \tag{727}$$

$$= -\frac{1}{2}F_{JK}\gamma^{IJK}. \tag{728}$$

Substituting these back, I get

$$\gamma^{IJ}A_J = -E^I\gamma^0 - \frac{1}{2}F_{JK}\gamma^{IJK} + \mathrm{i}a_J\gamma^{IJ}. \tag{729}$$

$$\therefore (\gamma^{IJ}A_J)^\dagger = -E^I\gamma^0 + \frac{1}{2}F_{JK}\gamma^{KJI} - \mathrm{i}a_I\gamma^{JI} = -E^I\gamma^0 - \frac{1}{2}F_{JK}\gamma^{IJK} + \mathrm{i}a_J\gamma^{IJ} = \gamma^{IJ}A_J. \tag{730}$$

$\therefore \gamma^{IJ}A_J$ is indeed hermitian.
Next, consider $\mathbb{M}$ for this example. By definition,

$$\mathbb{M} = 4\pi T_{0\mu}\gamma^0\gamma^\mu + \gamma^{IJ}D_IA_J + \mathrm{i}(\gamma^IA_I + A_I^\dagger\gamma^I) - A_I^\dagger\gamma^{IJ}A_J \tag{731}$$

$$= 4\pi T^{\text{other}}_{0\mu}\gamma^0\gamma^\mu + F_0{}^\nu F_{\mu\nu}\gamma^0\gamma^\mu - \frac{1}{4}\eta_{0\mu}F^{\nu\rho}F_{\nu\rho}\gamma^0\gamma^\mu + \gamma^{IJ}D_IA_J + \mathrm{i}(\gamma^IA_I + A_I^\dagger\gamma^I) - A_I^\dagger\gamma^{IJ}A_J. \tag{732}$$

Consider this expression term by term.

$$F_0{}^\nu F_{\mu\nu}\gamma^0\gamma^\mu = F_0{}^I F_{\mu I}\gamma^0\gamma^\mu \tag{733}$$

$$= -E^I(-E_I(\gamma^0)^2 + F_{JI}\gamma^0\gamma^J) \tag{734}$$

$$= E^IE_II - F_{IJ}E^J\gamma^0\gamma^I. \tag{735}$$

$$-\frac{1}{4}\eta_{0\mu}F^{\nu\rho}F_{\nu\rho}\gamma^0\gamma^\mu = \frac{1}{4}F^{\mu\nu}F_{\mu\nu}(\gamma^0)^2 = -\frac{1}{2}E^IE_II + \frac{1}{4}F^{IJ}F_{IJ}I. \tag{736}$$

$$\gamma^{IJ}D_IA_J = D_I(\gamma^{IJ}A_J) \tag{737}$$

$$= D_I\left(-E^I\gamma^0 - \frac{1}{2}F_{JK}\gamma^{IJK} + \mathrm{i}a_J\gamma^{IJ}\right) \text{ by equation 729} \tag{738}$$

$$= -D_I(E^I)\gamma^0 - \frac{1}{2}D_{[I}F_{JK]}\gamma^{IJK} + \mathrm{i}D_{[I}a_{J]}\gamma^{IJ} \tag{739}$$

$$= -4\pi\rho\gamma^0 - 0 + \frac{\mathrm{i}}{2}F_{IJ}\gamma^{IJ} \text{ by the equations of motion.} \tag{740}$$

$$\gamma^I A_I = \frac{1}{2}E_J\gamma^I\gamma^0\gamma^J\gamma_I - \frac{1}{4}F_{JK}\gamma^I\gamma^{JK}\gamma_I + \mathrm{i}a_I\gamma^I \tag{741}$$

$$= -\frac{1}{2}E_J\gamma^0\gamma^I\gamma^J\gamma_I - \frac{1}{4}F_{JK}(\gamma^{JK}\gamma^I - 2\delta^{IJ}\gamma^K + 2\delta^{IK}\gamma^J)\gamma_I + \mathrm{i}a_I\gamma^I \tag{742}$$

$$= \frac{1}{2}E_J\gamma^0\gamma^J\gamma^I\gamma_I + E_J\gamma^0\delta^{IJ}\gamma_I + \frac{3}{4}F_{IJ}\gamma^{IJ} - F_{IJ}\gamma^I\gamma^J + \mathrm{i}a_I\gamma^I \tag{743}$$

$$= -\frac{3}{2}E_I\gamma^0\gamma^I + E_I\gamma^0\gamma^I + \frac{3}{4}F_{IJ}\gamma^{IJ} - F_{IJ}\gamma^{IJ} + \mathrm{i}a_I\gamma^I \tag{744}$$

$$= -\frac{1}{2}E_I\gamma^0\gamma^I - \frac{1}{4}F_{IJ}\gamma^{IJ} + \mathrm{i}a_I\gamma^I. \tag{745}$$

$$\therefore \gamma^I A_I + A_I^\dagger\gamma^I = \gamma^I A_I - (\gamma^I A_I)^\dagger \tag{746}$$

$$= -\frac{1}{2}E_I\gamma^0\gamma^I - \frac{1}{4}F_{IJ}\gamma^{IJ} + \mathrm{i}a_I\gamma^I - \left(\frac{1}{2}E_I\gamma^I\gamma^0 - \frac{1}{4}F_{IJ}\gamma^{JI} + \mathrm{i}a_I\gamma^I\right) \tag{747}$$

$$= -\frac{1}{2}F_{IJ}\gamma^{IJ}. \tag{748}$$

The most tedious term to simplify is

$$A_I^\dagger\gamma^{IJ}A_J = \left(\frac{1}{2}E_J\gamma_I\gamma^J\gamma^0 + \frac{1}{4}F_{JK}\gamma_I\gamma^{KJ} - \mathrm{i}a_I I\right)\left(-E^I\gamma^0 - \frac{1}{2}F_{LM}\gamma^{ILM} + \mathrm{i}a_L\gamma^{IL}\right) \tag{749}$$

$$\begin{aligned}&= -\frac{1}{2}E_JE^I\gamma_I\gamma^J\gamma^0\gamma^0 - \frac{1}{4}E_JF_{LM}\gamma_I\gamma^J\gamma^0\gamma^{ILM} + \frac{\mathrm{i}}{2}E_Ja_L\gamma_I\gamma^J\gamma^0\gamma^{IL}\\ &\quad - \frac{1}{4}E^IF_{JK}\gamma_I\gamma^{KJ}\gamma^0 - \frac{1}{8}F_{JK}F_{LM}\gamma_I\gamma^{KJ}\gamma^{ILM} + \frac{\mathrm{i}}{4}F_{JK}a_L\gamma_I\gamma^{KJ}\gamma^{IL}\\ &\quad + \mathrm{i}a_IE^I\gamma^0 + \frac{\mathrm{i}}{2}a_IF_{LM}\gamma^{ILM} + a_Ia_L\gamma^{IL}.\end{aligned} \tag{750}$$

Consider each set of similar terms in this expression individually.

$$-\frac{1}{2}E_JE^I\gamma_I\gamma^J\gamma^0\gamma^0 = -\frac{1}{2}E_IE_J\gamma^I\gamma^J = \frac{1}{2}E^IE_II. \tag{751}$$

$$-\frac{1}{4}E_JF_{LM}\gamma_I\gamma^J\gamma^0\gamma^{ILM} - \frac{1}{4}E^IF_{JK}\gamma_I\gamma^{KJ}\gamma^0 \tag{752}$$

$$= \frac{1}{4}E_IF_{JK}(-\gamma_L\gamma^I\gamma^0\gamma^{LJK} - \gamma^I\gamma^{KJ}\gamma^0) \tag{753}$$

$$= \frac{1}{4}E_IF_{JK}\gamma^0(\gamma^I\gamma_L\gamma^{LJK} + 2\delta^I{}_L\gamma^{LJK} - \gamma^I\gamma^{JK}) \tag{754}$$

$$= \frac{1}{4}E_IF_{JK}\gamma^0(-\gamma^I\gamma^{JK} + 2\gamma^{IJK} - \gamma^I\gamma^{JK}) \tag{755}$$

$$= \frac{1}{2}E_IF_{JK}\gamma^0(-\gamma^I\gamma^{JK} + \gamma^I\gamma^{JK} + \delta^{IJ}\gamma^K - \delta^{IK}\gamma^J) \tag{756}$$

$$= E^IF_{IJ}\gamma^0\gamma^J. \tag{757}$$

$$\frac{\mathrm{i}}{2}E_Ja_L\gamma_I\gamma^J\gamma^0\gamma^{IL} + \mathrm{i}a_IE^I\gamma^0 = -\frac{\mathrm{i}}{2}E_Ja_K\gamma^J\gamma_I\gamma^{IK}\gamma^0 - \mathrm{i}E_Ja_K\delta^J{}_I\gamma^{IK}\gamma^0 + \mathrm{i}a_IE^I\gamma^0 \tag{758}$$

$$= \mathrm{i}E_Ia_J(\gamma^I\gamma^J - \gamma^{IJ})\gamma^0 + \mathrm{i}a_IE^I\gamma^0 \tag{759}$$

$$= -\mathrm{i}E_Ia_J\delta^{IJ}\gamma^0 + +\mathrm{i}a_IE^I\gamma^0 \tag{760}$$

$$= 0. \tag{761}$$

$$
\begin{aligned}
&-\frac{1}{8}F_{JK}F_{LM}\gamma_I\gamma^{KJ}\gamma^{ILM} && (762)\\
&=-\frac{1}{8}F_{JK}F_{LM}(\gamma^{KJ}\gamma_I-2\delta^K{}_I\gamma^J+2\delta^J{}_I\gamma^K)\gamma^{ILM} && (763)\\
&=-\frac{1}{8}F_{IJ}F_{KL}\gamma^{IJ}\gamma^{KL}-\frac{1}{2}F_{IJ}F_{KL}\gamma^J\gamma^{IKL} && (764)\\
&=-\frac{1}{8}F_{IJ}F_{KL}(\gamma^{IJKL}+\delta^{IK}\gamma^J\gamma^L-\delta^{IL}\gamma^J\gamma^K-\delta^{JK}\gamma^I\gamma^L+\delta^{JL}\gamma^I\gamma^K+\delta^{IK}\delta^{JL}I-\delta^{IL}\delta^{JK}I\\
&\quad+4\gamma^{JIKL}-4\delta^{JI}\gamma^{KL}+4\delta^{JK}\gamma^{IL}-4\delta^{JL}\gamma^{IK}) && (765)\\
&=0-\frac{1}{8}F_{IJ}F^I{}_L\gamma^J\gamma^L+\frac{1}{8}F_{IJ}F_K{}^I\gamma^J\gamma^K+\frac{1}{8}F_{IJ}F^J{}_L\gamma^I\gamma^L-\frac{1}{8}F_{IJ}F_K{}^J\gamma^I\gamma^K-\frac{1}{8}F^{IJ}F_{IJ}I\\
&\quad+\frac{1}{8}F^{IJ}F_{JI}I-0+0-\frac{1}{2}F_{IJ}F^J{}_L\gamma^{IL}+\frac{1}{2}F_{IJ}F_K{}^J\gamma^{IK}\ \text{ as }\gamma^{IJKL}=0\text{ when }n=4 && (766)\\
&=\frac{1}{2}F_{IJ}F^{IJ}I-\frac{1}{4}F^{IJ}F_{IJ}I-0-0 && (767)\\
&=\frac{1}{4}F^{IJ}F_{IJ}I. && (768)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\mathrm{i}}{4}F_{JK}a_L\gamma_I\gamma^{KJ}\gamma^{IL}+\frac{\mathrm{i}}{2}a_IF_{LM}\gamma^{ILM}\\
&=\frac{\mathrm{i}}{4}a_IF_{JK}(-\gamma_L\gamma^{JK}\gamma^{LI}+2\gamma^{IJK}) && (769)\\
&=\frac{\mathrm{i}}{4}a_IF_{JK}(-\gamma^{JK}\gamma_L\gamma^{LI}+2\delta^J{}_L\gamma^K\gamma^{LI}-2\delta^K{}_L\gamma^J\gamma^{LI}+2\gamma^{IJK}) && (770)\\
&=\frac{\mathrm{i}}{2}a_IF_{JK}(\gamma^{JK}\gamma^I+2\gamma^K\gamma^{JI}+\gamma^{IJK}) && (771)\\
&=\frac{\mathrm{i}}{2}a_IF_{JK}(\gamma^{JKI}-\delta^{KI}\gamma^J+\delta^{IJ}\gamma^K+2\gamma^{KJI}-2\delta^{KJ}\gamma^I+2\delta^{KI}\gamma^J+\gamma^{IJK}) && (772)\\
&=0. && (773)
\end{aligned}
$$

$$a_Ia_L\gamma^{IL}=0. \tag{774}$$

Substituting these results back up, I get

$$A_I^\dagger\gamma^{IJ}A_J=\frac{1}{2}E^IE_II+E^IF_{IJ}\gamma^0\gamma^J+\frac{1}{4}F^{IJ}F_{IJ}I. \tag{775}$$

