# Static black hole uniqueness theorems

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# Contents

# Chapter 1

# Introduction

Perhaps the oldest uniqueness result in general relativity is the famous Birkhoff theorem, discovered merely a few years after general relativity itself. Birkhoff's theorem states that the only spherically symmetric solution of the vacuum Einstein equation is the Schwarzschild solution. All subsequent uniqueness theorems essentially follow the same format - assume some symmetry and prove the symmetry to be so restrictive that Einstein's equation has only one solution.

Despite not being assumed a priori, the Schwarszschild solution possesses the property that it is static. Naturally, one may wonder if a kind of converse to Birkhoff's theorem is true. Does every static, vacuum spacetime have to be spherically symmetric and thus Schwarzschild? If not, what further assumptions are required? These issues are more interesting when studied in the context of spacetimes containing black holes and this has been the main topic I've explored in the first year of my PhD. In short, the task is to prove that the class of static, asymptotically flat black hole spacetimes contains only the Schwarzschild spacetime, or its analogues like the Reissner-Nordstrom spacetime when matter fields are involved.

More than forty years after Birkhoff's theorem was established, the static black hole uniqueness conjecture was resolved in the affirmative by Israel [1]. Although Israel's proof made some serious assumptions, most were relaxed soon afterwards [2, 3] and Israel's work effectively gave birth to a new industry of black hole uniqueness research - see [4] for a review. Given it all began with Israel, I spent some time studying Israel's proof and I've re-presented his proof in full in chapter 3. Probably the simplest and most comprehensive of the early proofs was by Robinson [3, 4]. Although I studied this proof, I have not included it in this report. Like Israel's proof and the others around in the 1960s and 1970s, the proof was built on constructing some seemingly ad hoc divergence, observing the result to be a sum of squares, integrating using Stokes' theorem and then using the vanishing integrand to detect spherical symmetry. Despite the successes of the early proofs - including their generalisations to accommodate source-free electromagnetic fields [4] - three main difficulties remained when it came to generalisations.

1. Dealing with disconnected horizons.
2. Integrating the Ricci scalar of a 'constant time' slice of the event horizon.
3. Constraining the Riemann tensor of spacelike hypersurfaces given only the Ricci tensor of those surfaces.

The 2nd and 3rd problems are only issues when one attempts the static uniqueness problem in higher dimensions[1], where as the 1st problem arises in all dimensions. All three problems were simultaneously solved by Bunting and Masood-ul-Alam [5] through an ingenious proof utilising the positive energy theorem. Their method has been adapted and generalised to several other scenarios such as higher dimensions and various matter fields - e.g. see [6, 7, 8, 9, 10, 11]. Note this is not a purely technical exercise. For example, for spacetimes that are merely stationary, but not static, the existence of black rings means uniqueness doesn't hold in higher dimensions [12]. Given the Bunting and Masood-ul-Alam method is the most comprehensive and I gave a talk on it earlier this year, it would have been remiss of me not to include it in this report. Chapter 4 is dedicated to the higher dimensional version of Bunting and Masood-ul-Alam's proof. As far as I know, this positive energy theorem approach explained in chapter 4 forms the basis for all static black hole uniqueness theorems not assuming connected event horizons or set in dimensions higher than four. While very elegant, these proofs are in some sense a little unsatisfying, because the difficulties of overcoming the aforementioned problems have been outsourced to the difficulties of proving the positive energy theorem.

I tried for some time in this past year to take up the problem of proving static black hole uniqueness without recourse to the positive energy theorem. I focused on one particular paper by Agostiniani and Mazzieri [13] for several months. In [13, 14], the authors found a new way of detecting spherical symmetry by conformally scaling the problem to an asymptotically cylindrical one. Their paper manages to circumvent the third problem listed above and provides a new proof of static, vacuum, asymptotically flat, connected black hole uniqueness in 4 dimensions. For higher dimensions, they are forced to deal with the second problem by assuming a particular inequality between the constants parameterising the solution and the Ricci scalar's integral.

I have extended the work of [13] by adding a source-free Maxwell field. Following [13], I have succeeded in finding a new proof of Reissner-Nordstrom's uniqueness among 4D, static, asymptotically flat, connected black holes in the Einstein-Maxwell system. In higher dimensions, I had less success. Initially, I considered a Maxwell field with purely electric components. In this case, by judicially choosing the variables in which the problem is expressed, I have a complete proof given the same inequality that's already assumed in [13]. In the course of this analysis, I managed to find the electric field fully in terms of the lapse function - as far as I know this has not been done before in higher dimensions without relying on the positive energy theorem. When magnetic components are included in higher dimensions, I had to rely on an additional, auxiliary inequality between the constants parameterising the problem. In some sense this is not surprising - after all [13] already required one such inequality in higher dimensions. The full drama of my endeavours on this topic is the subject of chapter 5.

I started the academic year though - as every PhD student does - with background reading. And that's exactly where I'll start the main body of this report, in chapter 2.

**How to read this report**

This report is written with the philosophy that as far as practical, the reader should never have to put 'pen to paper' to verify any equation or claim I make. As such, the report is quite long. However, I've tried to write in a format such that a reader who skips all the proofs should still be able to follow the story I'm trying to tell. The report is deliberately written in a somewhat casual style. I never enjoyed reading terse, austere papers that made me want to gouge my eyes out a la Oedipus Rex and so I tried to avoid that style myself. Only chapter 5 contains any new results and the absence of detailed citations should not be taken as a claim to originality.

[1] In 4 dimensions, a constant time slice of the event horizon is 2D, allowing one to use the Gauss-Bonnet theorem and spacelike hypersurfaces are 3D, meaning the Riemann tensor is determined by the Ricci tensor.

However, many known results do not have satisfyingly detailed proofs or properly quantified assumptions stated in the literature. I have sought to fill in such gaps whenever I could and in many ways it is a more natural reflection of the real experience of the first year in a PhD. On the whole, I intend this report to a non-exhaustive tour of my work for the past year. Finally, I would also advise readers to read appendix D as required.

# Chapter 2

# Foundational results

I once spent two weeks being inculcated with the philosophy that it pays handsomely to "think deeply about simple things." That idea is perhaps the motto underlying this chapter. Here, I collate many well known results about static spacetimes in a pedagogical style. These results are almost exclusively taken as assumed knowledge in academic papers and in the later chapters of this report. While much of the content is based off a book I spent many months studying, *Black Hole Uniqueness Theorems* by Markus Heusler [15], I believe it helps to unify results that are scattered across various books and various chapters of each book. Furthermore, there are several results - like the no ergoregion theorem and the 4D spherical horizon topology theorem - which are well known, say from Hawking and Ellis' seminal monograph [16], but don't possess satisfyingly detailed written proofs in the literature. Even for content available in textbooks like [15], I believe I have filled in many missing steps.

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Harvey Reall once told us in a lecture that one of the ironies of general relativity is that it's counterintuitive and yet makes perfect sense. An example of this in action is that one of the central ideas of general relativity is to be coordinate independent, and yet one rarely makes any progress in practice without choosing the right coordinate system first. Indeed, such is life in analysing static spacetimes. I'll start by constructing a coordinate system in the domain of outer communication, $C$, that is well adapted to the static geometry. To be clear, I'll be adopting the following definition of static.

**Definition 2.1** (Static)**.** *A spacetime, $(M, g)$, which is asymptotically flat at null infinity is called static if and only if it possesses a Killing vector field, $k^a$, such that $k^a$ is timelike near $\mathcal{I}^{\pm}$ and $k^a$ is hypersurface orthogonal.*

Note that the definition relies on having some asymptotics. I will always consider asymptotically flat spacetimes, so it suffices for me to take this somewhat strict definition.

For any Killing vector field, it's a standard result that one can define local coordinates such that the Killing vector field is a coordinate vector field and the metric's components are independent of that coordinate - e.g. see sections C.2 and C.3 of [17]. In the case of $k^a$, the Killing vector field making $(M, g)$ static, let $t$ be the corresponding local coordinate, i.e. $k^a = \partial/\partial t$. In analysing static uniqueness, most people tacitly assume there is no ergoregion. This is in fact well known to be true for static spacetimes, but I have yet to find a proof in the literature that goes into sufficiently satisfying or convincing detail. Hence, I have written my own based on the results in [16] and [18].

**Theorem 2.2** (Carter and Hawking & Ellis)**.** *Let $(M, g)$ be an asymptotically flat, static, spacetime. Let $k^a$ denote the Killing vector field making $(M, g)$ static. Assume $\nabla_a(k^b k_b) \neq 0$ whenever $k^a k_a = 0$. Then, $(M, g)$ has no ergoregion.*

*Proof.* Assume the ergoregion is non-empty, for a contradiction.
Let $V = -k^a k_a$ and let $Z = \{p \in M | V = 0 \text{ at } p\}$.
$\therefore Z$ is a closed set, since any convergent sequence of points in $Z$ has $V = 0$ at each point and thus $V = 0$ at the limit point from $k^a k_a$'s continuity.
$k^a$ is static $\implies k^a$ is hypersurface orthogonal $\implies k \wedge \mathrm{d}k = 0$, where I'll also use $k$ to denote $k_a$ when it's self-evident that the expression only makes sense if $k$ is a 1-form.

$$
\begin{aligned}
0 &= k \wedge \mathrm{d}k. && (2.1)\\
\therefore 0 &= k_a \nabla_b k_c + k_b \nabla_c k_a + k_c \nabla_a k_b && (2.2)\\
&= k_a \nabla_b k_c - k_b \nabla_a k_c + k_c \nabla_a k_b \text{ by the Killing equation.} && (2.3)\\
\therefore k_a \nabla_b V - k_b \nabla_a V &= -k_a \nabla_b (k^c k_c) + k_b \nabla_a (k^c k_c) && (2.4)\\
&= -2k^c (k_a \nabla_b k_c - k_b \nabla_a k_c) && (2.5)\\
&= 2k^c k_c \nabla_a k_b \text{ by equation 2.3} && (2.6)\\
&= -2V \nabla_a k_b. && (2.7)
\end{aligned}
$$

By the hypersurface orthogonality condition, $\exists$ a function, $f$, such that $k_a = -\alpha \mathrm{d}f$. Let $C_0$ be a constant $f$ hypersurface.
Let $p \in Z$. Let $\gamma(\lambda)$ be a curve in $C_0$ passing through $p$[1] and let $T^a$ be $\gamma(\lambda)$'s tangent vector.
Assume I can choose $\gamma(\lambda)$ such that $\mathrm{d}V/\mathrm{d}\lambda \neq 0$ in an open neighbourhood of $p$ within $\gamma(\lambda)$ (i.e. an interval), for a contradiction.

$$
\begin{aligned}
\frac{1}{2V} k^a \frac{\mathrm{d}V}{\mathrm{d}\lambda} &= \frac{1}{2V} k^a T^b \nabla_b V && (2.8)\\
&= \frac{1}{2V} T^b (k_b \nabla^a V - 2V \nabla^a k_b) \text{ by equation 2.7} && (2.9)\\
&= \frac{1}{2V} T^b k_b \nabla^a V - T^b \nabla^a k_b. && (2.10)
\end{aligned}
$$

$T^b$ is tangent to a curve in $C_0$ and $k_a$ is normal to $C_0$ by $k_a = -\alpha \mathrm{d}f$ and $C_0$ being a constant $f$ surface.
$\therefore T^b k_b = 0$. Then, upon applying the Killing equation I get

$$
\frac{1}{2V} k^a \frac{\mathrm{d}V}{\mathrm{d}\lambda} = T^b \nabla_b k^a. \tag{2.11}
$$

Since I assumed $\mathrm{d}V/\mathrm{d}\lambda \neq 0$ in an open interval around $p$, I can use $V$ itself as the parameter, $\lambda$. This is essentially the inverse function theorem.
$\therefore \frac{1}{2V} k^a = T^b \nabla_b k^a$. The RHS is continuous, but the LHS diverges at $p$ since $V = 0$ at $p$. $\lightning$
This contradiction means $\mathrm{d}V/\mathrm{d}\lambda = 0$ around $p$ for every curve in $C_0$.
$\therefore C_0 \subseteq Z$.
However, if $V$ were zero in an open neighbourhood of $p$, $p$'s arbitrariness in $Z$ would mean $Z$ was an open set.
But, $Z$ is also closed, so $M$'s connectedness would make $Z = M$, contradicting $V = -k^a k_a > 0$ near $\mathcal{I}^\pm$.
$\therefore V$ cannot remain zero along directions perpendicular to $C_0$.
$\therefore$ The connected component of $Z$ including $p$ is a connected component of $C_0$, say $E_0$.
$\therefore E_0$ is a connected null hypersurface (since $k$ is null on $E_0$ and perpendicular to $E_0$).
$\mathrm{d}V \neq 0$ when $V = 0 \implies V > 0$ on one side of $E_0$ and $V < 0$ on the other side.
Since the ergoregion is assumed to be non-empty, I can choose $p$ to be outside the event horizon.
$\therefore$ The $V < 0$ side is in the ergoregion's interior.

[1] $C_0$ is fixed by the value of $f$ at $p$.

Choose a point, $q$, just off $p$ in the ergoregion's interior.
Since $q$ is not inside the black hole, $q \in J^-(\mathcal{I}^+)$.
If $q \notin J^+(\mathcal{I}^-)$, then by definition, $q$ would be inside a white hole region. It's strange to define the ergoregion to include white holes and white holes are unphysical anyway, so either way, I can assume $q \in J^+(\mathcal{I}^-)$.
$\therefore \exists$ a future directed causal curve, say $s(\lambda)$, from $\mathcal{I}^-$ to $\mathcal{I}^+$ passing through $q$.
$E_0$ being a connected null hypersurface means the outgoing normal to $E_0$ is everywhere future directed or everywhere past directed (it can't stay null and flip from one light cone to the other without passing through zero).
$\therefore s(\lambda)$ can only be future directed while going into the $V < 0$ region considered here, or only while leaving it, but not both. $\lightning$.
This contradicts $s(\lambda)$ being future directed all the way from $\mathcal{I}^-$ to $\mathcal{I}^+$.
$\therefore$ The ergoregion must have been empty. □

**Corollary 2.2.1.** *In a static spacetime with $\nabla_a(k^b k_b) \neq 0$ whenever $k^a k_a = 0$, the event horizon of a black hole must be a Killing horizon of $k^a$.*

*Proof.* The event horizon is a causal boundary. As such, it must be a null hypersurface, because all the light cones are tangent to a causal boundary.
Next, let $\Phi^t$ denote flows along $k^a$.
$k^a$ is Killing $\iff (\Phi^t)^* g = g$.
$\therefore$ Causal structure is unaffected by flows along $k^a$.
$\therefore$ Since the event horizon is a causal boundary, it too must be unaffected by flows along $k^a$.
$\therefore k^a$ is tangent to the event horizon.
$\therefore k^a$ is null or spacelike, since the event horizon is a null hypersurface[2].
The theorem says that in the case considered here, $k^a$ is timelike just outside the event horizon. Thus, by continuity, $k^a$ can only become null, not spacelike, on the event horizon, thereby making the event horizon a Killing horizon of $k^a$. □

**Definition 2.3** (Adapted coordinate system)**.** *For a static spacetime, the metric in the domain of outer communication, $C$, will be written as*

$$g = -S^2 \mathrm{d}t \otimes \mathrm{d}t + h, \tag{2.12}$$

*where $h$ is a metric on the surfaces of constant $t$ (i.e. it only depends on the other $n-1$ coordinate 1-forms, $\mathrm{d}x^i$), $S > 0$ and neither $S$ nor $h$ depends on $t$.*

*Proof.* Theorem 2.2 implies $g(k,k) < 0$ in $C$, justifying the coefficient of $\mathrm{d}t \otimes \mathrm{d}t$ being $-S^2$.
Neither $S$ nor $h$ depending on $t$ follows from $k^a = \partial/\partial t$ being Killing.
$k^a$ is static $\implies$ it is hypersurface orthogonal. Let $\Sigma$ denote the relevant hypersurfaces.
The coordinate, $t$, is constructed by flowing along $k^a$ from $\Sigma$ and letting $(t, x^i)$ be the coordinates of the point reached by flowing a parameter distance, $t$, from a point with local coordinates, $x^i$, in $\Sigma$.
$\therefore \Sigma$ are constant $t$ surfaces and thus $k_a \propto \mathrm{d}t$.
$\therefore g_{0i} = 0$ and I get the metric in equation 2.12. □

It will help to decompose the Ricci tensor in an analogous way to the metric.

**Theorem 2.4.** *$R_{tt} = S\Box^{(h)}S$ and $R_{ij} = R^{(h)}_{ij} - \frac{1}{S}\nabla^{(h)}_i \nabla^{(h)}_j S$, where subscript $(h)$s denote quantities with respect to the metric on each $\Sigma_t$.*

[2] The argument so far actually applies to any Killing vector in any black hole spacetime.

*Proof.* Let $\{\theta^\mu\}_{\mu=0}^{n-1}$ be an orthonormal basis of 1-forms, with with $\theta^0 = S\mathrm{d}t$. Then, the structure equations imply the following.
$\mathrm{d}\theta^\mu = -\omega^\mu{}_\nu \wedge \theta^\nu \Rightarrow \mathrm{d}\theta^0 = \mathrm{d}(S\mathrm{d}t) = -\omega^0{}_\mu \wedge \theta^\mu \Leftrightarrow \frac{1}{S}\nabla_\mu(S)dx^\mu \wedge \theta^0 = -\omega^0{}_\mu \wedge \theta^\mu$
$\nabla_0(S) = \partial_t(S) = 0$, $\omega^0{}_0 = 0$ by $\omega_{\mu\nu} = -\omega_{\nu\mu}$ and $\nabla_i(S) = \nabla_i^{h_{ij}\text{ connection}}(S) = \partial_i(S)$.
Hence, the $\mathrm{d}\theta^0$ condition reduces to

$$-\omega^0{}_i \wedge \theta^i = \frac{1}{S}\nabla_i^{(h)}(S)\,\mathrm{d}x^i \wedge \theta^0 \tag{2.13}$$

$$= \frac{1}{S}\nabla_i^{(h,\theta)}(S)\,\theta^i \wedge \theta^0 \tag{2.14}$$

$$\Longleftrightarrow \omega^0{}_i \wedge \theta^i = \frac{1}{S}\nabla_i^{(h,\theta)}(S)\,\theta^0 \wedge \theta^i \tag{2.15}$$

where $\nabla_i^{(h,\theta)}$ denotes the connection components with respect to the tetrad, $\{\theta^i\}_{i=1}^{n-1}$, on constant $t$ hypersurfaces, $(\Sigma_t, h)$.
$\therefore \omega^0{}_i = \frac{1}{S}\nabla_i^{(h,\theta)}(S)\theta^0 + f\theta^i$ for some function, $f$.
The other three tetrad exterior derivatives say

$$\mathrm{d}\theta^i = -\omega^i{}_\mu \wedge \theta^\mu \tag{2.16}$$

$$= -\omega^i{}_0 \wedge \theta^0 - \omega^i{}_j \wedge \theta^i \tag{2.17}$$

$$= -\frac{1}{S}\nabla_i^{(h,\theta)}(S)\theta^0 \wedge \theta^0 - f\theta^i \wedge \theta^0 - \omega^i{}_j \wedge \theta^i \tag{2.18}$$

$$= -f\theta^i \wedge \theta^0 - \omega^i{}_j \wedge \theta^i. \tag{2.19}$$

Since everything is $t$ independent and there are no $t - x^i$ cross terms in the metric, $\mathrm{d}\theta^i$ cannot have a $\theta^0$ factor.
$\therefore f = 0$ and $\mathrm{d}\theta^i = -\omega^i{}_j \wedge \theta^i$.
The latter equation is the same as the structure equation on $(\Sigma_t, h)$.
$\therefore$ By the uniqueness of structure equation solutions, $\omega^i{}_j = \omega^{(h)i}{}_j$.
Next, I apply the other structure equation, $\frac{1}{2}R_{\mu\nu\rho\sigma}\theta^\rho \wedge \theta^\sigma = \mathrm{d}\omega_{\mu\nu} + \omega_{\mu\rho} \wedge \omega^\rho{}_\nu$.

$$\mathrm{d}\omega^0{}_i = \mathrm{d}\left(\frac{1}{S}\nabla_i^{(h,\theta)}(S)\theta^0\right) \tag{2.20}$$

$$= \mathrm{d}\left(\frac{1}{S}\nabla_i^{(h,\theta)}(S)\right) \wedge \theta^0 + \frac{1}{S}\nabla_i^{(h,\theta)}(S)\mathrm{d}\theta^0 \tag{2.21}$$

$$= \mathrm{d}\left(\frac{1}{S}\nabla_i^{(h,\theta)}(S)\right) \wedge \theta^0 - \frac{1}{S^2}\nabla_i^{(h,\theta)}(S)\nabla_j^{(h,\theta)}(S)\theta^0 \wedge \theta^j. \tag{2.22}$$

$\frac{1}{S}\nabla_i^{(h,\theta)}(S)$ is just a scalar as far the 1st term's exterior derivative is concerned.
However, from the definition of the covariant derivative, if $\{e_\mu\}_{\mu=0}^{n-1}$ is the inverse tetrad in the tangent spaces, then

$$\nabla_j^{(h,\theta)}\left(\frac{1}{S}\nabla_i^{(h,\theta)}S\right) = e_j{}^k\partial_k\left(\frac{1}{S}\nabla_i^{(h,\theta)}S\right) - (\omega^k{}_i)_j\frac{1}{S}\nabla_k^{(h,\theta)}(S). \tag{2.23}$$

$$\therefore \nabla_j^{(h,\theta)}\left(\frac{1}{S}\nabla_i^{(h,\theta)}S\right)\theta^j = (\mathrm{d}x^k)\partial_k\left(\frac{1}{S}\nabla_i^{(h,\theta)}S\right) - \omega^k{}_i\frac{1}{S}\nabla_k^{(h,\theta)}(S) \tag{2.24}$$

$$= \mathrm{d}\left(\frac{1}{S}\nabla_i^{(h,\theta)}S\right) - \frac{1}{S}\nabla_j^{(h,\theta)}(S)\omega^j{}_i. \tag{2.25}$$

Using this in equation 2.22, I get

$$\mathrm{d}\omega^0{}_i = \nabla_j^{(h,\theta)}\left(\frac{1}{S}\nabla_i^{(h,\theta)}S\right)\theta^j\wedge\theta^0 + \frac{1}{S}\nabla_j^{(h,\theta)}(S)\omega^j{}_i\wedge\theta^0 - \frac{1}{S^2}\nabla_i^{(h,\theta)}(S)\nabla_j^{(h,\theta)}(S)\theta^0\wedge\theta^j \quad (2.26)$$

$$= \frac{1}{S}\nabla_j^{(h,\theta)}\nabla_i^{(h,\theta)}(S)\theta^j\wedge\theta^0 + \frac{1}{S}\nabla_j^{(h,\theta)}(S)\omega^j{}_i\wedge\theta^0. \quad (2.27)$$

Inserting this into the structure equation, I get

$$\frac{1}{2}R^0{}_{i\mu\nu}\theta^\mu\wedge\theta^\nu = \mathrm{d}\omega^0{}_i + \omega^0{}_\mu\wedge\omega^\mu{}_i \quad (2.28)$$

$$= \frac{1}{S}\nabla_j^{(h,\theta)}\nabla_i^{(h,\theta)}(S)\theta^j\wedge\theta^0 + \frac{1}{S}\nabla_j^{(h,\theta)}(S)\omega^j{}_i\wedge\theta^0 + \omega^0{}_j\wedge\omega^j{}_i \text{ as } \omega_{00}=0 \quad (2.29)$$

$$= \frac{1}{S}\nabla_j^{(h,\theta)}\nabla_i^{(h,\theta)}(S)\theta^j\wedge\theta^0 + \frac{1}{S}\nabla_j^{(h,\theta)}(S)\omega^j{}_i\wedge\theta^0 + \frac{1}{S}\nabla_j^{(h,\theta)}(S)\theta^0\wedge\omega^j{}_i \quad (2.30)$$

$$= \frac{1}{S}\nabla_j^{(h,\theta)}\nabla_i^{(h,\theta)}(S)\theta^j\wedge\theta^0. \quad (2.31)$$

$\therefore$ I can read off that $R_{0i0j} = \frac{1}{S}\nabla_i^{(h,\theta)}\nabla_j^{(h,\theta)}(S)$ and $R_{0ijk} = 0$ (these two determine other (anti)symmetry related index permutations too).
The remaining Riemann tensor components follow from

$$\frac{1}{2}R^i{}_{jkl}\theta^k\wedge\theta^l = \mathrm{d}\omega^i{}_j + \omega^i{}_\mu\wedge\omega^\mu{}_j \quad (2.32)$$

$$= \mathrm{d}\omega^{(h)i}{}_j + \omega^i{}_0\wedge\omega^0{}_j + \omega^i{}_k\wedge\omega^k{}_j \quad (2.33)$$

$$= \mathrm{d}\omega^{(h)i}{}_j + \frac{1}{S}\nabla_i^{(h,\theta)}(S)\theta^0\wedge\frac{1}{S}\nabla_j^{(h,\theta)}(S)\theta^0 + \omega^{(h)i}{}_k\wedge\omega^{(h)k}{}_j \quad (2.34)$$

$$= \mathrm{d}\omega^{(h)i}{}_j + \omega^{(h)i}{}_k\wedge\omega^{(h)k}{}_j. \quad (2.35)$$

The RHS is the same as the corresponding structure equation on $(\Sigma_t, h)$.
$\therefore$ By the uniqueness of the solutions to the structure equations, it follows that $R_{ijkl} = R^{(h)}_{ijkl}$.
These expressions were in the tetrad indices. I can go back to the $\{t, x^i\}$ indices as follows.
$t$ is completely decoupled from the $x^i$ in $g_{\mu\nu}$, and likewise $\theta^0$ from $\theta^i$.
$\therefore$ Only $R_{0i0j}$ and $R_{ijkl}$ being non-zero in the tetrad basis will imply that only $R_{0i0j}$ and $R_{ijkl}$ will be non-zero in the $\{t, x^i\}$ basis too.

$$R^{(\text{coor.})}_{0i0j} = (\theta^\mu)_0(\theta^\nu)_i(\theta^\rho)_0(\theta^\sigma)_j R^{(\text{tetrad})}_{\mu\nu\rho\sigma} \quad (2.36)$$

$$= (\theta^0)_0(\theta^k)_i(\theta^0)_0(\theta^l)_j R^{(\text{tetrad})}_{0k0l} \text{ by the decoupling} \quad (2.37)$$

$$= S(\theta^k)_i S(\theta^l)_j \frac{1}{S}\nabla_i^{(h,\theta)}\nabla_j^{(h,\theta)}(S) \quad (2.38)$$

$$= S\nabla_i^{(h)}\nabla_j^{(h)}S. \quad (2.39)$$

For the other index combination, $R_{ijkl} = R^{(h)}_{ijkl}$ is automatically inherited by the coordinate basis components.
From hereon, I no longer need the tetrad.
$\therefore$ All quantities, unless otherwise stated, are in the $\{t, x^i\}$ coordinate basis.
Finally, I can compute the Ricci tensor components,

$$R_{00} = g^{\mu\nu}R_{\mu0\nu0} \quad (2.40)$$

$$= h^{ij}S\nabla_i^{(h)}\nabla_j^{(h)}S \text{ noting the relevant non} - \text{zero components} \quad (2.41)$$

$$= S\Box^{(h)}S, \quad (2.42)$$

$$R_{0i} = g^{\mu\nu}R_{\mu0\nu i} = 0 \text{ and} \quad (2.43)$$

$$R_{ij} = g^{\mu\nu}R_{\mu i\nu j} = -\frac{1}{S^2}R_{0i0j} + h^{kl}R_{kilj} = -\frac{1}{S}\nabla_i^{(h)}\nabla_j^{(h)}S + R^{(h)}_{ij}, \quad (2.44)$$

which are the expressions claimed in the theorem. □

**Corollary 2.4.1.** *The Ricci scalar is* $R = R^{(h)} - \frac{2}{S}\Box^{(h)} S$.

*Proof.* $R = g^{\mu\nu} R_{\mu\nu} = -\frac{1}{S^2} S\Box^{(h)} S + h^{ij}\left(R^{(h)}_{ij} - \frac{1}{S}\nabla^{(h)}_i \nabla^{(h)}_j S\right) = R^{(h)} - \frac{2}{S}\Box^{(h)} S$. □

**Corollary 2.4.2.** *The components of the Einstein tensor are* $G_{tt} = \frac{1}{2}S^2 R^{(h)}$, $G_{ti} = 0$ *and* $G_{ij} = G^{(h)}_{ij} + \frac{1}{S}(h_{ij}\Box^{(h)} S - \nabla^{(h)}_i \nabla^{(h)}_j S)$.

Although theorem 2.4 was proven using tetrads, the Christoffel symbols for adapted coordinates will also be needed briefly later.

**Lemma 2.5.** *The Christoffel symbols for adapted coordinates are*

$$\Gamma^0{}_{00} = 0,\ \Gamma^0{}_{0i} = \frac{1}{S}\nabla^{(h)}_i S,\ \Gamma^0{}_{ij} = 0,\ \Gamma^i{}_{00} = S\nabla^{(h)i} S,\ \Gamma^i{}_{0j} = 0 \text{ and } \Gamma^i{}_{jk} = \Gamma^{(h)i}{}_{jk}. \tag{2.45}$$

*Proof.* The proof is straightforward by direct evaluation.

$$\Gamma^i{}_{\mu\nu} = \frac{1}{2} g^{i\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}) \tag{2.46}$$
$$= \frac{1}{2} h^{ij}(\partial_\mu g_{\nu j} + \partial_\nu g_{j\mu} - \partial_j g_{\mu\nu}). \tag{2.47}$$

$\therefore \Gamma^i{}_{0j} = 0$ by $\partial_t g_{\mu\nu} = 0$ & $g_{0k} = 0$, $\Gamma^i{}_{00} = -\frac{1}{2}h^{ij}\partial_j(-S^2) = S\nabla^{(h)i} S$ and $\Gamma^i{}_{jk} = \Gamma^{(h)i}{}_{jk}$.

$$\Gamma^0{}_{\mu\nu} = \frac{1}{2} g^{0\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}) \tag{2.48}$$
$$= -\frac{1}{2}\frac{1}{S^2}(\partial_\mu g_{\nu 0} + \partial_\nu g_{0\mu} - \partial_0 g_{\mu\nu}) \tag{2.49}$$
$$= -\frac{1}{2S^2}(\partial_\mu g_{\nu 0} + \partial_\nu g_{0\mu} + 0) \tag{2.50}$$

$\therefore \Gamma^0{}_{00} = 0$, $\Gamma^0{}_{0i} = -\frac{1}{2S^2}\partial_i(-S^2) = \frac{1}{S}\nabla^{(h)}_i S$ and $\Gamma^0{}_{ij} = 0$. □

Thus far, I have analysed some implications of the spacetime being static. However, I am also assuming asymptotic flatness and this too imposes some very strict constraints on $(M, g)$. I'll be formulating the static uniqueness problems as systems of partial differential equations on $\Sigma_t$ in the domain of outer communication. Thus, $\Sigma_t$ has two boundaries, namely its intersection with the event horizon, $H^+$, and the boundary at infinity.
Let $\mathcal{H} = H^+ \cap \Sigma_t$ denote the inner boundary.
For the boundary at infinity, the construction of $(M, g)$ as an asymptotically flat manifold (at least locally) foliated by constant $t$ surfaces makes $\Sigma_t$ an asymptotically flat end (for each $t$).
$\therefore$ The boundary at infinity is a sphere, $S^{n-2}_\infty$, because an asymptotically flat end is diffeomorphic to $\mathbb{R}^n$ with a compact set removed and has metric & extrinsic curvature,

$$h_{ij} = \delta_{ij} + O(1/r^{n-3}) \tag{2.51}$$
$$K_{ij} = O(1/r^{n-2}), \tag{2.52}$$

where $r = \sqrt{x_i x_i}$ and $x_i$ are "almost Cartesian" coordinates arising from the diffeomorphism. The asymptotics at $S^{n-2}_\infty$ contain a lot of physically important information.

**Definition 2.6** (Mass and charge)**.** *The ADM mass*[3] *is defined to be*

$$M = \frac{1}{16\pi} \int_{S^{n-2}_\infty} n_i(\partial_j h_{ij} - \partial_i h_{jj}) \mathrm{d}A. \tag{2.53}$$

*When the matter fields include a Maxwell field, $F$, the electric charge is defined to be*

$$Q = \frac{1}{4\pi} \int_{S^{n-2}_\infty} \star F. \tag{2.54}$$

*Note that $\int_{S^{n-2}_\infty}$ should be interpreted as $\lim_{r\to\infty} \int_{S^{n-2}_r}$ and one has to use the almost Cartesian coordinates from the asymptotically flat end. With the benefit of hindsight, I will also define the quantities*

$$q = \frac{4\pi Q}{(n-3)\omega_{n-2}} \quad \text{and} \tag{2.55}$$

$$m = \frac{16\pi M}{(n-2)\omega_{n-2}}, \tag{2.56}$$

*where $\omega_{n-2}$ is the area of a unit radius $S^{n-2}$.*

I will assume the following asymptotic expansions, which are standard in the literature - e.g. compare with [11, 8, 15, 13, 9, 10].

**Definition 2.7** (Asymptotics)**.** *To leading order near $S^{n-2}_\infty$,*

$$S = 1 - \frac{m}{2r^{n-3}} \quad \text{and} \tag{2.57}$$

$$h_{ij} = \left(1 + \frac{m}{(n-3)r^{n-3}}\right)\delta_{ij}. \tag{2.58}$$

It has to actually be proven there exist coordinates near $S^{n-2}_\infty$ such that these asymptotics are valid. Unfortunately, I never got around to studying these proofs, so I'll have to defer to [19] and applicable results in subsequent work, e.g. [5, 20]. An interesting fact to note though is that [19, 5, 20] all assume the vacuum Einstein equations in their proof of equations 2.57 and 2.58. I haven't yet found an explicit generalisation to the Einstein-Maxwell system, which will be studied in chapter 5.

Observe that $\mathrm{d}S \neq 0$ near $S^{n-2}_\infty$ by equation 2.57. When this happens, the coordinate system can be further refined. Indeed in chapter 3, it will be assumed this is always possible.

**Definition 2.8** (Israel coordinates)**.** *When $\mathrm{d}S \neq 0$, one can use the Israel coordinates*[4]*. These use $S$ itself as a coordinate, since $\mathrm{d}S \neq 0$. The full coordinate system is $\{t, S, x^A\}$ where $A$ runs from 2 to $n-1$ and the $x^A$ are coordinates on constant $t$ and $S$ surfaces. In Israel coordinates, the metric is*

$$g = -S^2 \mathrm{d}t \otimes \mathrm{d}t + \rho^2 \mathrm{d}S \otimes \mathrm{d}S + \tilde{h}_{AB} \mathrm{d}x^A \otimes \mathrm{d}x^B. \tag{2.59}$$

[3]Perhaps this is more aptly called ADM energy, but it doesn't really matter for static spacetimes.

[4]I have given them this name because of their successful use by Israel in [1], which I'll discuss in chapter 3.

*Proof.* It's worth saying a bit more about why the metric can be written in this form.
Let $\{x^A\}_{A=2}^{n-1}$ be local coordinates on a particular constant $S$ surface.
$\therefore$ $\mathrm{d}S$ is normal to that hypersurface.
Extend $x^A$ off that hypersurface by keeping $x^A$ constant along flows of $(\mathrm{d}S)^a$.
That way only $S$ changes along flows of $(\mathrm{d}S)^a$, meaning $(\mathrm{d}S)^a \propto (\partial/\partial S)^a$.
$\therefore$ There are no $\mathrm{d}S$-$\mathrm{d}x^A$ cross terms in the metric. Then, I just define $\rho^2$ ($\rho > 0$ without loss of generality) to be whatever the coefficient of $\mathrm{d}S \otimes \mathrm{d}S$ is in the metric[5]. I know that coefficient must be positive because $\Sigma_t$ is Riemannian. □

I've already mentioned $\mathrm{d}S \neq 0$ near $S^{n-2}_\infty$. It is also true in another region.

**Lemma 2.9.** $\mathrm{d}S \neq 0$ *just outside the event horizon of a non-extremal black hole in a spacetime satisfying the conditions of theorem 2.2.*

*Proof.* By continuity, the value of $\mathrm{d}(k^a k_a)$ on the event horizon equals the limit of $\mathrm{d}(k^a k_a)$ while approaching the event horizon from any path. Outside the event horizon, I can evaluate $\mathrm{d}(k^a k_a)$ in the adapted coordinates.
$\therefore$ $\mathrm{d}(k^a k_a) = \lim \mathrm{d}(-S^2) = -2\lim(S\mathrm{d}S)$.
Corollary 2.2.1 means $S$ tends to zero as one approaches the event horizon.
$\therefore$ Since $(\mathrm{d}(k^b k_b))_a = -2\kappa k_a \neq 0$ on the event horizon of a non-extremal black hole, $\mathrm{d}S$ must diverge for $\mathrm{d}(k^a k_a) = -2\lim(S\mathrm{d}S)$ to hold.
$\therefore$ $\mathrm{d}S$ must be non-zero just outside the event horizon. □

Note that I'm effectively always dealing with non-extremal black holes because of the assumptions that I made for theorem 2.2. To deduce some significant results about the event horizon using Israel coordinates, I'll first need a few auxiliary lemmas.