Substituting this expression, and the others above, into the earlier expression for $\mathbb{M}$ says

$$
\begin{aligned}
\mathbb{M}&=4\pi T^{\text{other}}_{0\mu}\gamma^0\gamma^\mu+E^IE_II-F_{IJ}E^J\gamma^0\gamma^I-\frac{1}{2}E^IE_II+\frac{1}{4}F^{IJ}F_{IJ}I-4\pi\rho\gamma^0+\frac{\mathrm{i}}{2}F_{IJ}\gamma^{IJ}\\
&\quad-\frac{\mathrm{i}}{2}F_{IJ}\gamma^{IJ}-\frac{1}{2}E^IE_II-E^IF_{IJ}\gamma^0\gamma^J-\frac{1}{4}F^{IJ}F_{IJ}I && (776)\\
&=4\pi T^{\text{other}}_{0\mu}\gamma^0\gamma^\mu-4\pi\rho\gamma^0 && (777)\\
&=4\pi(T^{\text{other}}_{00}I+T^{\text{other}}_{0I}\gamma^0\gamma^I-\rho\gamma^0). && (778)
\end{aligned}
$$

This matrix's eigenvalues (e.g. found by computer algebra) are $4\pi(T^{\text{other}}_{00}\pm\sqrt{T^{\text{other}}_{0I}T^{\text{other}I}_{0}+\rho^2})$. Hence, $\mathbb{M}$ is non-negative definite by the condition I've assumed on T^{other}_{ab} and ρ.
The $||\mathbb{M}||_0$ decay condition assumed in definition 3.1 corresponds exactly to the decay conditions I'm assuming for $T^{\text{other}}_{0\mu}$ and ρ.
The $||A_I||_0$ decay condition assumed in definition 3.1 is merely required for the boundary

integrals in equation 715 or theorem 3.19 to be finite and is stronger than the decay required for convergence properties in section 3.1.
Finally, there is the assumption regarding $\gamma^I A_I = -\tilde{A}_I^\dagger \gamma^I$. This assumption is used only in theorem 3.10, when proving G's surjectivity.
From equation 748 and the computations leading to it, one can see that $\tilde{A}_I$ exists. In particular, it is the same as A_I, except F_{IJ} and a_I are replaced with $-F_{IJ}$ and $-a_I$ respectively.
These sign changes don't affect decay rates or $\gamma^{IJ} A_J$ being hermitian.
$\therefore$ The assumptions of definition 3.1 hold.
Having established theorem 3.19 is valid in the present scenario, it remains only to simplify the integrals there.
$\frac{n-1}{2} \mathrm{e}^{-r} \int_{\partial_\infty \Sigma_t} p_M \bar{\varepsilon}_k \gamma^M \varepsilon_k \sqrt{\iota^* f_{(0)}} \, \mathrm{d}^{n-2} x$ is unchanged and the $\int_{\Sigma_t} \mathrm{d}V$ integral follows immediately from equation 778. The other boundary integrals require finding $\gamma^1 \gamma^A A_A + A_A^\dagger \gamma^A \gamma^1$.

$$\gamma^1 \gamma^A A_A = \gamma^1 \gamma^A \left(\frac{1}{2} E_J \gamma^0 \gamma^J \gamma_A - \frac{1}{4} F_{JK} \gamma^{JK} \gamma_A + \mathrm{i} a_A I \right) \tag{779}$$

$$= \frac{1}{2} E_J \gamma^0 \gamma^1 \gamma^A \gamma^J \gamma_A - \frac{1}{4} F_{JK} \gamma^1 \gamma^A \gamma^{JK} \gamma_A + \mathrm{i} a_A \gamma^1 \gamma^A. \tag{780}$$

Consider the 1st two terms individually.

$$\frac{1}{2} E_J \gamma^0 \gamma^1 \gamma^A \gamma^J \gamma_A = -\frac{1}{2} E_J \gamma^0 \gamma^1 \gamma^J \gamma^A \gamma_A - E_J \gamma^0 \gamma^1 \delta^{AJ} \gamma_A \tag{781}$$

$$= E_J \gamma^0 \gamma^1 \gamma^J - E_A \gamma^0 \gamma^1 \gamma^A \tag{782}$$

$$= E_1 \gamma^0 \gamma^1 \gamma^1 + E_A \gamma^0 \gamma^1 \gamma^A - E_A \gamma^0 \gamma^1 \gamma^A \tag{783}$$

$$= -E_1 \gamma^0. \tag{784}$$

$$-\frac{1}{4} F_{JK} \gamma^1 \gamma^A \gamma^{JK} \gamma_A = -\frac{1}{4} F_{JK} \gamma^1 (\gamma^{JK} \gamma^A - 2\delta^{JA} \gamma^K + 2\delta^{KA} \gamma^J) \gamma_A \tag{785}$$

$$= \frac{1}{2} F_{IJ} \gamma^1 \gamma^{IJ} + F_{AI} \gamma^1 \gamma^I \gamma^A \tag{786}$$

$$= F_{1A} \gamma^1 \gamma^1 \gamma^A + \frac{1}{2} F_{AB} \gamma^1 \gamma^{AB} + F_{A1} \gamma^1 \gamma^1 \gamma^A + F_{AB} \gamma^1 \gamma^B \gamma^A \tag{787}$$

$$= -\frac{1}{2} F_{AB} \gamma^1 \gamma^{AB} \tag{788}$$

$$= -F_{23} \gamma^1 \gamma^2 \gamma^3. \tag{789}$$

Substituting these two expressions back,

$$\gamma^1 \gamma^A A_A = -E_1 \gamma^0 - F_{23} \gamma^1 \gamma^2 \gamma^3 + \mathrm{i} a_A \gamma^1 \gamma^A. \tag{790}$$

$$\therefore \gamma^1 \gamma^A A_A + A_A^\dagger \gamma^A \gamma^1 = \gamma^1 \gamma^A A_A + (\gamma^1 \gamma^A A_A)^\dagger \tag{791}$$

$$= -E_1 \gamma^0 - F_{23} \gamma^1 \gamma^2 \gamma^3 + \mathrm{i} a_A \gamma^1 \gamma^A + (-E_1 \gamma^0 - F_{23} \gamma^1 \gamma^2 \gamma^3 + \mathrm{i} a_A \gamma^1 \gamma^A)^\dagger \tag{792}$$

$$= 2(-E_1 \gamma^0 - F_{23} \gamma^1 \gamma^2 \gamma^3 + \mathrm{i} a_A \gamma^1 \gamma^A). \tag{793}$$

Finally, substituting this into theorem 3.19 completes the claimed result. □

Corollary 5.3.1. *If $f_{(0)0\alpha} = 0$ and the extrinsic curvature, K_{IJ}, of Σ_t is less than $O(\mathrm{e}^{-r})$ near*

$\partial_\infty \Sigma_t$, *then*

$$Q(\varepsilon) = \frac{n-1}{2}\mathrm{e}^{-r}\int_{\partial_\infty\Sigma_t} p_M \bar{\varepsilon}_k \gamma^M \varepsilon_k \sqrt{\iota^* f_{(0)}}\, \mathrm{d}^{n-2}x - 2\int_{\partial_\infty\Sigma_t} E_1 \bar{\varepsilon}_k \varepsilon_k \mathrm{d}A \tag{794}$$

$$= 2\int_{\Sigma_t} \left((\nabla_I \varepsilon)^\dagger \nabla^I \varepsilon + 4\pi T^{0\mu}_{\text{other}} \varepsilon^\dagger \gamma_0 \gamma_\mu \varepsilon - 4\pi\rho \varepsilon^\dagger \gamma^0 \varepsilon\right) \mathrm{d}V \tag{795}$$

$$\geq 0, \tag{796}$$

i.e. under the extra assumptions made, the magnetic and gauge field boundary integrals cancel.

Proof. Start by re-writing the magnetic integral as

$$\int_{\partial_\infty\Sigma_t} F_{23} \varepsilon_k^\dagger \gamma^1 \gamma^2 \gamma^3 \varepsilon_k \mathrm{d}A = \frac{1}{2}\int_{\partial_\infty\Sigma_t} F_{AB} \varepsilon_k^\dagger \gamma^1 \gamma^{AB} \varepsilon_k \mathrm{d}A \tag{797}$$

$$= \frac{1}{2}\int_{\partial_\infty\Sigma_t} l_I F_{JK} \varepsilon_k^\dagger \gamma^{IJK} \varepsilon_k \mathrm{d}A. \tag{798}$$

I've assumed $f_{(0)0\alpha} = 0$. Hence, to leading order $e_I{}^{\mu'} = \delta^{\mu'}{}_i e_I^{(h)i}$.
$\therefore F_{JK} \to e_J^{(h)j} e_K^{(h)k} F_{jk} = e_J^{(h)j} e_K^{(h)k} (\partial_j a_k - \partial_k a_j) \to D_J^{(h)} a_K - D_K^{(h)} a_J$.
The decay conditions I've assumed mean only leading order contributions survive the integral.

$$\therefore \int_{\partial_\infty\Sigma_t} F_{23} \varepsilon_k^\dagger \gamma^1 \gamma^2 \gamma^3 \varepsilon_k \mathrm{d}A = \int_{\partial_\infty\Sigma_t} l_I D_J^{(h)}(a_K) \varepsilon_k^\dagger \gamma^{IJK} \varepsilon_k \mathrm{d}A \tag{799}$$

$$= \int_{\partial_\infty\Sigma_t} l_I D_J^{(h)}(a_K \varepsilon_k^\dagger \gamma^{IJK} \varepsilon_k) \mathrm{d}A - \int_{\partial_\infty\Sigma_t} l_I a_K \left(D_J^{(h)}(\varepsilon_k)^\dagger \gamma^{IJK} \varepsilon_k + \varepsilon_k^\dagger \gamma^{IJK} D_J^{(h)} \varepsilon_k\right) \mathrm{d}A. \tag{800}$$

The covariant derivatives on Σ_t and M are related by $D_I^{(h)} \varepsilon_k = D_I \varepsilon_k + \frac{1}{2} K_{IJ} \gamma^J \gamma^0 \varepsilon_k$ when acting on spinors.
$\therefore$ The assumed K_{IJ} decay implies $D_I^{(h)} \varepsilon_k = D_I \varepsilon_k$ to leading order.
$\therefore$ By the Killing spinor equation, $D_I \varepsilon_k = -\frac{\mathrm{i}}{2} \gamma_I \varepsilon_k$ to leading order.

$$\therefore D_J^{(h)}(\varepsilon_k)^\dagger \gamma^{IJK} \varepsilon_k + \varepsilon_k^\dagger \gamma^{IJK} D_J^{(h)} \varepsilon_k \to \left(-\frac{\mathrm{i}}{2}\gamma_J \varepsilon_k\right)^\dagger \gamma^{IJK} \varepsilon_k - \frac{\mathrm{i}}{2} \varepsilon_k^\dagger \gamma^{IJK} \gamma_J \varepsilon_k \tag{801}$$

$$= -\frac{\mathrm{i}}{2} \varepsilon_k^\dagger \gamma_J \gamma^{IJK} \varepsilon_k - \frac{\mathrm{i}}{2} \varepsilon_k^\dagger \gamma^{IJK} \gamma_J \varepsilon_k \tag{802}$$

$$= -\mathrm{i} \varepsilon_k^\dagger \gamma^{IK} \varepsilon_k. \tag{803}$$

Meanwhile, for the other integral in equation 800, if $\tilde{h}$ is the metric on constant t and r surfaces, then by the same logic as lemma 3.6,

$$\int_{\partial_\infty\Sigma_t} l_I D_J^{(h)}(a_K \varepsilon_k^\dagger \gamma^{IJK} \varepsilon_k) \mathrm{d}A = \int_{\partial_\infty\Sigma_t} D_J^{(\tilde{h})}(l_I a_K \varepsilon_k^\dagger \gamma^{IJK} \varepsilon_k) \mathrm{d}A \tag{804}$$

$$= 0 \quad \text{by Stokes' theorem.} \tag{805}$$

Thus, equation 800 reduces to

$$\int_{\partial_\infty\Sigma_t} F_{23} \varepsilon_k^\dagger \gamma^1 \gamma^2 \gamma^3 \varepsilon_k \mathrm{d}A = 0 + \mathrm{i}\int_{\partial_\infty\Sigma_t} l_I a_K \varepsilon_k^\dagger \gamma^{IK} \varepsilon_k \mathrm{d}A = \mathrm{i}\int_{\partial_\infty\Sigma_t} a_A \varepsilon_k^\dagger \gamma^1 \gamma^A \varepsilon_k \mathrm{d}A. \tag{806}$$

$\therefore$ The last two integrals in equation 715 precisely cancel. □

5.1.1 Toroidal boundary

Consider again the toroidal boundary, $f_{(0)} = -\mathrm{d}t \otimes \mathrm{d}t + \delta_{\alpha\beta}\,\mathrm{d}\theta^\alpha \otimes \mathrm{d}\theta^\beta$.