**Lemma 2.10.** *In Israel coordinates, the Christoffel symbols are*

$$\begin{aligned} &\Gamma^{(h)S}{}_{SS} = \frac{1}{\rho}\partial_S\rho, \;\; \Gamma^{(h)S}{}_{SA} = \frac{1}{\rho}\partial_A\rho, \;\; \Gamma^{(h)S}{}_{AB} = -\frac{1}{\rho}K_{AB}, \\ &\Gamma^{(h)A}{}_{SS} = -\rho\tilde{h}^{AB}\partial_B\rho, \;\; \Gamma^{(h)A}{}_{SB} = \rho K^A{}_B \;\text{ and }\; \Gamma^{(h)A}{}_{BC} = \Gamma^{(\tilde{h})A}{}_{BC}, \end{aligned} \tag{2.60}$$

*where $K_{ij}$ is the extrinsic curvature tensor for constant $S$ surfaces.*

*Proof.* $\mathrm{d}S$ is normal to constant $S$ surfaces. From equation 2.59, $(\mathrm{d}S)^i(\mathrm{d}S)_i = \frac{1}{\rho^2}$.
$\therefore$ A unit normal is $n_a = \rho\,\mathrm{d}S \iff n^a = \frac{1}{\rho}\frac{\partial}{\partial S}$.
$\therefore$ The induced metric, $\hat{h}$, is $\hat{h}_{ij} = h_{ij} - n_i n_j = h_{ij} - \rho^2\delta_{i1}\delta_{j1}$. Then, the extrinsic curvature is

$$K_{ij} = \frac{1}{2}(\mathcal{L}_n\hat{h})_{ij} \tag{2.61}$$

$$= \frac{1}{2}(n^k\partial_k\hat{h}_{ij} + \hat{h}_{kj}\partial_i n^k + \hat{h}_{ik}\partial_j n^k) \tag{2.62}$$

$$\begin{aligned} &= \frac{1}{2}\bigg(\frac{1}{\rho}\delta^{k1}\partial_k(h_{ij} - \rho^2\delta_{i1}\delta_{j1}) + (h_{kj} - \rho^2\delta_{k1}\delta_{j1})\partial_i\left(\frac{1}{\rho}\delta^{k1}\right) \\ &\quad + (h_{ik} - \rho^2\delta_{i1}\delta_{k1})\partial_j\left(\frac{1}{\rho}\delta^{k1}\right)\bigg). \end{aligned} \tag{2.63}$$

$$\therefore K_{AB} = \frac{1}{2\rho}\partial_S\tilde{h}_{AB} \;\text{ since } h_{A1} = 0 \text{ by equation 2.59.} \tag{2.64}$$

[5]Note that all the metric components can depend on $S$, unlike the similar rigmarole with $t$ in equation 2.12, where $k^a$ being static meant all the metric components were $t$ independent.

The Christoffel symbols can be found using the Euler-Lagrange equations of

$$\mathcal{L} = \frac{1}{2}h_{ij}\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\lambda\lambda} \tag{2.65}$$

$$= \frac{1}{2}\rho^2\frac{\mathrm{d}S}{\mathrm{d}\lambda}\frac{\mathrm{d}S}{\mathrm{d}\lambda} + \frac{1}{2}\tilde{h}_{AB}\frac{\mathrm{d}x^A}{\mathrm{d}\lambda}\frac{\mathrm{d}x^B}{\mathrm{d}\lambda}. \tag{2.66}$$

For $S$, I have

$$\frac{\partial\mathcal{L}}{\partial S} = \rho\frac{\mathrm{d}S}{\mathrm{d}\lambda}\frac{\mathrm{d}S}{\mathrm{d}\lambda}\frac{\partial\rho}{\partial S} + \frac{1}{2}\partial_S(\tilde{h}_{AB})\frac{\mathrm{d}x^A}{\mathrm{d}\lambda}\frac{\mathrm{d}x^B}{\mathrm{d}\lambda} = \rho\frac{\mathrm{d}S}{\mathrm{d}\lambda}\frac{\mathrm{d}S}{\mathrm{d}\lambda}\frac{\partial\rho}{\partial S} + \rho K_{AB}\frac{\mathrm{d}x^A}{\mathrm{d}\lambda}\frac{\mathrm{d}x^B}{\mathrm{d}\lambda} \quad \text{and} \tag{2.67}$$

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{\mathrm{d}S}{\mathrm{d}\lambda}\right)}\right) = \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\rho^2\frac{\mathrm{d}S}{\mathrm{d}\lambda}\right) = \rho^2\frac{\mathrm{d}^2S}{\mathrm{d}\lambda^2} + 2\rho\left(\frac{\partial\rho}{\partial S}\frac{\mathrm{d}S}{\mathrm{d}\lambda} + \partial_A(\rho)\frac{\mathrm{d}x^A}{\mathrm{d}\lambda}\right)\frac{\mathrm{d}S}{\mathrm{d}\lambda}. \tag{2.68}$$

$$\therefore 0 = \frac{\mathrm{d}^2S}{\mathrm{d}\lambda^2} + \frac{1}{\rho}\frac{\partial\mathrm{d}}{\partial S}\frac{\mathrm{d}S}{\mathrm{d}\lambda}\frac{\mathrm{d}S}{\mathrm{d}\lambda} + \frac{2\partial A(\rho)}{\rho}\frac{\mathrm{d}x^A}{\mathrm{d}\lambda}\frac{\mathrm{d}S}{\mathrm{d}\lambda} - \frac{1}{\rho}K_{AB}\frac{\mathrm{d}x^A}{d\lambda}\frac{\mathrm{d}x^B}{\mathrm{d}\lambda}. \tag{2.69}$$

Hence $\Gamma^{(h)S}{}_{ij}$ takes the values claimed in the lemma. Next, $\Gamma^{(h)A}{}_{ij}$.

$$\frac{\partial\mathcal{L}}{\partial x^A} = \rho\frac{\mathrm{d}S}{\mathrm{d}\lambda}\frac{\mathrm{d}S}{\mathrm{d}\lambda}\frac{\partial\rho}{\partial x^A} + \frac{1}{2}\partial_A(\tilde{h}_{BC})\frac{\mathrm{d}x^B}{\mathrm{d}\lambda}\frac{\mathrm{d}x^C}{\mathrm{d}\lambda}. \tag{2.70}$$

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{\mathrm{d}x^A}{\mathrm{d}\lambda}\right)}\right) = \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\tilde{h}_{AB}\frac{\mathrm{d}x^B}{\mathrm{d}\lambda}\right) \tag{2.71}$$

$$= \tilde{h}_{AB}\frac{\mathrm{d}^2x^B}{\mathrm{d}\lambda^2} + \partial_S\left(\tilde{h}_{AB}\right)\frac{\mathrm{d}S}{\mathrm{d}\lambda}\frac{\mathrm{d}x^B}{\mathrm{d}\lambda} + \partial_C\left(\tilde{h}_{AB}\right)\frac{\mathrm{d}x^B}{\mathrm{d}\lambda}\frac{\mathrm{d}x^C}{\mathrm{d}\lambda} \tag{2.72}$$

$$= \tilde{h}_{AB}\frac{\mathrm{d}^2x^B}{\mathrm{d}\lambda^2} + 2\rho K_{AB}\frac{\mathrm{d}S}{\mathrm{d}\lambda}\frac{\mathrm{d}x^B}{\mathrm{d}\lambda} + \partial_C\left(\tilde{h}_{AB}\right)\frac{\mathrm{d}x^B}{\mathrm{d}\lambda}\frac{\mathrm{d}x^C}{\mathrm{d}\lambda}. \tag{2.73}$$

$$\therefore 0 = \tilde{h}^{AD}\Bigg(\tilde{h}_{DB}\frac{\mathrm{d}^2x^B}{\mathrm{d}\lambda^2} + 2\rho K_{DB}\frac{\mathrm{d}x^B}{\mathrm{d}\lambda}\frac{\mathrm{d}S}{\mathrm{d}\lambda} + \partial_C\left(\tilde{h}_{DB}\right)\frac{\mathrm{d}x^B}{\mathrm{d}\lambda}\frac{\mathrm{d}x^C}{\mathrm{d}\lambda} - \rho\frac{\partial\rho}{\partial x^D}\frac{\mathrm{d}S}{\mathrm{d}\lambda}\frac{\mathrm{d}S}{\mathrm{d}\lambda}$$

$$- \frac{1}{2}\partial_D\left(\tilde{h}_{BC}\right)\frac{\mathrm{d}x^B}{\mathrm{d}\lambda}\frac{\mathrm{d}x^C}{\mathrm{d}\lambda}\Bigg) \tag{2.74}$$

$$= \frac{\mathrm{d}^2x^A}{\mathrm{d}\lambda^2} + 2\rho K^A{}_B\frac{\mathrm{d}x^B}{\mathrm{d}\lambda}\frac{\mathrm{d}S}{\mathrm{d}\lambda} + \frac{1}{2}\tilde{h}^{AD}\left(\partial_B\tilde{h}_{CD} + \partial_C\tilde{h}_{DB} - \partial_D\tilde{h}_{BC}\right)\frac{\mathrm{d}x^B}{\mathrm{d}\lambda}\frac{\mathrm{d}x^C}{\mathrm{d}\lambda}$$

$$- \rho\tilde{h}^{AB}\partial_B(\rho)\frac{\mathrm{d}S}{\mathrm{d}\lambda}\frac{\mathrm{d}S}{\mathrm{d}\lambda} \tag{2.75}$$

$$= \frac{\mathrm{d}^2x^A}{\mathrm{d}\lambda^2} + 2\rho K^A{}_B\frac{\mathrm{d}x^B}{\mathrm{d}\lambda}\frac{\mathrm{d}S}{\mathrm{d}\lambda} + \Gamma^{(\tilde{h})A}{}_{BC}\frac{\mathrm{d}x^B}{\mathrm{d}\lambda}\frac{\mathrm{d}x^C}{\mathrm{d}\lambda} - \rho\tilde{h}^{AB}\partial_B(\rho)\frac{\mathrm{d}S}{\mathrm{d}\lambda}\frac{\mathrm{d}S}{\mathrm{d}\lambda}. \tag{2.76}$$

$\therefore \Gamma^{(h)A}{}_{ij}$ take the values claimed. □

**Corollary 2.10.1.** *The Ricci tensor components are*

$$R^{(h)}_{SS} = -\rho(\Box^{(\tilde{h})}\rho + \partial_S K + \rho K_{AB}K^{AB}),\ \ R^{(h)}_{AS} = \rho(\nabla^{(\tilde{h})}_b K^B{}_A - \partial_A K),\ \ \text{and}$$

$$R^{(h)}_{AB} = R^{(\tilde{h})}_{AB} - \frac{1}{\rho}\nabla^{(\tilde{h})}_A\nabla^{(\tilde{h})}_B\rho - KK_{AB} - \frac{1}{\rho}\tilde{h}_{AC}\partial_S K^C{}_B, \tag{2.77}$$

*where* $K = h^{ij}K_{ij} = \tilde{h}^{AB}K_{AB}$.

*Proof.* From the Christoffel symbols I calculated,

$$R^{(h)i}{}_{jkl} = \partial_k \Gamma^{(h)i}{}_{jl} - \partial_l \Gamma^{(h)i}{}_{jk} + \Gamma^{(h)i}{}_{mk}\Gamma^{(h)m}{}_{jl} - \Gamma^{(h)i}{}_{ml}\Gamma^{(h)m}{}_{jk}. \tag{2.78}$$

$$\therefore R_{ij} = R^{(h)k}{}_{ikj} = \partial_k \Gamma^{(h)k}{}_{ij} - \partial_j \Gamma^{(h)k}{}_{ik} + \Gamma^{(h)k}{}_{mk}\Gamma^{(h)m}{}_{ij} - \Gamma^{(h)k}{}_{mj}\Gamma^{(h)m}{}_{ik}. \tag{2.79}$$

$$\therefore R^{(h)}_{SS} = \partial_k \Gamma^{(h)k}{}_{SS} - \partial_S \Gamma^{(h)k}{}_{Sk} + \Gamma^{(h)k}{}_{mk}\Gamma^{(h)m}{}_{SS} - \Gamma^{(h)k}{}_{mS}\Gamma^{(h)m}{}_{Sk} \tag{2.80}$$

$$\begin{aligned}
&= \partial_S\left(\frac{1}{\rho}\partial_S\rho\right) + \partial_A\left(-\rho\tilde{h}^{AB}\partial_B\rho\right) - \partial_S\left(\frac{1}{\rho}\partial_S\rho\right) - \partial_S(\rho K) + \frac{1}{\rho^2}\partial_S(\rho)\partial_S(\rho) + K\partial_S\rho \\
&\quad - \tilde{h}^{AB}\partial_A(\rho)\partial_B(\rho) - \rho\Gamma^{(\tilde{h})A}{}_{BA}\tilde{h}^{BC}\partial_C\rho - \frac{1}{\rho^2}\partial_S(\rho)\partial_S(\rho) + \tilde{h}^{AB}\partial_A(\rho)\partial_B(\rho) \\
&\quad + \tilde{h}^{AB}\partial_A(\rho)\partial_B(\rho) - \rho^2 K^A{}_B K^B{}_A
\end{aligned} \tag{2.81}$$

$$\begin{aligned}
&= -\tilde{h}^{AB}\partial_A(\rho)\partial_B(\rho) - \rho\partial_A(\tilde{h}^{AB}\partial_B\rho) - \rho\partial_S K - K\partial_S\rho + K\partial_S\rho \\
&\quad - \rho\Gamma^{(\tilde{h})A}{}_{BA}\tilde{h}^{BC}\partial_C\rho + \tilde{h}^{AB}\partial_A(\rho)\partial_B(\rho) - \rho^2 K^A{}_B K^B{}_A
\end{aligned} \tag{2.82}$$

$$= -\rho\partial_A(\tilde{h}^{AB}\nabla^{(\tilde{h})}_B\rho) - \rho\partial_S K - \rho\Gamma^{(\tilde{h})A}{}_{BA}\tilde{h}^{BC}\nabla^{(\tilde{h})}_C\rho - \rho^2 K_{AB}K^{AB} \tag{2.83}$$

$$= -\rho\partial_A(\nabla^{(\tilde{h})A}\rho) - \rho\partial_S K - \rho\Gamma^{(\tilde{h})A}{}_{BA}\tilde{\nabla}^{(\tilde{h})B}\rho - \rho^2 K_{AB}K^{AB} \tag{2.84}$$

$$= -\rho\Box^{(\tilde{h})}\rho - \rho\partial_S K - \rho^2 K_{AB}K^{AB}. \tag{2.85}$$

$$R^{(h)}_{Sa} = \partial_k \Gamma^{(h)k}{}_{SA} - \partial_A \Gamma^{(h)k}{}_{Sk} + \Gamma^{(h)k}{}_{mk}\Gamma^{(h)m}{}_{SA} - \Gamma^{(h)k}{}_{mA}\Gamma^{(h)m}{}_{Sk} \tag{2.86}$$

$$\begin{aligned}
&= \partial_S\left(\frac{1}{\rho}\partial_A\rho\right) + \partial_B\left(\rho K^B{}_A\right) - \partial_A\left(\frac{1}{\rho}\partial_S\rho\right) - \partial_A(\rho K) + \frac{1}{\rho^2}\partial_S(\rho)\partial_A(\rho) + K\partial_A\rho \\
&\quad + K^B{}_A\partial_B\rho + \rho K^B{}_A\Gamma^{(\tilde{h})C}{}_{BC} - \frac{1}{\rho^2}\partial_A(\rho)\partial_S(\rho) - K^B{}_A\partial_B\rho + K^B{}_A\partial_B\rho \\
&\quad - \rho K^B{}_C\Gamma^{(\tilde{h})C}{}_{BA}
\end{aligned} \tag{2.87}$$

$$\begin{aligned}
&= -\frac{1}{\rho^2}\partial_S(\rho)\partial_A(\rho) + \frac{1}{\rho}\partial_S\partial_A\rho + \rho\partial_B K^B{}_A + \frac{1}{\rho^2}\partial_A(\rho)\partial_S(\rho) - \frac{1}{\rho}\partial_A\partial_S\rho - K\partial_A\rho - \rho\partial_A K \\
&\quad + K\partial_A\rho + \rho K^B{}_A\Gamma^{(\tilde{h})C}{}_{BC} - \rho K^B{}_C\Gamma^{(\tilde{h})C}{}_{BA}
\end{aligned} \tag{2.88}$$

$$= \rho\partial_B K^B{}_A + \rho K^B{}_A\Gamma^{(\tilde{h})C}{}_{BC} - \rho K^B{}_C\Gamma^{(\tilde{h})C}{}_{BA} - \rho\partial_A K \tag{2.89}$$

$$= \rho\nabla^{(\tilde{h})}_B K^B{}_A - \rho\nabla^{(\tilde{h})}_A K. \tag{2.90}$$

$$R^{(h)}_{AB} = \partial_k \Gamma^{(h)k}{}_{AB} - \partial_B \Gamma^{(h)k}{}_{Ak} + \Gamma^{(h)k}{}_{mk}\Gamma^{(h)m}{}_{AB} - \Gamma^{(h)k}{}_{mB}\Gamma^{(h)m}{}_{Ak} \tag{2.91}$$

$$\begin{aligned}
&= \partial_S\left(-\frac{1}{\rho}K_{AB}\right) + \partial_C\Gamma^{(\tilde{h})C}{}_{AB} - \partial_B\left(\frac{1}{\rho}\partial_A\rho\right) - \partial_B\Gamma^{(\tilde{h})C}{}_{AC} - \frac{1}{\rho^2}\partial_S(\rho)K_{AB} - KK_{AB} \\
&\quad + \frac{1}{\rho}\partial_C(\rho)\Gamma^{(\tilde{h})C}{}_{AB} + \Gamma^{(\tilde{h})C}{}_{DC}\Gamma^{(\tilde{h})D}{}_{AB} - \frac{1}{\rho^2}\partial_B(\rho)\partial_A(\rho) + K^C{}_B K_{AC} + K_{CB}K^C{}_A \\
&\quad - \Gamma^{(\tilde{h})C}{}_{DB}\Gamma^{(\tilde{h})D}{}_{AC}
\end{aligned} \tag{2.92}$$

$$\begin{aligned}
&= R^{(\tilde{h})}_{AB} - \frac{1}{\rho}\partial_S K_{AB} + \frac{1}{\rho^2}\partial_S(\rho)K_{AB} - \frac{1}{\rho}\partial_B\partial_A\rho + \frac{1}{\rho^2}\partial_B(\rho)\partial_A(\rho) - \frac{1}{\rho^2}\partial_S(\rho)K_{AB} - KK_{AB} \\
&\quad + \frac{1}{\rho}\partial_C(\rho)\Gamma^{(\tilde{h})C}{}_{AB} - \frac{1}{\rho^2}\partial_B(\rho)\partial_A(\rho) + 2K_{AC}K^C{}_B
\end{aligned} \tag{2.93}$$

$$= R^{(\tilde{h})}_{AB} - \frac{1}{\rho}\partial_S K_{AB} - \frac{1}{\rho}\nabla^{(\tilde{h})}_A\nabla^{(\tilde{h})}_B\rho - KK_{AB} + 2K_{AC}K^C{}_B. \tag{2.94}$$

This can be re-written in the form stated in the lemma as follows.

$$\frac{1}{\rho}\tilde{h}_{AC}\partial_S K^C{}_B = \frac{1}{\rho}\tilde{h}_{AC}\partial_S\left(\tilde{h}^{CD}K_{DB}\right) \tag{2.95}$$

$$= \frac{1}{\rho}\tilde{h}_{AC}\tilde{h}^{CD}\partial_S K_{DB} + \frac{1}{\rho}\tilde{h}_{AC}K_{DB}\partial_S\tilde{h}^{CD} \tag{2.96}$$

$$= \frac{1}{\rho}\partial_S K_{AB} + \frac{1}{\rho}\tilde{h}_{AC}K_{DB}\partial_S\tilde{h}^{CD}. \tag{2.97}$$

The second term can be re-written in terms of the extrinsic curvature because $0 = \partial_S\left(\delta^A{}_B\right) = \partial_S\left(\tilde{h}^{AC}\tilde{h}_{CB}\right) = \tilde{h}^{AC}\partial_S\tilde{h}_{CB} + \tilde{h}_{CB}\partial_S\tilde{h}^{AC} = 2\rho K^A{}_B + \tilde{h}_{CB}\partial_S\tilde{h}^{AC}$ implies $0 = \tilde{h}^{BD}2\rho K^A{}_B + \tilde{h}^{BD}\tilde{h}_{CB}\partial_S\tilde{h}^{AC} \implies \partial_S\tilde{h}^{AB} = -2\rho K^{AB}$.

$$\therefore \frac{1}{\rho}\tilde{h}_{AC}\partial_S K^C{}_B = \frac{1}{\rho}\partial_S K_{AB} + \frac{1}{\rho}\tilde{h}_{AC}K_{DB}(-2\rho K^{CD}) \tag{2.98}$$

$$= \frac{1}{\rho}\partial_S K_{AB} - 2\rho K_{AC}K^C{}_B. \tag{2.99}$$

Hence, I finally get $R^{(h)}_{AB} = R^{(\tilde{h})}_{AB} - \frac{1}{\rho}\nabla^{(\tilde{h})}_A\nabla^{(\tilde{h})}_B\rho - KK_{AB} - \frac{1}{\rho}\tilde{h}_{AC}\partial_S K^C{}_B$. □

**Corollary 2.10.2.** *The Ricci scalar is* $R^{(h)} = R^{(\tilde{h})} - K^2 - K_{AB}K^{AB} - \frac{2}{\rho}\Box^{(\tilde{h})}\rho - \frac{2}{\rho}\partial_S K$.

*Proof.* $R^{(h)} = h^{ij}R^{(h)}_{ij} = \frac{1}{\rho^2}R^{(h)}_{SS} + \tilde{h}^{AB}R^{(h)}_{AB}$. Then, using the components I just calculated,

$$\begin{aligned} R^{(h)} &= -\frac{1}{\rho^2}\rho(\Box^{(\tilde{h})}\rho + \partial_S K + \rho K_{AB}K^{AB}) \\ &\quad + \tilde{h}^{AB}\left(R^{(\tilde{h})}_{AB} - \frac{1}{\rho}\nabla^{(\tilde{h})}_A\nabla^{(\tilde{h})}_B\rho - KK_{AB} - \frac{1}{\rho}\tilde{h}_{AC}\partial_S K^C{}_B\right) \end{aligned} \tag{2.100}$$

$$= -\frac{1}{\rho}\Box^{(\tilde{h})}\rho - \frac{1}{\rho}\partial_S K - K_{AB}K^{AB} + R^{(\tilde{h})} - \frac{1}{\rho}\Box^{(\tilde{h})}\rho - K^2 - \frac{1}{\rho}\partial_S K \tag{2.101}$$

$$= -\frac{2}{\rho}\Box^{(\tilde{h})}\rho - \frac{2}{\rho}\partial_S K - K_{AB}K^{AB} + R^{(\tilde{h})} - K^2, \tag{2.102}$$

which is the expression claimed. □

**Corollary 2.10.3.** *The components of the Einstein tensor are*

$$\begin{aligned} &G^{(h)}_{SS} = \frac{1}{2}\rho^2(-R^{(\tilde{h})} + K^2 - K_{AB}K^{AB}),\ \ G^{(h)}_{SA} = \rho(\nabla^{(\tilde{h})}_b K^B{}_A - \partial_A K)\ \text{ and} \\ &G^{(h)}_{AB} = G^{(\tilde{h})}_{AB} - KK_{AB} + \frac{1}{2}\tilde{h}_{AB}(K^2 + K_{AB}K^{AB}) \\ &\qquad + \frac{1}{\rho}(\tilde{h}_{AB}\Box^{(\tilde{h})}\rho + \tilde{h}_{AB}\partial_S K - \nabla^{(\tilde{h})}_A\nabla^{(\tilde{h})}_B\rho - \tilde{h}_{AC}\partial_S K^C{}_B). \end{aligned} \tag{2.103}$$

*Proof.* Follows directly from $G^{(h)}_{ij} = R^{(h)}_{ij} - \frac{1}{2}R^{(h)}h_{ij}$ and the earlier corollaries. □

**Lemma 2.11.** *Under the conditions of theorem 2.2, the surface gravity, $\kappa$, is positive.*

*Proof.* The conditions of theorem 2.2 imply $\kappa \neq 0$. Assume $\kappa < 0$ for a contradiction.
In corollary 2.2.1, I have already shown the event horizon, $H^+$, is a Killing horizon of $k^a$.
Let $V^a$ be tangent to a future directed timelike curve passing through $H^+$ such that $V^a$ is not parallel to $k^a$ (at any point where the curve hits $H^+$).

$k^a$ is chosen to be future directed at null infinity without loss of generality.
$\therefore k^a$ remains future directed all through the domain of outer communication because theorem 2.2 says it's timelike throughout that region and $k^a$ can't flip from one light cone to the other while remaining timelike.
$\therefore k^a$ is null and non-zero on the $H^+ \implies k^a$ is still future directed on $H^+$.
$\therefore V^a k_a < 0$.
Then, $V^a \nabla_a(k^b k_b) = -2\kappa V^a k_a$ on $H^+ \implies \mathrm{sgn}(V^a \nabla_a(k^b k_b)) = \mathrm{sgn}(\kappa) = -$.
Timelike curves fall into the black hole, so $\mathrm{sgn}(V^a \nabla_a(k^b k_b)) = -$ and $k^a k_a = 0$ on $H^+$ together imply $k^a k_a < 0$ inside the black hole.
But, theorem 2.2 says $k^a k_a < 0$ outside the black hole too.
$\therefore \nabla_a(k^b k_b) = 0$ on $H^+$. ↯
This is a contradiction because $\nabla_a(k^b k_b) = 0$ on $H^+$ implies $\kappa = 0$, contradicting the $\kappa < 0$ assumption. □

**Lemma 2.12.** *As one approaches the event horizon, $\rho \to \frac{1}{\kappa}$, where $\kappa$ is the surface gravity.*

*Proof.* It follows pretty much from the definition of the Hodge star that for a $p$-form, $\alpha$, and a vector field or one-form, $X$ (i.e. $X^a$ or $X_a$), $\star(\alpha \wedge X) = \iota_X \star \alpha$.
Let $\omega = \frac{1}{2} \star (k \wedge \mathrm{d}k)$.
$\therefore \star(k \wedge \omega) = - \star (\omega \wedge k) = -\iota_k \star \omega = -\frac{1}{2}(-1)^n \iota_k (k \wedge \mathrm{d}k)$.
Let $N = k^a k_a$. Then, using the Killing equation liberally,

$$
\begin{aligned}
&2(-1)^n \star (k \wedge \omega) + k \wedge \mathrm{d}N \\
&= -\iota_k(k \wedge \mathrm{d}k) + k \wedge \mathrm{d}N && (2.104)\\
&= -k^c (k \wedge \mathrm{d}k)_{cab} + (k \wedge \nabla(k^c k_c))_{ab} && (2.105)\\
&= -k^c (k_c (\mathrm{d}k)_{ab} + k_a (\mathrm{d}k)_{bc} + k_b (\mathrm{d}k)_{ca}) + 2k_a k^c \nabla_b k_c - 2k_b k^c \nabla_a k_c && (2.106)\\
&= -N(\mathrm{d}k)_{ab} - k_a k^c (\mathrm{d}k)_{bc} - k_b k^c (\mathrm{d}k)_{ca} + k_a k^c (\mathrm{d}k)_{bc} - k_b k^c (\mathrm{d}k)_{ac} && (2.107)\\
&= -N(\mathrm{d}k)_{ab} && (2.108)\\
&= -2N \nabla_a k_b. && (2.109)
\end{aligned}
$$

For a static vector field, $k \wedge \mathrm{d}k$ is zero though[6].

$$
\begin{aligned}
\therefore 4N^2 \nabla^a(k^b)\nabla_a(k_b) &= (k \wedge \mathrm{d}N)^{ab}(k \wedge \mathrm{d}N)_{ab} && (2.110)\\
&= (k^a \nabla^b(N) - k^b \nabla^a(N))(k_a \nabla_b(N) - k_b \nabla_a(N)) && (2.111)\\
&= 2N \nabla^a(N)\nabla_a(N) - 2(k^a \nabla_a(N))^2 && (2.112)\\
&= 2N \nabla^a(N)\nabla_a(N) - 2(2k^a k^b \nabla_a(k_b))^2 && (2.113)\\
&= 2N \nabla^a(N)\nabla_a(N) - 0. && (2.114)\\
\therefore 2N \nabla^a(k^b)\nabla_a(k_b) &= \nabla^a(N)\nabla_a(N). && (2.115)\\
\therefore -2S^2 \nabla^a(k^b)\nabla_a(k_b) &= \nabla^a(-S^2)\nabla_a(-S^2) \text{ using adapted coordinates.} && (2.116)\\
\therefore \nabla^a(k^b)\nabla_a(k_b) &= -2\nabla^a(S)\nabla_a(S) && (2.117)\\
&= -\frac{2}{\rho^2} \text{ using Israel coordinates.} && (2.118)
\end{aligned}
$$

[6]This is true on a Killing horizon of $k^a$ regardless of further assumptions.

Next, observe that

$$(\mathrm{d}k|\mathrm{d}k) \star k = \iota_k \varepsilon(\mathrm{d}k|\mathrm{d}k) \tag{2.119}$$
$$= \iota_k(\mathrm{d}k \wedge \star \mathrm{d}k) \tag{2.120}$$
$$= \iota_k(\mathrm{d}k) \wedge \star\mathrm{d}k + \mathrm{d}k \wedge \iota_k(\star\mathrm{d}k) \tag{2.121}$$
$$= \iota_k(\mathrm{d}k) \wedge \star\mathrm{d}k + 2\mathrm{d}k \wedge \omega \tag{2.122}$$
$$= \iota_k(\mathrm{d}k) \wedge \star\mathrm{d}k \text{ by } k \wedge \mathrm{d}k = 0 \tag{2.123}$$
$$= -\mathrm{d}N \wedge \star\mathrm{d}k. \tag{2.124}$$

On the event horizon, $\mathrm{d}N = -2\kappa k$ by the definition of surface gravity. So, on the event horizon, $(\mathrm{d}k|\mathrm{d}k) \star k = 2\kappa k \wedge \star\mathrm{d}k$. Next, observe that

$$(\star(k \wedge \star\mathrm{d}k))_a = \frac{1}{(n-1)!}\epsilon_{bc_1\cdots c_{n-2}a}(k \wedge \star\mathrm{d}k)^{bc_1\cdots c_{n-2}} \tag{2.125}$$
$$= \frac{1}{(n-2)!}\varepsilon_{bc_1\cdots c_{n-2}a}k^b\varepsilon^{efc_1\cdots c_{n-2}}\nabla_e k_f \tag{2.126}$$
$$= -2(-1)^{n-2}\delta^e{}_{[b}\delta^f{}_{a]}k^b\nabla_e k_f \tag{2.127}$$
$$= -2(-1)^n k^b \nabla_b k_a \tag{2.128}$$
$$= -2(-1)^n \kappa k_a. \tag{2.129}$$
$$\therefore k \wedge \star\mathrm{d}k = -2\kappa \star k \tag{2.130}$$

Substituting this back up, I get $(\mathrm{d}k|\mathrm{d}k) \star k = -4\kappa^2 \star k \iff (\mathrm{d}k|\mathrm{d}k) = -4\kappa^2 \iff 2\nabla^a(k^b)\nabla_a(k_b) = -4\kappa^2$. Substituting this back into equation 2.118 says $\rho^2 = \frac{1}{\kappa^2}$. I can take the positive square root by lemma 2.11, thereby completing the proof. □

The main upshot of these last three lemmas is the following theorem.

**Theorem 2.13.** *Near the event horizon a static black hole,*

$$R_{abcd}R^{abcd} = \frac{4}{S^2\rho^6}(\partial_S\rho)^2 + \frac{8}{S^2\rho^4}\nabla^{(\tilde{h})}_A(\rho)\nabla^{(\tilde{h})A}(\rho) + \frac{4}{\rho^2 S^2}K_{AB}K^{AB} + R^{(h)}_{ijkl}R^{(h)ijkl}. \tag{2.131}$$

*Proof.* From the proof of theorem 2.4, the only non-zero components of $R_{\mu\nu\rho\sigma}$ in this coordinate system are $R_{0i0j} = S\nabla^{(h)}_i\nabla^{(h)}_j S$, $R_{ijkl} = R^{(h)}_{ijkl}$ and components related by the Riemann tensor's symmetries.

$$\therefore R_{abcd}R^{abcd} = 4R_{0i0j}R^{0i0j} + R_{ijkl}R^{ijkl} \tag{2.132}$$
$$= 4S\nabla^{(h)}_i\nabla^{(h)}_j(S)\frac{1}{S^3}\nabla^{(h)i}\nabla^{(h)j}(S) + R^{(h)}_{ijkl}R^{(h)ijkl} \tag{2.133}$$
$$= \frac{4}{S^2}\nabla^{(h)}_i\nabla^{(h)}_j(S)\nabla^{(h)i}\nabla^{(h)j}(S) + R^{(h)}_{ijkl}R^{(h)ijkl}. \tag{2.134}$$

In Israel coordinates,

$$\nabla^{(h)}_i\nabla^{(h)}_j S = \partial_i\partial_j S - \Gamma^{(h)k}{}_{ji}\partial_k S = 0 - \Gamma^{(h)1}{}_{ji}. \tag{2.135}$$

Then, by lemma 2.10,

$$\nabla^{(h)}_i\nabla^{(h)}_j(S)\nabla^{(h)i}\nabla^{(h)j}(S)$$
$$= (h^{11})^2(\Gamma^{(h)1}{}_{11})^2 + 2h^{11}h^{AB}\Gamma^{(h)1}{}_{1A}\Gamma^{(h)1}{}_{1B} + h^{AC}h^{BD}\Gamma^{(h)1}{}_{AB}\Gamma^{(h)1}{}_{CD} \tag{2.136}$$
$$= \left(\frac{1}{\rho^2}\right)^2\left(\frac{1}{\rho}\partial_S\rho\right)^2 + 2\frac{1}{\rho^2}\tilde{h}^{AB}\frac{1}{\rho}\partial_A(\rho)\frac{1}{\rho}\partial_B(\rho) + \tilde{h}^{AC}\tilde{h}^{BD}\left(-\frac{1}{\rho}K_{AB}\right)\left(-\frac{1}{\rho}K_{CD}\right) \tag{2.137}$$
$$= \frac{1}{\rho^6}(\partial_S\rho)^2 + \frac{2}{\rho^4}\nabla^{(\tilde{h})}_A(\rho)\nabla^{(\tilde{h})A}(\rho) + \frac{1}{\rho^2}K_{AB}K^{AB}. \tag{2.138}$$

Substituting this into equation 2.134 gives the claimed result. □

**Corollary 2.13.1.** *$\partial_S \rho$, $\nabla_A^{(\tilde{h})}(\rho)$ and $K_{AB}$ are all zero on the event horizon.*

*Proof.* $S \to 0^+$ as one approaches $H^+$ and by lemma 2.12, $\rho$ approaches the non-zero, finite value, $\frac{1}{\kappa}$, as one approaches $H^+$.
$\therefore$ Since every term on the RHS of equation 2.131 is positive definite, the event horizon can only be regular (in the sense that $R_{abcd}R^{abcd} \to \infty$ would be a curvature singularity) if $\partial_S \rho$, $\nabla_A^{(\tilde{h})}(\rho)$ and $K_{AB}$ are all zero. ∎

In chapter 3, some very strong assumptions will be made about the topology of constant $t$ and $S$ surfaces. In particular, such surfaces will be assumed to be spheres. This is true near $S_\infty^{n-2}$ from the definition of an asymptotically flat end. However, it is also a true statement about $\mathcal{H} = \Sigma_t \cap H^+$, which is a surface with constant $t$ and constant $S$, namely $S = 0$. For the remainder of this chapter, I will work towards the proof of this latter claim.

**Lemma 2.14.** *Let $\mathcal{H}$ be the intersection of the event horizon, $H$, with a spacelike hypersurface. Then, the Ricci scalar of $\mathcal{H}$ is related to the Riemann tensor of the overall manifold by*

$$R^{(\mathcal{H})} = \beta^{ac}\beta^{bd}R_{abcd}, \tag{2.139}$$

*where $\beta_{ab}$ is the induced metric on $\mathcal{H}$.*

*Proof.* $R^{(\mathcal{H})}$ will be found by first finding the Riemann tensor of $\mathcal{H}$.
Let $X^a$ be a vector tangent to $\mathcal{H}$ and hence invariant under projection, i.e. $\beta^a{}_b X^b = X^a$.
Let $K^a$ be tangent to the affinely parameterised null generators of the event horizon, $H$. Let $n^a$ be another set of null vectors on $\mathcal{H}$ such that $K^a n_a = -1$. Extend $n^a$ off $\mathcal{H}$ by parallel transport along $K^a$.
Then, $\beta_{ab} = g_{ab} + K_a n_b + K_b n_a$.
Let $D_a$ be the Levi-Civita covariant derivative on $\mathcal{H}$. Then, by definition,

$$\begin{aligned}
D_a D_b X^c &= \beta^d{}_a \beta^e{}_b \beta^c{}_f \nabla_d D_e X^f &(2.140)\\
&= \beta^d{}_a \beta^e{}_b \beta^c{}_f \nabla_d (\beta^g{}_e \beta^f{}_h \nabla_g X^h) &(2.141)\\
&= \beta^d{}_a \beta^e{}_b \beta^c{}_f \beta^g{}_e \beta^f{}_h \nabla_d \nabla_g X^h + \beta^d{}_a \beta^e{}_b \beta^c{}_f \beta^g{}_e \nabla_d(\beta^f{}_h) \nabla_g X^h \\
&\quad + \beta^d{}_a \beta^e{}_b \beta^c{}_f \nabla_d(\beta^g{}_e) \beta^f{}_h \nabla_g X^h &(2.142)\\
&= \beta^d{}_a \beta^e{}_b \beta^c{}_f \nabla_d \nabla_e X^f + \beta^d{}_a \beta^e{}_b \beta^c{}_f \nabla_d(\beta^f{}_g) \nabla_e X^g + \beta^d{}_a \beta^e{}_b \beta^c{}_f \nabla_d(\beta^g{}_e) \nabla_g X^f. &(2.143)
\end{aligned}$$

$\beta^e{}_b \nabla_d(\beta^g{}_e) = \beta^e{}_b(K^g \nabla_d n_e + n^g \nabla_d K_e)$ since $\beta^e{}_b K_e = \beta^e{}_b n_e = 0$.
Since $\mathcal{H}$ is a submanifold, there always exist functions, $a, b, c$ & $e$ and 1-forms, $y_a$ & $z_a$, such that $n_a = a(\mathrm{d}b)_a + b y_a$, $K_a = c(\mathrm{d}e)_a + e z_a$ and on $\mathcal{H}$, $b = e = 0$.

$$\begin{aligned}
\therefore \beta^d{}_a \beta^e{}_b \nabla_d(\beta^g{}_e) &= \beta^d{}_a \beta^e{}_b (K^g \nabla_d n_e + n^g \nabla_d K_e) &(2.144)\\
&= \beta^d{}_a \beta^e{}_b (K^g \nabla_d(a \nabla_e b + b y_e) + n^g \nabla_d(c \nabla_e e + e z_e)) &(2.145)\\
&= \beta^d{}_a \beta^e{}_b (K^g \nabla_d(a) \nabla_e b + a K^g \nabla_d \nabla_e b + K^g \nabla_d(b) y_e + b K^g \nabla_d y_e \\
&\quad + n^g \nabla_d(c) \nabla_e e + c n^g \nabla_d \nabla_e e + n^g \nabla_d(e) z_e + e n^g \nabla_d z_e) &(2.146)\\
&= \beta^d{}_a \beta^e{}_b \bigg( \frac{1}{a} K^g \nabla_d(a) n_e + a K^g \nabla_d \nabla_e b + \frac{1}{a} K^g n_d y_e \\
&\quad + \frac{1}{c} n^g \nabla_d(c) K_e + c n^g \nabla_d \nabla_e e + \frac{1}{c} n^g K_d z_e \bigg) \text{ on } \mathcal{H} &(2.147)\\
&= \beta^d{}_a \beta^e{}_b (a K^g \nabla_d \nabla_e b + c n^g \nabla_d \nabla_e e) \quad \text{by } \beta^{ab} K_b = \beta^{ab} n_b = 0, &(2.148)
\end{aligned}$$

which is now manifestly symmetric in $a$ and $b$ on $\mathcal{H}$.
Meanwhile, $\beta^c{}_f\nabla_d(\beta^f{}_g) = \beta^c{}_f(K_g\nabla_d n^f + n_g\nabla_d K^f)$ since $\beta^c{}_f K^f = \beta^c{}_f n^f = 0$.
$\therefore$ By equation 2.143 and the subsequent results,

$$R^{(\mathcal{H})c}{}_{dab}X^d$$
$$= [D_a, D_b]X^c \tag{2.149}$$
$$= \beta^d{}_a\beta^e{}_b\beta^c{}_f[\nabla_d,\nabla_e]X^f + 2\beta^d{}_a\beta^e{}_b\beta^c{}_f(K_g\nabla_{[d}(n^f)\nabla_{e]}X^g + n_g\nabla_{[d}(K^f)\nabla_{e]}X^g) \tag{2.150}$$
$$= \beta^d{}_a\beta^e{}_b\beta^c{}_f[\nabla_d,\nabla_e]X^f$$
$$- 2\beta^d{}_a\beta^e{}_b\beta^c{}_f X_g(\nabla_{[d}(n^f)\nabla_{e]}K^g + \nabla_{[d}(K^f)\nabla_{e]}n^g) \text{ as } K^aX_a = n^aX_a = 0 \tag{2.151}$$
$$= \beta^d{}_a\beta^e{}_b\beta^c{}_f R^f{}_{gde}X^g - 2\beta^d{}_a\beta^e{}_b\beta^c{}_f X_g(\nabla_{[d}(n^f)\nabla_{e]}K^g + \nabla_{[d}(K^f)\nabla_{e]}n^g). \tag{2.152}$$
$$\therefore R^{(\mathcal{H})c}{}_{dab}$$
$$= \beta^g{}_a\beta^e{}_b\beta^c{}_f(R^f{}_{dge} - 2\nabla_{[g}(n^f)\nabla_{e]}K_d - 2\nabla_{[g}(K^f)\nabla_{e]}n_d) \text{ as } X^a \text{ is arbitrary.} \tag{2.153}$$
$$\therefore R^{(\mathcal{H})} = \beta^{db}R^{(\mathcal{H})a}{}_{dab} \tag{2.154}$$
$$= \beta^{db}\beta^g{}_a\beta^e{}_b\beta^a{}_f(R^f{}_{dge} - 2\nabla_{[g}(n^f)\nabla_{e]}K_d - 2\nabla_{[g}(K^f)\nabla_{e]}n_d) \tag{2.155}$$
$$= \beta^{ac}\beta^{bd}(R_{abcd} - \nabla_a(n_c)\nabla_d(K_b) - \nabla_a(K_c)\nabla_d(n_b) + \nabla_d(n_c)\nabla_a(K_b)$$
$$+ \nabla_d(K_c)\nabla_a(n_b)). \tag{2.156}$$

$H$ is a Killing horizon. Since the expansion, shear and rotation are all zero on a Killing horizon, $\beta^{ac}\beta^{bd}\nabla_c K_d = 0$.

$$\therefore R^{(\mathcal{H})} = \beta^{ac}\beta^{bd}(R_{abcd} - \nabla_a(n_c)\nabla_d(K_b) - \nabla_a(K_c)\nabla_d(n_b)). \tag{2.157}$$

Also observe that $\beta^{ab}\nabla_a K_b = 0$ because $g^{ab}\nabla_a K_b = \nabla_a K^a = 0$ as the expansion is zero, $K^a n^b\nabla_a K_b = 0$ as affinely parameterised implies $K^a\nabla_a K_b = 0$ and $n^aK^b\nabla_a K_b = 0$ by $K^bK_b = 0$.
That leaves $R^{(\mathcal{H})} = \beta^{ac}\beta^{bd}R_{abcd}$. □

**Theorem 2.15** (Hawking and Ellis [16]). *For each black hole in a stationary, regular predictable, four dimensional spacetime with matter satisfying the dominant energy condition, the intersection of the event horizon with a spacelike hypersurface is homeomorphic to either a sphere or a torus.*

*Proof.* Let $\mathcal{H}$ be the intersection of the event horizon, $H$, with a spacelike hypersurface, typically some Cauchy slice, $\Sigma_t$.
$\therefore$ $\mathcal{H}$ is a 2D spacelike surface. $\mathcal{H}$ must be compact, because otherwise $\mathcal{H}$ would be extended and thus incompatible with the $M\backslash J^-(\mathcal{I}^+)$ definition of a black hole.
$\therefore$ I can introduce two null normals, say $K_a$ and $n_a$, to $\mathcal{H}$. Without loss of generality, assume that on $H$, $K^a$ is an affinely parameterised generator of null geodesics and that on $\mathcal{H}$, $K_an^a = -1$. By multiplying both $K^a$ and $n^a$ by $-1$ if required, choose $K^a$ and $n^a$ to both be future directed and with $n^a$ pointing into the region bounded by $\mathcal{H}$.
Consider a null geodesic congruence generated by $n^a$.
Let $\sigma$ be an affine parameter for flows along $n^a$. Let $\Omega$ be the surface generated by flows along $n^a$ and off $\mathcal{H}$.
Let $\mathcal{H}_\sigma$ be the images of the points on $\mathcal{H}$ after flowing by $\sigma$.
Since $\mathcal{H}$ is a spacelike 2-surface, for small enough $\sigma$, so is $\mathcal{H}_\sigma$.
Extend $K^a$ off $H$ as follows.
$\mathcal{H}_\sigma$ is a spacelike 2-surface with $n^a$ as a null normal.
$\therefore$ Enforcing $K^an_a = -1$ on $\mathcal{H}_\sigma$, $K^aK_a = 0$ on $\mathcal{H}_\sigma$ and $K_a$ is normal to $\mathcal{H}_\sigma$ uniquely determines

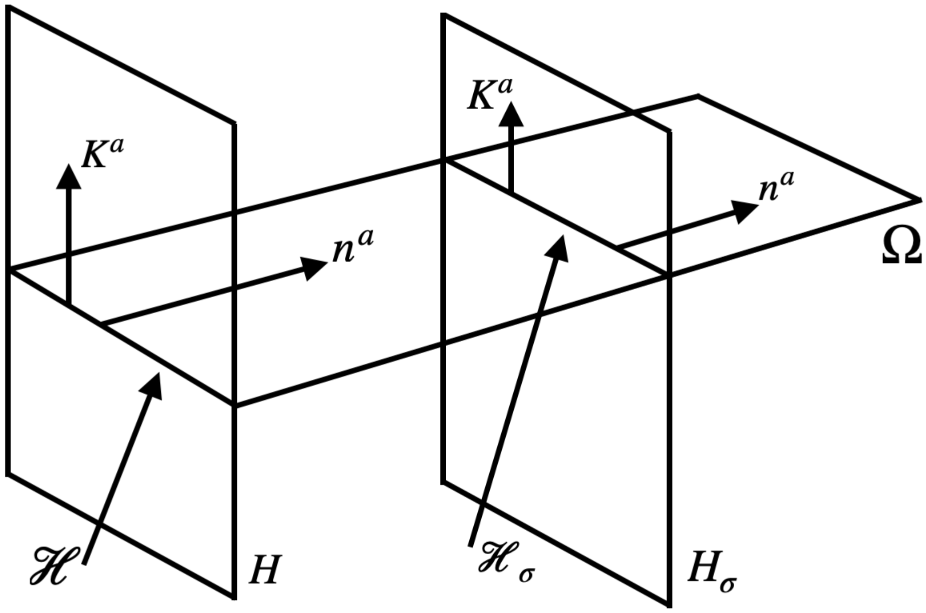


Figure 2.1: The set-up for theorem 2.15.

$K^a$ on $\mathcal{H}_\sigma$. This way, $K^a$ is uniquely determined on $\Omega$.
Extend $K^a$ off $\mathcal{H}_\sigma$ to create a $H_\sigma$ in an arbitrary way while keeping $K^a$ null.
Extend $n^a$ off $\mathcal{H}_\sigma$ and into $H_\sigma$ in any way that keeps $K_a n^a = -1$.
This construction is depicted visually in figure 2.1.
Since I've enforced $K^a n_a = -1$, and $K^a K_a = n^a n_a = 0$ on each $\mathcal{H}_\sigma$, the induced metric on each $\mathcal{H}_\sigma$ must be $\beta_{ab} = g_{ab} + K_a n_b + K_b n_a$.
My construction of the $K^a$ and $n^a$ also imposes a differential constraint on them. To see how, let $T^a$ be an arbitrary tangent vector to $\mathcal{H}_\sigma$.
Adopting the equivalence class of curves definition of tangent vectors, let $T^a = [\gamma(t)]$.
Let $\Phi^\sigma : M \to M$ denote flows along $n^a$. Since $\mathcal{H}_\sigma$ is defined by flowing off the points on $\mathcal{H}$, it follows that $[\gamma(t)]$ goes to $[\Phi^{\sigma'}(\gamma(t))]$ as one goes from $\mathcal{H}_\sigma$ to $\mathcal{H}_{\sigma+\sigma'}$.
Choose $\sigma'$ to be infinitesimally close to 0.
Then, $[\Phi^{\sigma'}(\gamma(t))] = (\Phi^{\sigma'})_*[\gamma(t)] = (\Phi^{-\sigma'})^*[\gamma(t)] = (\Phi^{-\sigma'})^*(T^a) = T^a - \sigma'(\mathcal{L}_n T)^a$.
Since $K_a$ is defined so that it stays normal to $\mathcal{H}_\sigma$, $K_a$ must adapt to these changes in $T^a$.

$$\therefore 0 = K_a(\mathcal{L}_n T)^a \tag{2.158}$$
$$= K_a \left(n^b \nabla_b T^a - T^b \nabla_b n^a\right) \tag{2.159}$$
$$= n^b \nabla_b \left(K_a T^a\right) - T^a n^b \nabla_b K_a - K_a T^b \nabla_b n^a. \tag{2.160}$$

$K_a T^a = 0$ on all $\mathcal{H}_\sigma$ and $n^b \nabla_b$ is a direction derivative between $\mathcal{H}_\sigma$ when acting on a scalar, like $K^a T_a$.

$$\therefore 0 = n^b \nabla_b (K_a T^a). \tag{2.161}$$
$$\therefore 0 = T^a n^b \nabla_b K_a + K_a T^b \nabla_b n^a \tag{2.162}$$
$$= T^a \left(n^b \nabla_b K_a + K_b \nabla_a n^b\right). \tag{2.163}$$

Since $T^a$ is an arbitrary tangent vector, it follows that $n^b \nabla_b K_a + K_b \nabla_a n^b$ must be orthogonal to $\mathcal{H}_\sigma$ $\forall\sigma$. Thus, I get the differential constraint,

$$0 = \beta^{ab}\left(n^c\nabla_c K_b + K_c\nabla_b n^c\right). \tag{2.164}$$

$$\therefore \beta^{ab}n^c\nabla_c K_b = -\beta^{ab}K^c\nabla_b n_c \tag{2.165}$$

$$= \beta^{ab}n^c\nabla_b K_c \quad \text{since I enforced } K^a n_a = -1. \tag{2.166}$$

Let $p^a = \beta^{ab}n^c\nabla_b K_c$. Hence, the identity says $p^a = \beta^{ab}n^c\nabla_b K_c = -\beta^{ab}K^c\nabla_b n_c = \beta^{ab}n^c\nabla_c K_b$. I will first try to prove

$$\begin{aligned}\nabla_g({\beta^e}_c{\beta^d}_f\nabla_d K^c)n^g{\beta^a}_e{\beta^f}_b &= \beta^{ac}\nabla_d(p_c){\beta^d}_b + p^a p_b - {\beta^a}_c\nabla_f(K^c)\beta^{fe}\nabla_d(n_e){\beta^d}_b \\ &\quad + {R^c}_{def}n^e K^d{\beta^a}_c{\beta^f}_b.\end{aligned} \tag{2.167}$$

The LHS expands out to

$$\begin{aligned}\text{LHS} &= \nabla_g({\beta^e}_c){\beta^d}_f\nabla_d(K^c)n^g{\beta^a}_e{\beta^f}_b + {\beta^e}_c\nabla_g({\beta^d}_f)\nabla_d(K^c)n^g{\beta^a}_e{\beta^f}_b \\ &\quad + {\beta^e}_c{\beta^d}_f\nabla_g\nabla_d(K^c)n^g{\beta^a}_e{\beta^f}_b \end{aligned}\tag{2.168}$$

$$= n^f\nabla_f({\beta^e}_c)\nabla_d(K^c){\beta^a}_e{\beta^d}_b + n^f\nabla_f({\beta^d}_e)\nabla_d(K^c){\beta^a}_c{\beta^e}_b + n^e\nabla_e\nabla_d(K^c){\beta^a}_c{\beta^d}_b. \tag{2.169}$$

In the 1st term, if $K^e$ or $n^e$ don't get differentiated, then ${\beta^a}_e K^e = {\beta^a}_e n^e = 0$.
However, when $n^e$ does get differentiated, $n^f\nabla_f n^e = 0$ from $n^e$ being tangent to affinely parameterised null geodesics.
$\therefore$ The first term in equation 2.169 can be simplified to

$$n^f\nabla_f({\beta^e}_c)\nabla_d(K^c){\beta^a}_e{\beta^d}_b = n^f n_c\nabla_f(K^e)\nabla_d(K^c){\beta^a}_e{\beta^d}_b \tag{2.170}$$

$$= p^a p_b. \tag{2.171}$$

The 3rd term in equation 2.169 is

$$n^e\nabla_e\nabla_d\left(K^c\right){\beta^a}_c{\beta^d}_b = n^e\nabla_d\nabla_e\left(K^c\right){\beta^a}_c{\beta^d}_b + n^e[\nabla_e,\nabla_d]\left(K^c\right){\beta^a}_c{\beta^d}_b \tag{2.172}$$

$$= n^e\nabla_d\nabla_e\left(K^c\right){\beta^a}_c{\beta^d}_b + {R^c}_{fed}n^e K^f{\beta^a}_c{\beta^d}_b. \tag{2.173}$$

$\therefore$ So far, my analysis of equation 2.167 simplifies to

$$\begin{aligned}\text{LHS} - \text{RHS} &= n^f\nabla_f({\beta^d}_e)\nabla_d(K^c){\beta^a}_c{\beta^e}_b + n^e\nabla_d\nabla_e(K^c){\beta^a}_c{\beta^d}_b - \beta^{ac}\nabla_d(p_c){\beta^d}_b \\ &\quad + \nabla_f(K^c)\nabla_d(n_e){\beta^a}_c\beta^{ef}{\beta^d}_b \end{aligned}\tag{2.174}$$

$$\begin{aligned}&= n^f\nabla_f({\beta^d}_e)\nabla_d(K^c){\beta^a}_c{\beta^e}_b + n^e\nabla_d\nabla_e(K^c){\beta^a}_c{\beta^d}_b - \beta^{ac}\nabla_d({\beta_c}^e n^f\nabla_f K_e){\beta^d}_b \\ &\quad + \nabla_f(K^c)\nabla_d(n_e){\beta^a}_c\beta^{ef}{\beta^d}_b \end{aligned}\tag{2.175}$$

$$\begin{aligned}&= n^f\nabla_f({\beta^d}_e)\nabla_d(K^c){\beta^a}_c{\beta^e}_b + n^e\nabla_d\nabla_e(K^c){\beta^a}_c{\beta^d}_b - \beta^{ac}\nabla_d({\beta_c}^e)n^f\nabla_f(K_e){\beta^d}_b \\ &\quad - \beta^{ac}{\beta_c}^e\nabla_d(n^f)\nabla_f(K_e){\beta^d}_b - \beta^{ac}{\beta_c}^e n^f\nabla_d\nabla_f(K_e){\beta^d}_b \\ &\quad + \nabla_f(K^c)\nabla_d(n_e){\beta^a}_c\beta^{ef}{\beta^d}_b \end{aligned}\tag{2.176}$$

$$\begin{aligned}&= n^f\nabla_f({\beta^d}_e)\nabla_d(K^c){\beta^a}_c{\beta^e}_b - \nabla_d({\beta^c}_e)n^f\nabla_f(K^e){\beta^a}_c{\beta^d}_b - \nabla_d(n^e)\nabla_e(K^c){\beta^a}_c{\beta^d}_b \\ &\quad + \nabla_f(K^c)\nabla_d(n_e){\beta^a}_c\beta^{ef}{\beta^d}_b. \end{aligned}\tag{2.177}$$

In the 1st term, if $K_e$ or $n_e$ don't get differentiated, then ${\beta^e}_b K_e = {\beta^e}_b n_e = 0$. Also, if $n_e$ does get differentiated, then $n^f\nabla_f n_e = 0$ from the geodesic property.
$\therefore n^f\nabla_f({\beta^d}_e)\nabla_d(K^c){\beta^a}_c{\beta^e}_b = n^f n^d\nabla_f\left(K_e\right)\nabla_d\left(K^c\right){\beta^a}_c{\beta^e}_b$.
Then, $K^e K_e = 0 \Rightarrow K^e\nabla_f K_e = 0$, removing the $K^e n_b$ term from ${\beta^e}_b$.
For the $n^e K_b$ term, $n^f\nabla_f(K_e)n^e = -n^f\nabla_f(n^e)K_e = 0$.

$\therefore n^f \nabla_f(\beta^d{}_e)\nabla_d(K^c)\beta^a{}_c\beta^e{}_b = n^f n^d \nabla_f\left(K_e\right)\nabla_d\left(K^c\right)\beta^a{}_c\delta^e{}_b = n^f n^d \nabla_f\left(K_b\right)\nabla_d\left(K^c\right)\beta^a{}_c$
Similarly, in the 2nd term of equation 2.177, if $K^c$ or $n^c$ don't get differentiated, then $\beta^a{}_c K^c$ and $\beta^a{}_c n^c$ equal 0.

$$\begin{aligned}
\therefore \nabla_d(\beta^c{}_e)n^f\nabla_f(K^e)\beta^a{}_c\beta^d{}_b &= n_e\nabla_d(K^c)n^f\nabla_f(K^e)\beta^a{}_c\beta^d{}_b + K_e\nabla_d(n^c)n^f\nabla_f(K^e)\beta^a{}_c\beta^d{}_b && (2.178)\\
&= n_e\nabla_d(K^c)n^f\nabla_f(K^e)\beta^a{}_c\beta^d{}_b + 0 \quad \text{from } k_e\nabla_f K^e = 0 && (2.179)\\
&= -K_e\nabla_d(K^c)n^f\nabla_f(n^e)\beta^a{}_c\beta^d{}_b && (2.180)\\
&= 0 \quad \text{from the geodsic property.} && (2.181)
\end{aligned}$$

Hence, equation 2.177 simplifies to

$$\begin{aligned}
&\text{LHS} - \text{RHS}\\
&= n^f n^d \nabla_f\left(K_b\right)\nabla_d\left(K^c\right)\beta^a{}_c - \nabla_d(n^e)\nabla_e(K^c)\beta^a{}_c\beta^d{}_b\\
&\quad + \nabla_f(K^c)\nabla_d(n_e)\beta^a{}_c\beta^{ef}\beta^d{}_b && (2.182)\\
&= n^f n^d \nabla_f\left(K_b\right)\nabla_d\left(K^c\right)\beta^a{}_c - \nabla_d(n^e)\nabla_e(K^c)\beta^a{}_c\beta^d{}_b\\
&\quad + \nabla_f(K^c)\nabla_d(n_e)\beta^a{}_c\beta^d{}_b(g^{ef} + K^e n^f + K^f n^e) && (2.183)\\
&= n^f n^d \nabla_f\left(K_b\right)\nabla_d\left(K^c\right)\beta^a{}_c - \nabla_d(n^e)\nabla_e(K^c)\beta^a{}_c\beta^d{}_b + \nabla_e(K^c)\nabla_d(n_e)\beta^a{}_c\beta^d{}_b\\
&\quad + K^e n^f\nabla_f(K^c)\nabla_d(n_e)\beta^a{}_c\beta^d{}_b + 0 \quad \text{from } n^e\nabla_d n_e = 0 && (2.184)\\
&= n^f n^d \nabla_f\left(K_b\right)\nabla_d\left(K^c\right)\beta^a{}_c + K^e n^f\nabla_f(K^c)\nabla_d(n_e)\beta^a{}_c\beta^d{}_b && (2.185)\\
&= n^f n^d \nabla_f\left(K_b\right)\nabla_d\left(K^c\right)\beta^a{}_c - n^e n^f\nabla_f(K^c)\nabla_e(K_d)\beta^a{}_c\beta^d{}_b \text{ by the } p_b \text{ expressions.} && (2.186)
\end{aligned}$$

Now, in the 2nd term, $K^d\nabla_e K_d = 0$ and $n^e\nabla_e\left(K_d\right)n^d = -n^e\nabla_e\left(n_d\right)K^d = 0$
$\therefore \text{LHS} - \text{RHS} = n^f n^d \nabla_f\left(K_b\right)\nabla_d\left(K^c\right)\beta^a{}_c - n^e n^f\nabla_f(K^c)\nabla_e(K_d)\beta^a{}_c\delta^d{}_b = 0.$
In summary, the proposition of equation 2.167 is true.
Next, contract both sides of equation 2.167 with $\beta^b{}_a$.

$$\begin{aligned}
\therefore \text{LHS} &= \beta^b{}_a\nabla_g(\beta^e{}_c\beta^d{}_f\nabla_d K^c)n^g\beta^a{}_e\beta^f{}_b && (2.187)\\
&= \beta^b{}_a n^e\nabla_e(\beta^a{}_c\beta^d{}_b\nabla_d K^c) && (2.188)\\
&= n^e\nabla_e(\beta^b{}_a\beta^a{}_c\beta^d{}_b\nabla_d K^c) - n^e\nabla_e(\beta^b{}_a)\beta^a{}_c\beta^d{}_b\nabla_d K^c && (2.189)\\
&= n^e\nabla_e(\beta^d{}_c\nabla_d K^c) - n^e\nabla_e(\beta^b{}_a)\beta^a{}_c\beta^d{}_b\nabla_d K^c. && (2.190)
\end{aligned}$$

On the horizon, $H$, $\beta^a{}_c\beta^d{}_b\nabla_d K^c = 0$ because the expansion, shear and rotation all vanish on a Killing horizon. Furthermore, in general, $\beta^d{}_c\nabla_d K^c = \theta_K$, the expansion along the $K^a$ direction. Hence, on $H$, I'm left with LHS $= n^a\nabla_a\theta_K = \frac{\mathrm{d}\theta_K}{\mathrm{d}\sigma}$.
On the RHS, I'll go term by term.

$$\begin{aligned}
\beta^{ac}\nabla_d(p_c)\beta^d{}_b\beta^b{}_a &= \beta^{ab}\nabla_a p_b. && (2.191)\\
p^a p_b\beta^b{}_a &= \beta^{ac}n^d\nabla_c(K_d)p_b\beta^b{}_a = \beta^{bc}n^d\nabla_c(K_d)p_b = p^a p_a. && (2.192)\\
\beta^a{}_c\nabla_f(K^c)\beta^{fe}\nabla_d(n_e)\beta^d{}_b\beta^b{}_a &= \beta^c{}_a\beta^b{}_d\nabla_c(K^d)\nabla_b(n^a) && (2.193)\\
&= 0 \text{ on } H, \text{ as above.} && (2.194)\\
R^c{}_{def}n^e K^d\beta^a{}_c\beta^f{}_b\beta^b{}_a &= R^a{}_{bcd}n^c K^b\beta^d{}_a && (2.195)\\
&= R^a{}_{bcd}n^c K^b(\delta^d{}_a + n^d K_a + K^d n_a) && (2.196)\\
&= -R_{ab}K^a n^b + 0 + R_{abcd}K^a K^c n^b n^d. && (2.197)
\end{aligned}$$

Hence, putting all the pieces together, on $H$ I have

$$\frac{\mathrm{d}\theta_K}{\mathrm{d}\sigma} = \beta^{ab}\nabla_a p_b + p^a p_a - R_{ab}K^a n^b + R_{abcd}K^a K^c n^b n^d. \qquad (2.198)$$

Next, let $K' = \mathrm{e}^f K$ and $n' = \mathrm{e}^{-f} n$ for some function, $f$.
$\therefore K^a n_a = -1, K^a K_a = 0$ and $n^a n_a = 0$ remain true everywhere.
In what follows, only $f$'s value and variation on $\mathcal{H}$ will matter, so I'll assume without loss of generality that $n^a \nabla_a f = K^a \nabla_a f = 0$.
$\therefore n'^a \nabla_a n'^b = \mathrm{e}^{-f} n^a \nabla_a \left(\mathrm{e}^{-f} n^b\right) = \mathrm{e}^{-2f} n^a \nabla_a n^b - \mathrm{e}^{-2f} n^b n^a \nabla_a f = 0 + 0 = 0$.
$\therefore n'^a$ remains a tangent to affinely parameterised geodesics.
Products like $K^a n^b$ remain invariant because the $\mathrm{e}^f$ and $\mathrm{e}^{-f}$ cancel.
The shear, rotation and expansion likewise remain zero on $\mathcal{H}$ because
$\beta^a{}_c \beta^d{}_b \nabla_d (K'^c) = \beta^a{}_c \beta^d{}_b \nabla_d (\mathrm{e}^f K^c) = \mathrm{e}^f \beta^a{}_c \beta^d{}_b \nabla_d (K^c) + \mathrm{e}^f \beta^a{}_c \beta^d{}_b K^c \nabla_d (f) = 0 + 0 = 0$.
$\therefore$ On $H$, I can use equation 2.198 to say

$$\frac{\mathrm{d}\theta_{K'}}{\mathrm{d}\sigma'} = \beta^{ab} \nabla_a p'_b + p'^a p'_a - R_{ab} K^a n^b + R_{abcd} K^a K^c n^b n^d. \tag{2.199}$$

Re-writing in terms of the old variables,
$p'^a = \beta^{ab} n'^c \nabla_b K'_c = \beta^{ab} \mathrm{e}^{-f} n^c \nabla_b \left(\mathrm{e}^f K_c\right) = p^a + \beta^{ab} n^c K_c \nabla_b (f) = p^a - \beta^{ab} \nabla_b f$.
Then, $K^b \nabla_b f = n^b \nabla_b f = 0 \Rightarrow p'^a = p^a - g^{ab} \nabla_b f = p^a - \nabla^a f$.
$\therefore$ On $\mathcal{H}$,

$$\frac{\mathrm{d}\theta_{K'}}{\mathrm{d}\sigma'} = \beta^{ab} \nabla_a p_b - \beta^{ab} \nabla_a \nabla_b f + p'^a p'_a - R_{ab} K^a n^b + R_{abcd} K^a K^c n^b n^d. \tag{2.200}$$

Let $D$ be the (Levi-Civita) covariant derivative on $\mathcal{H}$.
$\therefore$ From the properties of the induced metric, $D^a D_a f = \beta^a{}_b \beta^c{}_a \nabla^b \nabla_c f = \beta^{ab} \nabla_a \nabla_b f$. So, on $\mathcal{H}$,

$$\frac{\mathrm{d}\theta_{K'}}{\mathrm{d}\sigma'} = \beta^{ab} \nabla_a p_b - D^a D_a f + p'^a p'_a - R_{ab} K^a n^b + R_{abcd} K^a K^c n^b n^d. \tag{2.201}$$

$D^a D_a f$ is the Laplacian of $f$ on $\mathcal{H}$.
Since $\mathcal{H}$ is compact, there applies a theorem of Hodge that for a function, $F$, $\exists f$ such that $D^a D_a f = F$ if and only if $\int_{\mathcal{H}} F \, \mathrm{d}A = 0$.
Let $\langle F \rangle = \frac{1}{A} \int_{\mathcal{H}} F \, \mathrm{d}A$, with the area, $A$, well-defined because $\mathcal{H}$ is compact. Note that $\langle F \rangle$ is just a constant on $\mathcal{H}$.
$\therefore \int_{\mathcal{H}} (F - \langle F \rangle) \mathrm{d}A = \int_{\mathcal{H}} F \mathrm{d}A - \langle F \rangle \int_{\mathcal{H}} \mathrm{d}A = A\langle F \rangle - A\langle F \rangle = 0$.
$\therefore \exists f$ such that $D^a D_a f - F$ is a constant (namely $-\langle F \rangle$) for any $F$.
$\therefore$ I can choose $f$ in equation 2.201 so that $\beta^{ab} \nabla_a p_b - D^a D_a f - R_{ab} K^a n^b + R_{abcd} K^a K^c n^b n^d$ is a constant on $\mathcal{H}$.
$\therefore \frac{\mathrm{d}\theta_{K'}}{\mathrm{d}\sigma'} - p'^a p'_a$ is constant on $\mathcal{H}$ with that choice of $f$.
I can now finally get towards the topology part of the proof. Let $\mathcal{H}_0$ be an arbitrary connected component of $\mathcal{H}$, i.e. $\mathcal{H}_0$ is what one intuitively thinks of as the boundary of an individual black hole. Since $\mathcal{H}_0$ is a 2D compact, connected, closed surface, the Gauss-Bonnet theorem says $\chi(\mathcal{H}_0) = \frac{1}{4\pi} \int_{\mathcal{H}_0} R^{(\mathcal{H})} \mathrm{d}A$. By lemma 2.14,

$$\begin{aligned} R^{(\mathcal{H})} &= \beta^{ac} \beta^{bd} R_{abcd} && (2.202) \\ &= (g^{ac} + K^a n^c + K^c n^a)(g^{bd} + K^b n^d + K^d n^b) R_{abcd} && (2.203) \\ &= R + R_{bd} K^b n^d + R_{bd} K^d n^b + R_{ac} K^a n^c + R_{abcd} K^a K^b n^c n^d + R_{abcd} K^a K^d n^c n^b \\ &\quad + R_{ac} K^c n^a + R_{abcd} K^c K^b n^a n^d + R_{abcd} K^c K^d n^a n^b && (2.204) \\ &= R + 4 R_{ab} K^a n^b - 2 R_{abcd} K^a K^c n^b n^d. && (2.205) \end{aligned}$$

This expression is invariant under the $\mathrm{e}^{\pm f}$ scalings I performed earlier, so I can use it liberally.
The Einstein equation says $R_{ab} - \frac{1}{2} R g_{ab} = 8\pi T_{ab}$.

$\therefore$ Contracting with $K^a n^b$ and using $K^a n_a = -1$, $R_{ab}K^a n^b + \frac{1}{2}R = 8\pi T_{ab}K^a n^b$.

$$\therefore R^{(\mathcal{H})} = 16\pi T_{ab}K^a n^b + 2(R_{ab}K^a n^b - R_{abcd}K^a K^c n^b n^d). \tag{2.206}$$

$$\therefore \chi(\mathcal{H}_0) = 4\int_{\mathcal{H}_0} T_{ab}K^a n^b \, \mathrm{d}A + \frac{1}{2\pi}\int_{\mathcal{H}_0} (R_{ab}K^a n^b - R_{abcd}K^a K^c n^b n^d)\mathrm{d}A. \tag{2.207}$$

Next, observe that since $\mathcal{H}_0$ has no boundary of its own, $\int_{\mathcal{H}_0} D^a D_a f \, \mathrm{d}A = 0$ by Stokes' theorem. Likewise, in proving $\beta^b{}_a p^a p_b = p^a p_a$ earlier, I actually showed $\beta^a{}_b p^b = p^a$.
$\therefore D^a p_a = \beta^a{}_c \beta^b{}_a \nabla^c p_b = \beta^{ab}\nabla_a p_b \implies \int_{\mathcal{H}_0} \beta^{ab}\nabla_a p_b \, \mathrm{d}A = \int_{\mathcal{H}_0} D^a p_a \, \mathrm{d}A = 0$ too.

$$\begin{aligned}\therefore \chi(\mathcal{H}_0) &= 4\int_{\mathcal{H}_0} T_{ab}K^a n^b \, \mathrm{d}A \\ &\quad + \frac{1}{2\pi}\int_{\mathcal{H}_0} (R_{ab}K^a n^b - R_{abcd}K^a K^c n^b n^d + D^a D_a f - \beta^{ab}\nabla_a p_b)\mathrm{d}A \qquad (2.208) \\ &= 4\int_{\mathcal{H}_0} T_{ab}K^a n^b \, \mathrm{d}A + \frac{1}{2\pi}\int_{\mathcal{H}_0} \left(p'_a p'^a - \frac{\mathrm{d}\theta_{K'}}{\mathrm{d}\sigma}'\right) \mathrm{d}A \qquad (2.209) \\ &= 4\int_{\mathcal{H}_0} T_{ab}K^a n^b \, \mathrm{d}A + \frac{A_0}{2\pi}\left(p'_a p'^a - \frac{\mathrm{d}\theta_{K'}}{\mathrm{d}\sigma'}\right), \qquad (2.210)\end{aligned}$$

where $A_0$ is the area of $\mathcal{H}_0$.
Since $K^a$ and $n^a$ are null vectors and I've assumed the dominant energy condition, $T_{ab}K^a n^b \geq 0$.
$\therefore$ The only way $\chi(\mathcal{H}_0)$ can be negative is if $\frac{\mathrm{d}\theta_{K'}}{\mathrm{d}\sigma'} > p'_a p'^a$.
$\beta^a{}_b p^b = p^a$ and $\beta$ being a projection operator means $p^a$ is a valid tensor on $\mathcal{H}_0$. Since $\mathcal{H}_0$ is spacelike, it follows that $p'_a p'^a \geq 0$.
By construction, $n^a$ pointed into the black hole region. Hence, negative values of $\sigma$ correspond to points outside the black hole.
$\therefore \theta_{K'} = 0$ on a Killing horizon - as $H$ is - and $\frac{\mathrm{d}\theta_{K'}}{\mathrm{d}\sigma'} > 0$ on $\mathcal{H}_0 \implies \theta_{K'} < 0$ at all points just off $\mathcal{H}_0$ outside the black hole.
$\therefore$ There is an outer trapped surface.
However, in a regular predictable spacetime, outer trapped surfaces must be contained inside the black hole region.
$\therefore \chi(\mathcal{H}_0) \geq 0$.
Since the only compact, closed, connected 2D surfaces with non-negative Euler characteristic are the sphere and the torus, the proof is complete. □

The torus case has subsequently been ruled out using "topological censorship" methods - e.g. see [21, 22, 23]. However, these methods are completely different to what I've built so far in this chapter, my understanding of those topics is a little unsatisfactory and it's also quite the diversion to explain topological censorship here. Hence, I won't elaborate on those results any further.