Theorem 5.4 (4D, $\mathcal{N} = 2$, toroidal boundary supergravity BPS inequality)**.** *If the equations of motion hold, $T^{0\mu}_{\text{other}}$ decays faster than $O(\mathrm{e}^{-3r})$ near $\partial_\infty\Sigma_t$, $T^{\text{other}}_{00} \geq \sqrt{T^{\text{other}}_{0I} T^{\text{other}I}_{0} + \rho^2}$ and (M, g)'s spin structure is compatible with having periodic spinors near $\partial_\infty\Sigma_t$ then*

$$E \geq \sqrt{\mathbb{J}_A\mathbb{J}^A} = \sqrt{\mathbb{J}_2\mathbb{J}^2 + \mathbb{J}_3\mathbb{J}^3}. \tag{807}$$

Proof. Equation 328 applies once again as does the calculation in theorem 4.3 to show

$$\frac{n-1}{2}\mathrm{e}^{-r}\int_{\partial_\infty\Sigma_t} p_M\bar{\varepsilon}_k\gamma^M\varepsilon_k\sqrt{\iota^* f_{(0)}}\,\mathrm{d}^{n-2}x = 8\pi\varepsilon_0^\dagger P_1^-\left(EI + \mathbb{J}_A\gamma^0\gamma^A\right)P_1^-\varepsilon_0. \tag{808}$$

It remains to calculate the electromagnetic boundary terms in theorem 5.3.
The chosen $f_{(0)}$ suffices to apply corollary 5.3.1, so only the electric field integral remains. However, equation 328, implies

$$\bar{\varepsilon}_k\varepsilon_k = \mathrm{e}^r\varepsilon_0^\dagger P_1^-\gamma^0 P_1^-\varepsilon_0 = \mathrm{e}^r\varepsilon_0^\dagger\gamma^0 P_1^+ P_1^-\varepsilon_0 = 0. \tag{809}$$

∴ The electric field integral vanishes and the situation reduces to theorem 4.3. □

5.1.2 Asymptotically AdS

Theorem 5.5 (4D, $\mathcal{N} = 2$, asymptotically AdS supergravity BPS inequality)**.** *If the equations of motion hold, $T^{0\mu}_{\text{other}}$ decays faster than $O(\mathrm{e}^{-3r})$ near $\partial_\infty\Sigma_t$, $T^{\text{other}}_{00} \geq \sqrt{T^{\text{other}}_{0I} T^{\text{other}I}_{0} + \rho^2}$, $A_I = \frac{1}{2}E_J\gamma^0\gamma^J\gamma_I - \frac{1}{4}F_{JK}\gamma^{JK}\gamma_I + \mathrm{i}a_I I$ decays[37] faster than $O(\mathrm{e}^{-3r/2})$, E_1 decays as $O(\mathrm{e}^{-2r})$, a_A decays as $O(\mathrm{e}^{-2r})$ and F_{23} decays[38] as $O(\mathrm{e}^{-3r})$, then*

$$EI - \mathrm{i}P_I\gamma^I + \frac{\mathrm{i}}{2}J_{IJ}\gamma^0\gamma^{IJ} + K_I\gamma^0\gamma^I - q_e\gamma^0 \tag{810}$$

is a non-negative definite matrix.

Proof. The proof is essentially just substituting the present data into theorem 5.3 and noting that corollary 5.3.1 applies for the present boundary geometry.
The p_M boundary integral in equation 715 is identical to what I've already analysed in theorem 4.8. Hence, I immediately get

$$\begin{aligned}&\frac{n-1}{2}\mathrm{e}^{-r}\int_{\partial_\infty\Sigma_t} p_M\bar{\varepsilon}_k\gamma^M\varepsilon_k\sqrt{\iota^* f_{(0)}}\,\mathrm{d}^{n-2}x\\ &= 8\pi\varepsilon_0^\dagger\mathrm{e}^{-\mathrm{i}\gamma^0 t/2}\left(EI - \mathrm{i}P_I\gamma^I + \frac{\mathrm{i}}{2}J_{IJ}\gamma^0\gamma^{IJ} + K_I\gamma^0\gamma^I\right)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0.\end{aligned} \tag{811}$$

[37]This is a weaker decay condition than in definition 3.1. The decay there ensures the boundary integrals in theorem 3.19 are convergent. However, I am dealing with that issue separately with specific decay conditions on E_1, F_{23} and a_A. Hence, I can assume this weaker decay condition, which suffices for the analysis is section 3.1 - in particular, lemma 3.17.

[38]This decay is in fact automatically implied by the assumed decay on a_A.

Next, for the electromagnetic boundary integrals, first consider $\bar{\varepsilon}_k\varepsilon_k$. Since $\bar{\varepsilon}_k\varepsilon_k$ is a Lorentz scalar, I can evaluate it any frame. Thus, by lemma 4.6,

$$\bar{\varepsilon}_k\varepsilon_k = \frac{1}{1-\rho^2}\varepsilon_0^\dagger e^{-i\gamma^0 t/2}(I - ix_I\gamma^I)\gamma^0(I - ix_J\gamma^J)e^{i\gamma^0 t/2}\varepsilon_0 \tag{812}$$

$$= \frac{1}{1-\rho^2}\varepsilon_0^\dagger e^{-i\gamma^0 t/2}\gamma^0(I + ix_I\gamma^I)(I - ix_J\gamma^J)e^{i\gamma^0 t/2}\varepsilon_0 \tag{813}$$

$$= \frac{1}{1-\rho^2}\varepsilon_0^\dagger e^{-i\gamma^0 t/2}\gamma^0(I + x_Ix_J\gamma^I\gamma^J)e^{i\gamma^0 t/2}\varepsilon_0 \tag{814}$$

$$= \frac{1}{1-\rho^2}\varepsilon_0^\dagger e^{-i\gamma^0 t/2}\gamma^0(I - \rho^2 I)e^{i\gamma^0 t/2}\varepsilon_0 \tag{815}$$

$$= \varepsilon_0^\dagger e^{-i\gamma^0 t/2}\gamma^0 e^{i\gamma^0 t/2}\varepsilon_0. \tag{816}$$

$\therefore$ The relevant integral in equation 715 is

$$-2\int_{\partial_\infty\Sigma_t} E_1\bar{\varepsilon}_k\varepsilon_k dA = -2\int_{\partial_\infty\Sigma_t} E_1\varepsilon_0^\dagger e^{-i\gamma^0 t/2}\gamma^0 e^{i\gamma^0 t/2}\varepsilon_0 dA \tag{817}$$

$$= -2\varepsilon_0^\dagger e^{-i\gamma^0 t/2}\int_{\partial_\infty\Sigma_t} E_1 dA\gamma^0 e^{i\gamma^0 t/2}\varepsilon_0 \tag{818}$$

$$= -8\pi\varepsilon_0^\dagger e^{-i\gamma^0 t/2}q_e\gamma^0 e^{i\gamma^0 t/2}\varepsilon_0. \tag{819}$$

Hence, from corollary 5.3.1, I get

$$8\pi\varepsilon_0^\dagger e^{-i\gamma^0 t/2}\left(EI - iP_I\gamma^I + \frac{i}{2}J_{IJ}\gamma^0\gamma^{IJ} + K_I\gamma^0\gamma^I - q_e\gamma^0\right)e^{i\gamma^0 t/2}\varepsilon_0 \tag{820}$$

is non-negative definite. By the same logic I used in section 4, I can conclude the matrix in between $\varepsilon_0^\dagger e^{-i\gamma^0 t/2}$ and $e^{i\gamma^0 t/2}\varepsilon_0$ is non-negative definite. □

5.2 5D, $\mathcal{N} = 2$, gauged supergravity

The bosonic sector of 5D, $\mathcal{N} = 2$, gauged supergravity is described by the action,

$$S = \frac{1}{16\pi}\int_M \left(R - 2\Lambda - F_{ab}F^{ab} - \frac{2}{3\sqrt{3}}\varepsilon^{abcde}F_{ab}F_{cd}a_e\right)dV(g) + S^{\text{other}}_{\text{matter}}, \tag{821}$$

where ε^{abcde} is the Levi-Civita tensor and $F_{ab} = D_a a_b - D_b a_a$, for some locally defined gauge field, a_a. This theory could also be described as Einstein-Maxwell-Chern-Simons with cosmological constant[39]. Once again, $S^{\text{other}}_{\text{matter}}$ should be zero for the supergravity theory, but for completeness, I've left open the possibility of having further matter fields. I will again assume $S^{\text{other}}_{\text{matter}}$ couples to the Maxwell field at most through a term of the form, $\int_M j^a a_a dV(g)$.

The equations of motion in this theory are well known to be

$$R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = 8\pi T_{ab} = 2\left(F_a{}^cF_{bc} - \frac{1}{4}g_{ab}F^{cd}F_{cd}\right) + 8\pi T^{\text{other}}_{ab}, \tag{822}$$

$$D_bF^{ba} = -4\pi j^a - \frac{1}{2\sqrt{3}}\varepsilon^{abcde}F_{bc}F_{de}, \tag{823}$$

$$D_{[a}F_{bc]} = 0 \tag{824}$$

[39]Alas, this is just as big a mouthful as saying "bosonic sector of 5D, $\mathcal{N} = 2$, gauged supergravity."

and whatever equations the fields in $S^{\text{other}}_{\text{matter}}$ solve.

It is known that in this supergravity theory, the gravitino, ψ_μ, transforms under local supersymmetry transformations as

$$\delta\psi_\mu = D_\mu\varepsilon - \frac{1}{4\sqrt{3}}F_{\nu\rho}\gamma^{\nu\rho}\gamma_\mu\varepsilon - \frac{1}{2\sqrt{3}}F_{\mu\nu}\gamma^\nu\varepsilon + \mathrm{i}\sqrt{3}a_\mu\varepsilon + \frac{\mathrm{i}}{2}\gamma_\mu\varepsilon, \tag{825}$$

for a given spinor parameter, ε. Hence, I will choose

$$A_\mu = -\frac{1}{4\sqrt{3}}F_{\nu\rho}\gamma^{\nu\rho}\gamma_\mu - \frac{1}{2\sqrt{3}}F_{\mu\nu}\gamma^\nu + \mathrm{i}\sqrt{3}a_\mu I \tag{826}$$

$$= -\frac{1}{2\sqrt{3}}E_I\gamma^I\gamma^0\gamma_\mu - \frac{1}{4\sqrt{3}}F_{IJ}\gamma^{IJ}\gamma_\mu - \frac{1}{2\sqrt{3}}F_{\mu\nu}\gamma^\nu + \mathrm{i}\sqrt{3}a_\mu I \tag{827}$$

This time, I get the following BPS inequality.

Theorem 5.6 (5D, $\mathcal{N}=2$, supergravity BPS inequality). *If the equations of motion hold, $T^{0\mu}_{\text{other}}$ decays faster than $O(\mathrm{e}^{-(n-1)r})$ near $\partial_\infty\Sigma_t$, $T^{\text{other}}_{00} \geq \sqrt{T^{\text{other}}_{0I}T^{\text{other}I}_0 + 3\rho^2/4}$ and one uses the 5D Clifford algebra representation where*[40] *$\gamma^4 = \gamma^0\gamma^1\gamma^2\gamma^3$, then theorem 3.19 implies*

$$\begin{aligned}Q(\varepsilon) = {} & \frac{n-1}{2}\mathrm{e}^{-r}\int_{\partial_\infty\Sigma_t} p_M\bar{\varepsilon}_k\gamma^M\varepsilon_k\sqrt{\iota^* f_{(0)}}\,\mathrm{d}^{n-2}x \\ & - \sqrt{3}\int_{\partial_\infty\Sigma_t} E_1\bar{\varepsilon}_k\varepsilon_k\mathrm{d}A - \frac{\sqrt{3}}{2}\int_{\partial_\infty\Sigma_t} F_{AB}\varepsilon_k^\dagger\gamma^1\gamma^{AB}\varepsilon_k\mathrm{d}A \\ & + 2\mathrm{i}\sqrt{3}\int_{\partial_\infty\Sigma_t} a_A\varepsilon_k^\dagger\gamma^1\gamma^A\varepsilon_k\mathrm{d}A\end{aligned} \tag{828}$$

$$= 2\int_{\Sigma_t}\left((\nabla_I\varepsilon)^\dagger\nabla^I\varepsilon + 4\pi T^{0\mu}_{\text{other}}\varepsilon^\dagger\gamma_0\gamma_\mu\varepsilon - 2\pi\sqrt{3}\rho\varepsilon^\dagger\gamma^0\varepsilon\right)\mathrm{d}V \tag{829}$$

$$\geq 0 \tag{830}$$

Proof. Again, I start by checking the assumptions of definition 3.1 apply. From equation 827,

$$A_I = -\frac{1}{2\sqrt{3}}E_J\gamma^J\gamma^0\gamma_I - \frac{1}{4\sqrt{3}}F_{JK}\gamma^{JK}\gamma_I - \frac{1}{2\sqrt{3}}E_I\gamma^0 - \frac{1}{2\sqrt{3}}F_{IJ}\gamma^J + \mathrm{i}\sqrt{3}a_I I. \tag{831}$$

$$\begin{aligned}\therefore \gamma^{IJ}A_J = {} & -\frac{1}{2\sqrt{3}}E_K\gamma^{IJ}\gamma^K\gamma^0\gamma_J - \frac{1}{4\sqrt{3}}F_{KL}\gamma^{IJ}\gamma^{KL}\gamma_J - \frac{1}{2\sqrt{3}}E_J\gamma^{IJ}\gamma^0 \\ & - \frac{1}{2\sqrt{3}}F_{JK}\gamma^{IJ}\gamma^K + \mathrm{i}\sqrt{3}a_J\gamma^{IJ}.\end{aligned} \tag{832}$$

The first term simplifies as

$$-\frac{1}{2\sqrt{3}}E_K\gamma^{IJ}\gamma^K\gamma^0\gamma_J = \frac{1}{2\sqrt{3}}E_K\gamma^{IJ}\gamma^K\gamma_J\gamma^0 \tag{833}$$

$$= -\frac{1}{2\sqrt{3}}E_K\gamma^{IJ}\gamma_J\gamma^K\gamma^0 - \frac{1}{\sqrt{3}}E_K\gamma^{IJ}\delta^K{}_J\gamma^0 \tag{834}$$

$$= \frac{\sqrt{3}}{2}E_J\gamma^I\gamma^J\gamma^0 - \frac{1}{\sqrt{3}}E_J\gamma^{IJ}\gamma^0 \tag{835}$$

$$= \frac{\sqrt{3}}{2}E_J(-\delta^{IJ}I + \gamma^{IJ})\gamma^0 - \frac{1}{\sqrt{3}}E_J\gamma^{IJ}\gamma^0 \tag{836}$$

$$= -\frac{\sqrt{3}}{2}E^I\gamma^0 + \frac{1}{2\sqrt{3}}E_J\gamma^{IJ}\gamma^0, \tag{837}$$

[40] In 5D, there are two inequivalent, irreducible, Clifford algebra representations. They can be constructed by taking γ^0, γ^1, γ^2 & γ^3 the same as in 4D and then defining γ^4 to be $\gamma^0\gamma^1\gamma^2\gamma^3$ or $-\gamma^0\gamma^1\gamma^2\gamma^3$. In the present context, I have the freedom to choose which representation I use.

while the 2nd term simplifies as

$$-\frac{1}{4\sqrt{3}}F_{KL}\gamma^{IJ}\gamma^{KL}\gamma_J = -\frac{1}{4\sqrt{3}}F_{KL}\gamma^{IJ}(\gamma_J\gamma^{KL} - 2\delta^L{}_J\gamma^K + 2\delta^K{}_J\gamma^L) \tag{838}$$

$$= \frac{\sqrt{3}}{4}F_{JK}\gamma^I\gamma^{JK} - \frac{1}{\sqrt{3}}F_{JK}\gamma^{IJ}\gamma^K. \tag{839}$$

Substituting these back, I get

$$\gamma^{IJ}A_J = -\frac{\sqrt{3}}{2}E^I\gamma^0 + \frac{\sqrt{3}}{4}F_{JK}\gamma^I\gamma^{JK} - \frac{\sqrt{3}}{2}F_{JK}\gamma^{IJ}\gamma^K + \mathrm{i}\sqrt{3}a_J\gamma^{IJ}. \tag{840}$$

$$\therefore (\gamma^{IJ}A_J)^\dagger = -\frac{\sqrt{3}}{2}E^I\gamma^0 - \frac{\sqrt{3}}{4}F_{JK}\gamma^{KJ}\gamma^I + \frac{\sqrt{3}}{2}F_{JK}\gamma^K\gamma^{JI} - \mathrm{i}\sqrt{3}a_J\gamma^{JI} \tag{841}$$

$$= -\frac{\sqrt{3}}{2}E^I\gamma^0 + \frac{\sqrt{3}}{4}F_{JK}(\gamma^I\gamma^{JK} - 2\delta^{KI}\gamma^J + 2\delta^{JI}\gamma^K)$$
$$- \frac{\sqrt{3}}{2}F_{JK}(\gamma^{IJ}\gamma^K - 2\delta^{KI}\gamma^J + 2\delta^{KJ}\gamma^I) + \mathrm{i}\sqrt{3}a_J\gamma^{IJ} \tag{842}$$

$$= -\frac{\sqrt{3}}{2}E^I\gamma^0 + \frac{\sqrt{3}}{4}F_{JK}\gamma^I\gamma^{JK} + \sqrt{3}F^I{}_J\gamma^J$$
$$- \frac{\sqrt{3}}{2}F_{JK}\gamma^{IJ}\gamma^K + \sqrt{3}F_J{}^I\gamma^J - 0 + \mathrm{i}\sqrt{3}a_J\gamma^{IJ} \tag{843}$$

$$= -\frac{\sqrt{3}}{2}E^I\gamma^0 + \frac{\sqrt{3}}{4}F_{JK}\gamma^I\gamma^{JK} - \frac{\sqrt{3}}{2}F_{JK}\gamma^{IJ}\gamma^K + \mathrm{i}\sqrt{3}a_J\gamma^{IJ} \tag{844}$$

$$= \gamma^{IJ}A_J. \tag{845}$$

Hence, $\gamma^{IJ}A_J$ is indeed hermitian.
Next, consider $\mathbb{M}$ for this theory. By definition,

$$\mathbb{M} = 4\pi T_{0\mu}\gamma^0\gamma^\mu + \gamma^{IJ}D_IA_J + \frac{3\mathrm{i}}{2}(\gamma^IA_I + A_I^\dagger\gamma^I) - A_I^\dagger\gamma^{IJ}A_J. \tag{846}$$

Following the same analysis as in the proof of theorem 5.3, the energy-momentum term can be expanded to get

$$\mathbb{M} = 4\pi T_{0\mu}^{\text{other}}\gamma^0\gamma^\mu + \frac{1}{2}E^IE_II + \frac{1}{4}F^{IJ}F_{IJ}I - F_{IJ}E^J\gamma^0\gamma^I + \gamma^{IJ}D_IA_J$$
$$+ \frac{3\mathrm{i}}{2}(\gamma^IA_I + A_I^\dagger\gamma^I) - A_I^\dagger\gamma^{IJ}A_J. \tag{847}$$

Individually consider each to the terms containing A_I in this expression.

$$\gamma^IA_I = -\frac{1}{2\sqrt{3}}E_J\gamma^I\gamma^J\gamma^0\gamma_I - \frac{1}{4\sqrt{3}}F_{JK}\gamma^I\gamma^{JK}\gamma_I - \frac{1}{2\sqrt{3}}E_I\gamma^I\gamma^0 - \frac{1}{2\sqrt{3}}F_{IJ}\gamma^I\gamma^J$$
$$+ \mathrm{i}\sqrt{3}a_I\gamma^I. \tag{848}$$

The 1st and 3rd terms combine to give

$$-\frac{1}{2\sqrt{3}}E_I(\gamma^J\gamma^I\gamma^0\gamma_J + \gamma^I\gamma^0) = -\frac{1}{2\sqrt{3}}E_I(-\gamma^J\gamma^I\gamma_J + \gamma^I)\gamma^0 \tag{849}$$

$$= -\frac{1}{2\sqrt{3}}E_I(\gamma^J\gamma_J\gamma^I + 2\gamma^J\delta^I{}_J + \gamma^I)\gamma^0 \tag{850}$$

$$= \frac{1}{2\sqrt{3}}E_I\gamma^I\gamma^0, \tag{851}$$

while the 2nd term is

$$-\frac{1}{4\sqrt{3}}F_{JK}\gamma^I\gamma^{JK}\gamma_I = -\frac{1}{4\sqrt{3}}F_{JK}\gamma^I(\gamma_I\gamma^{JK} - 2\delta^K{}_I\gamma^J + 2\delta^J{}_I\gamma^K) \tag{852}$$

$$= -\frac{1}{4\sqrt{3}}F_{JK}(-4\gamma^{JK} - 2\gamma^K\gamma^J + 2\gamma^J\gamma^K) \tag{853}$$

$$= 0. \tag{854}$$

That leaves

$$\gamma^I A_I = \frac{1}{2\sqrt{3}}E_I\gamma^I\gamma^0 - \frac{1}{2\sqrt{3}}F_{IJ}\gamma^{IJ} + \mathrm{i}\sqrt{3}a_I\gamma^I. \tag{855}$$

$$\therefore A_I^\dagger\gamma^I = -(\gamma^I A_I)^\dagger \tag{856}$$

$$= -\left(-\frac{1}{2\sqrt{3}}E_I\gamma^0\gamma^I - \frac{1}{2\sqrt{3}}F_{IJ}\gamma^{JI} + \mathrm{i}\sqrt{3}a_I\gamma^I\right) \tag{857}$$

$$= -\frac{1}{2\sqrt{3}}E_I\gamma^I\gamma^0 - \frac{1}{2\sqrt{3}}F_{IJ}\gamma^{IJ} - \mathrm{i}\sqrt{3}a_I\gamma^I. \tag{858}$$

$$\therefore \frac{3\mathrm{i}}{2}(\gamma^I A_I + A_I^\dagger\gamma^I) = -\frac{\mathrm{i}\sqrt{3}}{2}F_{IJ}\gamma^{IJ}. \tag{859}$$

From equation 840,

$$\gamma^{IJ}D_I A_J = D_I\left(-\frac{\sqrt{3}}{2}E^I\gamma^0 + \frac{\sqrt{3}}{4}F_{JK}\gamma^I\gamma^{JK} - \frac{\sqrt{3}}{2}F_{JK}\gamma^{IJ}\gamma^K + \mathrm{i}\sqrt{3}a_J\gamma^{IJ}\right) \tag{860}$$

$$= -\frac{\sqrt{3}}{2}D_I(E^I)\gamma^0 + \frac{\sqrt{3}}{4}D_I(F_{JK})\gamma^I\gamma^{JK} - \frac{\sqrt{3}}{2}D_I(F_{JK})\gamma^{IJ}\gamma^K + \mathrm{i}\sqrt{3}D_I(a_J)\gamma^{IJ}. \tag{861}$$

The 2nd and 3rd term combine to

$$\frac{\sqrt{3}}{4}D_I(F_{JK})\gamma^I\gamma^{JK} - \frac{\sqrt{3}}{2}D_I(F_{JK})\gamma^{IJ}\gamma^K$$

$$= \frac{\sqrt{3}}{4}D_I(F_{JK})(\gamma^{IJK} - \delta^{IJ}\gamma^K + \delta^{IK}\gamma^J) - \frac{\sqrt{3}}{2}D_I(F_{JK})(\gamma^{IJK} - \delta^{KJ}\gamma^I + \delta^{KI}\gamma^J) \tag{862}$$

$$= \frac{\sqrt{3}}{4}D_{[I}F_{JK]}\gamma^{IJK} - \frac{\sqrt{3}}{2}D^I(F_{IJ})\gamma^J - \frac{\sqrt{3}}{2}D_{[I}F_{JK]}\gamma^{IJK} + 0 - \frac{\sqrt{3}}{2}D^I(F_{JI})\gamma^J \tag{863}$$

$$= -\frac{\sqrt{3}}{4}D_{[I}F_{JK]}\gamma^{IJK} \tag{864}$$

$$= 0 \quad \text{by the equations of motion,} \tag{865}$$

while the 4th term is

$$\mathrm{i}\sqrt{3}D_I(a_J)\gamma^{IJ} = \frac{\mathrm{i}\sqrt{3}}{2}(D_I a_J - D_J a_I)\gamma^{IJ} = \frac{\mathrm{i}\sqrt{3}}{2}F_{IJ}\gamma^{IJ} \tag{866}$$

and the 1st term is

$$-\frac{\sqrt{3}}{2}D_I(E^I)\gamma^0 = -2\pi\sqrt{3}\rho\gamma^0 - \frac{1}{4}\varepsilon^{IJKL}F_{IJ}F_{KL}\gamma^0 \quad \text{by the equations of motion.} \tag{867}$$

$$\therefore \gamma^{IJ}D_I A_J = -2\pi\sqrt{3}\rho\gamma^0 - \frac{1}{4}\varepsilon^{IJKL}F_{IJ}F_{KL}\gamma^0 + \frac{\mathrm{i}\sqrt{3}}{2}F_{IJ}\gamma^{IJ}. \tag{868}$$

The most tedious term to simplify is again $A_I^\dagger \gamma^{IJ} A_J$.

$$\begin{aligned} A_I^\dagger \gamma^{IJ} A_J = &\left(-\frac{1}{2\sqrt{3}} E_J \gamma_I \gamma^0 \gamma^J + \frac{1}{4\sqrt{3}} F_{JK} \gamma_I \gamma^{KJ} - \frac{1}{2\sqrt{3}} E_I \gamma^0 + \frac{1}{2\sqrt{3}} F_{IJ} \gamma^J - \mathrm{i}\sqrt{3} a_I I\right) \\ &\times \left(-\frac{\sqrt{3}}{2} E^I \gamma^0 + \frac{\sqrt{3}}{4} F_{LM} \gamma^I \gamma^{LM} - \frac{\sqrt{3}}{2} F_{LM} \gamma^{IL} \gamma^M + \mathrm{i}\sqrt{3} a_L \gamma^{IL}\right) \end{aligned} \tag{869}$$

$$\begin{aligned} &= \frac{1}{4} E_J E^I \gamma_I \gamma^0 \gamma^J \gamma^0 - \frac{1}{8} E_J F_{LM} \gamma_I \gamma^0 \gamma^J \gamma^I \gamma^{LM} + \frac{1}{4} E_J F_{LM} \gamma_I \gamma^0 \gamma^J \gamma^{IL} \gamma^M \\ &\quad - \frac{\mathrm{i}}{2} E_J a_L \gamma_I \gamma^0 \gamma^J \gamma^{IL} - \frac{1}{8} E^I F_{JK} \gamma_I \gamma^{KJ} \gamma^0 + \frac{1}{16} F_{LM} F_{JK} \gamma_I \gamma^{KJ} \gamma^I \gamma^{LM} \\ &\quad - \frac{1}{8} F_{LM} F_{JK} \gamma_I \gamma^{KJ} \gamma^{IL} \gamma^M + \frac{\mathrm{i}}{4} a_L F_{JK} \gamma_I \gamma^{KJ} \gamma^{IL} + \frac{1}{4} E^I E_I (\gamma^0)^2 - \frac{1}{8} E_I F_{LM} \gamma^0 \gamma^I \gamma^{LM} \\ &\quad + \frac{1}{4} E_I F_{LM} \gamma^0 \gamma^{IL} \gamma^M - \frac{\mathrm{i}}{2} E_I a_L \gamma^0 \gamma^{IL} - \frac{1}{4} E^I F_{IJ} \gamma^J \gamma^0 + \frac{1}{8} F_{LM} F_{IJ} \gamma^J \gamma^I \gamma^{LM} \\ &\quad - \frac{1}{4} F_{LM} F_{IJ} \gamma^J \gamma^{IL} \gamma^M + \frac{\mathrm{i}}{2} a_L F_{IJ} \gamma^J \gamma^{IL} + \frac{3\mathrm{i}}{2} a_I E^I \gamma^0 - \frac{3\mathrm{i}}{4} a_I F_{LM} \gamma^I \gamma^{LM} \\ &\quad + \frac{3\mathrm{i}}{2} a_I F_{LM} \gamma^{IL} \gamma^M + 3 a_I a_L \gamma^{IL}. \end{aligned} \tag{870}$$