# Chapter 3

# The beginning - Israel's original proof

In this chapter I will present a full account of the original black hole uniqueness result, due to Israel [1]. Many of the details of my presentation are taken from [15]. However, across all areas of the literature, it's a little unclear exactly what assumptions are made in proving this theorem, so I've put together what I believe to be a comprehensive list and proof.

**Theorem 3.1** (Israel [1, 15])**.** *Let $(M, g)$ be a spacetime with the following properties:*

1. *$M$ is 4 dimensional.*
2. *$M$ is time orientable.*
3. *$(M, g)$ is asymptotically flat.*
4. *$(M, g)$ is static.*
5. *$\mathrm{d}(k^a k_a) \neq 0$ at all points, where $k^a$ is the Killing vector field making $(M, g)$ static.*
6. *The event horizon is non-empty and connected.*
7. *The energy-momentum tensor is $T_{ab} = 0$.*
8. *$\mathrm{d}S \neq 0$ at all times and in Israel coordinates (see equation 2.59 earlier), constant $t$ and $S$ surfaces are diffeomorphic to spheres.*

*Then, $(M, g)$ is isometric to the Schwarzschild spacetime, with metric*

$$g = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t \otimes \mathrm{d}t + \frac{1}{1 - \frac{2M}{r}}\mathrm{d}r \otimes \mathrm{d}r + r^2\mathrm{d}\theta \otimes \mathrm{d}\theta + r^2 \sin^2(\theta)\mathrm{d}\phi \otimes \mathrm{d}\phi. \tag{3.1}$$

Assumptions 2, 4 and 5 justify the use of adapted coordinates, while assumption 8 justifies the use of Israel coordinates. From assumption 7, the Einstein equation is $R_{ab} = 0$, meaning the equations of motion are $\Box^{(h)}S = 0$ and $R^{(h)}_{ij} = \frac{1}{S}\nabla^{(h)}_i \nabla^{(h)}_j S$, by theorem 2.4. Assumption 3 justifies the use of asymptotics as per definition 2.7. Finally, I will also need the following property.

**Lemma 3.2.** *$0 \leq S < 1$ everywhere on $\Sigma_t$.*

*Proof.* Since $\Box^{(h)}S = 0$, the Hopf principle says $S$ is extremised on the boundaries of $\Sigma_t$. On the inner boundary, $\mathcal{H}$, $S = 0$ and by equation 2.57, $S \to 1^-$ on the outer boundary, $S^{n-2}_\infty$. Hence, the claim follows. □

In summary, the task is to solve the following problem.

**Definition 3.3** (Problem summary)**.** *Given the pair,* $(\Sigma_t, h)$*, the problem studied in this chapter is summarised by the equations,*

$$\Box^{(h)} S = 0 \text{ and } R^{(h)}_{ij} = \frac{1}{S}\nabla^{(h)}_i \nabla^{(h)}_j S, \tag{3.2}$$

*and the boundary conditions,* $S = 0$ *on* $\mathcal{H}$*,* $0 \leq S < 1$ *everywhere and*

$$S = 1 - \frac{m}{2r} = 1 - \frac{M}{r} \;\; \& \;\; h_{ij} = \left(1 + \frac{m}{r}\right)\delta_{ij} = \left(1 + \frac{2M}{r}\right)\delta_{ij} \tag{3.3}$$

*at* $S^{n-2}_\infty$*.*

*Proof of Israel's theorem.* First, I'll need two expressions in Israel coordinates, namely

$$\begin{aligned}
\nabla^{(h)}_i \nabla^{(h)}_j S &= \partial_i \partial_j S - \Gamma^{(h)k}{}_{ji} \nabla^{(h)}_k S &(3.4)\\
&= 0 - \Gamma^{(h)k}{}_{ji}\delta_{k1} \text{ in Israel coordinates} &(3.5)\\
&= -\Gamma^{(h)S}{}_{ji} \text{ and therefore} &(3.6)\\
\Box^{(h)} S &= -h^{ij}\Gamma^{(h)S}{}_{ji} &(3.7)\\
&= -\frac{1}{\rho^2}\frac{1}{\rho}\partial_S \rho + \tilde{h}^{AB}\frac{1}{\rho}K_{AB} \text{ by lemma 2.10} &(3.8)\\
&= -\frac{1}{\rho^3}\partial_S \rho + \frac{K}{\rho}. &(3.9)
\end{aligned}$$

Then, since $\Box^{(h)} S = 0$,

$$\partial_S \rho = \rho^2 K. \tag{3.10}$$

The main substance of the proof proceeds by constructing some seemingly bizarre linear combinations of the Einstein equations that when integrated miraculously conspire to prove the theorem.
The Einstein tensor vanishing means $\frac{1}{S^2}G_{tt} + \frac{1}{\rho^2}G_{SS} = 0$ too.

$$\begin{aligned}
\therefore 0 &= \frac{1}{2}R^{(h)} + \frac{1}{\rho^2}G^{(h)}_{SS} + \frac{1}{\rho^2 S}(h_{SS}\Box^{(h)} S - \nabla^{(h)}_S \nabla^{(h)}_S S) \text{ by corollary 2.4.2} &(3.11)\\
&= \frac{1}{2}R^{(h)} + \frac{1}{\rho^2}G^{(h)}_{SS} + \frac{1}{\rho^2 S}\left(\rho^2\left(-\frac{1}{\rho^3}\partial_S\rho + \frac{K}{\rho}\right) + \frac{1}{\rho}\partial_S \rho\right) \text{ by equations 3.6 and 3.9} &(3.12)\\
&= \frac{1}{2}R^{(h)} + \frac{1}{\rho^2}G^{(h)}_{SS} + \frac{1}{\rho S}K &(3.13)\\
&= \frac{1}{2}\left(R^{(\tilde{h})} - K^2 - K_{AB}K^{AB} - \frac{2}{\rho}\Box^{(\tilde{h})}\rho - \frac{2}{\rho}\partial_S K\right)\\
&\quad + \frac{1}{2}(-R^{(\tilde{h})} + K^2 - K_{AB}K^{AB}) + \frac{1}{\rho S}K \text{ by corollaries 2.10.2 and 2.10.3} &(3.14)\\
&= -K_{AB}K^{AB} - \frac{1}{\rho}\Box^{(\tilde{h})}\rho - \frac{1}{\rho}\partial_S K + \frac{1}{\rho S}K. &(3.15)
\end{aligned}$$

The $\Box^{(\tilde{h})}$ term is re-written by

$$\frac{2}{\sqrt{\rho}}\Box^{(\tilde{h})}\sqrt{\rho} = \frac{2}{\sqrt{\rho}}\nabla^{(\tilde{h})A}\left(\frac{1}{2\sqrt{\rho}}\nabla^{(\tilde{h})}_A\rho\right) \tag{3.16}$$

$$= \frac{1}{\rho}\Box^{(\tilde{h})}\rho - \frac{1}{2\rho^2}\nabla^{(\tilde{h})A}(\rho)\nabla^{(\tilde{h})}_A(\rho). \tag{3.17}$$

$$\therefore 0 = -K_{AB}K^{AB} - \frac{2}{\sqrt{\rho}}\Box^{(\tilde{h})}\sqrt{\rho} - \frac{1}{2\rho^2}\nabla^{(\tilde{h})A}(\rho)\nabla^{(\tilde{h})}_A(\rho) - \frac{1}{\rho}\partial_S K + \frac{1}{\rho S}K. \tag{3.18}$$

Let $L_{AB} = K_{AB} - \frac{1}{2}\tilde{h}_{AB}K$ be the traceless part of the extrinsic curvature. In terms of this, the identity I've been working with becomes

$$0 = -\left(L_{AB} + \frac{1}{2}\tilde{h}_{AB}K\right)\left(L^{AB} + \frac{1}{2}\tilde{h}^{AB}K\right) - \frac{2}{\sqrt{\rho}}\Box^{(\tilde{h})}\sqrt{\rho} - \frac{1}{2\rho^2}\nabla^{(\tilde{h})A}(\rho)\nabla^{(\tilde{h})}_A(\rho)$$
$$- \frac{1}{\rho}\partial_S\rho + \frac{1}{\rho S}K \tag{3.19}$$

$$= -L_{AB}L^{AB} - \frac{1}{2}K^2 - \frac{2}{\sqrt{\rho}}\Box^{(\tilde{h})}\sqrt{\rho} - \frac{1}{2\rho^2}\nabla^{(\tilde{h})A}(\rho)\nabla^{(\tilde{h})}_A(\rho) - \frac{1}{\rho}\partial_S K + \frac{1}{\rho S}K. \tag{3.20}$$

The second, equally obscure, identity I'll need is $0 = \frac{1}{S^2}G_{tt} + \frac{3}{\rho^2}G_{SS}$.

$$\therefore 0 = \frac{1}{2}R^{(h)} + \frac{3}{\rho^2}G^{(h)}_{SS} + \frac{3}{\rho^2 S}(h_{SS}\Box^{(h)}S - \nabla^{(h)}_S\nabla^{(h)}_S S) \text{ by corollary 2.4.2} \tag{3.21}$$

$$= \frac{1}{2}R^{(h)} + \frac{3}{\rho^2}G^{(h)}_{SS} + \frac{3}{\rho^2 S}\left(\rho^2\left(-\frac{1}{\rho^3}\partial_S\rho + \frac{1}{\rho}K\right) + \frac{1}{\rho}\partial_S\rho\right) \text{ by equations 3.6 and 3.9} \tag{3.22}$$

$$= \frac{1}{2}R^{(h)} + \frac{3}{\rho^2}G^{(h)}_{SS} + \frac{3}{\rho S}K \tag{3.23}$$

$$= \frac{1}{2}\left(R^{(\tilde{h})} - K^2 - K_{AB}K^{AB} - \frac{2}{\rho}\Box^{(\tilde{h})}\rho - \frac{2}{\rho}\partial_S K\right)$$
$$+ \frac{3}{2}(-R^{(\tilde{h})} + K^2 - K_{AB}K^{AB}) + \frac{3}{\rho S}K \text{ by corollaries 2.10.2 and 2.10.3} \tag{3.24}$$

$$= -R^{(\tilde{h})} + K^2 - 2K_{AB}K^{AB} - \frac{1}{\rho}\Box^{(\tilde{h})}\rho - \frac{1}{\rho}\partial_S K + \frac{3}{\rho S}K \tag{3.25}$$

$$= -R^{(\tilde{h})} - 2L_{AB}L^{AB} - \frac{1}{\rho}\Box^{(\tilde{h})}\rho - \frac{1}{\rho}\partial_S K + \frac{3}{\rho S}K. \tag{3.26}$$

This time, the $\Box^{(\tilde{h})}$ term will be re-written using

$$\Box^{(\tilde{h})}\ln\rho = \nabla^{(\tilde{h})A}\left(\frac{1}{\rho}\nabla^{(\tilde{h})}_A\rho\right) = \frac{1}{\rho}\Box^{(\tilde{h})}\rho - \frac{1}{\rho^2}\nabla^{(\tilde{h})A}(\rho)\nabla^{(\tilde{h})}_A\rho. \tag{3.27}$$

$$\therefore 0 = -R^{(\tilde{h})} - 2L_{AB}L^{AB} - \Box^{(\tilde{h})}\ln\rho - \frac{1}{\rho^2}\nabla^{(\tilde{h})A}(\rho)\nabla^{(\tilde{h})}_A\rho - \frac{1}{\rho}\partial_S K + \frac{3}{\rho S}K. \tag{3.28}$$

Next, observe that

$$-\frac{S\sqrt{\rho}}{\sqrt{\tilde{h}}}\partial_S\left(\frac{K\sqrt{\tilde{h}}}{S\sqrt{\rho}}\right) = -\frac{S\sqrt{\rho}}{\sqrt{\tilde{h}}}\left(\frac{\sqrt{\tilde{h}}}{S\sqrt{\rho}}\partial_S K - \frac{K\sqrt{\tilde{h}}}{S^2\sqrt{\rho}} + \frac{K\partial_S\sqrt{\tilde{h}}}{S\sqrt{\rho}} - \frac{K\sqrt{\tilde{h}}\partial_S\rho}{2S\rho^{3/2}}\right) \tag{3.29}$$

$$= -\partial_S K + \frac{K}{S} - \frac{K\partial_S\sqrt{\tilde{h}}}{\sqrt{\tilde{h}}} + \frac{K\partial_S\rho}{2\rho} \tag{3.30}$$

$$= -\partial_S K + \frac{K}{S} - \rho K^2 + \frac{1}{2}\rho K^2 \text{ by equations 3.10 and 2.64} \tag{3.31}$$

$$= -\partial_S K + \frac{K}{S} - \frac{1}{2}\rho K^2. \tag{3.32}$$

Substituting this back into equation 3.20, I get

$$L_{AB}L^{AB} + \frac{1}{2\rho^2}\nabla^{(\tilde{h})A}(\rho)\nabla^{(\tilde{h})}_A(\rho) = -\frac{2}{\sqrt{\rho}}\Box^{(\tilde{h})}\sqrt{\rho} - \frac{S}{\sqrt{\rho\tilde{h}}}\partial_S\left(\frac{K\sqrt{\tilde{h}}}{S\sqrt{\rho}}\right). \tag{3.33}$$

$$\therefore -\frac{2\sqrt{\tilde{h}}}{S}\Box^{(\tilde{h})}\sqrt{\rho} \geq \partial_S\left(\frac{K\sqrt{\tilde{h}}}{S\sqrt{\rho}}\right) \text{ as } \tilde{h}_{AB} \text{ is Riemannian,} \tag{3.34}$$

with equality if and only if $L_{AB} = 0$ and $\nabla^{(\tilde{h})}_A\rho = 0$.
Similarly, again using $\partial_S\sqrt{\tilde{h}} = \sqrt{\tilde{h}}K\rho$ and $\partial_S\rho = \rho^2 K$ from equations 2.64 and 3.10 along the way, observe that

$$\partial_S\left(\frac{KS\sqrt{\tilde{h}}}{\rho} + \frac{4\sqrt{\tilde{h}}}{\rho^2}\right) = \frac{K\sqrt{\tilde{h}}}{\rho} + \frac{S\sqrt{\tilde{h}}\partial_S K}{\rho} - K^2S\sqrt{\tilde{h}} + K^2S\sqrt{\tilde{h}} + \frac{4K\sqrt{\tilde{h}}}{\rho} - \frac{8K\sqrt{\tilde{h}}}{\rho} \tag{3.35}$$

$$= -\frac{3K\sqrt{\tilde{h}}}{\rho} + \frac{S\sqrt{\tilde{h}}\partial_S K}{\rho}. \tag{3.36}$$

Substituting this into equation 3.28, I get

$$0 = -R^{(\tilde{h})} - 2L_{AB}L^{AB} - \Box^{(\tilde{h})}\ln\rho - \frac{1}{\rho^2}\nabla^{(\tilde{h})A}(\rho)\nabla^{(\tilde{h})}_A\rho - \frac{1}{S\sqrt{\tilde{h}}}\partial_S\left(\frac{KS\sqrt{\tilde{h}}}{\rho} + \frac{4\sqrt{\tilde{h}}}{\rho^2}\right). \tag{3.37}$$

$$\therefore \partial_S\left(\frac{KS\sqrt{\tilde{h}}}{\rho} + \frac{4\sqrt{\tilde{h}}}{\rho^2}\right) \leq -S\sqrt{\tilde{h}}(R^{(\tilde{h})} + \Box^{(\tilde{h})}\ln\rho), \tag{3.38}$$

where, again, equality occurs if and only if $L_{AB} = 0$ and $\nabla^{(\tilde{h})}_A\rho = 0$.
Then, by equation 3.34,

$$\int_{\Sigma_t}\partial_S\left(\frac{K\sqrt{\tilde{h}}}{S\sqrt{\rho}}\right)\mathrm{d}S\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3 \leq -2\int_{\Sigma_t}\frac{\sqrt{\tilde{h}}}{S}\Box^{(\tilde{h})}\sqrt{\rho}\,\mathrm{d}S\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3. \tag{3.39}$$

$$\therefore \int\int_0^1\partial_S\left(\frac{K\sqrt{\tilde{h}}}{S\sqrt{\rho}}\right)\mathrm{d}S\mathrm{d}x^2\wedge\mathrm{d}x^3 \leq -2\int_0^1\frac{1}{S}\int\sqrt{\tilde{h}}\Box^{(\tilde{h})}(\sqrt{\rho})\,\mathrm{d}x^2\wedge\mathrm{d}x^3\mathrm{d}S. \tag{3.40}$$

$$\therefore \int\left[\frac{K\sqrt{\tilde{h}}}{S\sqrt{\rho}}\right]_0^1\mathrm{d}x^2\wedge\mathrm{d}x^3 \leq -2\int_0^1\frac{1}{S}\int\Box^{(\tilde{h})}(\sqrt{\rho})\varepsilon^{(\tilde{h})}\,\mathrm{d}S \tag{3.41}$$

$$= 0, \tag{3.42}$$

with the last line following by Stokes'/divergence theorem and spheres having no boundary. Similarly, by equation 3.38,

$$\int_{\Sigma_t} \partial_S\left(\frac{KS\sqrt{\tilde{h}}}{\rho} + \frac{4\sqrt{\tilde{h}}}{\rho^2}\right)\mathrm{d}S \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$$

$$\leq -\int_{\Sigma_t} S\sqrt{\tilde{h}}(R^{(\tilde{h})} + \Box^{(\tilde{h})}\ln\rho)\mathrm{d}S \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3. \tag{3.43}$$

$$\therefore \int \left[\frac{KS\sqrt{\tilde{h}}}{\rho} + \frac{4\sqrt{\tilde{h}}}{\rho^2}\right]_0^1 \mathrm{d}x^2 \wedge \mathrm{d}x^3$$

$$\leq -\int_0^1 S \int \sqrt{\tilde{h}}(R^{(\tilde{h})} + \Box^{(\tilde{h})}\ln\rho)\mathrm{d}x^2 \wedge \mathrm{d}x^3\mathrm{d}S \tag{3.44}$$

$$= -\int_0^1 S \int \sqrt{\tilde{h}}R^{(\tilde{h})}\mathrm{d}x^2 \wedge \mathrm{d}x^3\mathrm{d}S \quad \text{by the Stokes}'\text{/divergence theorem again} \tag{3.45}$$

$$= -8\pi \int_0^1 S\,\mathrm{d}S \quad \text{by the Gauss} - \text{Bonnet theorem} \tag{3.46}$$

$$= -4\pi. \tag{3.47}$$

Now I have to evaluate the integrals on the LHSs of these inequalities. First consider $S = 0$, the event horizon. Then, by lemma 2.12, $\rho = 1/\kappa$. From equation 3.13,

$$\frac{K}{\rho S} = -\frac{1}{2}R^{(h)} - \frac{1}{\rho^2}G^{(h)}_{SS} \tag{3.48}$$

$$= -\frac{1}{\rho^2}G^{(h)}_{SS} \text{ since the Einstein equation implies } R^{(h)} = 0 \tag{3.49}$$

$$= -\frac{1}{2}(-R^{(\tilde{h})} + K^2 - K_{AB}K^{AB}) \text{ by corollary 2.10.3.} \tag{3.50}$$

By corollary 2.13.1, it follows that $\frac{K}{\rho S} = \frac{1}{2}R^{(\tilde{h})}$ on the event horizon.

$$\therefore \int \left.\frac{K\sqrt{\tilde{h}}}{S\sqrt{\rho}}\right|_{S=0} \mathrm{d}x^2 \wedge \mathrm{d}x^3 = \frac{1}{2\sqrt{\kappa}}\int R^{(\tilde{h})}\sqrt{\tilde{h}}\,\mathrm{d}x^2 \wedge \mathrm{d}x^3 \tag{3.51}$$

$$= \frac{4\pi}{\sqrt{\kappa}} \quad \text{by the Gauss} - \text{Bonnet theorem.} \tag{3.52}$$

For the other integral inequality,

$$\int \left.\left(\frac{KS\sqrt{\tilde{h}}}{\rho} + \frac{4\sqrt{\tilde{h}}}{\rho^2}\right)\right|_{S=0} \mathrm{d}x^2 \wedge \mathrm{d}x^3 = \int (0 + 4\kappa^2\sqrt{\tilde{h}})\mathrm{d}x^2 \wedge \mathrm{d}x^3 = 4\kappa^2 A \tag{3.53}$$

where $A$ is the area of the event horizon.
The other part is $S = 1$, the asymptotically flat end.
By definition 2.7, to leading order

$$S = 1 - \frac{M}{r}. \tag{3.54}$$

By definition, $1/\rho^2 = \nabla^{(h)}_i(S)\nabla^{(h)i}(S)$. Near infinity, I can raise and lower indices with $\delta$ and I can use the asymptotically Cartesian coordinates.

$\therefore \nabla^{(h)}_i S \equiv \mathrm{d}S = \frac{M}{r^2}\mathrm{d}r \implies \nabla^{(h)}_i(S)\nabla^{(h)i}(S) = \frac{M^2}{r^4}h^{rr} = \frac{M^2}{r^4} \implies \rho = r^2/M.$
Then, for the extrinsic curvature, by equation 3.10,

$$K = \frac{\partial_S \rho}{\rho^2} = \frac{1}{\mathrm{d}S/\mathrm{d}r}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{M}\right)\frac{M^2}{r^4} = \frac{r^2}{M}\frac{2r}{M}\frac{M^2}{r^4} = \frac{2}{r}. \tag{3.55}$$

Now, I'm ready to evaluate the $S = 1$ integrals.

$$\int \frac{K\sqrt{\tilde{h}}}{S\sqrt{\rho}}\bigg|_{S=1} \mathrm{d}x^2 \wedge \mathrm{d}x^3 = \lim_{r\to\infty}\int_0^{2\pi}\int_0^{\pi} \frac{2}{r}\frac{\sqrt{M}}{r} r^2 \sin(\theta)\mathrm{d}\theta\mathrm{d}\phi \tag{3.56}$$

$$= 8\pi\sqrt{M}. \tag{3.57}$$

$$\int \left(\frac{KS\sqrt{\tilde{h}}}{\rho} + \frac{4\sqrt{\tilde{h}}}{\rho^2}\right)\bigg|_{S=1} \mathrm{d}x^2 \wedge \mathrm{d}x^3 = \lim_{r\to\infty}\int_0^{2\pi}\int_0^{\pi} \left(\frac{2}{r}\frac{M}{r^2} + \frac{4M^2}{r^4}\right) r^2 \sin(\theta)\mathrm{d}\theta\mathrm{d}\phi \tag{3.58}$$

$$= 0. \tag{3.59}$$

In summary, the integral inequalities I derived reduce to $8\pi\sqrt{M} \le \frac{4\pi}{\sqrt{\kappa}}$ and $-4\kappa^2 A \le -4\pi$.
$\therefore M \le \frac{1}{4\kappa}$ and $A \ge \frac{\pi}{\kappa^2}$.
However, the Smarr relation says $M = \frac{\kappa A}{4\pi}$, so the first inequality becomes $A \le \frac{\pi}{\kappa^2}$.
$\therefore$ The two inequalities can be consistent only if they were actually equalities to begin with.
As I proved earlier, equality occurs if and only if $L_{AB} = K_{AB} - \frac{1}{2}\tilde{h}_{AB}K = 0$ and $\nabla^{(\tilde{h})}_A \rho = 0$.
The latter equation means $\rho$ depends only on $S$, since covariant and partial derivatives are identical on scalars.
$\therefore$ Equation 3.10 becomes $K = \frac{1}{\rho^2}\frac{\mathrm{d}\rho}{\mathrm{d}S} = -\frac{\mathrm{d}}{\mathrm{d}S}\left(\frac{1}{\rho}\right)$.
$\therefore K$ also depends only on $S$.
With the results so far, equation 3.20 reduces to

$$0 = -\frac{1}{2\rho^4}\left(\frac{\mathrm{d}\rho}{\mathrm{d}S}\right)^2 - \frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}S}\left(\frac{1}{\rho^2}\frac{\mathrm{d}\rho}{\mathrm{d}S}\right) + \frac{1}{\rho^3 S}\frac{\mathrm{d}\rho}{\mathrm{d}S} \tag{3.60}$$

$$= \frac{3}{2\rho^4}\left(\frac{\mathrm{d}\rho}{\mathrm{d}S}\right)^2 - \frac{1}{\rho^3}\frac{\mathrm{d}^2\rho}{\mathrm{d}S^2} + \frac{1}{\rho^3 S}\frac{\mathrm{d}\rho}{\mathrm{d}S}. \tag{3.61}$$

$$\therefore 0 = \frac{S}{\rho^{3/2}}\frac{\mathrm{d}^2\rho}{\mathrm{d}S^2} - \frac{1}{\rho^{3/2}}\frac{\mathrm{d}\rho}{\mathrm{d}S} - \frac{3S}{2\rho^{5/2}}\left(\frac{\mathrm{d}\rho}{\mathrm{d}S}\right)^2 \tag{3.62}$$

$$= \frac{\mathrm{d}}{\mathrm{d}S}\left(\frac{S}{\rho^{3/2}}\frac{\mathrm{d}\rho}{\mathrm{d}S}\right) - \frac{2}{\rho^{3/2}}\frac{\mathrm{d}\rho}{\mathrm{d}S} \tag{3.63}$$

$$= \frac{\mathrm{d}}{\mathrm{d}S}\left(\frac{S}{\rho^{3/2}}\frac{\mathrm{d}\rho}{\mathrm{d}S} + \frac{4}{\rho^{1/2}}\right). \tag{3.64}$$

$$\therefore C_1 = \frac{S}{\rho^{3/2}}\frac{\mathrm{d}\rho}{\mathrm{d}S} + \frac{4}{\rho^{1/2}} \tag{3.65}$$

for some integration constant, $C_1$[1]. The ODE left is separable. I get

$$\frac{\mathrm{d}S}{S} = -\frac{\mathrm{d}\rho}{C_1\rho^{3/2} + 4\rho}. \tag{3.66}$$

$$\therefore -\ln(S) = \int \frac{\mathrm{d}\rho}{C_1\rho^{3/2} + 4\rho}. \tag{3.67}$$

[1] I will be arbitrarily relabelling and scaling the integration constants without any further mention.

Let $p = \sqrt{\rho}$. Then, $\mathrm{d}\rho = 2p\,\mathrm{d}p$.

$$\therefore -\ln(S) = \int \frac{2\mathrm{d}p}{4p + C_1 p^2} \tag{3.68}$$

$$= \int \left( \frac{1}{2p} - \frac{C_1}{2(4 + C_1 p)} \right) \mathrm{d}p \tag{3.69}$$

$$= \frac{1}{2}\ln(p) - \frac{C_1}{2}\ln(4 + C_1 p) + C_2 \quad \text{for some constant } C_2. \tag{3.70}$$

$$\therefore S^2 = \frac{4 + C_1 p}{C_2 p} \iff p = \frac{C_1}{S^2 - C_2}. \tag{3.71}$$

$$\therefore \rho = \frac{C_1}{(S^2 - C_2)^2}. \tag{3.72}$$

I showed earlier that near the asymptotically flat end, $\rho = \frac{r^2}{M}$ and $S^2 = 1 - \frac{2M}{r}$ to leading order.

$$\therefore \frac{M}{r^2} = \frac{1}{C_1}\left(1 - \frac{2M}{r} - C_2\right)^2 = \frac{(1 - C_2)^2}{C_1} - \frac{4(1 - C_2)M}{rC_1} + \frac{4M^2}{C_1 r^2}. \tag{3.73}$$

$$\therefore C_2 = 1 \text{ and } C_1 = 4M. \tag{3.74}$$

$$\therefore \rho = \frac{4M}{(1 - S^2)^2}. \tag{3.75}$$

$$\therefore K = -\frac{\mathrm{d}}{\mathrm{d}S}\left(\frac{1}{\rho}\right) = \frac{S(1 - S^2)}{M}. \tag{3.76}$$

Then, by equation 3.50 and $L_{AB} = 0$, I get

$$R^{(\tilde{h})} = \frac{2K}{\rho S} + K^2 - \frac{1}{4}\tilde{h}_{AB} K \tilde{h}^{AB} K \tag{3.77}$$

$$= \frac{2K}{\rho S} + \frac{1}{2}K^2 \tag{3.78}$$

$$= \frac{2}{S}\frac{(1 - S^2)^2}{4M}\frac{S(1 - S^2)}{M} + \frac{1}{2}\frac{S^2(1 - S^2)^2}{M^2} \tag{3.79}$$

$$= \frac{(1 - S^2)^2}{2M^2}. \tag{3.80}$$

Since $R^{(\tilde{h})}$ depends only on $S$ and the constant, $M$, and every constant $S$ surface is assumed to be a sphere, each constant $S$ surface is a sphere with a constant scalar curvature.
A corollary of the solution to Liouville's equation says that the only metric on the sphere that yields a constant scalar curvature is the round metric.
$\therefore \tilde{h}_{AB}\,\mathrm{d}x^A \otimes \mathrm{d}x^B = r^2(\mathrm{d}\theta \otimes \mathrm{d}\theta + \sin^2(\theta)\,\mathrm{d}\phi \otimes \mathrm{d}\phi)$, where $r$ is the area-radius of the sphere. A sphere with this metric is known to have Ricci scalar, $R^{(\tilde{h})} = 2/r^2$.
$\therefore \frac{2}{r^2} = \frac{(1-S^2)^2}{2M^2} \iff S^2 = 1 - \frac{2M}{r}$.
Since this equation between $S$ and $r$ is one-to-one, I can swap $S$ out for $r$ in the adapted coordinates. Then, the term in the metric becomes

$$\rho^2 \mathrm{d}S \otimes \mathrm{d}S = \frac{16M^2}{(1 - (1 - \frac{2M}{r}))^4}\mathrm{d}\left(\sqrt{1 - \frac{2M}{r}}\right) \otimes \mathrm{d}\left(\sqrt{1 - \frac{2M}{r}}\right) \tag{3.81}$$

$$= \frac{r^4}{M^2}\frac{\frac{2M}{r^2}}{2\sqrt{1 - \frac{2M}{r}}}\mathrm{d}r \otimes \frac{\frac{2M}{r^2}}{2\sqrt{1 - \frac{2M}{r}}}\mathrm{d}r \tag{3.82}$$

$$= \frac{\mathrm{d}r \otimes \mathrm{d}r}{1 - 2M/r}. \tag{3.83}$$

Hence, I can finally conclude that the metric is

$$g = -S^2 \mathrm{d}t \otimes \mathrm{d}t + \rho^2 \mathrm{d}S \otimes \mathrm{d}S + \tilde{h}_{AB} \mathrm{d}x^A \otimes \mathrm{d}x^B \tag{3.84}$$

$$= -\left(1 - \frac{2M}{r}\right) \mathrm{d}t \otimes \mathrm{d}t + \frac{\mathrm{d}r \otimes \mathrm{d}r}{1 - 2M/r} + r^2 (\mathrm{d}\theta \otimes \mathrm{d}\theta + \sin^2(\theta)\, \mathrm{d}\phi \otimes \mathrm{d}\phi), \tag{3.85}$$

which is the Schwarzschild solution. □

# Chapter 4

# The end - the most comprehensive proof

In this chapter I'll present the most comprehensive proof of the Schwarzschild solution's uniqueness among static, asymptotically flat spacetimes. My exposition is based on [7].

**Theorem 4.1** ([5, 6, 7]). *Let $(M, g)$ be a spacetime with the following properties:*

1. *$M$ is $n$ dimensional.*
2. *$M$ is time orientable.*
3. *$(M, g)$ is asymptotically flat.*
4. *$(M, g)$ is static.*
5. *$\mathrm{d}(k^a k_a) \neq 0$ whenever $k^a k_a = 0$, where $k^a$ is the Killing vector field making $(M, g)$ static.*
6. *The energy-momentum tensor is $T_{ab} = 0$.*

*Then, $(M, g)$ is isometric to the Schwarzschild spacetime, with metric*

$$g = -\left(1 - \frac{m}{r^{n-3}}\right)\mathrm{d}t \otimes \mathrm{d}t + \frac{1}{1 - \frac{m}{r^{n-3}}}\mathrm{d}r \otimes \mathrm{d}r + r^2 g_{S^{n-2}}. \tag{4.1}$$

Most saliently, theorem 4.1 doesn't make assumptions about event horizon connectedness or spacetime dimension, unlike [1, 2, 3]. The actual PDE problem to be solved is ultimately exactly the same as chapter 3.

**Definition 4.2** (Problem summary). *Given the pair, $(\Sigma_t, h)$, the problem studied in this chapter is summarised by the equations,*

$$\Box^{(h)} S = 0 \text{ and } R^{(h)}_{ij} = \frac{1}{S}\nabla^{(h)}_i \nabla^{(h)}_j S, \tag{4.2}$$

*and the boundary conditions, $S = 0$ on $\mathcal{H}$, $0 \leq S < 1$ everywhere and*

$$S = 1 - \frac{m}{2r^{n-3}} \ \& \ h_{ij} = \left(1 + \frac{m}{(n-3)r^{n-3}}\right)\delta_{ij} \tag{4.3}$$

*at $S^{n-2}_\infty$.*

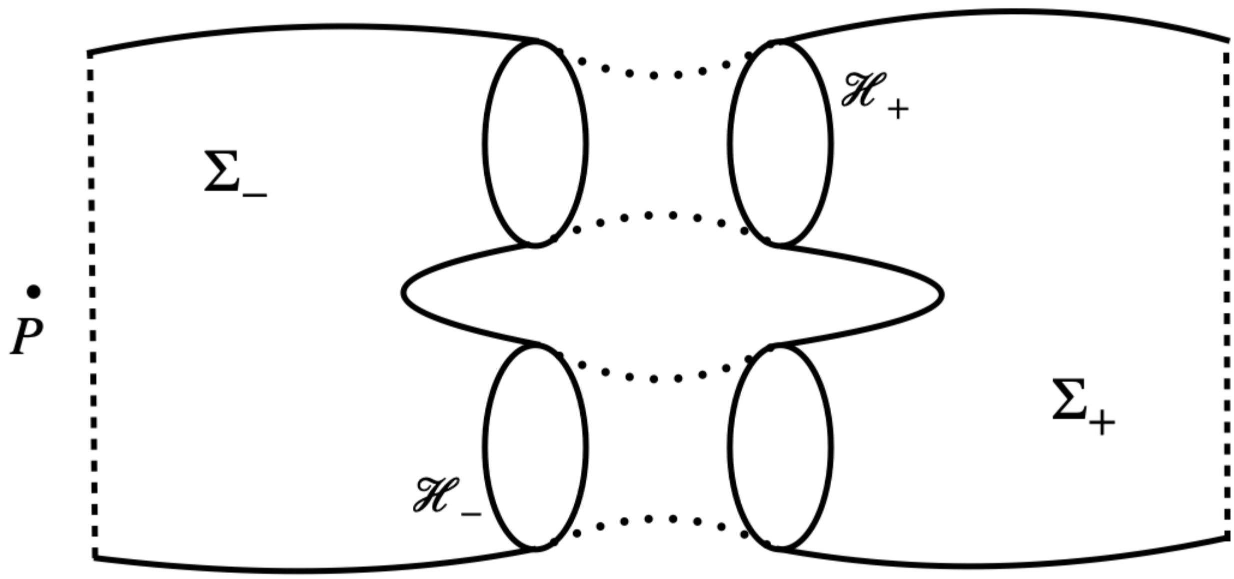


Figure 4.1: The construction of the manifold, $\hat{\Sigma}$, invented by [5].