Consider each set of terms with similar fields separately.

$$3 a_I a_L \gamma^{IL} = 0. \tag{871}$$

$$\begin{aligned} &- \frac{\mathrm{i}}{2} E_J a_L \gamma_I \gamma^0 \gamma^J \gamma^{IL} - \frac{\mathrm{i}}{2} E_I a_L \gamma^0 \gamma^{IL} + \frac{3\mathrm{i}}{2} a_I E^I \gamma^0 \\ &= \frac{\mathrm{i}}{2} E_I a_J \gamma^0 (\gamma_K \gamma^I \gamma^{KJ} - \gamma^{IJ} + 3\delta^{IJ} I) &(872) \\ &= \frac{\mathrm{i}}{2} E_I a_J \gamma^0 (-\gamma^I \gamma_K \gamma^{KJ} - 2\delta^I{}_K \gamma^{KJ} - \gamma^{IJ} + 3\delta^{IJ} I) &(873) \\ &= \frac{\mathrm{i}}{2} E_I a_J \gamma^0 (3\gamma^I \gamma^J - 2\gamma^{IJ} - \gamma^{IJ} + 3\delta^{IJ} I) &(874) \\ &= \frac{\mathrm{i}}{2} E_I a_J \gamma^0 (3(\gamma^{IJ} - \delta^{IJ} I) - 3\gamma^{IJ} + 3\delta^{IJ} I) &(875) \\ &= 0. &(876) \end{aligned}$$

$$\begin{aligned} \frac{1}{4} E_J E^I \gamma_I \gamma^0 \gamma^J \gamma^0 + \frac{1}{4} E^I E_I (\gamma^0)^2 &= -\frac{1}{4} E_I E_J \gamma^I \gamma^J + \frac{1}{4} E^I E_I I &(877) \\ &= \frac{1}{2} E^I E_I I. &(878) \end{aligned}$$

$$\begin{aligned} &\frac{\mathrm{i}}{4} a_L F_{JK} \gamma_I \gamma^{KJ} \gamma^{IL} + \frac{\mathrm{i}}{2} a_L F_{IJ} \gamma^J \gamma^{IL} - \frac{3\mathrm{i}}{4} a_I F_{LM} \gamma^I \gamma^{LM} + \frac{3\mathrm{i}}{2} a_I F_{LM} \gamma^{IL} \gamma^M \\ &= \frac{\mathrm{i}}{4} a_I F_{JK} (\gamma_L \gamma^{KJ} \gamma^{LI} + 2\gamma^K \gamma^{JI} - 3\gamma^I \gamma^{JK} + 6\gamma^{IJ} \gamma^K) &(879) \\ &= \frac{\mathrm{i}}{4} a_I F_{JK} \left((\gamma^{KJ} \gamma_L - 2\delta^K{}_L \gamma^J + 2\delta^J{}_L \gamma^K) \gamma^{LI} + 2\gamma^K \gamma^{JI} - 3\gamma^I \gamma^{JK} + 6\gamma^{IJ} \gamma^K\right) &(880) \\ &= \frac{\mathrm{i}}{4} a_I F_{JK} \left(3\gamma^{JK} \gamma^I - 6\gamma^J \gamma^{KI} - 3\gamma^I \gamma^{JK} + 6\gamma^{IJ} \gamma^K\right) &(881) \\ &= \frac{3\mathrm{i}}{4} a_I F_{JK} \big(\gamma^{JKI} - \delta^{IK} \gamma^J + \delta^{IJ} \gamma^K - 2\gamma^{JKI} + 2\delta^{JK} \gamma^I - 2\delta^{JI} \gamma^K - \gamma^{IJK} + \delta^{IJ} \gamma^K - \delta^{IK} \gamma^J \\ &\quad + 2\gamma^{IJK} - 2\delta^{KJ} \gamma^I + 2\delta^{KI} \gamma^J\big) &(882) \\ &= 0. &(883) \end{aligned}$$

$$
\begin{aligned}
&\frac{1}{16}F_{LM}F_{JK}\gamma_I\gamma^{KJ}\gamma^I\gamma^{LM} - \frac{1}{8}F_{LM}F_{JK}\gamma_I\gamma^{KJ}\gamma^{IL}\gamma^M + \frac{1}{8}F_{LM}F_{IJ}\gamma^J\gamma^I\gamma^{LM} - \frac{1}{4}F_{LM}F_{IJ}\gamma^J\gamma^{IL}\gamma^M \\
&= \frac{1}{16}F_{IJ}F_{KL}\big(- \gamma_M\gamma^{KL}\gamma^M\gamma^{IJ} + 2\gamma_M\gamma^{KL}\gamma^{MI}\gamma^J - 2\gamma^{IJ}\gamma^{KL} - 4\gamma^J\gamma^{IK}\gamma^L\big) && (884)\\
&= \frac{1}{16}F_{IJ}F_{KL}\big(- (\gamma^{KL}\gamma_M - 2\delta^K{}_M\gamma^L + 2\delta^L{}_M\gamma^K)\gamma^M\gamma^{IJ} \\
&\quad + 2(\gamma^{KL}\gamma_M - 2\delta^K{}_M\gamma^L + 2\delta^L{}_M\gamma^K)\gamma^{MI}\gamma^J - 2\gamma^{IJ}\gamma^{KL} - 4\gamma^J\gamma^I\gamma^K\gamma^L - 4\gamma^J\delta^{IK}\gamma^L\big) && (885)\\
&= \frac{1}{16}F_{IJ}F_{KL}\big(- \gamma_M\gamma^{KL}\gamma^M\gamma^{IJ} + 2\gamma_M\gamma^{KL}\gamma^{MI}\gamma^J - 2\gamma^{IJ}\gamma^{KL} - 4\gamma^J\gamma^{IK}\gamma^L\big) && (886)\\
&= \frac{1}{16}F_{IJ}F_{KL}\big(4\gamma^{KL}\gamma^{IJ} + 4\gamma^L\gamma^K\gamma^{IJ} - 6\gamma^{KL}\gamma^I\gamma^J - 8\gamma^L\gamma^{KI}\gamma^J - 2\gamma^{IJ}\gamma^{KL} + 4\gamma^{IJ}\gamma^{KL} \\
&\quad - 4\gamma^J\delta^{IK}\gamma^L\big) && (887)\\
&= \frac{1}{16}F_{IJ}F_{KL}\big(- 4\gamma^{IJ}\gamma^{KL} - 8\gamma^L\gamma^K\gamma^I\gamma^J - 8\gamma^L\delta^{KI}\gamma^J - 4\gamma^J\delta^{IK}\gamma^L\big) && (888)\\
&= \frac{1}{4}F_{IJ}F_{KL}\gamma^{IJ}\gamma^{KL} - \frac{1}{2}F_{IJ}F^I{}_K\gamma^K\gamma^J - \frac{1}{4}F_{IJ}F^I{}_K\gamma^J\gamma^K && (889)\\
&= \frac{1}{4}F_{IJ}F_{KL}\big(\gamma^{IJKL} + \delta^{IK}\gamma^J\gamma^L - \delta^{IL}\gamma^J\gamma^K - \delta^{JK}\gamma^I\gamma^L + \delta^{JL}\gamma^I\gamma^K + \delta^{IK}\delta^{JL}I - \delta^{IL}\delta^{JK}I\big) \\
&\quad + \frac{3}{4}F^{IJ}F_{IJ}I && (890)\\
&= \frac{1}{4}F_{IJ}F_{KL}\gamma^{IJKL} + F_{IJ}F^I{}_K\gamma^J\gamma^K + \frac{1}{2}F^{IJ}F_{IJ}I + \frac{3}{4}F^{IJ}F_{IJ}I && (891)\\
&= \frac{1}{4}F_{IJ}F_{JK}\varepsilon^{IJKL}\gamma^1\gamma^2\gamma^3\gamma^4 + \frac{1}{4}F^{IJ}F_{IJ}I && (892)\\
&= -\frac{1}{4}F_{IJ}F_{JK}\varepsilon^{IJKL}\gamma^0 + \frac{1}{4}F^{IJ}F_{IJ}I \ \text{ as I've chosen } \gamma^4 = \gamma^0\gamma^1\gamma^2\gamma^3. && (893)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{8}E_JF_{LM}\gamma_I\gamma^0\gamma^J\gamma^I\gamma^{LM} + \frac{1}{4}E_JF_{LM}\gamma_I\gamma^0\gamma^J\gamma^{IL}\gamma^M - \frac{1}{8}E^IF_{JK}\gamma_I\gamma^{KJ}\gamma^0 \\
&\quad - \frac{1}{8}E_IF_{LM}\gamma^0\gamma^I\gamma^{LM} + \frac{1}{4}E_IF_{LM}\gamma^0\gamma^{IL}\gamma^M - \frac{1}{4}E^IF_{IJ}\gamma^J\gamma^0 && (894)\\
&= \frac{1}{8}E_IF_{JK}\big(- \gamma_L\gamma^0\gamma^I\gamma^L\gamma^{JK} + 2\gamma_L\gamma^0\gamma^I\gamma^{LJ}\gamma^K + \gamma^I\gamma^{JK}\gamma^0 - \gamma^0\gamma^I\gamma^{JK} + 2\gamma^0\gamma^{IJ}\gamma^K \\
&\quad - 2\delta^{IJ}\gamma^K\gamma^0\big) && (895)\\
&= \frac{1}{8}E_IF_{JK}\gamma^0\big(\gamma_L\gamma^I\gamma^L\gamma^{JK} - 2\gamma_L\gamma^I\gamma^{LJ}\gamma^K - \gamma^I\gamma^{JK} - \gamma^I\gamma^{JK} + 2\gamma^{IJ}\gamma^K + 2\delta^{IJ}\gamma^K\big) && (896)\\
&= \frac{1}{8}E_IF_{JK}\gamma^0\big(- \gamma^I\gamma_L\gamma^L\gamma^{JK} - 2\delta^I{}_L\gamma^L\gamma^{JK} + 2\gamma^I\gamma_L\gamma^{LJ}\gamma^K + 4\delta^I{}_L\gamma^{LJ}\gamma^K - 2\gamma^I\gamma^{JK} \\
&\quad + 2\gamma^{IJ}\gamma^K + 2\delta^{IJ}\gamma^K\big) && (897)\\
&= \frac{1}{8}E_IF_{JK}\gamma^0\big(4\gamma^I\gamma^{JK} - 2\gamma^I\gamma^{JK} - 6\gamma^I\gamma^J\gamma^K + 4\gamma^{IJ}\gamma^K - 2\gamma^I\gamma^{JK} + 2\gamma^{IJ}\gamma^K + 2\delta^{IJ}\gamma^K\big) && (898)\\
&= \frac{1}{8}E_IF_{JK}\gamma^0\big(- 6\gamma^I\gamma^{JK} + 6\gamma^{IJ}\gamma^K + 2\delta^{IJ}\gamma^K\big) && (899)\\
&= \frac{1}{8}E_IF_{JK}\gamma^0\big(- 6\gamma^{IJK} + 6\delta^{IJ}\gamma^K - 6\delta^{IK}\gamma^J + 6\gamma^{IJK} - 6\delta^{JK}\gamma^I + 6\delta^{KI}\gamma^J + 2\delta^{IJ}\gamma^K\big) && (900)\\
&= E^IF_{IJ}\gamma^0\gamma^J. && (901)
\end{aligned}
$$

Substituting all these expressions back,

$$
A_I^\dagger\gamma^{IJ}A_J = \frac{1}{2}E^IE_II + E^IF_{IJ}\gamma^0\gamma^J - \frac{1}{4}F_{IJ}F_{JK}\varepsilon^{IJKL}\gamma^0 + \frac{1}{4}F^{IJ}F_{IJ}I. \qquad (902)
$$

That, along with the previous calculations, in turn implies

$$
\begin{aligned}
\mathbb{M} &= 4\pi T^{\text{other}}_{0\mu}\gamma^0\gamma^\mu + \frac{1}{2}E^I E_I I + \frac{1}{4}F^{IJ}F_{IJ}I - F_{IJ}E^J\gamma^0\gamma^I - 2\pi\sqrt{3}\rho\gamma^0 - \frac{1}{4}\varepsilon^{IJKL}F_{IJ}F_{KL}\gamma^0 \\
&\quad + \frac{\mathrm{i}\sqrt{3}}{2}F_{IJ}\gamma^{IJ} - \frac{\mathrm{i}\sqrt{3}}{2}F_{IJ}\gamma^{IJ} - \frac{1}{2}E^I E_I I - E^I F_{IJ}\gamma^0\gamma^J + \frac{1}{4}F_{IJ}F_{JK}\varepsilon^{IJKL}\gamma^0 \\
&\quad - \frac{1}{4}F^{IJ}F_{IJ}I \qquad (903) \\
&= 4\pi T^{\text{other}}_{0\mu}\gamma^0\gamma^\mu - 2\pi\sqrt{3}\rho\gamma^0 \qquad (904) \\
&= 4\pi T^{\text{other}}_{00}I + 4\pi T^{\text{other}}_{0I}\gamma^0\gamma^I - 2\pi\sqrt{3}\rho\gamma^0. \qquad (905)
\end{aligned}
$$

It can be checked, e.g. by computer algebra, that $\mathbb{M}$'s eigenvalues are

$$
4\pi\left(T^{\text{other}}_{00} \pm \sqrt{T^{\text{other}}_{0I}T^{\text{other}I}_{0} + \frac{3}{4}\rho^2}\right). \qquad (906)
$$

Thus, the condition I've assumed on T^{other}_{ab} ensures $\mathbb{M}$ is non-negative definite.
The $||\mathbb{M}||_0$ decay condition assumed in definition 3.1 corresponds exactly to the decay conditions I'm assuming for $T^{\text{other}}_{0\mu}$ and ρ.
The $||A_I||_0$ decay condition assumed in definition 3.1 is merely required for the boundary integrals in equation 828 or theorem 3.19 to be finite and is stronger than the decay required for convergence properties in section 3.1.
Finally, there is the assumption regarding $\gamma^I A_I = -\tilde{A}_I^\dagger\gamma^I$. Just as in the proof of theorem 5.3, equation 858 implies $\tilde{A}_I$ exists and it is identical to A_I except $F_{IJ} \to -F_{IJ}$ and $a_I \to -a_I$. Just as in theorem 5.3, decay rates and $\gamma^{IJ}A_J$ being hermitian are unaffected by this change, meaning the assumptions of definition 3.1 hold.
Having established theorem 3.19 is valid in the present scenario, it remains only to simplify the integrals there.
$\frac{n-1}{2}\mathrm{e}^{-r}\int_{\partial_\infty\Sigma_t} p_M\bar{\varepsilon}_k\gamma^M\varepsilon_k\sqrt{\iota^* f_{(0)}}\,\mathrm{d}^{n-2}x$ is unchanged and the $\int_{\Sigma_t}\mathrm{d}V$ integral follows immediately from equation 905. The other boundary integrals require finding $\gamma^1\gamma^A A_A + A_A^\dagger\gamma^A\gamma^1$.

$$
\begin{aligned}
&\gamma^1\gamma^A A_A \\
&= \gamma^1\gamma^A\Big(-\frac{1}{2\sqrt{3}}E_I\gamma^I\gamma^0\gamma_A - \frac{1}{4\sqrt{3}}F_{IJ}\gamma^{IJ}\gamma_A - \frac{1}{2\sqrt{3}}E_A\gamma^0 - \frac{1}{2\sqrt{3}}F_{AI}\gamma^I + \mathrm{i}\sqrt{3}a_A I\Big) \qquad (907) \\
&= -\frac{1}{2\sqrt{3}}E_1\gamma^1\gamma^A\gamma^1\gamma^0\gamma_A - \frac{1}{2\sqrt{3}}E_B\gamma^1\gamma^A\gamma^B\gamma^0\gamma_A - \frac{1}{2\sqrt{3}}F_{1B}\gamma^1\gamma^A\gamma^1\gamma^B\gamma_A \\
&\quad - \frac{1}{4\sqrt{3}}F_{BC}\gamma^1\gamma^A\gamma^{BC}\gamma_A - \frac{1}{2\sqrt{3}}E_A\gamma^1\gamma^A\gamma^0 - \frac{1}{2\sqrt{3}}F_{A1}\gamma^1\gamma^A\gamma^1 - \frac{1}{2\sqrt{3}}F_{AB}\gamma^1\gamma^A\gamma^B \\
&\quad + \mathrm{i}\sqrt{3}a_A\gamma^1\gamma^A \qquad (908)
\end{aligned}
$$

Again, consider terms with the same fields separately.

$$
\begin{aligned}
&-\frac{1}{2\sqrt{3}}E_1\gamma^1\gamma^A\gamma^1\gamma^0\gamma_A - \frac{1}{2\sqrt{3}}E_B\gamma^1\gamma^A\gamma^B\gamma^0\gamma_A - \frac{1}{2\sqrt{3}}E_A\gamma^1\gamma^A\gamma^0 \\
&= -\frac{\sqrt{3}}{2}E_1\gamma^0 + \frac{1}{2\sqrt{3}}E_B\gamma^1\gamma^0\gamma^B\gamma^A\gamma_A + \frac{1}{\sqrt{3}}E_B\gamma^1\delta^{AB}\gamma^0\gamma_A - \frac{1}{2\sqrt{3}}E_A\gamma^1\gamma^A\gamma^0 \qquad (909) \\
&= -\frac{\sqrt{3}}{2}E_1\gamma^0 - \frac{\sqrt{3}}{2}E_A\gamma^1\gamma^0\gamma^A + \frac{1}{\sqrt{3}}E_A\gamma^1\gamma^0\gamma^A - \frac{1}{2\sqrt{3}}E_A\gamma^1\gamma^A\gamma^0 \qquad (910) \\
&= -\frac{\sqrt{3}}{2}E_1\gamma^0. \qquad (911)
\end{aligned}
$$

$$-\frac{1}{2\sqrt{3}}F_{1B}\gamma^1\gamma^A\gamma^1\gamma^B\gamma_A - \frac{1}{4\sqrt{3}}F_{BC}\gamma^1\gamma^A\gamma^{BC}\gamma_A - \frac{1}{2\sqrt{3}}F_{A1}\gamma^1\gamma^A\gamma^1 - \frac{1}{2\sqrt{3}}F_{AB}\gamma^1\gamma^A\gamma^B$$

$$= -\frac{1}{2\sqrt{3}}F_{1B}\gamma^A\gamma^B\gamma_A - \frac{1}{4\sqrt{3}}F_{BC}\gamma^1\gamma^A(\gamma_A\gamma^{BC} - 2\delta^C{}_A\gamma^B + 2\delta^B{}_A\gamma^C) - \frac{1}{2\sqrt{3}}F_{A1}\gamma^A - \frac{1}{2\sqrt{3}}F_{AB}\gamma^1\gamma^{AB} \tag{912}$$

$$= \frac{1}{2\sqrt{3}}F_{1B}\gamma^B\gamma^A\gamma_A + \frac{1}{\sqrt{3}}F_{1B}\delta^{AB}\gamma_A + \frac{\sqrt{3}}{4}F_{AB}\gamma^1\gamma^{AB} - \frac{1}{\sqrt{3}}F_{AB}\gamma^1\gamma^{AB} - \frac{1}{2\sqrt{3}}F_{A1}\gamma^A - \frac{1}{2\sqrt{3}}F_{AB}\gamma^1\gamma^{AB} \tag{913}$$

$$= -\frac{\sqrt{3}}{2}F_{1A}\gamma^A + \frac{1}{\sqrt{3}}F_{1A}\gamma^A - \frac{\sqrt{3}}{4}F_{AB}\gamma^1\gamma^{AB} - \frac{1}{2\sqrt{3}}F_{A1}\gamma^A \tag{914}$$

$$= -\frac{\sqrt{3}}{4}F_{AB}\gamma^1\gamma^{AB}. \tag{915}$$

Altogether, I get

$$\gamma^1\gamma^A A_A = -\frac{\sqrt{3}}{2}E_1\gamma^0 - \frac{\sqrt{3}}{4}F_{AB}\gamma^1\gamma^{AB} + \mathrm{i}\sqrt{3}a_A\gamma^1\gamma^A. \tag{916}$$

$$\therefore \gamma^1\gamma^A A_A + A_A^\dagger\gamma^1\gamma^A = -\sqrt{3}E_1\gamma^0 - \frac{\sqrt{3}}{2}F_{AB}\gamma^1\gamma^{AB} + 2\mathrm{i}\sqrt{3}a_A\gamma^1\gamma^A, \tag{917}$$

which corresponds exactly to the integrand in the theorem. □

Corollary 5.6.1. *If the extrinsic curvature, K_{IJ}, of Σ_t is less than $O(1)$ near $\partial_\infty\Sigma_t$ and $f_{(0)0\alpha} = 0$, then*

$$Q(\varepsilon) = \frac{n-1}{2}\mathrm{e}^{-r}\int_{\partial_\infty\Sigma_t} p_M\bar{\varepsilon}_k\gamma^M\varepsilon_k\sqrt{\iota^* f_{(0)}}\,\mathrm{d}^{n-2}x - \sqrt{3}\int_{\partial_\infty\Sigma_t} E_1\bar{\varepsilon}_k\varepsilon_k \mathrm{d}A \tag{918}$$

$$= 2\int_{\Sigma_t}\Big((\nabla_I\varepsilon)^\dagger\nabla^I\varepsilon + 4\pi T^{0\mu}_{\text{other}}\varepsilon^\dagger\gamma_0\gamma_\mu\varepsilon - 2\pi\sqrt{3}\rho\varepsilon^\dagger\gamma^0\varepsilon\Big)\,\mathrm{d}V \tag{919}$$

$$\geq 0, \tag{920}$$

i.e. under the extra assumptions made, the magnetic and gauge field boundary integrals cancel.

Proof. The proof is identical to that of corollary 5.3.1, except that $\gamma_J\gamma^{JIK} = -2\gamma^{IK}$ now, instead of just $-\gamma^{IK}$. This factor of 2 exactly matches the extra factor of 2 between the coefficients of the magnetic and gauge field boundary integrals compared to theorem 5.3. □

5.2.1 Toroidal boundary

Consider again the toroidal boundary, $f_{(0)} = -\mathrm{d}t\otimes\mathrm{d}t + \delta_{\alpha\beta}\,\mathrm{d}\theta^\alpha\otimes\mathrm{d}\theta^\beta$.

Theorem 5.7 (5D, $\mathcal{N}=2$, toroidal boundary supergravity BPS inequality)**.** *If the equations of motion hold, $T^{0\mu}_{\text{other}}$ decays faster than $O(\mathrm{e}^{-(n-1)r})$ near $\partial_\infty\Sigma_t$, $T^{\text{other}}_{00} \geq \sqrt{T^{\text{other}}_{0I}T^{\text{other}I}_0 + 3\rho^2/4}$ and (M,g)'s spin structure is compatible with having periodic spinors near $\partial_\infty\Sigma_t$, then*

$$E \geq \sqrt{\mathbb{J}_A\mathbb{J}^A}. \tag{921}$$

Proof. The proof is identical to theorem 5.4. □

5.2.2 Asymptotically AdS

Theorem 5.8 (5D, $\mathcal{N}=2$, asymptotically AdS supergravity BPS inequality)**.** *If the equations of motion hold,* $T^{0\mu}_{\text{other}}$ *decays faster than* $O(\mathrm{e}^{-3r})$ *near* $\partial_\infty\Sigma_t$*,* $T^{\text{other}}_{00} \geq \sqrt{T^{\text{other}}_{0I}T^{\text{other}I}_{0} + 3\rho^2/4}$*,* A_I *decays*[41] *faster than* $O(\mathrm{e}^{2r})$ *and* E_1 *decays as* $O(\mathrm{e}^{-3r})$*, then*

$$EI - \mathrm{i}P_I\gamma^I + \frac{\mathrm{i}}{2}J_{IJ}\gamma^0\gamma^{IJ} + K_I\gamma^0\gamma^I - \frac{\sqrt{3}}{2}q_e\gamma^0 \tag{922}$$

is a non-negative definite matrix.

Proof. As in 4D, corollary 5.6.1 is applicable for this boundary geometry. Both integrals in corollary 5.6.1 are analysed identically to their analogues in theorem 5.5. □

This theorem generalises the result in [20], while also allowing for non-zero magnetic fields and non-zero spacelike components to the gauge field.

5.2.3 Charged, equal angular momenta Myers-Perry solution example

As an example of the BPS inequalities proven in this section, I'll apply theorem 5.8 to the 5D, minimal, gauged supergravity analogue [56] of the example in section of 4.2.1. In the form presented in [57, 58],

$$\begin{aligned} g = &-\frac{R^2W}{4b^2}\mathrm{d}t\otimes\mathrm{d}t + \frac{1}{W}\mathrm{d}R\otimes\mathrm{d}R + \frac{1}{4}R^2(\mathrm{d}\theta\otimes\mathrm{d}\theta + \sin^2(\theta)\mathrm{d}\phi\otimes\mathrm{d}\phi) \\ &+ b^2(\mathrm{d}\psi + \cos(\theta)\mathrm{d}\phi + f\mathrm{d}t)\otimes(\mathrm{d}\psi + \cos(\theta)\mathrm{d}\phi + f\mathrm{d}t) \end{aligned} \tag{923}$$

$$\text{and } a = -\frac{Q\sqrt{3}}{2R^2}\left(\mathrm{d}t - \frac{j}{2}(\mathrm{d}\psi + \cos(\theta)\mathrm{d}\phi)\right), \tag{924}$$

$$\text{where } W = 1 + 4b^2 - \frac{2P-2Q}{R^2} + \frac{Q^2+2Pj^2}{R^4}, \tag{925}$$

$$f = -\frac{j}{2b^2}\left(\frac{2P-Q}{R^2} - \frac{Q^2}{R^4}\right), \tag{926}$$

$$b^2 = \frac{1}{4}R^2\left(1 + \frac{2j^2P}{R^4} - \frac{j^2Q^2}{R^6}\right), \tag{927}$$

P, Q & j are constants and the angles, ψ, θ & ϕ, are the same as in section 4.2.1.
Start with the Fefferman-Graham coordinate. $\frac{1}{W}$ plays the same role here as f^2 in section 4.2.1.

$$W = 1 + R^2\left(1 + \frac{2j^2P}{R^4} - \frac{j^2Q^2}{R^6}\right) - \frac{2P-2Q}{R^2} + \frac{Q^2+2Pj^2}{R^4} \tag{928}$$

$$= 1 + R^2 + \frac{2((j^2-1)P+Q)}{R^2} + \frac{(1-j^2)Q + 2j^2P}{R^4}. \tag{929}$$

Comparing with section 4.2.1, the analogue of MZ in equation 435 is $(1-j^2)P - Q$.
∴ From the work there, I immediately get

$$\mathrm{e}^r \to \frac{1}{2}(R + \sqrt{1+R^2})\left(1 - \frac{(1-j^2)P - Q}{4R^4}\right) \text{ and} \tag{930}$$

$$R^2 \to \mathrm{e}^{2r}\left(\left(1 - \frac{1}{4}\mathrm{e}^{-2r}\right)^2 + \frac{1}{2}\left((1-j^2)P - Q\right)\mathrm{e}^{-4r}\right). \tag{931}$$

[41] This is a weaker decay condition than in definition 3.1. The decay there ensures the boundary integrals in theorem 3.19 are convergent. However, I am dealing with that issue separately with a specific decay condition on E_1. Hence, I can assume this weaker decay condition, which suffices for the analysis is section 3.1 - in particular, lemma 3.17.