*Proof.* The genius of Bunting and Masood-ul-Alam's proof [5] lies in the construction of a specific new manifold, $\hat{\Sigma}$, I'll describe below.
It begins by considering two conformal transformations,

$$h_{\pm ij} = \Omega_{\pm}^2 h_{ij} \quad \text{where} \tag{4.4}$$

$$\Omega_{\pm} = \left(\frac{1 \pm S}{2}\right)^{2/(n-3)} . \tag{4.5}$$

Note that $\Omega_{\pm}$ are well defined conformal factors because $0 \leq S < 1$ in definition 4.2.
Let the copies of $\Sigma_t$ with these metrics be denoted $\Sigma_{\pm} = (\Sigma_t, h_{\pm})$.
Since $S = 0$ on $\mathcal{H}$, $\Omega_+|_{\mathcal{H}} = \Omega_-|_{\mathcal{H}} \implies h_+|_{\mathcal{H}} = h_-|_{\mathcal{H}}$.
$\therefore$ I can glue $\Sigma_+$ and $\Sigma_-$ together along their inner boundaries, $\mathcal{H}_+$ and $\mathcal{H}_-$ respectively, to get a manifold which still has a continuous metric,

$$\hat{h} = \begin{cases} h_+ & \text{on } \Sigma_+ \\ h_- & \text{on } \Sigma_- \end{cases} . \tag{4.6}$$

The resulting manifold now has two asymptotically flat ends - one each from $\Sigma_+$ and $\Sigma_-$.
Let $P$ be a point at infinity and get rid of the asymptotically flat end from $\Sigma_-$ by performing a one point compactification with $P$ on that end.
The resulting manifold is $\hat{\Sigma}$ and is depicted in figure 4.1. In summary, $\hat{\Sigma} = \{P\} \sqcup \Sigma_- \sqcup \Sigma_+$, with a one point compactification between $P$ and $\Sigma_-$ and a gluing between the inner boundaries of $\Sigma_+$ and $\Sigma_-$.
$\hat{\Sigma}$ has a number of great properties.
It has just the one boundary at infinity and has no inner boundary.
Consider the one point compactification in more detail. Since an asymptotically flat end is diffeomorphic to $\mathbb{R}^{n-1}$ minus a compact set, the smooth structure is identical to the one point compactification of $\mathbb{R}^{n-1}$ to $S^{n-1}$.
However, this is Riemannian geometry, so one would need the metric to also extend well to $P$; I'll show this happens too.

From the asymptotics of $S$ in definition 4.2, to leading order, near $P$

$$\Omega_- = \left(\frac{1-S}{2}\right)^{2/(n-3)} = \left(\frac{m}{2r^{n-3}}\right)^{2/(n-3)} = \left(\frac{m}{2}\right)^{2/(n-3)} \frac{1}{r^2} \tag{4.7}$$

$$\implies h_{-ij} = \frac{1}{r^4}\left(\frac{m}{2}\right)^{2/(n-3)} \delta_{ij}. \tag{4.8}$$

The $r \to \infty$ limit, to get to $P$, cannot be defined in this coordinate system, but it can by changing to $z_i = \frac{1}{r^2}x_i$. $z_i = 0$ would then be $P$. I'll still raise and lower these indices near infinities by $\delta$, so all indices can just be subscripts.

$$r^2 = x_i x_i = r^4 z_i z_i \implies z_i z_i = \frac{1}{r^2}. \tag{4.9}$$

$$\therefore \mathrm{d}(z_i z_i) = \mathrm{d}(1/r^2) = -\frac{2}{r^3}\mathrm{d}r \implies \mathrm{d}r = -r^3 z_i \mathrm{d}z_i. \tag{4.10}$$

Hence, in the $z_i$ coordinates, to leading order $h_-$ is

$$\begin{aligned}
h_- &= \frac{1}{r^4}\left(\frac{m}{2}\right)^{2/(n-3)} \mathrm{d}x_i \otimes \mathrm{d}x_i && (4.11)\\
&= \frac{1}{r^4}\left(\frac{m}{2}\right)^{2/(n-3)} \mathrm{d}(r^2 z_i) \otimes \mathrm{d}(r^2 z_i) && (4.12)\\
&= \frac{1}{r^4}\left(\frac{m}{2}\right)^{2/(n-3)} (r^2\mathrm{d}z_i + 2rz_i\mathrm{d}r) \otimes (r^2\mathrm{d}z_i + 2rz_i\mathrm{d}r) && (4.13)\\
&= \frac{1}{r^4}\left(\frac{m}{2}\right)^{2/(n-3)} (r^2\mathrm{d}z_i - 2r^4 z_i z_j \mathrm{d}z_j) \otimes (r^2\mathrm{d}z_i - 2r^4 z_i z_k \mathrm{d}z_k) && (4.14)\\
&= \frac{1}{r^4}\left(\frac{m}{2}\right)^{2/(n-3)} (r^4\mathrm{d}z_i \otimes \mathrm{d}z_i - 4r^6 z_i z_j \mathrm{d}z_i \otimes \mathrm{d}z_j + 4r^8 z_i z_i z_j z_k \mathrm{d}z_j \otimes \mathrm{d}z_k) && (4.15)\\
&= \left(\frac{m}{2}\right)^{2/(n-3)} \mathrm{d}z_i \otimes \mathrm{d}z_i \ \text{ using } z_i z_i = 1/r^2. && (4.16)
\end{aligned}$$

This now is smoothly extendable to $z_i = 0 \iff r \to \infty$.
$\therefore \hat{h}$ is smoothly extendable to $P$.
Furthermore, $\hat{h}$ is smooth everywhere else too, except perhaps on the join between $\mathcal{H}_-$ and $\mathcal{H}_+$. Even there, on directions parallel to $\mathcal{H}_\pm$, smoothness is inherited from $\Omega_\pm$ and $\Sigma_t$. It only remains to see how the derivatives behave perpendicular to $\mathcal{H}_\pm$.
As a proxy for that, one typically uses the extrinsic curvature of $\mathcal{H}_\pm$.
Upon a conformal transformation, $h' = \Omega^2 h$, the extrinsic curvature transforms as [15]

$$K^{(h')}_{ij} = \Omega K^{(h)}_{ij} + n_k(h'_{ij} - n_i n_j)\nabla^{(h')k}(\ln(\Omega)), \tag{4.17}$$

where $n_i$ is a unit normal in the $h'$ metric.
By corollary 2.13.1, $K^{(h)}_{ij}$ is zero [1] on $\mathcal{H}$.

$$\begin{aligned}
\therefore K^{(h_\pm)}_{ij} &= n_k(h_{\pm ij} - n_i n_j)\nabla^{(h_\pm)k}(\ln(\Omega_\pm)) && (4.18)\\
&= n_k\left(\frac{1}{2^{4/(n-3)}}h_{ij} - n_i n_j\right) 2^{4/(n-3)}\nabla^{(h)k}(\ln(\Omega_\pm)) \quad \text{as } \Omega_\pm = \frac{1}{2^{2/(n-3)}} \text{ on } \mathcal{H} && (4.19)\\
&= n_k(h_{ij} - 2^{4/(n-3)} n_i n_j)\nabla^{(h)k}(\ln(\Omega_\pm)). && (4.20)
\end{aligned}$$

[1] Corollary 2.13.1 only says $K_{AB} = 0$, but in the Israel coordinates used there, $K_{0i} = 0$ automatically, so the whole tensor is indeed zero.

On $\mathcal{H}_\pm$, the derivative in this last expression is

$$\nabla_i^{(h)}(\ln(\Omega_\pm)) = \frac{1}{\Omega_\pm}\nabla_i^{(h)}\Omega_\pm \tag{4.21}$$

$$= 2^{2/(n-3)}\nabla_i^{(h)}\left(\left(\frac{1\pm S}{2}\right)^{2/(n-3)}\right)\Bigg|_{\mathcal{H}} \tag{4.22}$$

$$= 2^{2/(n-3)}\frac{1}{2^{2/(n-3)-1}}\frac{2}{n-3}\left(\pm\frac{1}{2}\right)\nabla_i^{(h)}(S)|_{\mathcal{H}} \tag{4.23}$$

$$= \pm\frac{2}{n-3}\nabla_i^{(h)}(S)|_{\mathcal{H}}. \tag{4.24}$$

Hence, for the extrinsic curvature I get

$$K_{ij}^{(h\pm)} = \pm\frac{2}{n-3}n_k(h_{ij} - 2^{4/(n-3)}n_i n_j)\nabla^{(h)k}(S)|_{\mathcal{H}}. \tag{4.25}$$

Since $\Omega_+ = \Omega_-$ on $\mathcal{H}$, the normal, $n_i$, is of the same magnitude for both $\mathcal{H}_+$ and $\mathcal{H}_-$; only the direction is flipped. This direction flip cancels the $\pm$ in the last expression to mean that the extrinsic curvatures do match when $\mathcal{H}_\pm$ is viewed as a single surface in $\hat{\Sigma}$.
$\therefore \hat{h}$ is at least continuously differentiable everywhere.
The next property of $\hat{\Sigma}$ I'll need is its Ricci scalar.
Upon a conformal transformation, $h' = \Omega^2 h$, the Ricci scalar changes as [17]

$$R^{(h')} = \frac{R^{(h)}}{\Omega^2} - \frac{(n-2)(n-5)}{\Omega^4}\nabla_i^{(h)}(\Omega)\nabla^{(h)i}(\Omega) - \frac{2(n-2)}{\Omega^3}\Box^{(h)}\Omega. \tag{4.26}$$

Since $R_{ij}^{(h)} = \frac{1}{S}\nabla_i^{(h)}\nabla_j^{(h)}S$ and $\Box^{(h)}S = 0$ in definition 4.2, $R^{(h)} = h^{ij}R^{(h)} = \frac{1}{S}\Box^{(h)}S = 0$.

$$\therefore R^{(h\pm)} = -\frac{n-2}{\Omega_\pm^3}\left(\frac{n-5}{\Omega_\pm}\nabla_i^{(h)}(\Omega_\pm)\nabla^{(h)i}(\Omega_\pm) + 2\Box^{(h)}\Omega_\pm\right). \tag{4.27}$$

The derivatives are

$$\nabla_i^{(h)}\Omega_\pm = \nabla_i^{(h)}\left(\left(\frac{1\pm S}{2}\right)^{2/(n-3)}\right) \tag{4.28}$$

$$= \frac{2}{n-3}\left(\pm\frac{1}{2}\right)\left(\frac{1\pm S}{2}\right)^{(5-n)/(n-3)}\nabla_i^{(h)}S \tag{4.29}$$

$$= \pm\frac{1}{n-3}\left(\frac{1\pm S}{2}\right)^{(5-n)/(n-3)}\nabla_i^{(h)}S \text{ and} \tag{4.30}$$

$$\therefore \Box^{(h)}\Omega_\pm = \pm\frac{1}{n-3}\nabla^{(h)i}\left(\left(\frac{1\pm S}{2}\right)^{(5-n)/(n-3)}\right)\nabla_i^{(h)}S \quad \text{as } \Box^{(h)}S = 0 \tag{4.31}$$

$$= \frac{5-n}{2(n-3)^2}\left(\frac{1\pm S}{2}\right)^{(8-2n)/(n-3)}\nabla_i^{(h)}(S)\nabla^{(h)i}(S). \tag{4.32}$$

Substituting these expressions back into equation 4.27 says

$$R^{(h\pm)} = -\frac{n-2}{\Omega_\pm^3}\Bigg((n-5)\left(\frac{1\pm S}{2}\right)^{-2/(n-3)}\frac{1}{(n-3)^2}\left(\frac{1\pm S}{2}\right)^{2(5-n)/(n-3)}\nabla_i^{(h)}(S)\nabla^{(h)i}(S)$$
$$+ \frac{5-n}{(n-3)^2}\left(\frac{1\pm S}{2}\right)^{(8-2n)/(n-3)}\nabla_i^{(h)}(S)\nabla^{(h)i}(S)\Bigg) \tag{4.33}$$
$$= 0. \tag{4.34}$$

The final property of $\hat{\Sigma}$ I'll need is that near its asymptotic end, to leading order, the metric is

$$\Omega_+^2 h_{ij} = \left(\frac{1 + 1 - \frac{m}{2r^{n-3}}}{2}\right)^{4/(n-3)} \left(1 + \frac{m}{(n-3)r^{n-3}}\right) \delta_{ij} \tag{4.35}$$

$$= \left(1 - \frac{m}{4r^{n-3}}\right)^{4/(n-3)} \left(1 + \frac{m}{(n-3)r^{n-3}}\right) \delta_{ij} \tag{4.36}$$

$$= \left(1 - \frac{m}{(n-3)r^{n-3}} + \frac{m}{(n-3)r^{n-3}} + O(1/r^{n-2})\right) \delta_{ij} \tag{4.37}$$

$$= \delta_{ij} + O(1/r^{n-2}). \tag{4.38}$$

$\therefore \hat{\Sigma}$ has zero ADM energy.
In summary, I've shown so far that $\hat{\Sigma}$ is a complete, asymptotically flat end with zero ADM mass, zero Ricci scalar and continuously differentiable metric.
$\therefore$ From a corollary to the positive energy theorem[2], $(\hat{\Sigma}, \hat{h})$ is in fact just $(\mathbb{R}^{n-1}, \delta)$.
$\therefore$ Taking $\Sigma_+$, without loss of generality, as the copy of $\Sigma_t$ within $\hat{\Sigma}$, it follows that the metric on $\Sigma_t$ is

$$h_{ij} = \frac{1}{\Omega_+^2}\delta_{ij} = \left(\frac{2}{1+S}\right)^{4/(n-3)} \delta_{ij}. \tag{4.39}$$

$S$ is still unknown though. It will be easier to find it by changing variable to

$$s = \frac{2}{1+S} \iff S = \frac{2}{s} - 1. \tag{4.40}$$

The $\Box^{(h)} S = 0$ condition says

$$0 = \Box^{(h)} S = h^{ij}\partial_i\partial_j S - h^{ij}\Gamma^{(h)k}{}_{ji}\partial_k S. \tag{4.41}$$

In terms of $s$, these quantities are as follows.

$$\partial_k S = \partial_k \left(\frac{2}{s} - 1\right) = -\frac{2}{s^2}\partial_k s. \tag{4.42}$$

$$\therefore \partial_i\partial_j S = \partial_i \left(-\frac{2}{s^2}\partial_j s\right) = \frac{4}{s^3}\partial_i(s)\partial_j(s) - \frac{2}{s^2}\partial_i\partial_j s. \tag{4.43}$$

$$\Gamma^{(h)k}{}_{ji} = \frac{1}{2}s^{-4/(n-3)}\delta^{kl}\left(\partial_j(s^{4/(n-3)}\delta_{il}) + \partial_i(s^{4/(n-3)}\delta_{lj}) - \partial_l(s^{4/(n-3)}\delta_{ji})\right) \tag{4.44}$$

$$= \frac{2}{(n-3)s}(\delta^k{}_i\partial_j s + \delta^k{}_j\partial_i s - \delta_{ji}\partial_k s). \tag{4.45}$$

[2]There is some subtlety here. There are several proofs of the positive energy theorem, with Schoen & Yau's [24] and Witten's [25] being the most popular. Witten's proof works in all dimensions, but requires the manifold to be spin. Schoen and Yau's does not, but was known not to work in arbitrarily large dimensions - although recently they claim to have generalised their 'spin-free' method to all dimensions [26]. I am going to brush these subtleties under the carpet and simply assume the theorem holds.

Putting these expressions back into the $\Box^{(h)}S = 0$ equation says

$$0 = s^{-4/(n-3)}\delta^{ij}\partial_i\partial_j S - s^{-4/(n-3)}\delta^{ij}\Gamma^{(h)k}{}_{ji}\partial_k S. \tag{4.46}$$

$$\therefore 0 = \delta^{ij}\left(\frac{4}{s^3}\partial_i(s)\partial_j(s) - \frac{2}{s^2}\partial_i\partial_j s\right) - \delta^{ij}\frac{2}{(n-3)s}(\delta^k{}_i\partial_j s + \delta^k{}_j\partial_i s - \delta_{ji}\partial_k s)\left(-\frac{2}{s^2}\partial_k s\right) \tag{4.47}$$

$$= \frac{4}{s^3}\delta^{ij}\partial_i(s)\partial)i(s) - \frac{2}{s^2}\nabla^2 s + \frac{4}{(n-3)s^3}\partial_i(s)\partial_i(s) + \frac{4}{(n-3)s^3}\partial_i(s)\partial_i(s)$$
$$- \frac{4(n-1)}{(n-3)s^3}\partial_i(s)\partial_i(s) \tag{4.48}$$

$$= -\frac{2}{s^2}\nabla^2 s. \tag{4.49}$$

$$\therefore 0 = \nabla^2 s. \tag{4.50}$$

There are three parts to any boundary value problem.

1. PDE: $\nabla^2 s = 0$.
2. Boundary conditions: $s = 0$ on $\mathcal{H}$ and $s \to 1 + \frac{m}{4r^{n-3}}$ to leading order near $S^{n-2}_\infty$.
3. The boundary itself: $\mathcal{H}$ and $S^{n-2}_\infty$.

The third condition is usually not worth mentioning - one can't have a boundary value problem without saying what the boundary actually is. Usually, the boundary is obvious. However, in this case, although I know how $s$ behaves at the boundary, I haven't yet actually determined what the boundary, $\mathcal{H}$, actually is. Its shape, its number of connected components etc. are all still unknown.
$\mathcal{H}$'s topology will be easier to determine by analysing $\mathcal{H}_+$, which has the same topology.
From equation 4.25,

$$K^{(h_+)}_{ij} = \frac{2}{n-3}n_k(h_{ij} - 2^{4/(n-3)}n_i n_j)\nabla^{(h)k}(S)|_{\mathcal{H}} \tag{4.51}$$

$$= \frac{2^{(n+1)/(n-3)}}{n-3}n_k(h_{+ij} - n_i n_j)\nabla^{(h)k}(S)|_{\mathcal{H}} \tag{4.52}$$

$$= \frac{2^{(n+1)/(n-3)}}{n-3}n_k\nabla^{(h)k}(S)|_{\mathcal{H}}\tilde{h}_{+ij}, \tag{4.53}$$

where $\tilde{h}_+$ is the induced metric on $\mathcal{H}_+$.
$\mathcal{H}$ is a constant $S$ surface, so $\nabla^{(h)}_i S \propto n_i$.
From lemma 2.12, $\nabla^{(h)}_i(S)|_{\mathcal{H}}\nabla^{(h)i}(S)|_{\mathcal{H}} = h^{ij}\nabla^{(h)}_i(S)|_{\mathcal{H}}\nabla^{(h)}_j(S)|_{\mathcal{H}} = \kappa^2$. Note that $\kappa$ is only a constant on each connected component of $\mathcal{H}_+$. Different horizon components could have different surface gravities.

$$\therefore h^{ij}_+\nabla^{(h_+)}_i(S)|_{\mathcal{H}}\nabla^{(h_+)}_j(S)|_{\mathcal{H}} = 2^{4/(n-3)}h^{ij}\nabla^{(h)}_i(S)|_{\mathcal{H}}\nabla^{(h)}_j(S)|_{\mathcal{H}} = 2^{4/(n-3)}\kappa^2. \tag{4.54}$$

$$\therefore n_i = \frac{1}{2^{2/(n-3)}\kappa}\nabla^{(h)}_i(S)|_{\mathcal{H}}. \tag{4.55}$$

$$\therefore K^{(h_+)}_{ij} = \frac{2^{(n+1)/(n-3)}}{n-3}n_k\nabla^{(h)k}(S)|_{\mathcal{H}}\tilde{h}_{+ij} = \frac{2^{(n-1)/(n-3)}\kappa}{n-3}\tilde{h}_{+ij}. \tag{4.56}$$

The fact $K^{(h_+)}_{ij}$ is a non-zero constant scaling of $\tilde{h}_{+ij}$ can be used to prove each connected component of $\mathcal{H}_+$ is a round sphere, using a theorem from [27] I'll describe below.
Let $X^i$ be the components of a vector tangent to $\mathcal{H}_+$ and let $D$ denote the covariant derivative

on $\mathcal{H}_+$. Furthermore, $h_+ = \delta$ from equation 4.39; let $x^i$ be the associated Cartesian variables. Then, $\nabla_i^{(h_+)} = \partial_i$ and

$$D_j\left(\frac{2^{(n-1)/(n-3)}\kappa}{n-3}x^i - n^i\right) = \tilde{h}_{+\ k}^{\ i}\tilde{h}_{+\ j}^{\ l}\nabla_l^{(h_+)}\left(\frac{2^{(n-1)/(n-3)}\kappa}{n-3}x^k - n^k\right) \tag{4.57}$$

$$= \frac{2^{(n-1)/(n-3)}\kappa}{n-3}\tilde{h}_{+\ k}^{\ i}\tilde{h}_{+\ j}^{\ l}\nabla_l^{(h_+)}x^k - \tilde{h}_{+\ k}^{\ i}\tilde{h}_{+\ j}^{\ l}\nabla_l^{(h_+)}n^k \tag{4.58}$$

$$= \frac{2^{(n-1)/(n-3)}\kappa}{n-3}\tilde{h}_{+\ k}^{\ i}\tilde{h}_{+\ j}^{\ l}\delta^k_{\ l} - K^{(h_+)i}_{\quad\ \ j}. \tag{4.59}$$

$$= \frac{2^{(n-1)/(n-3)}\kappa}{n-3}\tilde{h}_{+\ j}^{\ i} - K^{(h_+)i}_{\quad\ \ j} \tag{4.60}$$

$$= 0 \text{ by equation 4.56.} \tag{4.61}$$

Going back up the equation chain, this means

$$0 = \tilde{h}_{+\ k}^{\ i}\tilde{h}_{+\ j}^{\ l}\nabla_l^{(h_+)}\left(\frac{2^{(n-1)/(n-3)}\kappa}{n-3}x^k - n^k\right) = \tilde{h}_{+\ k}^{\ i}\tilde{h}_{+\ j}^{\ l}\partial_l\left(\frac{2^{(n-1)/(n-3)}\kappa}{n-3}x^k - n^k\right). \tag{4.62}$$

$\therefore \frac{2^{(n-1)/(n-3)}\kappa}{n-3}x^k - n^k$ is a constant on each connected component of $\mathcal{H}_+$. I'll call that constant $c^i$, where $c^i$ can depend on the connected component in question.
Then, $1 = n^i n_i = \delta^{ij}n_i n_j$ means

$$\left|\left|\frac{2^{(n-1)/(n-3)}\kappa}{n-3}x - c\right|\right|^2 = 1. \tag{4.63}$$

$\therefore$ The points, $x$, which lie on $\mathcal{H}_+$ lie on spheres of radius, $\frac{n-3}{\kappa}2^{-(n-1)/(n-3)}$, and centre[3], $c$.
Since $h_+ = \delta$, the induced metric on each of these spheres is the standard metric on the sphere.
Then, since $\Omega_+$ is just a constant on $\mathcal{H}$, the connected components of $\mathcal{H}$ are also just spheres with the round metric.
As long as $\mathcal{H}_+$ is geodesically complete, I can start at a point, $p \in \mathcal{H}_+$, view geodesics of $\mathcal{H}_+$ as being in $S^{n-2}$ and follow to arbitrary affine parameter to deduce that each connected component of $\mathcal{H}_+$ is a full sphere - not just contained within a sphere.
Indeed, $\mathcal{H}_+$ is geodesically complete because the event horizon is assumed to be non-singular and a spacetime is singular if and only if it's inextendable and geodesically incomplete[4].
The final step in completing requirement 3 of the boundary value problem is constraining the number of and relationship between $\mathcal{H}$'s connected components.
Suppose $\mathcal{H}$ is disconnected, for a contradiction.
$\mathcal{H}$, $\mathcal{H}_+$ and $\mathcal{H}_-$ all have identical topology and smooth structure - only the metrics are scaled by constants.
$\hat{\Sigma} = \Sigma_+ \sqcup \Sigma_- \sqcup \{P\} = \mathbb{R}^{n-1}$ from earlier.
$\therefore \Sigma_- \sqcup \{P\}$ is a disjoint union of multiple closed balls, with the surface of each ball being one of the connected components of $\mathcal{H}_+$.
$\therefore \Sigma_-$ is also disconnected, as $P$ is just a point at infinity. ↯
This contradicts $\Sigma_t$'s connectedness, because $\Sigma_-$ is topologically identical to $\Sigma_t$ by construction.
$\therefore \mathcal{H}$ is a sphere with radius, $\frac{n-3}{\kappa}2^{-(n-1)/(n-3)}$, and the standard metric[5].

[3] The centre's location can be arbitrarily adjusted by changing coordinates; it has no physical meaning.

[4] Note that $\mathcal{H}$ and $\mathcal{H}_+$ have the same geodesics because $\Omega_+$ is a constant on $\mathcal{H}$.

[5] Note that $\kappa$ is fully determined by $m$ because the radius determines the sphere's area and then I can apply the Smarr relation, $m = \frac{2\kappa A}{(n-3)\omega_{n-2}}$. Hence, there is just the one free parameter in play, $m$.

It remains to consider parts 1 and 2 of the boundary value problem.
Let $s_1$ and $s_2$ be two solutions of the boundary value problem. Then,

$$\nabla^2((s_1 - s_2)^2) = 2\partial_i((s_1 - s_2)\partial_i(s_1 - s_2)) \tag{4.64}$$
$$= 2\partial_i(s_1 - s_2)\partial_i(s_1 - s_2) + 2(s_1 - s_2)\nabla^2(s_1 - s_2) \tag{4.65}$$
$$= 2||\partial(s_1 - s_2)||^2 \text{ as } \nabla^2 s_1 = \nabla^2 s_2 = 0. \tag{4.66}$$
$$\therefore \int_{\Sigma_t} 2||\partial(s_1 - s_2)||^2 \mathrm{d}^{n-1}x = \int_{\Sigma_t} \nabla^2((s_1 - s_2)^2)\mathrm{d}^{n-1}x \tag{4.67}$$
$$= 2\int_{\Sigma_t} \partial_i((s_1 - s_2)\partial_i(s_1 - s_2))\mathrm{d}^{n-1}x \tag{4.68}$$
$$= 2\int_{S^\infty_{n-2}} n_i(s_1 - s_2)\partial_i(s_1 - s_2)\mathrm{d}^{n-2}x$$
$$- 2\int_{\mathcal{H}} n_i(s_1 - s_2)\partial_i(s_1 - s_2)\mathrm{d}^{n-2}x. \tag{4.69}$$

On $\mathcal{H}$, $s_1 = s_2 = 2$, so the second integral is zero.
Near $S^{n-2}_\infty$, the asymptotics mean $s_1 - s_2$ is $O(1/r^{n-3})$, so the integrand is $O(1/r^{2n-5})$. This is sufficiently fast decay for that integral to be zero.

$$\therefore \int_{\Sigma_t} 2||\partial(s_1 - s_2)||^2 \mathrm{d}^{n-1}x = 0 \iff ||\partial(s_1 - s_2)||^2 = 0 \iff s_1 = s_2. \tag{4.70}$$

$\therefore$ The solution is unique.
Since the Schwarzschild solution solves all the conditions in definition 4.2, it must be unique solution in question. $\square$

# Chapter 5

# The aftermath - contemporary research

The proof in chapter 4 and its subsequent extensions mean the static, asymptotically flat uniqueness problem has largely been solved. Modern researchers have mainly pursued three variations on the results I've discussed thus far.

1. Non-zero cosmological constant, $\Lambda$.
2. More exotic matter fields.
3. Avoiding the positive energy theorem.

Problem 1 is of significant physical importance and much remains unknown in this case - see [28, 29] for some comments and recent results for asymptotically de Sitter and asymptotically anti-de Sitter black holes respectively. I only fleetingly considered problem 1. Problem 2 is often motivated by theories of supergravity and I made some attempts at it by studying [10] and trying to apply it to the Einstein-Maxwell-Chern-Simons system. However, no progress has been made (yet). Problem 3, as explained in the introduction, appeals to a somewhat more aesthetic grievance. It's the only one where I can said to have made any progress though. In particular, I spent a long time studying the new work of Agostiniani and Mazzieri [13]. I've had some some success in generalising their work from vacuum spacetimes to those containing an electromagnetic field. That work is recounted here for the remainder of this chapter.

## 5.1 Purely electric

The substance of this chapter is dedicated to a new proof of Reissner-Nordstrom uniqueness I worked on for a large part of the first year of my PhD. Rather than start with full Einstein-Maxwell system, I started by adding only a purely electric field, i.e. one where $\iota_k \star F = 0$.

### 5.1.1 Background material

Chapter 2 contained many general results, most of which I'll call upon again in this chapter. However, I will also need some background material specific to the Einstein-Maxwell system - I'll start the chapter by presenting that.

**Definition 5.1** (Running assumptions)**.** *Until section 5.2, I'll assume the spacetime, $(M, g)$, satisfies the following properties:*

- *$M$ is $n$ dimensional.*
- *$M$ is time orientable.*

- *$(M, g)$ is asymptotically flat.*
- *$(M, g)$ is static, with the Killing vector field making $(M, g)$ static called $k^a$.*
- *$\mathrm{d}(k^a k_a) \neq 0$ whenever $k^a k_a = 0$.*
- *The event horizon is non-empty and connected.*
- *The matter action is electromagnetic, i.e. $S = -\frac{1}{16\pi} \int F^{ab} F_{ab} \sqrt{-g}\, \mathrm{d}^n x$ for a closed 2-form, $F$.*
- *$F$ is invariant under $k^a$, i.e. $\mathcal{L}_k F = 0$.*
- *$F$ is purely electric - or equivalently the magnetic component vanishes, $\iota_k \star F = 0$.*
- *$(M, g)$ is globally hyperbolic.*
- *The mass and charge parameters, $m$ and $q$, satisfy[1] $m > 2C|q|$, where $C = \sqrt{\frac{2(n-3)}{n-2}}$.*

The equations of motion for the Einstein-Maxwell system are well known and straightforward to derive. They are the Einstein equation,

$$R_{ab} = 2F_a{}^c F_{bc} - \frac{1}{n-2} g_{ab} F^{cd} F_{cd}, \tag{5.1}$$

and the Maxwell equation,

$$\nabla_b F^{ba} = 0. \tag{5.2}$$

**Lemma 5.2.** *Under the assumptions of definition 5.1, $F = \mathrm{d}\psi \wedge \mathrm{d}t$, for some function, $\psi$, that does not depend on $t$.*

Note that this lemma also serves to define $\psi$.

*Proof.* First, I assumed $\mathcal{L}_k F = 0$, to make $F$ compatible with the stationary nature of the problem.

$$\therefore 0 = k^\rho \partial_\rho F_{\mu\nu} + F_{\rho\nu} \partial_\mu k^\rho + F_{\mu\rho} \partial_\nu k^\rho \tag{5.3}$$

$$= \partial_t F_{\mu\nu} \text{ as } k = \partial/\partial t \text{ in adapted coordinates.} \tag{5.4}$$

Until section 5.2, I have also assumed the magnetic components, $B = \iota_k \star F$, vanish. In adapted coordinates, this says

$$0 = \varepsilon_{\rho\sigma\nu\mu_1\cdots\mu_{n-3}} k^\nu F^{\rho\sigma} = \varepsilon_{0\nu\rho\mu_1\cdots\mu_{n-3}} F^{\nu\rho} = \varepsilon_{0ij\mu_1\cdots\mu_{n-3}} F^{ij}. \tag{5.5}$$

$$\therefore F_{ij} = 0. \tag{5.6}$$

$$\therefore F = F_{0i}\, \mathrm{d}t \wedge \mathrm{d}x^i. \tag{5.7}$$

The $F_{0i}$ can be repackaged in the electric components as follows. By definition, they are

$$E_a = -k^b F_{ba} = -(\iota_k F)_a. \tag{5.8}$$

[1] The $m$ and $q$ I refer to are actually scaled versions of the electric charge and ADM mass. My exact definitions are given in definition 2.6. The key point is that $m > 2C|q|$ is exactly the well known relation that prevents naked singularities in the Reissner-Nordstrom solution.

In adapted coordinates, this says

$$E_\mu = -k^\nu F_{\nu\mu} = -F_{0\mu} \implies E_0 = 0 \text{ and } E_i = F_{i0}. \tag{5.9}$$

Then, by Cartan's magic formula,

$$\mathrm{d}E = \mathrm{d}(-\iota_k F) = -\mathcal{L}_k F + \iota_k \mathrm{d}F = 0 + 0 = 0. \tag{5.10}$$

The electric field is thus a closed 1-form. In definition 5.1, I have made all the assumptions required to apply the topological censorship results of [22, 21] and thereby conclude the domain of outer communication is simply connected.
∴ The first homology class of the domain of outer communication is also zero.
∴ $\mathrm{d}E = 0 \implies E = \mathrm{d}\psi$, for some function, $\psi$.
From equation 5.9, $\psi$ does not depend on $t$. Equation 5.9 also implies $F_{i0} = \partial_i \psi$.
Finally equation 5.7 then says $F = -\partial_i(\psi)\mathrm{d}t \wedge \mathrm{d}x^i = \mathrm{d}\psi \wedge \mathrm{d}t$. □

I'm now ready to write the equations of motion in terms of $\psi$, $h$ and $S$.

**Theorem 5.3.** *The equations of motion are*

$$S\Box^{(h)}S = C^2 \nabla^{(h)}_i(\psi)\nabla^{(h)i}(\psi), \tag{5.11}$$

$$0 = \nabla^{(h)}_i \left(\frac{1}{S}\nabla^{(h)i}\psi\right) \text{ and} \tag{5.12}$$

$$R^{(h)}_{ij} = \frac{1}{S}\nabla^{(h)}_i\nabla^{(h)}_j S + \frac{C^2}{(n-3)S^2} h_{ij}\nabla^{(h)}_k(\psi)\nabla^{(h)k}(\psi) - \frac{(n-2)C^2}{(n-3)S^2}\nabla^{(h)}_i(\psi)\nabla^{(h)}_j(\psi), \tag{5.13}$$

*where* $C = \sqrt{\frac{2(n-3)}{n-2}}$.

*Proof.* Let's start with the Einstein equation. First, every component involves

$$F^{ab}F_{ab} = 2F^{i0}F_{i0} = -\frac{2}{S^2}h^{ij}F_{i0}F_{j0} = -\frac{2}{S^2}h^{ij}\partial_i(\psi)\partial_j(\psi) = -\frac{2}{S^2}\nabla^{(h)i}(\psi)\nabla^{(h)}_i(\psi). \tag{5.14}$$

Applying lemma 2.4 and equation 5.1, from the $0-0$ component of $R_{\mu\nu}$, I get

$$S\Box^{(h)}S = 2F_0{}^\mu F_{0\mu} - \frac{1}{n-2}g_{00}F^{cd}F_{cd} \tag{5.15}$$

$$= 2F_0{}^i F_{0i} - \frac{1}{n-2}(-S^2)\left(-\frac{2}{S^2}\nabla^{(h)i}(\psi)\nabla^{(h)}_i(\psi)\right) \tag{5.16}$$

$$= 2\nabla^{(h)i}(\psi)\nabla^{(h)}_i(\psi) - \frac{2}{n-2}\nabla^{(h)i}(\psi)\nabla^{(h)}_i(\psi) \tag{5.17}$$

$$= \frac{2(n-3)}{n-2}\nabla^{(h)i}(\psi)\nabla^{(h)}_i(\psi) \tag{5.18}$$

$$= C^2\nabla^{(h)i}(\psi)\nabla^{(h)}_i(\psi). \tag{5.19}$$

Next, the $0-i$ components say

$$0 = 2F_0{}^\mu F_{i\mu} - \frac{1}{n-2}g_{0i}F^{ab}F_{ab} \tag{5.20}$$

$$= 2F_0{}^0 F_{i0} + 2F_0{}^j F_{ij} - 0 \tag{5.21}$$

$$= 0 + 0, \tag{5.22}$$

thus providing no new information. Finally, the $i-j$ components say

$$R^{(h)}_{ij} = \frac{1}{S}\nabla^{(h)}_i\nabla^{(h)}_j S + 2F_i{}^{\mu}F_{j\mu} - \frac{1}{n-2}g_{ij}F^{ab}F_{ab} \tag{5.23}$$

$$= \frac{1}{S}\nabla^{(h)}_i\nabla^{(h)}_j S + 2F_i{}^{0}F_{j0} - \frac{1}{n-2}h_{ij}\left(-\frac{2}{S^2}\nabla^{(h)k}(\psi)\nabla^{(h)}_k(\psi)\right) \tag{5.24}$$

$$= \frac{1}{S}\nabla^{(h)}_i\nabla^{(h)}_j S + 2\left(-\frac{1}{S^2}\right)\nabla^{(h)}_i(\psi)\nabla^{(h)}_j(\psi) - \frac{2}{(n-2)S^2}\nabla^{(h)k}(\psi)\nabla^{(h)}_k(\psi) \tag{5.25}$$

$$= \frac{1}{S}\nabla^{(h)}_i\nabla^{(h)}_j S - \frac{(n-2)C^2}{(n-3)S^2}\nabla^{(h)}_i(\psi)\nabla^{(h)}_j(\psi) + \frac{C^2}{(n-3)S^2}h_{ij}\nabla^{(h)}_k(\psi)\nabla^{(h)k}(\psi). \tag{5.26}$$

Meanwhile, the Maxwell equation says

$$0 = \nabla^{\nu}F_{\nu\mu} \tag{5.27}$$

$$= \nabla^0 F_{0\mu} + \nabla^i F_{i\mu} \tag{5.28}$$

$$= -\frac{1}{S^2}\nabla_0 F_{0\mu} + h^{ij}\nabla_j F_{i\mu} \tag{5.29}$$

$$= -\frac{1}{S^2}\partial_t F_{0\mu} + \frac{1}{S^2}\Gamma^{\nu}{}_{00}F_{\nu\mu} + \frac{1}{S^2}\Gamma^{\nu}{}_{\mu 0}F_{0\nu} + h^{ij}\partial_j F_{i\mu} - h^{ij}\Gamma^{\nu}{}_{ij}F_{\nu\mu} - h^{ij}\Gamma^{\nu}{}_{\mu j}F_{i\nu} \tag{5.30}$$

$$= 0 + \frac{1}{S^2}\Gamma^{\nu}{}_{00}F_{\nu\mu} + \frac{1}{S^2}\Gamma^{i}{}_{\mu 0}F_{0i} + h^{ij}\delta_{\mu 0}\partial_j F_{i0} - h^{ij}\Gamma^{\nu}{}_{ij}F_{\nu\mu} - h^{ij}\Gamma^{0}{}_{\mu j}F_{i0}. \tag{5.31}$$

Using lemma 2.5, I then get

$$0 = \frac{1}{S}\nabla^{(h)i}(S)F_{i\mu} + \frac{1}{S}\nabla^{(h)}(S)\delta_{\mu 0}F_{0i} + h^{ij}\delta_{\mu 0}\partial_j F_{i0} - h^{ij}\Gamma^{k}{}_{ij}F_{k\mu} - h^{ij}\frac{1}{S}\nabla^{(h)}_j(S)\delta_{\mu 0}F_{i0} \tag{5.32}$$

$$= \delta_{\mu 0}\Bigg(\frac{1}{S}\nabla^{(h)i}(S)\nabla^{(h)}_i(\psi) - \frac{1}{S}\nabla^{(h)i}(S)\nabla^{(h)}_i(\psi) + h^{ij}\partial_j(\nabla^{(h)}_i(\psi)) - h^{ij}\Gamma^{k}{}_{ij}\nabla^{(h)}_k(\psi)$$
$$- \frac{1}{S}\nabla^{(h)i}(S)\nabla^{(h)}_i(\psi)\Bigg) \tag{5.33}$$

$$= \delta_{\mu 0}\left(-\frac{1}{S}\nabla^{(h)i}(S)\nabla^{(h)}_i(\psi) + \Box^{(h)}(\psi)\right). \tag{5.34}$$

The final result, $-\frac{1}{S}\nabla^{(h)i}(S)\nabla^{(h)}_i(\psi) + \Box^{(h)}(\psi) = 0$, is equivalent to the total derivative, $\nabla^{(h)}_i(\frac{1}{S}\nabla^{(h)i}\psi) = 0$, by dividing by $S$. □

The equations of motion are partial differential equations; they must be supplemented by boundary conditions for any kind of uniqueness analysis. As in the earlier chapters, the problem is once again formulated on $\Sigma_t$ with inner boundary, $\mathcal{H}$, and outer boundary, $S^{n-2}_{\infty}$.
On the outer boundary, the asymptotics are only an extension of definition 2.7.

**Definition 5.4** (Asymptotics)**.** *To leading order near* $S^{n-2}_{\infty}$,

$$\psi = -\frac{q}{r^{n-3}}, \tag{5.35}$$

$$S = 1 - \frac{m}{2r^{n-3}} \quad \text{and} \tag{5.36}$$

$$h_{ij} = \left(1 + \frac{m}{(n-3)r^{n-3}}\right)\delta_{ij}. \tag{5.37}$$

The only addition to definition 2.7 is the assumption on $\psi$'s decay. The assumption I'm making is not rigorously justified here, but it is the standard one used in the literature - e.g. see [9, 10, 8, 11].

Meanwhile, at the inner boundary, $\mathcal{H}$, $S = 0$ again as per corollary 2.2.1. For $\psi$, I have the following result.

**Lemma 5.5.** $\nabla_i^{(h)}\psi = 0$ *on* $\mathcal{H}$.

*Proof.* From equation 5.11, $\Box^{(h)}S = \frac{C^2}{S}\nabla_i^{(h)}(\psi)\nabla^{(h)i}(\psi)$. As explained around lemma 2.9, I can use Israel coordinates near the event horizon. Then.

$$\Box^{(h)}S = h^{ij}\nabla_i^{(h)}\nabla_j^{(h)}S \tag{5.38}$$

$$= -h^{ij}\Gamma^{(h)1}{}_{ji} \text{ by equation 2.135} \tag{5.39}$$

$$= -\frac{1}{\rho^2}\frac{1}{\rho}\partial_S\rho + \tilde{h}^{AB}\frac{1}{\rho}K_{AB} \tag{5.40}$$

$$= -\frac{1}{\rho^3}\partial_S\rho + \frac{1}{\rho}K. \tag{5.41}$$

Then, since lemma 2.12 and corollary 2.13.1 say $\rho = \frac{1}{\kappa}$, $\partial_S\rho = 0$ and $K = 0$ on $\mathcal{H}$, it follows that $\frac{1}{S}\nabla_i^{(h)}(\psi)\nabla^{(h)i}(\psi) \to 0$ as one approaches $\mathcal{H}$. Since $S = 0$ on $\mathcal{H}$ (and $h$ is Riemannian), $\nabla_i^{(h)}\psi$ must be zero on $\mathcal{H}$. □

**Corollary 5.5.1.** *$\psi$ equals some constant, $\psi_0$, on $\mathcal{H}$.*

**Lemma 5.6.** *With the boundary conditions assumed, $0 \le S < 1$. Meanwhile, for $\psi$, if $q \ge 0$, then $\psi_0 \le \psi \le 0$ and if $q < 0$, then $0 < \psi \le \psi_0$. For both $S$ and $\psi$, equalities occur at the boundaries, $\mathcal{H}$ or $S_\infty^{n-2}$.*

*Proof.* From equation 5.11,

$$\Box^{(h)}S = \frac{C^2}{S}\nabla_i^{(h)}(\psi)\nabla^{(h)i}(\psi) \ge 0. \tag{5.42}$$

∴ The Hopf maximum principle can be applied, to conclude that $S$ is maximised on the boundary of $\Sigma_t$.
I've already shown $S = 0$ on the inner boundary, $\mathcal{H}$, and $S > 0$ elsewhere, so the maximum must be on $S_\infty^{n-2}$.
From equation 2.57, $S \to 1^-$ as one approaches $S_\infty^{n-2}$, thereby completely proving the claims about $S$.
Meanwhile, for $\psi$, equation 5.12 says

$$\Box^{(h)}(\psi) - \frac{1}{S}\nabla_i^{(h)}(S)\nabla^{(h)i}(\psi) = 0. \tag{5.43}$$

∴ The Hopf maximum principle can be applied again, this time to conclude $\psi$ must be extremised on the boundaries of $\Sigma_t$.
From equation 5.35, $\psi = 0$ on $S_\infty^{n-2}$ and $\psi = \psi_0$ on $\mathcal{H}$.
∴ One of 0 and $\psi_0$ must be the maximum and the other must be the minimum. By equation 5.35, $\psi < 0$ near $S_\infty^{n-2}$ when $q > 0$ and $\psi > 0$ near $S_\infty^{n-2}$ when $q < 0$.
∴ When $q > 0$, 0 is the maximum and $\psi_0$ is the minimum, while $\psi_0$ is the maximum and 0 is the minimum when $q < 0$.
When $q = 0$, the maximum and minimum must both be 0, meaning $\psi = 0$ and there's actually no electric field at all. □

In summary, the equations to solve are the following.

**Definition 5.7** (Problem summary)**.** *The problem studied in this section is summarised by the equations,*

$$S\Box^{(h)}S = C^2\nabla_i^{(h)}(\psi)\nabla^{(h)i}(\psi), \tag{5.44}$$

$$0 = \nabla_i^{(h)}\left(\frac{1}{S}\nabla^{(h)i}\psi\right) \quad \text{and} \tag{5.45}$$

$$R_{ij}^{(h)} = \frac{1}{S}\nabla_i^{(h)}\nabla_j^{(h)}S + \frac{C^2}{(n-3)S^2}h_{ij}\nabla_k^{(h)}(\psi)\nabla^{(h)k}(\psi) - \frac{(n-2)C^2}{(n-3)S^2}\nabla_i^{(h)}(\psi)\nabla_j^{(h)}(\psi), \tag{5.46}$$

*where* $C = \sqrt{\frac{2(n-3)}{n-2}}$ *and the boundary conditions are* $S = 0$ *on* $\mathcal{H}$*,* $\psi = \psi_0$ *(a constant) on* $\mathcal{H}$*,* $\nabla_i^{(h)}\psi = 0$ *on* $\mathcal{H}$*,* $0 \leq S < 1$ *everywhere,* $S \to 1 - \frac{m}{2r^{n-3}}$ *at* $S_\infty^{n-2}$ *and* $\psi \to -\frac{q}{r^{n-3}}$ *at* $S_\infty^{n-2}$*.*

### 5.1.2 Conformal reformulation

One of the key steps underpinning the proof strategy I'll explain in this chapter is re-writing the problem in the right variables and with the right conformal transformation.

**Definition 5.8** ($\varphi$ and $z$)**.** *Define new variables,* $\varphi$ *and* $z$*, to replace* $S$ *and* $\psi$*, by*

$$\varphi = \ln\left(\frac{(1+S)^2 - C^2\psi^2}{(1-S)^2 - C^2\psi^2}\right) \quad \text{and} \quad z = \ln\left(\frac{(1+C\psi)^2 - S^2}{(1-C\psi)^2 - S^2}\right). \tag{5.47}$$

*Proof.* It has to be checked that this change of variables is well-defined. The expressions for $\varphi$ and $z$ are manifestly independent, so it all just boils down to checking the arguments of the logarithms are positive. I will follow some techniques presented in [11]. Following [11], define

$$F_\pm = S \pm C\psi - 1. \tag{5.48}$$

Then, using equations 5.44 and 5.45,

$$\begin{aligned}
\Box^{(h)}F_\pm &= \Box^{(h)}S \pm C\Box^{(h)}\psi && (5.49)\\
&= \frac{C^2}{S}\nabla_i^{(h)}(\psi)\nabla^{(h)i}(\psi) \pm \frac{C}{S}\nabla_i^{(h)}(S)\nabla^{(h)i}(\psi) && (5.50)\\
\therefore \Box^{(h)}F_\pm \mp \frac{C}{S}\nabla_i^{(h)}(F_\pm)\nabla^{(h)i}(\psi) &= \frac{C^2}{S}\nabla_i^{(h)}(\psi)\nabla^{(h)i}(\psi) \pm \frac{C}{S}\nabla_i^{(h)}(S)\nabla^{(h)i}(\psi) \\
&\quad \mp \frac{C}{S}\nabla^{(h)i}(\psi)\left(\nabla_i^{(h)}(S) \pm C\nabla_i^{(h)}(\psi)\right) && (5.51)\\
&= 0. && (5.52)
\end{aligned}$$

This PDE for $F_\pm$ is of the form which the Hopf maximum principle applies to.
$\therefore F_\pm$ is extremised on the boundary of $\Sigma_t$.
At the $S_\infty^{n-2}$ boundary, the asymptotics in definition 5.7 mean that

$$F_\pm \to 1 - \frac{m}{2r^{n-3}} \pm C\left(-\frac{q}{r^{n-3}}\right) - 1 = -\frac{m \pm 2Cq}{2r^{n-3}}. \tag{5.53}$$

Since I'm assuming $m > 2C|q|$, it follows that $F_\pm \to 0^-$ at $S_\infty^{n-2}$.
Meanwhile, the other boundary is at $\mathcal{H}$.
Since $\mathcal{H} = \{S = 0\}$, $\nabla_i^{(h)}S$ is a normal to $S$ that points into $\Sigma_t$. Then, since $\nabla_i^{(h)}\psi = 0$ on $\mathcal{H}$,

$$\nabla_i^{(h)}(S)\nabla^{(h)i}(F_\pm)|_{\mathcal{H}} = \nabla_i^{(h)}(S)\nabla^{(h)i}(S)|_{\mathcal{H}} = \kappa^2 > 0. \tag{5.54}$$

$\therefore F_\pm$ is not maximised on $\mathcal{H}$.
Since $F_\pm$ has to be maximised on a boundary, that boundary must be $S^{n-2}_\infty$.
Since $F_\pm \to 0^-$ there, it follows that $F_\pm = S \pm C\psi - 1 < 0$ everywhere on $\Sigma_t$ itself.
$\therefore S < 1 \mp C\psi$.
Since $S \geq 0$, this inequality implies $S^2 < (1 \mp C\psi)^2 \iff (1 \mp C\psi)^2 - S^2 > 0$.
$\therefore z$ is well-defined.
Meanwhile, $F_\pm < 0$ can also be written as $1 - S > \pm C\psi$.
Since the RHS has both $+$ & $-$ and $S < 1$, the inequality implies $(1-S)^2 > C^2\psi^2$.
$\therefore \varphi$ is also well defined. □

**Lemma 5.9.** *In terms of the new variables, the old variables are*

$$S = \frac{\sinh(\varphi/2)}{\cosh(\varphi/2) + \cosh(z/2)} \text{ and } C\psi = \frac{\sinh(z/2)}{\cosh(\varphi/2) + \cosh(z/2)}. \tag{5.55}$$

*Proof.* Observe that

$$\coth(z/2) = \frac{e^z + 1}{e^z - 1} \tag{5.56}$$

$$= \frac{\frac{(1+C\psi)^2 - S^2}{(1-C\psi)^2 - S^2} + 1}{\frac{(1+C\psi)^2 - S^2}{(1-C\psi)^2 - S^2} - 1} \tag{5.57}$$

$$= \frac{(1+C\psi)^2 - S^2 + (1-C\psi)^2 - S^2}{(1+C\psi)^2 - S^2 - (1-C\psi)^2 + S^2} \tag{5.58}$$

$$= \frac{1 + C^2\psi^2 - S^2}{2C\psi}. \tag{5.59}$$

Then, from the definition of $\varphi$,

$$e^\varphi = \frac{(1+S)^2 - C^2\psi^2}{(1-S)^2 - C^2\psi^2} \tag{5.60}$$

$$= \frac{2(1+S) - 1 - C^2\psi^2 + S^2}{2(1-S) - 1 - C^2\psi^2 + S^2} \tag{5.61}$$

$$= \frac{1 + S - C\psi\coth(z/2)}{1 - S - C\psi\coth(z/2)}. \tag{5.62}$$

$$\therefore 1 + S - C\psi\coth(z/2) = e^\varphi - Se^\varphi - C\psi e^\varphi\coth(z/2). \tag{5.63}$$

$$\therefore 0 = S(1 + e^\varphi) + (1 - e^\varphi) + C\psi\coth(z/2)(e^\varphi - 1). \tag{5.64}$$

$$\therefore S = \tanh(\varphi/2)(1 - C\psi\coth(z/2)). \tag{5.65}$$

Substituting this back into equation 5.59, I get

$$\tanh^2(\varphi/2)(1 - C\psi\coth(z/2))^2 = 1 + C^2\psi^2 - 2C\psi\coth(z/2). \tag{5.66}$$

$$\therefore 0 = \frac{1}{\cosh^2(\varphi/2)} - \frac{2C\psi\coth(z/2)}{\cosh^2(\varphi/2)} + (1 - \tanh^2(\varphi/2)\coth^2(z/2))C^2\psi^2. \tag{5.67}$$

$$\therefore C\psi = \frac{\frac{2\coth(z/2)}{\cosh^2(\varphi/2)} \pm \sqrt{\frac{4\coth^2(z/2)}{\cosh^4(\varphi/2)} - \frac{4(1-\tanh^2(\varphi/2)\coth^2(z/2))}{\cosh^2(\varphi/2)}}}{2(1-\tanh^2(\varphi/2)\coth^2(z/2))} \tag{5.68}$$

$$= \frac{\coth(z/2) \pm \sqrt{\coth^2(z/2) - \cosh^2(\varphi/2) + \sinh^2(\varphi/2)\coth^2(z/2)}}{\cosh^2(\varphi/2) - \sinh^2(\varphi/2)\coth^2(z/2)} \tag{5.69}$$

$$= \frac{\coth(z/2) \pm \sqrt{\cosh^2(\varphi/2)(\coth^2(z/2) - 1)}}{\cosh^2(\varphi/2) - \sinh^2(\varphi/2)\coth^2(z/2)} \tag{5.70}$$

$$= \frac{\coth(z/2) \pm \frac{\cosh(\varphi/2)}{\sinh(z/2)}}{\cosh^2(\varphi/2) - \sinh^2(\varphi/2)\coth^2(z/2)} \tag{5.71}$$

$$= \frac{\sinh(z/2)(\cosh(z/2) \pm \cosh(\varphi/2))}{\cosh^2(\varphi/2)\sinh^2(z/2) - \sinh^2(\varphi/2)\cosh^2(z/2)}. \tag{5.72}$$

The denominator simplifies as

$$\cosh^2(\varphi/2)\sinh^2(z/2) - \sinh^2(\varphi/2)\cosh^2(z/2)$$

$$= \cosh^2(\varphi/2)(\cosh^2(z/2) - 1) - (\cosh^2(\varphi/2) - 1)\cosh^2(z/2) \tag{5.73}$$

$$= \cosh^2(z/2) - \cosh^2(\varphi/2) \tag{5.74}$$

$$= (\cosh(z/2) + \cosh(\varphi/2))(\cosh(z/2) - \cosh(\varphi/2)). \tag{5.75}$$

Substituting this back up,

$$C\psi = \frac{\sinh(z/2)}{\cosh(z/2) \mp \cosh(\varphi/2)}. \tag{5.76}$$

From the asymptotics - see definition 5.7 - of $S$ and $\psi$, $\varphi \to \infty$, $S \to 1 - \frac{m}{2r^{n-3}}$ and $\psi \to -\frac{q}{r^{n-3}}$ near spatial infinity.

$$\therefore z = \ln\left(\frac{(1+C\psi)^2 - S^2}{(1-C\psi)^2 - S^2}\right) \tag{5.77}$$

$$\to \ln\left(\frac{(1 - Cq/r^{n-3})^2 - (1 - m/2r^{n-3})^2}{(1 + Cq/r^{n-3})^2 - (1 - m/2r^{n-3})^2}\right) \tag{5.78}$$

$$\to \ln\left(\frac{m - 2Cq}{m + 2Cq}\right). \tag{5.79}$$

$\therefore z$ is negative when the electric charge is positive.
Hence, to get $\psi \to -q/r^{n-3} < 0$ in equation 5.76 when $q > 0$, I need to pick the $+$ in $\mp$, since $\cosh(\varphi/2)$ dominates in the denominator.

$$\therefore C\psi = \frac{\sinh(z/2)}{\cosh(z/2) + \cosh(\varphi/2)}. \tag{5.80}$$

Substituting this back into equation 5.65, I get

$$S = \tanh(\varphi/2)\left(1 - \coth(z/2)\frac{\sinh(z/2)}{\cosh(\varphi/2) + \cosh(z/2)}\right) \tag{5.81}$$

$$= \frac{\tanh(\varphi/2)(\cosh(\varphi/2) + \cosh(z/2) - \cosh(z/2))}{\cosh(\varphi/2) + \cosh(z/2)} \tag{5.82}$$

$$= \frac{\sinh(\varphi/2)}{\cosh(\varphi/2) + \cosh(z/2)}, \tag{5.83}$$

which completes the proof. □

**Definition 5.10** (Conformal scaling, $h'$). *Define a conformally scaled metric, $h' = \Omega^2 h$, where*

$$\Omega = \left( \frac{2\cosh(z/2)}{\cosh(z/2) + \cosh(\varphi/2)} \right)^{1/(n-3)}. \tag{5.84}$$

Note that $\cosh(x) \geq 1$ means $\Omega$ is manifestly smooth and non-zero.

**Lemma 5.11.** *In terms of the old variables,*

$$\Omega = (1 - S^2 + C^2\psi^2)^{1/(n-3)}. \tag{5.85}$$

*Proof.* By lemma 5.9,

$$\begin{aligned}
&1 - S^2 + C^2\psi^2 \\
&= 1 - \left( \frac{\sinh(\varphi/2)}{\cosh(\varphi/2) + \cosh(z/2)} \right)^2 + \left( \frac{\sinh(z/2)}{\cosh(\varphi/2) + \cosh(z/2)} \right)^2 && (5.86)\\
&= \frac{\cosh^2(\varphi/2) + \cosh^2(z/2) + 2\cosh(\varphi/2)\cosh(z/2) - \sinh^2(\varphi/2) + \sinh^2(z/2)}{(\cosh(\varphi/2) + \cosh(z/2))^2} && (5.87)\\
&= \frac{1 + \cosh^2(z/2) + 2\cosh(\varphi/2)\cosh(z/2) + \cosh^2(z/2) - 1}{(\cosh(\varphi/2) + \cosh(z/2))^2} && (5.88)\\
&= \frac{2\cosh(z/2)}{\cosh(z/2) + \cosh(\varphi/2)}, && (5.89)
\end{aligned}$$

which matches equation 5.84. □

**Theorem 5.12.** *In terms of $\varphi$, $z$ and $h'$, equations 5.44 and 5.45 are equivalent to*

$$\Box^{(h')}\varphi = \frac{1}{2}\tanh(z/2)\nabla_i^{(h')}(\varphi)\nabla^{(h')i}(z) + \frac{1}{2}\coth(\varphi/2)\nabla_i^{(h')}(z)\nabla^{(h')i}(z) \text{ and} \tag{5.90}$$

$$\Box^{(h')}z = \frac{1}{2}\coth(\varphi/2)\nabla_i^{(h')}(\varphi)\nabla^{(h')i}(z) + \frac{1}{2}\tanh(z/2)\nabla_i^{(h')}(z)\nabla^{(h')i}(z). \tag{5.91}$$

*Proof.* The proof is so tedious and uninsightful I have relegated it to appendix A. □

### 5.1.3 Uniqueness proof

**Theorem 5.13.** *$z$ is a constant.*

*Proof.* First observe that

$$\begin{aligned}
&\nabla_i^{(h')}\left( \frac{1}{\sinh(\varphi/2)\cosh(z/2)} \nabla^{(h')i} z \right) \\
&= \frac{1}{\sinh(\varphi/2)\cosh(z/2)}\Box^{(h')}z - \frac{\cosh(\varphi/2)}{2\sinh^2(\varphi/2)\cosh(z/2)}\nabla_i^{(h')}(\varphi)\nabla^{(h')i}(z) \\
&\quad - \frac{\sinh(z/2)}{2\sinh(\varphi/2)\cosh^2(z/2)}\nabla_i^{(h')}(z)\nabla^{(h')i}(z) && (5.92)\\
&= 0 \text{ by equation 5.91.} && (5.93)
\end{aligned}$$

Hence, when I integrate this total derivative across $\Sigma_t$, I get

$$0 = \int_{\Sigma_t} \nabla^{(h')}_i \left( \frac{1}{\sinh(\varphi/2)\cosh(z/2)} \nabla^{(h')i} z \right) \mathrm{d}V' \tag{5.94}$$

$$= \int_{S^{n-2}_\infty} n_i \frac{1}{\sinh(\varphi/2)\cosh(z/2)} \nabla^{(h')i}(z)\,\mathrm{d}A' - \int_{\mathcal{H}} n_i \frac{1}{\sinh(\varphi/2)\cosh(z/2)} \nabla^{(h')i}(z)\,\mathrm{d}A' \tag{5.95}$$

by Stokes' theorem, where $n_i$ is the appropriate normal for each integral. Note that my sign convention in equation 5.95 means $n_i$ points towards infinity in $\int_{S^{n-2}_\infty}$ and $n_i$ points into the domain of outer communication in $\int_{\mathcal{H}}$. Also note that since $\varphi = 0$ on $\mathcal{H}$, the 2nd surface integral should be seen as

$$\lim_{\varphi_0 \to 0} \int_{\{\varphi = \varphi_0\}} n_i \frac{1}{\sinh(\varphi/2)\cosh(z/2)} \nabla^{(h')i}(z)\,\mathrm{d}A'. \tag{5.96}$$

The domain of integration, $\{\varphi = \varphi_0\}$, is a regular hypersurface since $\mathrm{d}\varphi \neq 0$ on the horizon (I am only considering non-extremal black holes) and only $\varphi_0$ infinitesimally close to 0 matters in taking the limit. Likewise, the first surface integral should be interpreted as

$$\lim_{r \to \infty} \int_{S^{n-2}_r} n_i \frac{1}{\sinh(\varphi/2)\cosh(z/2)} \nabla^{(h')i}(z)\,\mathrm{d}A' \tag{5.97}$$

and the radius-$r$ spheres are defined by $\Sigma_t$ being an asymptotically flat end.
$\therefore n_i \propto (\mathrm{d}r)_i$ in the 1st surface integral. The requirement that $h'^{ij} n_i n_j = 1$ determines the proportionality constant.
Use the almost Cartesian coordinates, $x_i$ (I will always leave the $x_i$ indices down by convention), that arise from $\Sigma_t$ being an asymptotically flat end.
$\therefore (\mathrm{d}r)_i \propto x_i$ and thus $n_i \propto x_i$.
By lemma 5.11 and definition 5.7,

$$\Omega = (1 - S^2 - C^2\psi^2)^{1/(n-3)} \to \left(1 - \left(1 - \frac{m}{2r^{n-3}}\right)^2 - \frac{C^2 q^2}{r^{2(n-3)}}\right) \to \frac{m^{1/(n-3)}}{r}. \tag{5.98}$$

$$\therefore h_{ij} \to \delta_{ij} \implies h'_{ij} \to \frac{m^{2/(n-3)}}{r^2}\delta_{ij}. \tag{5.99}$$

$$\therefore h'^{ij} n_i n_j = 1 \implies n_i = \frac{m^{1/(n-3)}}{r^2} x_i. \tag{5.100}$$

From here, I can now determine the asymptotic behaviour of $\varphi$ and $z$ at $S^{n-2}_\infty$.

$$(1+S)^2 - C^2\psi^2 \to \left(1 + 1 - \frac{m}{2r^{n-3}}\right)^2 - \frac{C^2 q^2}{r^{2(n-3)}} \to 4. \tag{5.101}$$

$$(1-S)^2 - C^2\psi^2 \to \left(1 - 1 - \frac{m}{2r^{n-3}}\right)^2 - \frac{C^2 q^2}{r^{2(n-3)}} \to \frac{m^2 - 4C^2 q^2}{4r^{2(n-3)}}. \tag{5.102}$$

$$\therefore \sinh(\varphi/2) \to \frac{1}{2}(\mathrm{e}^{\varphi/2} - \mathrm{e}^{-\varphi/2}) \tag{5.103}$$

$$= \frac{1}{2}\left( \sqrt{\frac{(1+S)^2 - C^2\psi^2}{(1-S)^2 - C^2\psi^2}} - \sqrt{\frac{(1-S)^2 - C^2\psi^2}{(1+S)^2 - C^2\psi^2}} \right) \tag{5.104}$$

$$\to \frac{1}{2}\left( \sqrt{\frac{16}{m^2 - 4C^2 q^2} r^{2(n-3)}} - 0 \right) \tag{5.105}$$

$$= \frac{2}{\sqrt{m^2 - 4C^2 q^2}} r^{n-3}. \tag{5.106}$$

$$\therefore \frac{1}{\sinh(\varphi/2)} \to \frac{1}{2r^{n-3}}\sqrt{m^2 - 4C^2 q^2}. \tag{5.107}$$

Similarly, for $z$ I get the following.

$$(1 + C\psi)^2 - S^2 \to \left(1 - \frac{Cq}{r^{n-3}}\right)^2 - \left(1 - \frac{m}{2r^{n-3}}\right)^2 \to \frac{m - 2Cq}{r^{n-3}}. \tag{5.108}$$

$$(1 - C\psi)^2 - S^2 \to \left(1 + \frac{Cq}{r^{n-3}}\right)^2 - \left(1 - \frac{m}{2r^{n-3}}\right)^2 \to \frac{m + 2Cq}{r^{n-3}}. \tag{5.109}$$

$$\therefore z \to \ln\left(\frac{m - 2Cq}{m + 2Cq}\right) = z_1, \text{ say.} \tag{5.110}$$

While $z$ takes the constant value, $z_1$, at infinity, what I need for equation 5.97 is $\nabla^{(h')i}(z)$.

$$\nabla_i^{(h')}(z) = \frac{2C(1 + C\psi)\nabla_i^{(h')}\psi - 2S\nabla_i^{(h')}S}{(1 + C\psi)^2 - S^2} + \frac{2C(1 + C\psi)\nabla_i^{(h')}\psi + 2S\nabla_i^{(h')}S}{(1 - C\psi)^2 - S^2} \tag{5.111}$$

$$\to \frac{2r^{n-3}}{m - 2Cq}\left(\frac{(n-3)Cq}{r^{n-2}} - \frac{(n-3)m}{r^{n-2}}\right)\frac{x_i}{r} + \frac{2r^{n-3}}{m + 2Cq}\left(\frac{(n-3)Cq}{r^{n-2}} + \frac{(n-3)m}{r^{n-2}}\right)\frac{x_i}{r} \tag{5.112}$$

$$= -\frac{(n-3)x_i}{r^2} + \frac{(n-3)x_i}{r^2} \tag{5.113}$$

$$= 0. \tag{5.114}$$

$$\therefore \nabla_i^{(h')}(z) \to x_i f, \text{ for some function, } f, \text{ that's } O(1/r^3). \tag{5.115}$$

Putting the different parts together, the integrand goes as

$$n_i \frac{1}{\sinh(\varphi/2)\cosh(z/2)}\nabla^{(h')i}(z) \to \frac{m^{1/(n-3)}}{r^2} x_i \frac{1}{2r^{n-3}}\sqrt{m^2 - 4C^2q^2}\frac{1}{\cosh(z_0/2)}\frac{r^2}{m^{1/(n-3)}} f x_i \tag{5.116}$$

$$= O(1/r^{n-2}), \tag{5.117}$$

which goes to zero as $r \to \infty$.

$$\therefore \lim_{r\to\infty}\int_{S_r^{n-2}} n_i \frac{1}{\sinh(\varphi/2)\cosh(z/2)}\nabla^{(h')i}(z)\,\mathrm{d}A' = 0 \tag{5.118}$$

because in the $h'$ metric, $S_\infty^{n-2}$ has finite area,

$$A'_{S_\infty^{n-2}} = r^{n-2}\omega_{n-2}\Omega^{n-2} \tag{5.119}$$

$$= m^{(n-2)/(n-3)}\omega_{n-2}, \tag{5.120}$$

where $\omega_{n-2}$ is the area of a unit radius $S^{n-2}$.
Hence, by equation 5.95,

$$\int_{\mathcal{H}} n_i \frac{1}{\sinh(\varphi/2)\cosh(z/2)}\nabla^{(h')i}(z)\,\mathrm{d}A' = 0. \tag{5.121}$$

$\psi$ and $S$ are both constants on $\mathcal{H}$ though - namely $\psi_0$ and 0 - meaning $z$ is also a constant,

$$z|_{\mathcal{H}} = z_0 = 2\ln\left(\frac{1 + C\psi_0}{1 - C\psi_0}\right). \tag{5.122}$$

Hence, I can conclude

$$\int_{\mathcal{H}} n_i \frac{1}{\sinh(\varphi/2)} \nabla^{(h')i}(z) \mathrm{d}A' = 0. \tag{5.123}$$

This result is quite useful because of the next divergence I'll work with. Observe that

$$\nabla_i^{(h')} \left( \frac{1}{\sinh(\varphi/2)} \nabla^{(h')i} z \right) = \frac{1}{\sinh(\varphi/2)} \Box^{(h')} z - \frac{\cosh(\varphi/2)}{2\sinh^2(\varphi/2)} \nabla_i^{(h')}(\varphi) \nabla^{(h')i}(z) \tag{5.124}$$

$$= \frac{\tanh(z/2)}{2\sinh(\varphi/2)} ||\nabla^{(h')} z||^2 \text{ by equation 5.91,} \tag{5.125}$$

where $|| \cdot ||$ denotes the natural norm with respect to $h'$.

$$\therefore \int_{\Sigma_t} \frac{\tanh(z/2)}{2\sinh(\varphi/2)} ||\nabla^{(h')} z||^2 \mathrm{d}V'$$

$$= \int_{\Sigma_t} \nabla_i^{(h')} \left( \frac{1}{\sinh(\varphi/2)} \nabla^{(h')i} z \right) \mathrm{d}V' \tag{5.126}$$

$$= \int_{S_\infty^{n-2}} n_i \frac{1}{\sinh(\varphi/2)} \nabla^{(h')i}(z) \mathrm{d}A' - \int_{\mathcal{H}} n_i \frac{1}{\sinh(\varphi/2)} \nabla^{(h')i}(z) \mathrm{d}A'. \tag{5.127}$$

The $\mathcal{H}$ integral is zero by equation 5.123. The $S_\infty^{n-2}$ integral is zero by equations 5.110 and 5.118.

$$\therefore \int_{\Sigma_t} \frac{\tanh(z/2)}{2\sinh(\varphi/2)} ||\nabla^{(h')} z||^2 \mathrm{d}V' = 0. \tag{5.128}$$

Since the integrand has definite sign at all times, the integral being zero implies $||\nabla^{(h')} z||^2 = 0$ and thus $\nabla_i^{(h')} z = 0$.
$\therefore z$ is a constant. $\Box$

**Corollary 5.13.1.** $\Box^{(h')} \varphi = 0$.

*Proof.* $z$ being a constant means $\nabla_i^{(h')}(z) = 0$ in equation 5.90. $\Box$

**Corollary 5.13.2.** *$\psi$ is fully determined in terms of $S$. In particular,*

$$0 = 1 - S^2 + \frac{m}{q}\psi + C^2\psi^2. \tag{5.129}$$

*Proof.* By equation 5.110 and theorem 5.13,

$$z = \ln \left( \frac{m - 2Cq}{m + 2Cq} \right). \tag{5.130}$$

By definition 5.47, it follows that

$$\ln \left( \frac{m - 2Cq}{m + 2Cq} \right) = \ln \left( \frac{(1 + C\psi)^2 - S^2}{(1 - C\psi)^2 - S^2} \right) \tag{5.131}$$

$$\iff \frac{m - 2Cq}{m + 2Cq} = \frac{(1 + C\psi)^2 - S^2}{(1 - C\psi)^2 - S^2} \tag{5.132}$$

$$\iff m - 2Cm\psi + mC^2\psi^2 - mS^2 - 2Cq + 4C^2q\psi - 2C^3q\psi^2 + 2CqS^2$$
$$= m + 2Cm\psi + mC^2\psi^2 - mS^2 + 2Cq + 4C^2q\psi + 2C^3q\psi^2 - 2CqS^2 \tag{5.133}$$

$$\iff 0 = 4Cm\psi + 4Cq + 4C^3q\psi^2 - 4CqS^2 \tag{5.134}$$

$$\iff 0 = \frac{m}{q}\psi + 1 + C^2\psi^2 - S^2. \tag{5.135}$$

$\Box$

Note that corollary 5.13.2 holds in all dimensions, $n \geq 4$. Unlike previous work on this or similar problems - e.g. see [11, 8] and references therein - I did not have to appeal to the positive energy theorem to derive the the relationship between the electric potential, $\psi$, and the lapse, $S$. As far as I know, for $n \geq 5$, this is the first time the positive energy theorem has not been required to establish this relationship.

Having deduced $z$ to be a constant, the conformal transformation of equation 5.46 is greatly simplified.

**Theorem 5.14.** *In terms of $\varphi$, $z$ and $h'$, equation 5.46 is equivalent to*

$$R_{ij}^{(h')} = \frac{1}{2}\coth(\varphi/2)\nabla_i^{(h')}\nabla_j^{(h')}(\varphi) - \frac{1}{4(n-3)}\nabla_i^{(h')}(\varphi)\nabla_j^{(h')}(\varphi) + \frac{1}{4(n-3)}h'_{ij}\nabla_k^{(h')}(\varphi)\nabla^{(h')k}(\varphi). \tag{5.136}$$

*Proof.* The proof is still tedious and uninsightful, so I've placed it in appendix B. □

Note that despite the presence of the source-free electric field, the system of PDEs to solve - equation 5.136 & corollary 5.13.1 with $\varphi = 0$ on $\mathcal{H}$ & $\varphi = \infty$ at $S_\infty^{n-2}$ - are exactly the same as the one studied in [13] for the vacuum case. The rest of the uniqueness proof then works similarly[2] to [13]. I highlight the salient steps below. One of the novelties of [13] is their method of detecting spherical symmetry in the solutions. I confirm the same method works in the present scenario. While I focus below only on the modifications relevant to the black hole uniqueness problem, one could also follow the methods of [13] to generalise the Willmore-type inequalities and other results of [13] to the static bounded potentials arising from the Einstein-Maxwell system.

**Lemma 5.15.** *If $\nabla_i^{(h')}\nabla_j^{(h')}\varphi = 0$, then $(M, g)$ is isometric to the Reissner-Nordstrom solution.*

*Proof.* The key ideas of the proof are from [14, 13]. First observe that $\nabla_i^{(h')}\nabla_j^{(h')}\varphi = 0$ implies

$$\nabla_i^{(h')}(||\nabla^{(h')}\varphi||^2) = \nabla_i^{(h')}(\nabla_j^{(h')}(\varphi)\nabla^{(h')j}(\varphi)) = 2\nabla_i^{(h')}\nabla_j^{(h')}(\varphi)\nabla^{(h')j}(\varphi) = 0. \tag{5.137}$$

$\therefore ||\nabla^{(h')}\varphi||$ is a constant.

$$\nabla_i^{(h')}\varphi = \frac{2(1+S)\nabla_i^{(h')}S - 2C^2\psi\nabla_i^{(h')}\psi}{(1+S)^2 - C^2\psi^2} + \frac{2(1-S)\nabla_i^{(h')}S + 2C^2\psi\nabla_i^{(h')}\psi}{(1-S)^2 - C^2\psi^2} \tag{5.138}$$

$$\to \frac{1}{2}\left((1+S)\nabla_i^{(h')}S - C^2\psi\nabla_i^{(h')}\psi\right) + \frac{8r^{2(n-3)}}{m^2 - 4C^2q^2}\left((1-S)\nabla_i^{(h')}S + C^2\psi\nabla_i^{(h')}\psi\right) \tag{5.139}$$

by equations 5.101 and 5.102.
As summarised in definition 5.7, $S \to 1 - \frac{m}{2r^{n-3}}$ and $\psi \to -\frac{q}{r^{n-3}}$.

$$\therefore 1 + S \to 2, \tag{5.140}$$

$$1 - S \to \frac{m}{2r^{n-3}}, \tag{5.141}$$

$$\nabla_i^{(h')}S \to \frac{(n-3)m}{2r^{n-1}}x_i \text{ and} \tag{5.142}$$

$$\nabla_i^{(h')}\psi \to \frac{(n-3)q}{r^{n-1}}x_i. \tag{5.143}$$

[2]The proof is not quite identical because the presence of the extra field, $\psi$, changes some of the derivatives, and can change some of the asymptotics.