These expansions fully determine the other coefficients in the metric.

$$b^2 \to \frac{1}{4}\mathrm{e}^{2r}\left(\left(1-\frac{1}{4}\mathrm{e}^{-2r}\right)^2+\frac{1}{2}\left((1-j^2)P-Q\right)\mathrm{e}^{-4r}\right)\left(1+2j^2P\mathrm{e}^{-4r}-j^2Q^2\mathrm{e}^{-6r}\right) \tag{932}$$

$$\to \frac{1}{4}\mathrm{e}^{2r}\left(\left(1-\frac{1}{4}\mathrm{e}^{-2r}\right)^2+\frac{1}{2}\left((1-j^2)P-Q\right)\mathrm{e}^{-4r}+2j^2P\mathrm{e}^{-4r}\right) \tag{933}$$

$$= \frac{1}{4}\mathrm{e}^{2r}\left(\left(1-\frac{1}{4}\mathrm{e}^{-2r}\right)^2+\frac{1}{2}\left((1+3j^2)P-Q\right)\mathrm{e}^{-4r}\right). \tag{934}$$

$$\frac{R^2W}{4b^2} = \frac{R^2}{4}\frac{4}{R^2(1+2j^2P/R^4-j^2Q^2/R^6)} \times\left(1+R^2+\frac{2((j^2-1)P+Q)}{R^2}+\frac{(1-j^2)Q+2j^2P}{R^4}\right) \tag{935}$$

$$\to \left(1-2j^2P\mathrm{e}^{-4r}\right)\left(1+\mathrm{e}^{2r}\left(\left(1-\frac{1}{4}\mathrm{e}^{-2r}\right)^2+\frac{1}{2}\left((1-j^2)P-Q\right)\mathrm{e}^{-4r}\right) + 2((j^2-1)P+Q)\mathrm{e}^{-2r}\right) \tag{936}$$

$$= \mathrm{e}^{2r}\left(1-2j^2P\mathrm{e}^{-4r}\right)\left(\left(\left(1+\frac{1}{4}\mathrm{e}^{-2r}\right)^2+\frac{1}{2}\left((1-j^2)P-Q\right)\mathrm{e}^{-4r}\right) + 2((j^2-1)P+Q)\mathrm{e}^{-4r}\right) \tag{937}$$

$$\to \mathrm{e}^{2r}\left(\left(1+\frac{1}{4}\mathrm{e}^{-2r}\right)^2+\frac{1}{2}\left(3Q-(j^2+3)P\right)\mathrm{e}^{-4r}\right). \tag{938}$$

$$b^2f = -\frac{j}{2}\left(\frac{2P-Q}{R^2}-\frac{Q^2}{R^4}\right) \to -\frac{j(2P-Q)}{2}\mathrm{e}^{-2r}. \tag{939}$$

Substituting these into equation 923 gives

$$g = \mathrm{e}^{2r}\Bigg(-\left(1+\frac{1}{4}\mathrm{e}^{-2r}\right)^2\mathrm{d}t\otimes\mathrm{d}t+\frac{1}{4}\left(1-\frac{1}{4}\mathrm{e}^{-2r}\right)^2\big(\mathrm{d}\theta\otimes\mathrm{d}\theta+\sin^2(\theta)\mathrm{d}\phi\otimes\mathrm{d}\phi + (\mathrm{d}\psi+\cos(\theta)\mathrm{d}\phi)\otimes(\mathrm{d}\psi+\cos(\theta)\mathrm{d}\phi)\big) + \mathrm{e}^{-4r}\Big(-\frac{1}{2}(3Q-(J^2+3)P)\mathrm{d}t\otimes\mathrm{d}t+\frac{1}{8}((1-j^2)P-Q)(\mathrm{d}\theta\otimes\mathrm{d}\theta+\sin^2(\theta)\mathrm{d}\phi\otimes\mathrm{d}\phi) + \frac{1}{8}((1+3j^2)P-Q)(\mathrm{d}\psi+\cos(\theta)\mathrm{d}\phi)\otimes(\mathrm{d}\psi+\cos(\theta)\mathrm{d}\phi) - \frac{1}{2}j(2P-Q)(\mathrm{d}t\otimes(\mathrm{d}\psi+\cos(\theta)\mathrm{d}\phi)+(\mathrm{d}\psi+\cos(\theta)\mathrm{d}\phi)\otimes\mathrm{d}t)\Big)\Bigg)+\mathrm{d}r\otimes\mathrm{d}r. \tag{940}$$

∴ The metric is indeed asymptotically AdS in the sense of definition 2.3. Furthermore, it has

$$\begin{aligned} f_{(0)} &= -\mathrm{d}t\otimes\mathrm{d}t + \frac{1}{4}((\mathrm{d}\psi+\cos(\theta)\mathrm{d}\phi)\otimes(\mathrm{d}\psi+\cos(\theta)\mathrm{d}\phi)+\mathrm{d}\theta\otimes\mathrm{d}\theta+\sin^2(\theta)\mathrm{d}\phi\otimes\mathrm{d}\phi) && (941)\\ &= -\mathrm{d}t\otimes\mathrm{d}t + g_{S^3} \quad \text{and} && (942)\\ f_{(4)} &= -\frac{1}{2}(3Q-(J^2+3)P)\mathrm{d}t\otimes\mathrm{d}t + \frac{1}{8}((1-j^2)P-Q)(\mathrm{d}\theta\otimes\mathrm{d}\theta+\sin^2(\theta)\mathrm{d}\phi\otimes\mathrm{d}\phi)\\ &\quad + \frac{1}{8}((1+3j^2)P-Q)(\mathrm{d}\psi+\cos(\theta)\mathrm{d}\phi)\otimes(\mathrm{d}\psi+\cos(\theta)\mathrm{d}\phi)\\ &\quad - \frac{1}{2}j(2P-Q)(\mathrm{d}t\otimes(\mathrm{d}\psi+\cos(\theta)\mathrm{d}\phi)+(\mathrm{d}\psi+\cos(\theta)\mathrm{d}\phi)\otimes\mathrm{d}t). && (943)\end{aligned}$$

These are the same form as equations 460 and 461, so following the work there,

$$\begin{aligned} E &= \frac{1}{4\pi}\int_{S^3}\left(4(1+\cot^2(\theta))f_{(4)22} - \frac{8\cos(\theta)}{\sin^2(\theta)}f_{(4)24} + 4f_{(4)33} + \frac{4}{\sin^2(\theta)}f_{(4)44}\right)\mathrm{d}(g_{S^3}) && (944)\\ &= \frac{1}{4\pi}\int_{S^3}\left(\frac{1}{2}(1+\cot^2(\theta))((1+3j^2)P-Q) - \frac{\cos(\theta)}{\sin^2(\theta)}((1+3j^2)P-Q)\cos(\theta)\right.\\ &\quad + \frac{1}{2}((1-j^2)P-Q) + \frac{1}{2\sin^2(\theta)}((1-j^2)P-Q)\sin^2(\theta)\\ &\quad \left. + \frac{1}{2\sin^2(\theta)}((1+3j^2)P-Q)\cos^2(\theta)\right)\mathrm{d}(g_{S^3}) && (945)\\ &= \frac{(j^2+3)P-3Q}{8\pi}\int_{S^3}\mathrm{d}(g_{S^3}) && (946)\\ &= \frac{\pi}{4}((j^2+3)P-3Q), && (947)\end{aligned}$$

which matches the result calculated in [58] using the completely different methods of [59]. Similarly, from the work in section 4.2.1, I can also immediately read off $K_I = P_I = 0$ and

$$J_{IJ} \equiv \frac{\frac{1}{2}j(2P-Q)}{Ma}\frac{\pi Ma}{2}\begin{bmatrix}0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0\end{bmatrix} = \frac{\pi j(2P-Q)}{4}\begin{bmatrix}0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0\end{bmatrix}. \tag{948}$$

The only remaining quantity in equation 922 is the electric charge. For that,

$$\begin{aligned} F = \mathrm{d}a &= \frac{Q\sqrt{3}}{R^3}\mathrm{d}R\wedge\left(\mathrm{d}t - \frac{j}{2}(\mathrm{d}\psi+\cos(\theta)\mathrm{d}\phi)\right) - \frac{jQ\sqrt{3}}{4R^2}\sin(\theta)\mathrm{d}\theta\wedge\mathrm{d}\phi && (949)\\ &\to Q\sqrt{3}\mathrm{e}^{-2r}\mathrm{d}r\wedge\left(\mathrm{d}t - \frac{j}{2}(\mathrm{d}\psi+\cos(\theta)\mathrm{d}\phi)\right) - \frac{jQ\sqrt{3}}{4}\mathrm{e}^{-2r}\sin(\theta)\mathrm{d}\theta\wedge\mathrm{d}\phi. && (950)\\ \therefore E_1 &= F_{10} = Q\sqrt{3}\mathrm{e}^{-3r}. && (951)\\ \therefore q_e &= \frac{1}{4\pi}\int_{S^3_\infty}E_1\mathrm{d}A = \frac{Q\sqrt{3}}{4\pi}\int_{S^3_\infty}\mathrm{d}(g_{S^3}) = \frac{\pi Q\sqrt{3}}{2}. && (952)\end{aligned}$$

Substituting all these quantities into theorem 5.8 implies

$$\frac{\pi}{4}((j^2+3)P-3Q)I + \frac{\mathrm{i}\pi j(2P-Q)}{4}\gamma^0\left(\gamma^2\gamma^1+\gamma^4\gamma^3\right) - \frac{3\pi Q}{4}\gamma^0 \tag{953}$$

is non-negative definite. Using a computer algebra software, it can be checked the eigenvalues of this matrix are

$$\frac{\pi}{4}((j^2+3)P-3Q)+\frac{3\pi Q}{4},\ \frac{\pi}{4}((j^2+3)P-3Q)-\frac{3\pi Q}{4}+\frac{\pi j(2P-Q)}{2}$$
$$\text{and } \frac{\pi}{4}((j^2+3)P-3Q)-\frac{3\pi Q}{4}-\frac{\pi j(2P-Q)}{2}. \tag{954}$$

Which of these is the lowest eigenvalue depends on the choices of j, P and Q. Nonetheless, they all have to be non-negative. Therefore,

$$\frac{\pi}{4}((j^2+3)P-3Q)+\frac{3\pi Q}{4}\geq 0 \iff P\geq 0, \tag{955}$$
$$\frac{\pi}{4}((j^2+3)P-3Q)-\frac{3\pi Q}{4}+\frac{\pi j(2P-Q)}{2}\geq 0 \iff P\geq \frac{2Q}{j+1} \text{ and} \tag{956}$$
$$\frac{\pi}{4}((j^2+3)P-3Q)-\frac{3\pi Q}{4}-\frac{\pi j(2P-Q)}{2}\geq 0 \iff P\geq -\frac{2Q}{j-1}. \tag{957}$$

From [57], each inequality is saturated by a known supersymmetric solution. In particular, $P=0$ is the Klemm-Sabra solution [60] and $P=\pm\frac{2Q}{j\pm1}$ are the Gutowski-Reall solutions [61] with their $\epsilon=\mp1$ respectively[42].