To leading order, equation 5.139 then says

$$\nabla_i^{(h')}\varphi \to \frac{8r^{2(n-3)}}{m^2-4C^2q^2}\left(\frac{m}{2r^{n-3}}\frac{(n-3)m}{2r^{n-1}}x_i - C^2\frac{q}{r^{n-3}}\frac{(n-3)q}{r^{n-1}}x_i\right) \tag{5.144}$$

$$= \frac{2(n-3)}{r^2}x_i. \tag{5.145}$$

From equation 5.99, I then get

$$||\nabla^{(h')}\varphi||^2 = h'^{ij}\nabla_i^{(h')}(\varphi)\nabla_j^{(h')}(\varphi) \to \frac{r^2}{m^{2(n-3)}}\delta^{ij}\frac{2(n-3)}{r^2}x_i\frac{2(n-3)}{r^2}x_j = \frac{4(n-3)^2}{m^{2/(n-3)}}. \tag{5.146}$$

$\therefore ||\nabla^{(h')}\varphi|| = \frac{2(n-3)}{m^{1/(n-3)}}$ at $S^{n-2}_\infty$.
$\therefore ||\nabla^{(h')}\varphi|| = \frac{2(n-3)}{m^{1/(n-3)}}$ everywhere.
As this is non-zero everywhere, I can use $\varphi$ as a local coordinate, essentially by the implicit function theorem.
Let $\{x^A\}_{A=2}^{n-1}$ be local coordinates on a particular constant $\varphi$ surface.
$\therefore \mathrm{d}\varphi \equiv \nabla_i^{(h')}\varphi$ is normal to that hypersurface.
Extend $x^A$ off that hypersurface by keeping $x^A$ constant along flows of $(\mathrm{d}\varphi)^a$.
That way only $\varphi$ changes along flows of $(\mathrm{d}\varphi)^a$, meaning $(\mathrm{d}\varphi)^a \propto (\partial/\partial\varphi)^a$.
$\therefore$ There are no $\mathrm{d}\varphi$-$\mathrm{d}x^A$ cross terms in the metric.
From the value of $||\nabla^{(h')}\varphi||$, I can then conclude that

$$h' = \frac{m^{2/(n-3)}}{4(n-3)^2}\mathrm{d}\varphi\otimes\mathrm{d}\varphi + \tilde{h}'_{AB}\mathrm{d}x^A\otimes\mathrm{d}x^B, \tag{5.147}$$

for some invertible $\tilde{h}'_{AB}$. Then,

$$0 = \nabla_i^{(h')}\nabla_j^{(h')}\varphi = \partial_i\partial_j\varphi - \Gamma^{(h')k}{}_{ji}\nabla_k^{(h')}\varphi = -\Gamma^{(h')1}{}_{ji} \tag{5.148}$$

in the $(\varphi, x^A)$ coordinates.
Choose $(i,j) = (A,B)$.

$$\therefore 0 = \Gamma^{(h')1}{}_{BA} = \frac{1}{2}h'^{1i}(\partial_B h'_{A1} + \partial_A h'_{1B} - \partial_1 h'_{BA}) = -\frac{2(n-3)^2}{m^{2/(n-3)}}\partial_\varphi\tilde{h}'_{AB} \implies \partial_\varphi\tilde{h}'_{AB} = 0. \tag{5.149}$$

$$\therefore h' = \frac{m^{2/(n-3)}}{4(n-3)^2}\mathrm{d}\varphi\otimes\mathrm{d}\varphi + \tilde{h}'|_{\{\varphi=\varphi_0\}} \text{ for any } \varphi_0. \tag{5.150}$$

Choose $\varphi_0 \to \infty$. Then, $\tilde{h}' \to m^{2/(n-3)}g_{S^{n-2}}$ by equation 5.99. Hence,

$$h' = \frac{m^{2/(n-3)}}{4(n-3)^2}\mathrm{d}\varphi\otimes\mathrm{d}\varphi + m^{2/(n-3)}g_{S^{n-2}}. \tag{5.151}$$

The metric of the physical spacetime - which is what I'm actually interested in - is then

$$g = -S^2\mathrm{d}t\otimes\mathrm{d}t + h \tag{5.152}$$

$$= -S^2\mathrm{d}t\otimes\mathrm{d}t + \frac{1}{\Omega^2}h' \tag{5.153}$$

$$= -S^2\mathrm{d}t\otimes\mathrm{d}t + \frac{m^{2/(n-3)}}{4(n-3)^2\Omega^2}\mathrm{d}\varphi\otimes\mathrm{d}\varphi + \frac{m^{2/(n-3)}}{\Omega^2}g_{S^{n-2}}. \tag{5.154}$$

By equation 5.84, theorem 5.9 and theorem 5.13, $S$ and $\Omega$ only depend on $\varphi$.
$\therefore$ The metric in equation 5.154 is spherically symmetric. Since $||\nabla^{(h')}\varphi|| \neq 0$, the area-radius function, $r = m^{1/(n-3)}/\Omega$, is non-constant.
$\therefore$ By the version of Birkhoff's theorem applicable here, $(M, g)$ is isometric to the Reissner-Nordstrom solution. $\square$

The logical next step after this lemma is to prove that $\nabla_i^{(h')}\nabla_j^{(h')}(\varphi) = 0$ is actually true.

**Lemma 5.16.** *Let $R^{(h',\mathcal{H})}$ be the Ricci scalar of $\mathcal{H}$ in the $h'$ metric and let*

$$\kappa' = \frac{4\kappa}{(1 - C^2\psi_0^2)(1 + C^2\psi_0)^{1/(n-3)}}. \tag{5.155}$$

*Then,*

$$\int_{\Sigma_t} \frac{||\nabla^{(h')}\nabla^{(h')}\varphi||^2}{\sinh(\varphi/2)} \mathrm{d}V' = \kappa' \left( \int_{\mathcal{H}} R^{(h',\mathcal{H})} \mathrm{d}A' - \frac{(n-2)\kappa'^2}{4(n-3)} A' \right). \tag{5.156}$$

*Proof.* First observe that $\kappa'$ is the analogue of the surface gravity in the $h'$ and $\varphi$ variables. In particular, since $\nabla^{(h)}\psi = 0$ and $S = 0$ on $\mathcal{H}$,

$$\begin{aligned}
&\nabla_i^{(h')}(\varphi)\nabla^{(h')i}(\varphi)|_{\mathcal{H}} \\
&= \left( \frac{2(1+S)\nabla_i^{(h)}S - 2C^2\psi\nabla_i^{(h)}\psi}{(1+S)^2 - C^2\psi^2} + \frac{2(1-S)\nabla_i^{(h)}S + 2C^2\psi\nabla_i^{(h)}\psi}{(1-S)^2 - C^2\psi^2} \right)\Bigg|_{\mathcal{H}} \frac{1}{\Omega^2|_{\mathcal{H}}} \\
&\quad \times \left( \frac{2(1+S)\nabla^{(h)i}S - 2C^2\psi\nabla^{(h)i}\psi}{(1+S)^2 - C^2\psi^2} + \frac{2(1-S)\nabla^{(h)i}S + 2C^2\psi\nabla^{(h)i}\psi}{(1-S)^2 - C^2\psi^2} \right)\Bigg|_{\mathcal{H}} &(5.157) \\
&= \frac{16\nabla_i^{(h)}(S)\nabla^{(h)i}(S)|_{\mathcal{H}}}{1 - C^2\psi_0^2} \frac{1}{(1 + C^2\psi_0^2)^{2/(n-3)}} &(5.158) \\
&= \frac{16\kappa^2}{(1 - C^2\psi_0^2)(1 + C^2\psi_0^2)^{2/(n-3)}} &(5.159) \\
&= \kappa'^2. &(5.160)
\end{aligned}$$

Next, observe that

$$\begin{aligned}
&\nabla_i^{(h')}\left( \frac{1}{\sinh(\varphi/2)} \nabla^{(h')i}(||\nabla^{(h')}\varphi||^2) \right) \\
&= 2\nabla_i^{(h')}\left( \frac{1}{\sinh(\varphi/2)} \nabla^{(h')i}\nabla^{(h')j}(\varphi)\nabla_j^{(h')}(\varphi) \right) &(5.161) \\
&= -\frac{\cosh(\varphi/2)}{\sinh^2(\varphi/2)} \nabla_i^{(h')}\nabla_j^{(h')}(\varphi)\nabla^{(h')i}(\varphi)\nabla^{(h')j}(\varphi) + \frac{2}{\sinh(\varphi/2)} \nabla_i^{(h')}\nabla^{(h')j}\nabla^{(h')i}(\varphi)\nabla_j^{(h')}(\varphi) \\
&\quad + \frac{2}{\sinh(\varphi/2)} ||\nabla^{(h')}\nabla^{(h')}\varphi||^2 &(5.162) \\
&= -\frac{\cosh(\varphi/2)}{\sinh^2(\varphi/2)} \nabla_i^{(h')}\nabla_j^{(h')}(\varphi)\nabla^{(h')i}(\varphi)\nabla^{(h')j}(\varphi) + \frac{2}{\sinh(\varphi/2)} [\nabla_i^{(h')}, \nabla^{(h')j}]\nabla^{(h')i}(\varphi)\nabla_j^{(h')}(\varphi) \\
&\quad + \frac{2}{\sinh(\varphi/2)} ||\nabla^{(h')}\nabla^{(h')}\varphi||^2 \text{ by corollary 5.13.1} &(5.163) \\
&= -\frac{\cosh(\varphi/2)}{\sinh^2(\varphi/2)} \nabla_i^{(h')}\nabla_j^{(h')}(\varphi)\nabla^{(h')i}(\varphi)\nabla^{(h')j}(\varphi) + \frac{2}{\sinh(\varphi/2)} R_{ij}^{(h')}\nabla^{(h')i}(\varphi)\nabla^{(h')i}(\varphi) \\
&\quad + \frac{2}{\sinh(\varphi/2)} ||\nabla^{(h')}\nabla^{(h')}\varphi||^2 &(5.164)
\end{aligned}$$

Then, by theorem 5.14,

$$\begin{aligned}
&\nabla_i^{(h')}\left(\frac{1}{\sinh(\varphi/2)}\nabla^{(h')i}(||\nabla^{(h')}\varphi||^2)\right)\\
&= -\frac{\cosh(\varphi/2)}{\sinh^2(\varphi/2)}\nabla_i^{(h')}\nabla_j^{(h')}(\varphi)\nabla^{(h')i}(\varphi)\nabla^{(h')j}(\varphi) + \frac{2}{\sinh(\varphi/2)}||\nabla^{(h')}\nabla^{(h')}\varphi||^2\\
&\quad + \frac{2}{\sinh(\varphi/2)}\left(\frac{1}{2}\coth(\varphi/2)\nabla_i^{(h')}\nabla_j^{(h')}(\varphi) - \frac{1}{4(n-3)}\nabla_i^{(h')}(\varphi)\nabla_j^{(h')}(\varphi)\right.\\
&\quad \left. + \frac{1}{4(n-3)}h'_{ij}\nabla_k^{(h')}(\varphi)\nabla^{(h')k}(\varphi)\right)\nabla^{(h')i}(\varphi)\nabla^{(h')j}(\varphi) \qquad (5.165)\\
&= \frac{2}{\sinh(\varphi/2)}||\nabla^{(h')}\nabla^{(h')}\varphi||^2. \qquad (5.166)
\end{aligned}$$

Thus, by Stokes' theorem,

$$\begin{aligned}
&\int_{\Sigma_t}\frac{2}{\sinh(\varphi/2)}||\nabla^{(h')}\nabla^{(h')}\varphi||^2 \mathrm{d}V'\\
&= \int_{S_\infty^{n-2}} n_i\frac{1}{\sinh(\varphi/2)}\nabla^{(h')i}(||\nabla^{(h')}\varphi||^2)\mathrm{d}A' - \int_{\mathcal{H}} n_i\frac{1}{\sinh(\varphi/2)}\nabla^{(h')i}(||\nabla^{(h')}\varphi||^2)\mathrm{d}A'. \qquad (5.167)
\end{aligned}$$

On $\mathcal{H}$, by equation 5.160, $n_i = \frac{1}{\kappa'}\nabla_i^{(h')}(\varphi)$.

$$\therefore \int_{\mathcal{H}} n_i\frac{1}{\sinh(\varphi/2)}\nabla^{(h')i}(||\nabla^{(h')}\varphi||^2)\mathrm{d}A' = \int_{\mathcal{H}}\frac{2\nabla_i^{(h')}\nabla_j^{(h')}(\varphi)\nabla^{(h')i}(\varphi)\nabla^{(h')i}(\varphi)}{\kappa'\sinh(\varphi/2)}\mathrm{d}A'. \qquad (5.168)$$

Meanwhile at $S_\infty^{n-2}$, $\nabla_i^{(h')}(||\nabla^{(h')}\varphi||^2)$ is $O(1/r)$ by equation 5.146, $n_i = \frac{m^{1/(n-3)}}{r^2}x_i$ by equation 5.100, $h'_{ij} = \frac{m^{2/(n-3)}}{r^2}\delta_{ij}$ by equation 5.99, the area of $S_\infty^{n-2}$ is finite by equation 5.120 and $\varphi = \infty$.

$$\therefore \int_{S_\infty^{n-2}} n_i\frac{1}{\sinh(\varphi/2)}\nabla^{(h')i}(||\nabla^{(h')}\varphi||^2)\mathrm{d}A' = 0. \qquad (5.169)$$

Now, equation 5.167 says

$$\int_{\Sigma_t}\frac{||\nabla^{(h')}\nabla^{(h')}\varphi||^2}{\sinh(\varphi/2)}\mathrm{d}V' = -\int_{\mathcal{H}}\frac{\nabla_i^{(h')}\nabla_j^{(h')}(\varphi)\nabla^{(h')i}(\varphi)\nabla^{(h')i}(\varphi)}{\kappa'\sinh(\varphi/2)}\mathrm{d}A'. \qquad (5.170)$$

By the Gauss-Codacci equations, $R^{(h',\mathcal{H})}$, the Ricci scalar of $\mathcal{H}$, is

$$R^{(h',\mathcal{H})} = R^{(h')} - 2R_{ij}^{(h')}n^in^j + K^{(h')2} - K_{ij}^{(h')}K^{(h')ij}, \qquad (5.171)$$

where $K_{ij}^{(h')}$ is $\mathcal{H}$'s extrinsic curvature.
Upon a conformal transformation, the extrinsic curvature transforms as [15]

$$K_{ij}^{(h')} = \Omega K_{ij}^{(h)} + n_k(h'_{ij} - n_in_j)\nabla^{(h')k}(\ln(\Omega)). \qquad (5.172)$$

By corollary 2.13.1, $K_{ij}^{(h)}$ is zero[3].
Meanwhile, equation A.20 with $z$ constant and $\varphi = 0$ on $\mathcal{H}$ means $\nabla^{(h')k}(\ln(\Omega)) = 0$ on $\mathcal{H}$ too.
$\therefore K_{ij}^{(h')} = 0$ on $\mathcal{H}$.

$$\begin{aligned}
\therefore R^{(h',\mathcal{H})} &= R^{(h')} - 2R_{ij}^{(h')}n^in^j \qquad (5.173)\\
&= R^{(h')} - \frac{2}{\kappa'^2}R_{ij}^{(h')}\nabla^{(h')i}(\varphi)\nabla^{(h')j}(\varphi). \qquad (5.174)
\end{aligned}$$

[3]Corollary 2.13.1 only says $K_{AB} = 0$, but in the Israel coordinates used there, $K_{0i} = 0$ automatically, so the whole tensor is indeed zero.

By equation 5.136 and corollary 5.13.1, $R^{(h')} = \frac{n-2}{4(n-3)}\nabla_i^{(h')}(\varphi)\nabla^{(h')i}(\varphi)$, which equals $\frac{(n-2)\kappa'^2}{4(n-3)}$ on $\mathcal{H}$.
Also from equation 5.136, $R_{ij}^{(h')}\nabla^{(h')i}(\varphi)\nabla^{(h')j}(\varphi) = \frac{1}{2}\coth(\varphi/2)\nabla_i^{(h')}\nabla_j^{(h')}(\varphi)\nabla^{(h')i}(\varphi)\nabla^{(h')j}(\varphi)$.

$$\therefore \frac{\nabla_i^{(h')}\nabla_j^{(h')}(\varphi)\nabla^{(h')i}(\varphi)\nabla^{(h')j}(\varphi)}{\sinh(\varphi/2)} = \frac{2R_{ij}^{(h')}\nabla^{(h')i}(\varphi)\nabla^{(h')j}(\varphi)}{\cosh(\varphi/2)} \tag{5.175}$$

$$= \frac{\kappa'^2}{\cosh(\varphi/2)}\left(R^{(h')} - R^{(h',\mathcal{H})}\right) \tag{5.176}$$

$$= \frac{\kappa'^2}{\cosh(\varphi/2)}\left(\frac{(n-2)\kappa'^2}{4(n-3)} - R^{(h',\mathcal{H})}\right). \tag{5.177}$$

Substituting this back into equation 5.170 and noting that $\varphi = 0$ on $\mathcal{H}$,

$$\int_{\Sigma_t}\frac{||\nabla^{(h')}\nabla^{(h')}\varphi||^2}{\sinh(\varphi/2)}\mathrm{d}V' = \kappa'\int_{\mathcal{H}}\left(R^{(h',\mathcal{H})} - \frac{(n-2)\kappa'^2}{4(n-3)}\right)\mathrm{d}A' \tag{5.178}$$

$$= \kappa'\left(\int_{\mathcal{H}} R^{(h',\mathcal{H})}\mathrm{d}A' - \frac{(n-2)\kappa'^2}{4(n-3)}A'\right), \tag{5.179}$$

which is the claimed identity. □

**Corollary 5.16.1.** *If $\int_{\mathcal{H}} R^{(h',\mathcal{H})}\mathrm{d}A' \leq \frac{(n-2)\kappa'^2}{4(n-3)}A'$, then the spacetime is isometric to Reissner-Nordstrom.*

*Proof.* The LHS of equation 5.156 is $\geq 0$, but if the assumption of the corollary is true, the RHS would be $\leq 0$.
$\therefore$ Equation 5.156 can only hold if $\int_{\Sigma_t}\frac{||\nabla^{(h')}\nabla^{(h')}\varphi||^2}{\sinh(\varphi/2)}\mathrm{d}V' = 0$.
Since the integrand is non-negative and continuous, it must be that $||\nabla^{(h')}\nabla^{(h')}\varphi||^2 = 0$.
$\therefore \nabla_i^{(h')}\nabla_j^{(h')}\varphi = 0$.
The result then follows from lemma 5.15. □

The inability to actually do the integral, $\int_{\mathcal{H}} R^{(h',\mathcal{H})}\mathrm{d}A'$, in higher dimensions is a shortcoming of [13]'s method that I was not able to ameliorate. However, like [13], I can rephrase the inequality on $\int_{\mathcal{H}} R^{(h',\mathcal{H})}\mathrm{d}A'$ in a potentially more useful form and I can eliminate the issue entirely when $n = 4$.

**Lemma 5.17.** *The mass parameter, $m$, satisfies both*

$$m \leq \frac{1}{\omega_{n-2}}\sqrt{\frac{A'}{(n-2)(n-3)}\int_{\mathcal{H}} R^{(h',\mathcal{H})}\mathrm{d}A'} \text{ and } m \geq \left(\frac{A'}{\omega_{n-2}}\right)^{(n-3)/(n-2)}, \tag{5.180}$$

*where $\omega_{n-2}$ is the area of a unit $S^{n-2}$.*

*Proof.* First I'll prove the analogue of the Smarr relation in the variables deployed here. By corollary 5.13.1, $\Box^{(h')}\varphi = 0$.

$$\therefore 0 = \int_{\Sigma_t}\Box^{(h')}(\varphi)\mathrm{d}V' \tag{5.181}$$

$$= \int_{S_\infty^{n-2}} n_i\nabla^{(h')i}(\varphi)\mathrm{d}A' - \int_{\mathcal{H}} n_i\nabla^{(h')i}(\varphi)\mathrm{d}A'. \tag{5.182}$$

From equation 5.160, $n_i = \frac{1}{\kappa'}\nabla^{(h')}_i\varphi$ on $\mathcal{H}$.

$$\therefore \int_{\mathcal{H}} n_i \nabla^{(h')i}(\varphi)\mathrm{d}A' = \int_{\mathcal{H}} \frac{1}{\kappa'}\nabla^{(h')}_i(\varphi)\nabla^{(h')i}(\varphi)\mathrm{d}A' = \int_{\mathcal{H}} \kappa'\mathrm{d}A' = \kappa' A'. \tag{5.183}$$

For the other surface integral, equations 5.100, 5.99 and 5.145 say

$$\begin{aligned}
&\int_{S^{n-2}_\infty} n_i \nabla^{(h')i}(\varphi)\mathrm{d}A' \\
&= \int_{S^{n-2}_\infty} \frac{m^{1/(n-3)}}{r^2} x_i \frac{r^2}{m^{2/(n-3)}}\delta^{ij}\frac{2(n-3)}{r^2}x_j\sqrt{\left(\frac{m^{2/(n-3)}}{r^2}\right)^{n-2}} r^{n-2}\mathrm{d}\omega_{n-2} && (5.184)\\
&= 2m(n-3)\omega_{n-2}. && (5.185)
\end{aligned}$$

Hence, the new Smarr relation is

$$\kappa' A' = 2m(n-3)\omega_{n-2}. \tag{5.186}$$

Then, by equation 5.156,

$$\begin{aligned}
0 &\leq \int_{\mathcal{H}} R^{(h',\mathcal{H})}\mathrm{d}A' - \frac{(n-2)\kappa'^2}{4(n-3)}A' && (5.187)\\
&= \int_{\mathcal{H}} R^{(h',\mathcal{H})}\mathrm{d}A' - \frac{(n-2)A'}{4(n-3)}\left(\frac{2m(n-3)\omega_{n-2}}{A'}\right)^2 \quad \text{by equation 5.186.} && (5.188)\\
\therefore m &\leq \frac{1}{\omega_{n-2}}\sqrt{\frac{A'}{(n-2)(n-3)}\int_{\mathcal{H}} R^{(h',\mathcal{H})}\mathrm{d}A'}. && (5.189)
\end{aligned}$$

The other inequality requires a more scenic tour.
First observe that since $\Box^{(h')}\varphi = 0$ now,

$$\begin{aligned}
&\nabla^{(h')}_i\left(\frac{1}{\sinh(\varphi/2)}||\nabla^{(h')}\varphi||^2\nabla^{(h')i}\varphi\right)\\
&= \frac{1}{\sinh(\varphi/2)}\left(2\nabla^{(h')}_i\nabla^{(h')}_j(\varphi)\nabla^{(h')i}(\varphi)\nabla^{(h')j}(\varphi) - \frac{1}{2}\coth(\varphi/2)||\nabla^{(h')}\varphi||^4\right). && (5.190)
\end{aligned}$$

Hence, by Stokes' theorem, for any $\varphi_0$ such that $\{\varphi = \varphi_0\}$ is a regular set,

$$\begin{aligned}
&\int_{\{\varphi\geq\varphi_0\}} \frac{1}{\sinh(\varphi/2)}\left(2\nabla^{(h')}_i\nabla^{(h')}_j(\varphi)\nabla^{(h')i}(\varphi)\nabla^{(h')j}(\varphi) - \frac{1}{2}\coth(\varphi/2)||\nabla^{(h')}\varphi||^4\right)\mathrm{d}V'\\
&= \int_{S^{n-2}_\infty} \frac{n_i}{\sinh(\varphi/2)}||\nabla^{(h')}\varphi||^2\nabla^{(h')i}(\varphi)\mathrm{d}A' - \int_{\{\varphi=\varphi_0\}} \frac{n_i}{\sinh(\varphi/2)}||\nabla^{(h')}\varphi||^2\nabla^{(h')i}(\varphi)\mathrm{d}A'. && (5.191)
\end{aligned}$$

The domains of integration are valid, as follows. View $\varphi$ as a surjective function, $\varphi : M \to [0,\infty)$. By Sard's theorem, the set of regular values in $[0,\infty)$ is dense. Thus, although $\varphi_0$ cannot be chosen to be any number in $[0,\infty)$, it can be chosen arbitrarily close to any desired number in $[0,\infty)$, which suffices for my purposes.
The normal to $\{\varphi = \varphi_0\}$ is $n_i = \frac{1}{||\nabla^{(h')}\varphi||}\nabla^{(h')}_i\varphi$, so

$$\int_{\{\varphi=\varphi_0\}} \frac{n_i}{\sinh(\varphi/2)}||\nabla^{(h')}\varphi||^2\nabla^{(h')i}(\varphi)\mathrm{d}A' = \int_{\{\varphi=\varphi_0\}} \frac{||\nabla^{(h')}\varphi||^3}{\sinh(\varphi/2)}\mathrm{d}A'. \tag{5.192}$$

The integral at infinity is the same as equation 5.185, except that there is a $\sinh(\varphi/2) = \infty$ suppression in the integrand's denominator and the $||\nabla^{(h')}\varphi||^2$ in the numerator takes a finite value by equation 5.146.
$\therefore$ The integral at infinity is zero.
Hence we can define a new function, $\Phi(\varphi_0)$ as

$$\Phi(\varphi_0) = \int_{\{\varphi=\varphi_0\}} ||\nabla^{(h')}\varphi||^3 \mathrm{d}A' \tag{5.193}$$

$$= -\sinh(\varphi_0/2) \int_{\{\varphi\geq\varphi_0\}} \frac{1}{\sinh(\varphi/2)} \times \left(2\nabla_i^{(h')}\nabla_j^{(h')}(\varphi)\nabla^{(h')i}(\varphi)\nabla^{(h')j}(\varphi) - \frac{1}{2}\coth(\varphi/2)||\nabla^{(h')}\varphi||^4\right) \mathrm{d}V' \tag{5.194}$$

The coarea formula allows the volume integral to be re-written as an integral over constant $\varphi$ surface integrals[4]. It says

$$\Phi(\varphi_0) = -\sinh(\varphi_0/2) \int_{\varphi_0}^{\infty} \int_{\{\varphi=\tau\}} \frac{1}{||\nabla^{(h')}\varphi|| \sinh(\varphi/2)} \times \left(2\nabla_i^{(h')}\nabla_j^{(h')}(\varphi)\nabla^{(h')i}(\varphi)\nabla^{(h')j}(\varphi) - \frac{1}{2}\coth(\varphi/2)||\nabla^{(h')}\varphi||^4\right) \mathrm{d}A'\mathrm{d}\tau \tag{5.195}$$

$$= \sinh(\varphi_0/2) \int_{\varphi_0}^{\infty} \left(\frac{\cosh(\tau/2)}{2\sinh^2(\tau/2)}\Phi(\tau) - 2\int_{\{\varphi=\tau\}} \frac{\nabla_i^{(h')}\nabla_j^{(h')}(\varphi)\nabla^{(h')i}(\varphi)\nabla^{(h')j}(\varphi)}{||\nabla^{(h')}\varphi|| \sinh(\varphi/2)} \mathrm{d}A'\right)\mathrm{d}\tau. \tag{5.196}$$

Then, by the fundamental theorem of calculus and the product rule,

$$\Phi'(\varphi_0) = \frac{1}{2}\cosh(\varphi_0/2) \int_{\varphi_0}^{\infty} \left(\frac{\cosh(\tau/2)}{2\sinh^2(\tau/2)}\Phi(\tau) - 2\int_{\{\varphi=\tau\}} \frac{\nabla_i^{(h')}\nabla_j^{(h')}(\varphi)\nabla^{(h')i}(\varphi)\nabla^{(h')j}(\varphi)}{||\nabla^{(h')}\varphi|| \sinh(\varphi/2)} \mathrm{d}A'\right)\mathrm{d}\tau - \sinh(\varphi_0/2)\frac{\cosh(\varphi_0/2)}{2\sinh^2(\varphi_0/2)}\Phi(\varphi_0) + 2\sinh(\varphi_0/2) \int_{\{\varphi=\varphi_0\}} \frac{\nabla_i^{(h')}\nabla_j^{(h')}(\varphi)\nabla^{(h')i}(\varphi)\nabla^{(h')j}(\varphi)}{||\nabla^{(h')}\varphi|| \sinh(\varphi/2)} \mathrm{d}A' \tag{5.197}$$

$$= \frac{1}{2}\coth(\varphi_0/2)\Phi(\varphi_0) - \frac{1}{2}\coth(\varphi_0/2)\Phi(\varphi_0) + 2\int_{\{\varphi=\varphi_0\}} \frac{\nabla_i^{(h')}\nabla_j^{(h')}(\varphi)\nabla^{(h')i}(\varphi)\nabla^{(h')j}(\varphi)}{||\nabla^{(h')}\varphi||} \mathrm{d}A' \tag{5.198}$$

$$= 2\int_{\{\varphi=\varphi_0\}} \frac{\nabla_i^{(h')}\nabla_j^{(h')}(\varphi)\nabla^{(h')i}(\varphi)\nabla^{(h')j}(\varphi)}{||\nabla^{(h')}\varphi||} \mathrm{d}A'. \tag{5.199}$$

From the exact same logic that went into deriving equation 5.170, it follows that this last integral can be re-written as

$$\Phi'(\varphi_0) = -2\sinh(\varphi_0/2) \int_{\{\varphi\geq\varphi_0\}} \frac{||\nabla^{(h')}\nabla^{(h')}\varphi||^2}{\sinh(\varphi/2)} \mathrm{d}V' \tag{5.200}$$

$$\leq 0. \tag{5.201}$$

[4]Again, the Sard's theorem argument means the expression can be made sense of even though not every constant $\varphi$ surface may be regular.

$\therefore \Phi(\varphi_1) \geq \Phi(\varphi_2)$ whenever $\varphi_1 \leq \varphi_2$.
$\therefore \Phi(0) \geq \Phi(\infty)$.
Using equations 5.160, 5.146 and 5.120, it follows that

$$\Phi(0) = \int_{\{\varphi=0\}} ||\nabla^{(h')}\varphi||^3 \mathrm{d}A' = \int_{\mathcal{H}} ||\nabla^{(h')}\varphi||^3 \mathrm{d}A' = \kappa'^3 A' \text{ and} \tag{5.202}$$

$$\Phi(\infty) = \int_{\{\varphi=\infty\}} ||\nabla^{(h')}\varphi||^3 \mathrm{d}A' = \int_{S^{n-2}_\infty} ||\nabla^{(h')}\varphi||^3 \mathrm{d}A' = \frac{8(n-3)^3}{m^{3/(n-3)}} m^{(n-2)/(n-3)} \omega_{n-2}. \tag{5.203}$$

$$\therefore \kappa'^3 A' \geq 8(n-3)^3 m^{(n-5)/(n-3)} \omega_{n-2}. \tag{5.204}$$

Using equation 5.186, this inequality says

$$\frac{8m^3(n-3)^3(\omega_{n-2})^3}{A'^3} A' \geq 8(n-3)^3 m^{(n-5)/(n-3)} \omega_{n-2} \tag{5.205}$$

$$\iff m \geq \left(\frac{A'}{\omega_{n-2}}\right)^{(n-3)/(n-2)}. \tag{5.206}$$

This is exactly the second inequality claimed. ☐

**Corollary 5.17.1.** *If*

$$\int_{\mathcal{H}} R^{(h',\mathcal{H})} \mathrm{d}A' \leq (n-2)(n-3) A'^{(n-4)/(n-2)} (\omega_{n-2})^{2/(n-2)}, \tag{5.207}$$

*then the spacetime is isometric to Reissner-Nordstrom.*

*Proof.* The assumption implies that

$$\frac{1}{\omega_{n-2}} \sqrt{\frac{A'}{(n-2)(n-3)} \int_{\mathcal{H}} R^{(h',\mathcal{H})} \mathrm{d}A'}$$

$$\leq \frac{1}{\omega_{n-2}} \sqrt{\frac{A'}{(n-2)(n-3)} (n-2)(n-3) A'^{(n-4)/(n-2)} (\omega_{n-2})^{2/(n-2)}} \tag{5.208}$$

$$= \frac{1}{\omega_{n-2}} (A'^{2(n-3)/(n-2)} (\omega_{n-2})^{2/(n-2)})^{1/2} \tag{5.209}$$

$$= \left(\frac{A'}{\omega_{n-2}}\right)^{(n-3)/(n-2)}. \tag{5.210}$$

$\therefore$ Both inequalities in lemma 5.17 must actually be equalities.
From the proof of lemma 5.17, equality occurs if and only if $\nabla_i^{(h')}\nabla_j^{(h')}\varphi = 0$ everywhere; the latter condition implies the metric is isometric to Reissner-Nordstrom by lemma 5.15. ☐

**Theorem 5.18.** *When $n = 4$, the solution is isometric to Reissner-Nordstrom.*

*Proof.* When $n = 4$, the diagnostic of corollary 5.17.1 is $\int_{\mathcal{H}} R^{(h',\mathcal{H})} \mathrm{d}A' \leq 8\pi$.
Since I'm assuming the event horizon is connected and $\mathcal{H}$ is 2D when $n = 4$,

$$\int_{\mathcal{H}} R^{(h',\mathcal{H})} \mathrm{d}A' = 4\pi\chi(\mathcal{H}) \tag{5.211}$$

by the Gauss-Bonnet theorem.
The Euler characteristic of a closed 2-surface is at most 2, so one indeed gets $\int_{\mathcal{H}} R^{(h',\mathcal{H})} \mathrm{d}A' \leq 8\pi$. ☐

## 5.2 With magnetic fields

In this section I'll now drop the assumption in definition 5.1 that $\iota_k \star F = 0$, i.e. $F_{ab}$ is now allowed to have magnetic components as well.

**Lemma 5.19.** *The electromagnetic field is* $F = \mathrm{d}\psi \wedge \mathrm{d}t + \frac{1}{2}F_{ij}\mathrm{d}x^i \wedge \mathrm{d}x^j$, *for some* $\psi$ *and* $F_{ij}$ *such that* $\partial_t \psi = 0$ *and* $\partial_t F_{ij} = 0$.

*Proof.* The proof of lemma 5.2 carries through identically except that $\iota_k \star F = 0$ cannot be used to set $F_{ij}$ to zero. □

**Theorem 5.20.** *The equations of motion are now*

$$S\Box^{(h)}S = C^2\nabla^{(h)}_i(\psi)\nabla^{(h)i}(\psi) + \frac{S^2}{n-2}F_{ij}F^{ij}, \tag{5.212}$$

$$0 = \nabla^{(h)j}(SF_{ji}), \tag{5.213}$$

$$0 = F_{ij}\nabla^{(h)j}\psi, \tag{5.214}$$

$$0 = \partial_{[k}F_{ij]} = \nabla^{(h)}_{[k}F_{ij]}, \tag{5.215}$$

$$0 = \nabla^{(h)}_i\left(\frac{1}{S}\nabla^{(h)i}\psi\right) \quad \text{and} \tag{5.216}$$

$$\begin{aligned} R^{(h)}_{ij} = {} & \frac{1}{S}\nabla^{(h)}_i\nabla^{(h)}_j S + \frac{C^2}{(n-3)S^2}h_{ij}\nabla^{(h)}_k(\psi)\nabla^{(h)k}(\psi) - \frac{(n-2)C^2}{(n-3)S^2}\nabla^{(h)}_i(\psi)\nabla^{(h)}_j(\psi) \\ & + 2F_i{}^kF_{jk} - \frac{1}{n-2}h_{ij}F^{kl}F_{kl}, \end{aligned} \tag{5.217}$$

*where* $C = \sqrt{\frac{2(n-3)}{n-2}}$.