5.2.4 Lens spaces, $L(p,1)$

Theorem 5.9 (5D, $\mathcal{N}=2$ supergravity BPS inequality for spacetimes asymptotically Kottler with lens space cross-section)**.** *If the equations of motion hold, $T^{0\mu}_{\text{other}}$ decays faster than $O(\mathrm{e}^{-3r})$ near $\partial_\infty\Sigma_t$, $T^{\text{other}}_{00}\geq\sqrt{T^{\text{other}}_{0I}T^{\text{other}I}_{0}+3\rho^2/4}$, A_I decays[43] faster than $O(\mathrm{e}^{2r})$ and E_1 decays as $O(\mathrm{e}^{-3r})$, then*[44]

$$E\geq-\frac{\sqrt{3}}{2}q_e+\sqrt{J_2^2+J_3^2+J_4^2} \tag{958}$$

Proof. Corollary 5.6.1 applies again and the Killing spinor, ε_k, is constructed exactly as in section 4.3.2.
∴ The first boundary integral is identical to what I had to evaluate earlier; the result is

$$\frac{n-1}{2}\mathrm{e}^{-r}\int_{\partial_\infty\Sigma_t}p_M\bar{\varepsilon}_k\gamma^M\varepsilon_k\sqrt{\iota^*f_{(0)}}\,\mathrm{d}^{n-2}x \tag{959}$$
$$=16\pi x^\dagger\left(EI+J_2\sigma_1+J_3\sigma_2+J_4\sigma_3\right)x \tag{960}$$
$$\text{for } \varepsilon_k=\mathrm{e}^{r/2}P_1^-\left(\mathrm{e}^{\mathrm{i}\gamma^0t/2}-\mathrm{i}\mathrm{e}^{-\mathrm{i}\gamma^0t/2}\right)\varepsilon_H+\frac{1}{2}\mathrm{e}^{-r/2}P_1^+\left(\mathrm{e}^{\mathrm{i}\gamma^0t/2}+\mathrm{i}\mathrm{e}^{-\mathrm{i}\gamma^0t/2}\right)\varepsilon_H, \tag{961}$$
$$\varepsilon_H=\mathrm{e}^{\theta\gamma^3/4}\mathrm{e}^{\phi_1\gamma^2/2}\mathrm{e}^{\phi_2\gamma^3\gamma^4/2}\varepsilon_0 \tag{962}$$
$$\text{and } \varepsilon_0=[x,-x]^{\mathrm{T}}. \tag{963}$$

[42]The $\epsilon=\pm1$ solutions are not physically too dissimilar; it's merely that a rotation direction is reversed.

[43]This is a weaker decay condition than in definition 3.1. The decay there ensures the boundary integrals in theorem 3.19 are convergent. However, I am dealing with that issue separately with a specific decay condition on E_1. Hence, I can assume this weaker decay condition, which suffices for the analysis is section 3.1 - in particular, lemma 3.17.

[44]Note that the meaning of positive and negative charge depends on the choice of positive orientation on the lens space. I will be using the orientation defined by lemma 4.20.

It remains to evaluate the 2nd integral in corollary 5.6.1. The spinor factor is

$$\begin{aligned}
\overline{\varepsilon}_k\varepsilon_k &= \left(\mathrm{e}^{r/2}P_1^-\varepsilon_- + \mathrm{e}^{-r/2}P_1^+\varepsilon_+\right)^\dagger \gamma^0 \left(\mathrm{e}^{r/2}P_1^-\varepsilon_- + \mathrm{e}^{-r/2}P_1^+\varepsilon_+\right) && (964)\\
&= \mathrm{e}^r\varepsilon_-^\dagger\gamma^0P_1^+P_1^-\varepsilon_- + \varepsilon_+^\dagger\gamma^0P_1^-P_1^-\varepsilon_- + \varepsilon_-^\dagger\gamma^0P_1^+P_1^+\varepsilon_+ + \mathrm{e}^{-r}\varepsilon_+^\dagger\gamma^0P_1^-P_1^+\varepsilon_+ && (965)\\
&= 0 + \varepsilon_+^\dagger\gamma^0P_1^-\varepsilon_- + \varepsilon_-^\dagger\gamma^0P_1^+\varepsilon_+ + 0 && (966)\\
&= \frac{1}{2}\varepsilon_H^\dagger\left(\mathrm{e}^{-\mathrm{i}\gamma^0t/2} - \mathrm{i}\mathrm{e}^{\mathrm{i}\gamma^0t/2}\right)\gamma^0P_1^-\left(\mathrm{e}^{\mathrm{i}\gamma^0t/2} - \mathrm{i}\mathrm{e}^{-\mathrm{i}\gamma^0t/2}\right)\varepsilon_H\\
&\quad + \frac{1}{2}\varepsilon_H^\dagger\left(\mathrm{e}^{-\mathrm{i}\gamma^0t/2} + \mathrm{i}\mathrm{e}^{\mathrm{i}\gamma^0t/2}\right)\gamma^0P_1^+\left(\mathrm{e}^{\mathrm{i}\gamma^0t/2} + \mathrm{i}\mathrm{e}^{-\mathrm{i}\gamma^0t/2}\right)\varepsilon_H && (967)\\
&= \frac{(1-\mathrm{i})^2}{2}\varepsilon_H^\dagger\left(\cos(t/2)I + \sin(t/2)\gamma^0\right)\gamma^0P_1^-\left(\cos(t/2)I - \sin(t/2)\gamma^0\right)\varepsilon_H\\
&\quad + \frac{(1+\mathrm{i})^2}{2}\varepsilon_H^\dagger\left(\cos(t/2)I - \sin(t/2)\gamma^0\right)\gamma^0P_1^+\left(\cos(t/2)I + \sin(t/2)\gamma^0\right)\varepsilon_H. && (968)
\end{aligned}$$

Using computer algebra, one finds the result is $\overline{\varepsilon}_k\varepsilon_k = -2x^\dagger x$.

$$\therefore -\sqrt{3}\int_{\partial_\infty\Sigma_t} E_1\bar{\varepsilon}_k\varepsilon_k\mathrm{d}A = 2\sqrt{3}x^\dagger x\int_{\partial_\infty\Sigma_t} E_1\mathrm{d}A = 8\pi\sqrt{3}q_ex^\dagger x. \tag{969}$$

$\therefore$ Corollary 5.6.1 reduces to

$$0 \leq 16\pi x^\dagger\left(EI + \frac{\sqrt{3}}{2}q_eI + J_2\sigma_1 + J_3\sigma_2 + J_4\sigma_3\right)x. \tag{970}$$

The eigenvalues of the matrix inbetween $x^\dagger$ and x are

$$E + \frac{\sqrt{3}}{2}q_e \pm \sqrt{J_2^2 + J_3^2 + J_4^2} \tag{971}$$

and thus the result follows. □

There is a known soliton solution in this theory with $L(p,1)$ cross-section for $p \geq 3$. From [62],

$$\begin{aligned}
g &= -f^2(\mathrm{d}t+\omega)\otimes(\mathrm{d}t+\omega) + \frac{1}{f}h \text{ and} && (972)\\
F &= \frac{\sqrt{3}}{2}\mathrm{d}(f(\mathrm{d}t+\omega)) - \frac{1}{\sqrt{3}}G^+ - \frac{\sqrt{3}}{f}J \text{ for} && (973)\\
h &= \frac{1}{V}\mathrm{d}\rho\otimes\mathrm{d}\rho + \frac{\rho^2}{4}(\mathrm{d}\theta\otimes\mathrm{d}\theta + \sin^2(\theta)\mathrm{d}\phi\otimes\mathrm{d}\phi)\\
&\quad + \frac{V\rho^2}{4}(\mathrm{d}\psi + \cos(\theta)\mathrm{d}\phi)\otimes(\mathrm{d}\psi + \cos(\theta)\mathrm{d}\phi), && (974)\\
\omega &= \omega_3(\mathrm{d}\psi + \cos(\theta)\mathrm{d}\phi), && (975)\\
G^+ &= \frac{f}{2}(\mathrm{d}\omega + \star_h\mathrm{d}\omega), && (976)\\
J &= \frac{1}{4}\mathrm{d}(\rho^2(\mathrm{d}\omega + \star_h\mathrm{d}\omega)), && (977)\\
f &= \frac{3\rho^2}{c_0 - 1 + 3\rho^2}, && (978)\\
V &= \frac{1}{\rho^4}(\rho^2 - \rho_0^2)(a_0 + a_1\rho^2 + \rho^4), && (979)\\
\omega_3 &= \frac{1}{36\rho^4}(2(c_0-1)c_2 + (3(c_0-1)^2 + 9c_2)\rho^2 + 18(c_0-1)\rho^4 + 18\rho^6) && (980)
\end{aligned}$$

and all constants determined in terms of p by

$$c_0 = a_1 - \rho_0^2, \tag{981}$$

$$c_2 = a_0 - a_1\rho_0^2, \tag{982}$$

$$a_1 = p - \rho_0^2 - \frac{a_0}{\rho_0^2}, \tag{983}$$

$$a_0 = \frac{\rho_0^2}{p+1}(2p^2 - 4p + 3 + (p-8)\rho_0^2) \text{ and} \tag{984}$$

$$\rho_0^2 = \frac{p-2}{54}(p^2 + 14p - 5 + (p+1)\sqrt{(p+1)(25+p)}. \tag{985}$$

From [63] (as can be verified using the methods in this work), this solution has

$$E = -\frac{\pi(2p+5)(p-2)^2}{108p}, \tag{986}$$

$$q_e = -\frac{\pi(p-2)^2}{6p\sqrt{3}}, \tag{987}$$

$$J_1 = -\frac{\pi(p-2)^3}{108p} \text{ and } J_2 = J_3 = J_4 = 0. \tag{988}$$

This solution was constructed as a supersymmetric solution as per the methods of [64]. Hence, one would expect it to saturate the BPS inequality of theorem 5.9. However, it explicitly violates theorem 5.9. It turns out theorem 5.9 is not applicable to this solution. While this metric is locally constructed using methods from supersymmetry, there are global topological problems. The issues are the same as those discussed in [48] for solutions with analogous topological structure. In particular, when p is even, there are two inequivalent spin structures. The soliton described requires spinors to be antiperiodic in ψ, while the ϵ_k used in theorem 5.9 requires spinors to be periodic in ψ. This situation is somewhat similar to the AdS soliton with torioidal cross-section discussed earlier. Meanwhile, when p is odd, the soliton in fact admits no spin structure at all. The best that can be done is instead a spin^c structure. This soliton satisfies the tantalising BPS identity,

$$E = \frac{\sqrt{3}}{2}q_e + 2J_1. \tag{989}$$

It remains to be seen whether there exists a more general inequality of this sort and whether it can be proven using a variation of Witten's technique where one leverages spin^c structures instead of the familiar spinor methods discussed in this work.

A Conventions

I use nine different types of indices, as given below.

- $a, b, c, \cdots$ are abstract indices on the full spacetime.
- $\mu, \nu, \rho, \cdots$ are vielbein indices running $0, 1, \cdots, n-1$.
- $\mu', \nu', \rho', ...$ are coordinate indices running $0, 1, \cdots, n-1$.
- $M, N, P, \cdots$ are vielbein indices running $0, 2, 3, \cdots, n-1$.
- $m, n, p, \cdots$ are coordinate indices running $0, 2, 3, \cdots, n-1$.
- $A, B, C, \cdots$ are vielbein indices running $2, 3, \cdots, n-1$.
- $\alpha, \beta, \gamma, \cdots$ are coordinate indices running $2, 3, \cdots, n-1$.
- $I, J, K, \cdots$ are vielbein indices running $1, 2, \cdots, n-1$.
- $i, j, k, \cdots$ are coordinate indices running $1, 2, \cdots, n-1$.

I use a mostly pluses metric signature[45].

The gamma matrices are chosen to be unitary and satisfying $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = -2\eta^{\mu\nu}I$.

On occasion it may be convenient to choose a representation of the gamma matrices for practical calculations like finding eigenvalues, even though all equivalent representations will give the same result. When $n = 4$, I'll choose

$$\gamma^0 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad \gamma^1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \quad \gamma^2 = \begin{bmatrix} 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ -\mathrm{i} & 0 & 0 & 0 \end{bmatrix}$$
$$\text{and } \gamma^3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \tag{990}$$

When $n = 5$, I'll choose

$$\gamma^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad \gamma^1 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \gamma^2 = \begin{bmatrix} 0 & 0 & 0 & \mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \end{bmatrix},$$
$$\gamma^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \text{ and } \gamma^4 = \begin{bmatrix} 0 & 0 & \mathrm{i} & 0 \\ 0 & 0 & 0 & -\mathrm{i} \\ \mathrm{i} & 0 & 0 & 0 \\ 0 & -\mathrm{i} & 0 & 0 \end{bmatrix}. \tag{991}$$

When $n = 5$, there are two inequivalent representations of the Clifford algebra; it will matter in section 5.2 that I choose this particular equivalence class.

[45] This is the only sensible convention.

The cosmological constant is always taken to be negative and parameterised as $\Lambda = -\frac{1}{2l^2}(n-1)(n-2)$, for some length scale, l. It will then be convenient to work in $l = 1$ units; l can be restored in any equation on dimensional grounds.

I use the Riemann tensor convention where $[D_a, D_b]V^c = R^c{}_{dab}V^d$.

The following symbols have the meanings listed.

- M: The full spacetime
- g: The (Lorentzian) metric on M
- n: The dimension of M
- C_c^∞: The space of compactly supported, smooth spinors on M
- $\mathcal{H}$: The (metric space) completion of C_c^∞ under the metric corresponding to the inner product defined by equation 86.
- $\bar{\psi} = \psi^\dagger\gamma^0$ for an spinor, ψ
- D_a: The Levi-Civita connection of g
- $D_i^{(h)}$: The Levi-Civita connection of a metric, h
- $\nabla_\mu\psi = D_\mu\psi + \mathrm{i}\alpha\gamma_\mu\psi + A_\mu\psi$ for any spinor, ψ.
- $\nabla_\mu\bar{\psi} = D_\mu\bar{\psi} - \mathrm{i}\alpha\bar{\psi}\gamma_\mu + \bar{\psi}\gamma^0 A_\mu^\dagger\gamma^0 = (\nabla_\mu\psi)^\dagger\gamma^0$ for any spinor, ψ.
- $\omega_{\nu\rho\mu}$: Spin connection coefficients, with μ being the one-form index and ν & ρ being the $\mathfrak{o}(\mathfrak{n}-1,1)$ indices
- I: The identity matrix

References

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