*Proof.* I will follow the logic of theorem 5.3 and borrow its calculations liberally. This time,

$$F^{ab}F_{ab} = 2F^{i0}F_{i0} + F^{ij}F_{ij} = -\frac{2}{S^2}\nabla^{(h)i}(\psi)\nabla^{(h)}_i(\psi) + F^{ij}F_{ij}. \tag{5.218}$$

Applying lemma 2.4 and equation 5.1, from the $0-0$ component of $R_{\mu\nu}$, I get

$$S\Box^{(h)}S = 2F_0{}^{\mu}F_{0\mu} - \frac{1}{n-2}g_{00}F^{cd}F_{cd} \tag{5.219}$$

$$= 2F_0{}^{i}F_{0i} - \frac{1}{n-2}(-S^2)\left(-\frac{2}{S^2}\nabla^{(h)i}(\psi)\nabla^{(h)}_i(\psi) + F^{ij}F_{ij}\right) \tag{5.220}$$

$$= C^2\nabla^{(h)i}(\psi)\nabla^{(h)}_i(\psi) + \frac{S^2}{n-2} \quad \text{borrowing from the previous calculation.} \tag{5.221}$$

Next, the $0-i$ components say

$$0 = 2F_0{}^{\mu}F_{i\mu} - \frac{1}{n-2}g_{0i}F^{ab}F_{ab} \tag{5.222}$$

$$= 2F_0{}^{j}F_{ij} - 0 \tag{5.223}$$

$$= -2\nabla^{(h)j}(\psi)F_{ij} \iff F_{ij}\nabla^{(h)j}(\psi) = 0. \tag{5.224}$$

Lastly, the $i-j$ components say

$$R^{(h)}_{ij} = \frac{1}{S}\nabla^{(h)}_i\nabla^{(h)}_j S + 2F_i{}^{\mu}F_{j\mu} - \frac{1}{n-2}g_{ij}F^{ab}F_{ab} \tag{5.225}$$

$$= \frac{1}{S}\nabla^{(h)}_i\nabla^{(h)}_j S + 2F_i{}^{0}F_{j0} + 2F_i{}^{k}F_{jk} - \frac{1}{n-2}h_{ij}\left(-\frac{2}{S^2}\nabla^{(h)k}(\psi)\nabla^{(h)}_k(\psi) + F^{kl}F_{kl}\right) \tag{5.226}$$

$$= \frac{1}{S}\nabla^{(h)}_i\nabla^{(h)}_j S - \frac{(n-2)C^2}{(n-3)S^2}\nabla^{(h)}_i(\psi)\nabla^{(h)}_j(\psi) + \frac{C^2}{(n-3)S^2}h_{ij}\nabla^{(h)}_k(\psi)\nabla^{(h)k}(\psi)$$
$$+ 2F_i{}^{k}F_{jk} - \frac{1}{n-2}h_{ij}F^{kl}F_{kl} \text{ borrowing from the previous calculation.} \tag{5.227}$$

Meanwhile, the Maxwell equation says

$$0 = \nabla^{\nu}F_{\nu\mu} \tag{5.228}$$
$$= \nabla^0 F_{0\mu} + \nabla^i F_{i\mu} \tag{5.229}$$
$$= -\frac{1}{S^2}\nabla_0 F_{0\mu} + h^{ij}\nabla_j F_{i\mu} \tag{5.230}$$
$$= -\frac{1}{S^2}\partial_t F_{0\mu} + \frac{1}{S^2}\Gamma^{\nu}{}_{00}F_{\nu\mu} + \frac{1}{S^2}\Gamma^{\nu}{}_{\mu 0}F_{0\nu} + h^{ij}\partial_j F_{i\mu} - h^{ij}\Gamma^{\nu}{}_{ij}F_{\nu\mu} - h^{ij}\Gamma^{\nu}{}_{\mu j}F_{i\nu} \tag{5.231}$$
$$= 0 + \frac{1}{S^2}\Gamma^{\nu}{}_{00}F_{\nu\mu} + \frac{1}{S^2}\Gamma^{i}{}_{\mu 0}F_{0i} + h^{ij}\partial_j F_{i\mu} - h^{ij}\Gamma^{\nu}{}_{ij}F_{\nu\mu} - h^{ij}\Gamma^{\nu}{}_{\mu j}F_{i\nu}. \tag{5.232}$$

Using lemma 2.5, I then get

$$0 = \frac{1}{S}\nabla^{(h)i}(S)F_{i\mu} + \frac{1}{S}\nabla^{(h)}(S)\delta_{\mu 0}F_{0i} + h^{ij}\partial_j F_{i\mu} - h^{ij}\Gamma^{k}{}_{ij}F_{k\mu} - h^{ij}\Gamma^{\nu}{}_{\mu j}F_{i\nu}. \tag{5.233}$$

Noting $\Gamma^k{}_{0j} = 0$, it's seen that when $\mu = 0$, one gets the exact same equation as in theorem 5.3 earlier, so the $\mu = 0$ case again says $\nabla^{(h)}_i(\frac{1}{S}\nabla^{(h)i}\psi) = 0$.
For the $\mu = i$ case, I get

$$0 = \frac{1}{S}\nabla^{(h)j}(S)F_{ji} + h^{jk}\partial_k F_{ji} - h^{jk}\Gamma^{l}{}_{jk}F_{li} - h^{jk}\Gamma^{\nu}{}_{ik}F_{j\nu} \tag{5.234}$$
$$= \frac{1}{S}\nabla^{(h)j}(S)F_{ji} + h^{jk}\partial_k F_{ji} - h^{jk}\Gamma^{(h)l}{}_{jk}F_{li} - h^{jk}\Gamma^{(h)l}{}_{ik}F_{jl} \text{ by lemma 2.5} \tag{5.235}$$
$$= \frac{1}{S}\nabla^{(h)j}(S)F_{ji} + \nabla^{(h)j}F_{ji}. \tag{5.236}$$
$$\therefore 0 = \nabla^{(h)j}(S)F_{ji} + S\nabla^{(h)j}F_{ji} = \nabla^{(h)j}(SF_{ji}). \tag{5.237}$$

Finally, $\mathrm{d}F = 0 \implies 0 = \mathrm{d}(\mathrm{d}\psi \wedge \mathrm{d}t + \frac{1}{2}F_{ij}\mathrm{d}x^i \wedge \mathrm{d}x^j) \implies \partial_{[k}F_{ij]} = \nabla^{(h)}_{[k}F_{ij]} = 0.$ □

Next, I once again need boundary conditions.
$S = 0$ on $\mathcal{H}$ for the same reasons as before.
The steps that build to proving $\nabla^{(h)}_i\psi = 0$ on $\mathcal{H}$ are unaffected by the presence of the magnetic field, except during the proof of corollary 5.5. Instead of $\Box^{(h)}S = \frac{C^2}{S}\nabla^{(h)}_i(\psi)\nabla^{(h)i}(\psi)$, this time I have $\Box^{(h)}S = \frac{C^2}{S}\nabla^{(h)}_i(\psi)\nabla^{(h)i}(\psi) + \frac{S}{n-2}F_{ij}F^{ij}$. However, because $S = 0$ on $\mathcal{H}$, the extra term makes no difference and the same reasoning can be applied again.
As for the outer boundary, $S^{n-2}_{\infty}$, I once again assume the asymptotics of definition 5.4. $F_{ij}$ will also be assumed to decay to zero at $S^{n-2}_{\infty}$, but the details of the decay will not be important for what follows.
Note that since the boundary conditions are the same, the conformal transformations of section 5.1.2 are all still well-defined, for the same reasons as before.
Finally, since $\frac{S}{n-2}F_{ij}F^{ij} \geq 0$ and equation 5.45 is unchanged upon introducing $F_{ij}$, the proof of lemma 5.6 still works analogously.

**Definition 5.21** (Problem summary). *The problem studied in this section is summarised by the equations,*

$$S\square^{(h)}S = C^2\nabla^{(h)}_i(\psi)\nabla^{(h)i}(\psi) + \frac{S^2}{n-2}F_{ij}F^{ij}, \tag{5.238}$$
$$0 = \nabla^{(h)j}(SF_{ji}), \tag{5.239}$$
$$0 = F_{ij}\nabla^{(h)j}\psi, \tag{5.240}$$
$$0 = \nabla^{(h)}_i\left(\frac{1}{S}\nabla^{(h)i}\psi\right) \quad \text{and} \tag{5.241}$$
$$\begin{aligned} R^{(h)}_{ij} &= \frac{1}{S}\nabla^{(h)}_i\nabla^{(h)}_j S + \frac{C^2}{(n-3)S^2}h_{ij}\nabla^{(h)}_k(\psi)\nabla^{(h)k}(\psi) - \frac{(n-2)C^2}{(n-3)S^2}\nabla^{(h)}_i(\psi)\nabla^{(h)}_j(\psi) \\ &\quad + 2F_i{}^kF_{jk} - \frac{1}{n-2}h_{ij}F^{kl}F_{kl}, \end{aligned} \tag{5.242}$$

*where $C = \sqrt{\frac{2(n-3)}{n-2}}$ and the boundary conditions are $S = 0$ on $\mathcal{H}$, $\psi = \psi_0$ (a constant) on $\mathcal{H}$, $\nabla^{(h)}_i\psi = 0$ on $\mathcal{H}$, $0 \le S < 1$ everywhere, $S \to 1 - \frac{m}{2r^{n-3}}$ at $S^{n-2}_\infty$ and $\psi \to -\frac{q}{r^{n-3}}$ at $S^{n-2}_\infty$.*

### 5.2.1 $n = 4$

When $n = 4$, the fact that the Hodge dual of a 2-form is again a 2-form can be leveraged to very quickly redeploy the work of the previous sections.

**Definition 5.22** (Electric and magnetic 1-forms). *Define the electric and magnetic 1-form components of $F$ by $E_a = -k^bF_{ba}$ and $B_a = k^b(\star F)_{ba}$.*

**Lemma 5.23.** *$E = \mathrm{d}\psi$ and $B = a\mathrm{d}\psi$ for some function, $\psi$ and constant, $a$.*

*Proof.* The proof that $E = \mathrm{d}\psi$ is unchanged from the corresponding reasoning in lemma 5.2. The rest is based off proposition 9.8 in [15].
$k \wedge \mathrm{d}k = 0$ because static requires $k^a$ to be hypersurface orthogonal.

$$\therefore 0 = \mathrm{d}(\star(k\wedge \mathrm{d}k)) = -\star^2\mathrm{d}(\star(k\wedge\mathrm{d}k)). \tag{5.243}$$
$$\therefore 0 = \mathrm{d}^\dagger(k\wedge\mathrm{d}k). \tag{5.244}$$

Switching to abstract indices and using that $k^a$ is Killing, this last equation says

$$\begin{aligned} 0 &= \nabla^c(k_a\nabla_bk_c + k_b\nabla_ck_a + k_c\nabla_ak_b) &(5.245)\\ &= \nabla^c(k_a)\nabla_bk_c + k_a\nabla^c\nabla_bk_c + \nabla^c(k_b)\nabla_ck_a + k_b\nabla^c\nabla_ck_a + \nabla^c(k_c)\nabla_ak_b + k_c\nabla^c\nabla_ak_b &(5.246)\\ &= \nabla^c(k_a)\nabla_bk_c - k_a\nabla^c\nabla_ck_b - \nabla_b(k^c)\nabla_ck_a + k_b\nabla^c\nabla_ck_a + 0 + k_cR^{dc}{}_{ab}k_d &(5.247)\\ &= -k_aR^{dc}{}_{cb}k_d + k_bR^{dc}{}_{ca}k_d &(5.248)\\ &= k_aR_{bc}k^c - k_bR_{ac}k^c. &(5.249) \end{aligned}$$

With the energy momentum tensor considered here, $R_{ab} = 2F_a{}^cF_{bc} - \frac{1}{2}g_{ab}F^{cd}F_{cd}$.

$$\begin{aligned} \therefore 0 &= k_ak_c\left(2F_{bd}F^{cd} - \frac{1}{2}\delta_b{}^cF^{de}F_{de}\right) - k_bk_c\left(2F_{ad}F^{cd} - \frac{1}{2}\delta_a{}^cF^{de}F_{de}\right) &(5.250)\\ &= 2k_ak_cF_{bd}F^{cd} - \frac{1}{2}k_ak_bF^{bc}F_{bc} - 2k_bk_cF_{ad}F^{cd} + \frac{1}{2}k_bk_aF^{bc}F_{bc} &(5.251)\\ &= 2k_ak_cF_{bd}F^{cd} - 2k_bk_cF_{ad}F^{cd}. &(5.252) \end{aligned}$$

Then, observe that

$$(\star(E\wedge B))_{ab} = \frac{1}{2}\varepsilon_{cdab}E^cB^d \tag{5.253}$$
$$= -\frac{1}{4}\varepsilon_{cdab}k_eF^{ec}k_f\varepsilon^{ghfd}F_{gh} \tag{5.254}$$
$$= \frac{3}{2}\delta^g{}_{[c}\delta^h{}_a\delta^f{}_{b]}k_eF^{ec}k_fF_{gh} \tag{5.255}$$
$$= \frac{1}{2}\left(\delta^f{}_b\delta^g{}_{[c}\delta^h{}_{a]} + \delta^f{}_c\delta^g{}_{[a}\delta^h{}_{b]} + \delta^f{}_a\delta^g{}_{[b}\delta^h{}_{c]}\right)k_eF^{ec}k_fF_{gh} \tag{5.256}$$
$$= \frac{1}{2}(k_bk_dF^{dc}F_{ca} + k_ck_dF^{dc}F_{ab} + k_ak_dF^{dc}F_{bc}) \tag{5.257}$$
$$= \frac{1}{2}(k_bk_dF^{dc}F_{ca} + k_ak_dF^{dc}F_{bc}) \tag{5.258}$$
$$= 0 \text{ by equation 5.252.} \tag{5.259}$$

$\therefore E\wedge B = 0 \iff E \propto B \iff E = aB$ for some function, $a$. It remains to be shown that $a$ is a constant.
To that end, I'll first note that $B$ is also a closed form because

$$\mathrm{d}B = (\iota_k \star F) \tag{5.260}$$
$$= \mathcal{L}_k \star F - \iota_k(\mathrm{d}\star F) \tag{5.261}$$

by Cartan's magic formula. In this last line, $\mathrm{d}\star F = 0$ is Maxwell's equation and since $k^a = \partial/\partial t$ & everything is time independent, $\mathcal{L}_k \star F = \partial_t \star F = 0$. Then,

$$\mathrm{d}E = 0 \implies 0 = (a\mathrm{d}B) = \mathrm{d}a\wedge B. \tag{5.262}$$

$\therefore \mathrm{d}a \propto B \propto E$.
I'll need a few more obscure identities before the denouement.
Let $N = k^ak_a$. When adapted coordinates are valid, $N = -S^2$. From $k\wedge \mathrm{d}k = 0$,

$$0 = k_a\nabla_bk_c + k_b\nabla_ck_a + k_c\nabla_ak_b. \tag{5.263}$$
$$\therefore 0 = k^ck_a\nabla_bk_c + k^ck_b\nabla_ck_a + N\nabla_ak_b. \tag{5.264}$$
$$\therefore \nabla_ak_b = -\frac{1}{N}(k^ck_a\nabla_bk_c + k^ck_b\nabla_ck_a). \tag{5.265}$$

Using this, I get

$$\nabla^a\left(\frac{1}{N}E_a\right) = -\frac{1}{N^2}\nabla^a(N)E_a + \frac{1}{N}\nabla^aE_a \tag{5.266}$$
$$= \frac{2}{N^2}k^b\nabla_a(k_b)k_cF^{ca} - \frac{1}{N}\nabla_a(k_bF^{ba}) \tag{5.267}$$
$$= \frac{2}{N^2}k^b\nabla_a(k_b)k_cF^{ca} - \frac{1}{N}F^{ba}\nabla_ak_b - 0 \text{ by Maxwell's equation} \tag{5.268}$$
$$= \frac{2}{N^2}k^b\nabla_a(k_b)k_cF^{ca} + \frac{2}{N^2}F^{ba}k^ck_a\nabla_bk_c \text{ by equation 5.265} \tag{5.269}$$
$$= 0. \tag{5.270}$$

Similarly, for $B$,

$$\nabla^a\left(\frac{1}{N}B_a\right) = -\frac{1}{N^2}\nabla^a(N)B_a + \frac{1}{N}\nabla^a B_a \tag{5.271}$$
$$= -\frac{2}{N^2}B^a k^b \nabla_a k_b + \frac{1}{2N}\nabla_a\left(k_b \varepsilon^{cdba}F_{cd}\right) \tag{5.272}$$
$$= -\frac{2}{N^2}B^a k^b \nabla_a k_b + \frac{1}{2N}F_{cd}\varepsilon^{cdba}\nabla_a k_b \text{ as } \mathrm{d}F = 0 \implies \varepsilon^{cdba}\nabla_a F_{cd} = 0 \tag{5.273}$$
$$= -\frac{2}{N^2}B^a k^b \nabla_a k_b - \frac{1}{N^2}F_{cd}\varepsilon^{cdba}k^e k_a \nabla_b k_e \text{ by equation 5.265} \tag{5.274}$$
$$= 0. \tag{5.275}$$

Putting these last two identities together,

$$0 = \nabla^a\left(\frac{1}{N}E_a\right) \tag{5.276}$$
$$= \nabla^a\left(\frac{a}{N}B_a\right) \tag{5.277}$$
$$= \frac{1}{N}B^a\nabla_a(a). \tag{5.278}$$

$\therefore$ If $\nabla_a a \neq 0$, then $\nabla_a a = \mathrm{d}a$ is both parallel and perpendicular to $B$.
This is only possible if $\nabla_a a$ is null.
Then $B_a$ must be null too.
However, theorem 2.2 says $k^a$ is timelike in the domain of outer communication and so $k^a B_a = \frac{1}{2}\varepsilon_{cdba}F^{cd}k^b k^a = 0$ would imply $B_a$ is spacelike.
Hence, it must be that $\nabla_a a = 0$ to begin with. □

Note that when $E = 0$, it still holds that $B = \mathrm{d}\phi$, for some function, $\phi$, because of topological censorship and the $\mathrm{d}B = 0$ result in the proof. In that case, I can rename $\phi$ as $a\psi$ for my favourite non-zero constant, $a$. The next theorem also works fine when $E = 0$, except that instead of $\sqrt{1+a^2}$, one would just get $a$.

**Theorem 5.24.** *The problem considered in this section - i.e. definition 5.21 - reduces to the one previously considered - i.e. definition 5.7 - but with $\psi$ replaced by $\sqrt{1+a^2}\psi$.*

*Proof.* The boundary conditions in definition 5.7 and 5.21 are the same, so it only remains to check that the equations of motion transform as claimed.
By definition, $B_\mu = k^\nu(\star F)_{\nu\mu} = (\star F)_{0\mu}$.
$\therefore B_0 = 0$ and

$$B_i = (\star F)_{0i} = \frac{1}{2}\varepsilon_{\mu\nu 0i}F^{\mu\nu} = \frac{1}{2}\varepsilon_{jk0i}F^{jk} = \frac{S}{2}\varepsilon^{(h)}_{ijk}F^{jk}. \tag{5.279}$$
$$\therefore F_{ij} = \frac{1}{S}\varepsilon^{(h)}_{ijk}B^k \tag{5.280}$$
$$= \frac{a}{S}\varepsilon^{(h)}_{ijk}\nabla^{(h)k}(\psi) \text{ by lemma 5.23.} \tag{5.281}$$

Then, equation 5.238 becomes

$$S\square^{(h)}S = \nabla^{(h)}_i(\psi)\nabla^{(h)i}(\psi) + \frac{S^2}{2}\frac{a^2}{S^2}\varepsilon^{(h)}_{ijk}\nabla^{(h)k}(\psi)\varepsilon^{(h)ijl}\nabla^{(h)}_l(\psi) \tag{5.282}$$
$$= (1+a^2)\nabla^{(h)}_i(\psi)\nabla^{(h)i}(\psi), \tag{5.283}$$

which is equation 5.44 with $\psi \to \sqrt{1+a^2}\psi$.
Next, equation 5.239 is identically satisfied because

$$\nabla^{(h)j}(SF_{ji}) = \nabla^{(h)j}(a\varepsilon^{(h)}_{jik}\nabla^{(h)k}\psi) \tag{5.284}$$

$$= -a\varepsilon^{(h)}_{ijk}\nabla^{(h)j}\nabla^{(h)k}\psi \tag{5.285}$$

$$= 0 \text{ as } \nabla^{(h)j}\nabla^{(h)k}\psi = \nabla^{(h)k}\nabla^{(h)j}\psi \text{ for a scalar.} \tag{5.286}$$

Likewise, equation 5.240 is also identically satisfied because

$$F_{ij}\nabla^{(h)j}(\psi) = \frac{a}{S}\varepsilon^{(h)}_{ijk}\nabla^{(h)k}(\psi)\nabla^{(h)j}(\psi) = 0. \tag{5.287}$$

Equation 5.241 is the same as equation 5.45 and equation 5.242 now reads

$$R^{(h)}_{ij} = \frac{1}{S}\nabla^{(h)}_i\nabla^{(h)}_j S + \frac{1}{S^2}h_{ij}\nabla^{(h)}_k(\psi)\nabla^{(h)k}(\psi) - \frac{2}{S^2}\nabla^{(h)}_i(\psi)\nabla^{(h)}_j(\psi)$$
$$+ \frac{2a^2}{S^2}\varepsilon_i{}^{kl}\nabla^{(h)}_l(\psi)\varepsilon_{jkm}\nabla^{(h)m}(\psi) - \frac{a^2}{2S^2}h_{ij}\varepsilon^{klm}\nabla^{(h)}_m(\psi)\varepsilon_{kln}\nabla^{(h)n}(\psi) \tag{5.288}$$

$$= \frac{1}{S}\nabla^{(h)}_i\nabla^{(h)}_j S + \frac{1}{S^2}h_{ij}\nabla^{(h)}_k(\psi)\nabla^{(h)k}(\psi) - \frac{2}{S^2}\nabla^{(h)}_i(\psi)\nabla^{(h)}_j(\psi)$$
$$+ \frac{4a^2}{S^2}\nabla^{(h)}_l(\psi)\nabla^{(h)m}(\psi)h_{in}\delta^n{}_{[j}\delta^l{}_{m]} - \frac{a^2}{S^2}h_{ij}\nabla^{(h)}_k(\psi)\nabla^{(h)k}(\psi) \tag{5.289}$$

$$= \frac{1}{S}\nabla^{(h)}_i\nabla^{(h)}_j S + \frac{1+a^2}{S^2}h_{ij}\nabla^{(h)}_k(\psi)\nabla^{(h)k}(\psi) - \frac{2+2a^2}{S^2}\nabla^{(h)}_i(\psi)\nabla^{(h)}_j(\psi), \tag{5.290}$$

which is equation 5.46 with $\psi \to \sqrt{1+a^2}\psi$. □

**Corollary 5.24.1.** *All the results of the previous section carry over, except one now gets the Reissner-Nordstrom solution with both electric charge, $q$, and magnetic charge, $p = aq$.*

Hence, the uniqueness proof is complete in the $n = 4$ case.

### 5.2.2 $n > 4$

The method of [13] that I've generalised here works somewhat less satisfyingly with magnetic fields when $n > 4$. It's known from the positive energy theorem based proof in [11] that when $n > 4$, $F_{ij} = 0$, i.e. magnetic fields are not possible in higher dimensional static, vacuum, Einstein-Maxwell systems. Proving that result with the method in this paper is only possible after assuming $q \neq 0$ and the auxiliary inequality, 5.292, stated below. In some sense, it is not surprising an auxiliary inequality like inequality 5.292 is required in this method. After all, even without the Maxwell field, the proof in [13] relies on assuming inequality 5.207 when $n > 4$. Thus, it stands to reason that generalising to the Einstein-Maxwell system may require assuming further inequalities between the constants in the solution; it just happens that such additional inequalities are not required when purely electric fields are considered. But first, I'll need one generalisation of theorem 5.12 to account for magnetic fields.

**Lemma 5.25.** *With the magnetic field, instead of equation 5.91, one has*

$$\square^{(h')}(z) = \frac{1}{2}\coth(\varphi/2)\nabla^{(h')}_i(\varphi)\nabla^{(h')i}(z) + \frac{1}{2}\tanh(z/2)\nabla^{(h')}_i(z)\nabla^{(h')i}(z)$$
$$+ \frac{2C}{n-2}\Omega^2\sinh^2(\varphi/2)\psi h'^{ik}h'^{jl}F_{ij}F_{kl}. \tag{5.291}$$

*Proof.* This is still fairly tedious, so I've presented the proof in appendix C. □

**Theorem 5.26.** *If* $q \neq 0$ *and*

$$\kappa A \leq -\frac{1 - C^2\psi_0^2}{2\psi_0} q(n-3)\omega_{n-2}, \tag{5.292}$$

*then* $F_{ij} = 0$.

*Proof.* First observe that

$$\begin{aligned}
&\nabla_i^{(h')}\left(\frac{1}{\sinh(\varphi/2)\cosh(z/2)}\nabla^{(h')i}(z)\right) \\
&= \frac{1}{\sinh(\varphi/2)\cosh(z/2)}\Box^{(h)}z - \frac{\cosh(\varphi/2)}{2\sinh^2(\varphi/2)\cosh(z/2)}\nabla_i^{(h')}(\varphi)\nabla^{(h')i}(z) \\
&\quad - \frac{\sinh(z/2)}{\sinh(\varphi/2)\cosh^2(z/2)}\nabla_i^{(h')}(z)\nabla^{(h')i}(z) \qquad (5.293)\\
&= \frac{2C\Omega^2\sinh(\varphi/2)\psi}{(n-2)\cosh(z/2)} h'^{ik}h'^{jl}F_{ij}F_{kl} \text{ using lemma 5.25} \qquad (5.294)\\
&= \frac{2S\Omega^2\tanh(z/2)}{n-2} h'^{ik}h'^{jl}F_{ij}F_{kl}. \qquad (5.295)
\end{aligned}$$

Then, by Stokes' theorem,

$$\begin{aligned}
\int_{\Sigma_t}\frac{2S\Omega^2\tanh(z/2)}{n-2}h'^{ik}h'^{jl}F_{ij}F_{kl}\mathrm{d}V' &= \int_{S_\infty^{n-2}} n_i \frac{1}{\sinh(\varphi/2)\cosh(z/2)}\nabla^{(h')i}(z)\mathrm{d}A' \\
&\quad - \int_{\mathcal{H}} n_i \frac{1}{\sinh(\varphi/2)\cosh(z/2)}\nabla^{(h')i}(z)\mathrm{d}A'. \qquad (5.296)
\end{aligned}$$

The boundary conditions for $S$ and $\psi$ - thus also $\varphi$ and $z$ - are unchanged by introducing the magnetic field. Hence, equation 5.118 still holds. Likewise, $z$ takes the constant value, $z_0$, as before, on $\mathcal{H}$.

$$\therefore \int_{\Sigma_t}\frac{2S\Omega^2\tanh(z/2)}{n-2}h'^{ik}h'^{jl}F_{ij}F_{kl}\mathrm{d}V' = -\frac{1}{\cosh(z_0/2)}\int_{\mathcal{H}} n_i\frac{1}{\sinh(\varphi/2)}\nabla^{(h')i}(z)\mathrm{d}A'. \tag{5.297}$$

The integral on the RHS is evaluated as follows.
Equation 5.45 is unchanged upon introducing the magnetic field.
Hence, the derivation of equation A.42 in appendix A still holds. It says

$$\begin{aligned}
0 &= \frac{2(1+\cosh(\varphi/2)\cosh(z/2))}{\sinh(\varphi/2)}\Box^{(h')}(z) - 2\sinh(z/2)\Box^{(h')}(\varphi) - \frac{\tanh(z/2)}{\sinh(\varphi/2)}\nabla_i^{(h')}(z)\nabla^{(h')i}(z) \\
&\quad - \left(\frac{\cosh(z/2)}{\sinh^2(\varphi/2)} + \frac{1}{\cosh(z/2)} + \frac{\cosh(\varphi/2)}{\sinh^2(\varphi/2)}\right)\nabla_i^{(h')}(\varphi)\nabla^{(h')i}(z). \qquad (5.298)
\end{aligned}$$

Observe that dividing by $2\cosh(z/2)$ gives

$$\begin{aligned}
0 &= \left(\frac{1}{\sinh(\varphi/2)\cosh(z/2)} + \coth(\varphi/2)\right)\Box^{(h')}(z) - \tanh(z/2)\Box^{(h')}(\varphi) \\
&\quad - \frac{\sinh(z/2)}{2\sinh(\varphi/2)\cosh^2(z/2)}\nabla_i^{(h')}(z)\nabla^{(h')i}(z) \\
&\quad - \frac{1}{2}\left(\frac{1}{\sinh^2(\varphi/2)} + \frac{1}{\cosh^2(z/2)} + \frac{\cosh(\varphi/2)}{\sinh^2(\varphi/2)\cosh(z/2)}\right)\nabla_i^{(h')}(\varphi)\nabla^{(h')i}(z) \qquad (5.299)\\
&= \nabla_i^{(h')}\left(\left(\frac{1}{\sinh(\varphi/2)\cosh(z/2)} + \coth(\varphi/2)\right)\nabla^{(h')i}(z) - \tanh(z/2)\nabla^{(h')i}(\varphi)\right). \qquad (5.300)
\end{aligned}$$

Then, by Stokes' theorem,

$$0 = \int_{\Sigma_t} \nabla_i^{(h')} \left( \left( \frac{1}{\sinh(\varphi/2)\cosh(z/2)} + \coth(\varphi/2) \right) \nabla^{(h')i}(z) - \tanh(z/2)\nabla^{(h')i}(\varphi) \right) \mathrm{d}V' \tag{5.301}$$

$$= \int_{S_\infty^{n-2}} n_i \left( \left( \frac{1}{\sinh(\varphi/2)\cosh(z/2)} + \coth(\varphi/2) \right) \nabla^{(h')i}(z) - \tanh(z/2)\nabla^{(h')i}(\varphi) \right) \mathrm{d}A'$$
$$- \int_{\mathcal{H}} n_i \left( \left( \frac{1}{\sinh(\varphi/2)\cosh(z/2)} + \coth(\varphi/2) \right) \nabla^{(h')i}(z) - \tanh(z/2)\nabla^{(h')i}(\varphi) \right) \mathrm{d}A'. \tag{5.302}$$

As before, the boundary conditions mean $z$ takes the constant values, $z_0$ on $\mathcal{H}$ and $z_1$ at $S_\infty^{n-2}$. Likewise, $\varphi = 0$ on $\mathcal{H}$ and $\varphi = \infty$ at $S_\infty^{n-2}$.
Again, $\nabla_i^{(h')}(\varphi)\nabla^{(h')i}(\varphi) = \kappa'^2$.

$$\therefore \int_{\mathcal{H}} n_i \left( \left( \frac{1}{\sinh(\varphi/2)\cosh(z/2)} + \coth(\varphi/2) \right) \nabla^{(h')i}(z) - \tanh(z/2)\nabla^{(h')i}(\varphi) \right) \mathrm{d}A'$$
$$= \frac{1+\cosh(z_0/2)}{\cosh(z_0/2)} \int_{\mathcal{H}} n_i \frac{1}{\sinh(\varphi/2)} \nabla^{(h')i}(z)\mathrm{d}A'$$
$$- \tanh(z_0/2) \int_{\mathcal{H}} \frac{1}{\sqrt{\nabla_j^{(h')}(\varphi)\nabla^{(h')j}(\varphi)}} \nabla_i^{(h')}(\varphi)\nabla^{(h')i}(\varphi)\mathrm{d}A' \tag{5.303}$$

$$= \frac{1+\cosh(z_0/2)}{\cosh(z_0/2)} \int_{\mathcal{H}} n_i \frac{1}{\sinh(\varphi/2)} \nabla^{(h')i}(z)\mathrm{d}A' - \tanh(z_0/2)\kappa' A'. \tag{5.304}$$

Meanwhile, for the integral at infinity, the asymptotics are the same as without the magnetic field, so equations 5.115, 5.100, 5.120, 5.99, 5.118 and 5.185 still hold. Hence,

$$\int_{S_\infty^{n-2}} n_i \left( \left( \frac{1}{\sinh(\varphi/2)\cosh(z/2)} + \coth(\varphi/2) \right) \nabla^{(h')i}(z) - \tanh(z/2)\nabla^{(h')i}(\varphi) \right) \mathrm{d}A'$$
$$= 0 + \int_{S_\infty^{n-2}} n_i \nabla^{(h')i}(z)\mathrm{d}A' - \tanh(z_1/2) \int_{S_\infty^{n-2}} n_i \nabla^{(h')i}(\varphi)\mathrm{d}A' \tag{5.305}$$

$$= \int_{S_\infty^{n-2}} \frac{r^2}{m^{2/(n-3)}} \delta^{ij} \frac{m^{1/(n-3)}}{r^2} x_i f x_j \mathrm{d}A' - \tanh(z_1/2) 2m(n-3)\omega_{n-2} \tag{5.306}$$

$$= -2m(n-3)\omega_{n-2}\tanh(z_1/2). \tag{5.307}$$

Putting both integrals together, I get

$$0 = -2m(n-3)\omega_{n-2}\tanh(z_1/2) - \frac{1+\cosh(z_0/2)}{\cosh(z_0/2)} \int_{\mathcal{H}} n_i \frac{1}{\sinh(\varphi/2)} \nabla^{(h')i}(z)\mathrm{d}A'$$
$$+ \tanh(z_0/2)\kappa' A' \tag{5.308}$$

$$\iff \int_{\mathcal{H}} n_i \frac{1}{\sinh(\varphi/2)} \nabla^{(h')i}(z)\mathrm{d}A'$$
$$= \frac{\sinh(z_0/2)}{1+\cosh(z_0/2)}\kappa' A' - 2m(n-3)\omega_{n-2}\tanh(z_1/2)\frac{\cosh(z_0/2)}{1+\cosh(z_0/2)}. \tag{5.309}$$

From equation 5.122,

$$\cosh(z_0/2) = \frac{1}{2}\left( \frac{1+C\psi_0}{1-C\psi_0} + \frac{1-C\psi_0}{1+C\psi_0} \right) = \frac{1+C^2\psi_0^2}{1-C^2\psi_0^2} \text{ and} \tag{5.310}$$

$$\sinh(z_0/2) = \frac{1}{2}\left( \frac{1+C\psi_0}{1-C\psi_0} - \frac{1-C\psi_0}{1+C\psi_0} \right) = \frac{2C\psi_0}{1-C^2\psi_0^2}. \tag{5.311}$$

$$\therefore \frac{\sinh(z_0/2)}{1+\cosh(z_0/2)} = \frac{2C\psi_0}{1-C^2\psi_0^2+1+C^2\psi_0^2} = C\psi_0 \text{ and} \tag{5.312}$$

$$\frac{\cosh(z_0/2)}{1+\cosh(z_0/2)} = \frac{1+C^2\psi_0^2}{1-C^2\psi_0^2+1+C^2\psi_0^2} = \frac{1}{2}(1+C^2\psi_0^2). \tag{5.313}$$

Likewise, from equation 5.110,

$$\tanh(z_1/2) = \frac{\mathrm{e}^{z_1}-1}{\mathrm{e}^{z_1}+1} = \frac{\frac{m-2Cq}{m+2Cq}-1}{\frac{m-2Cq}{m+2Cq}+1} = -\frac{2Cq}{m}. \tag{5.314}$$

Substituting back into equation 5.309 gives

$$\int_{\mathcal{H}} n_i \frac{1}{\sinh(\varphi/2)} \nabla^{(h')i}(z)\mathrm{d}A' = C\psi_0\kappa' A' - 2m(n-3)\omega_{n-2}\left(-\frac{2Cq}{m}\right)\frac{1}{2}(1+C^2\psi_0^2) \tag{5.315}$$

$$= C(\psi_0\kappa' A' + 2q(n-3)\omega_{n-2}(1+C^2\psi_0^2)) \tag{5.316}$$

$$= C\Bigg(\psi_0 \frac{4\kappa}{(1-C^2\psi_0^2)(1+C^2\psi_0)^{1/(n-3)}}\Omega_{|\mathcal{H}}^{n-2} A + 2q(n-3)\omega_{n-2}(1+C^2\psi_0^2)\Bigg) \text{ using equation 5.155} \tag{5.317}$$

$$= C\Bigg(\psi_0 \frac{4\kappa}{(1-C^2\psi_0^2)(1+C^2\psi_0)^{1/(n-3)}}(1+C^2\psi_0^2)^{(n-2)/(n-3)} A + 2q(n-3)\omega_{n-2}(1+C^2\psi_0^2)\Bigg) \tag{5.318}$$

$$= 2C(1+C^2\psi_0^2)\left(2\kappa A\frac{\psi_0}{1-C^2\psi_0^2} + q(n-3)\omega_{n-2}\right). \tag{5.319}$$

Substituting this result back into equation 5.297 says

$$\int_{\Sigma_t} \frac{2S\Omega^2\tanh(z/2)}{n-2} h'^{ik}h'^{jl}F_{ij}F_{kl}\mathrm{d}V' = -\frac{2C(1+C^2\psi_0^2)}{\cosh(z_0/2)}\left(\frac{2\kappa A\psi_0}{1-C^2\psi_0^2} + q(n-3)\omega_{n-2}\right) \tag{5.320}$$

$$= -2C(2\kappa A\psi_0 + q(n-3)\omega_{n-2}(1-C^2\psi_0^2)) \tag{5.321}$$

$$= -4C\psi_0\left(\kappa A + \frac{1-C^2\psi_0^2}{2\psi_0} q(n-3)\omega_{n-2}\right). \tag{5.322}$$

First suppose that $q > 0$. Then, $z > 0$ and $\psi_0 < 0$. Then, inequality 5.292 implies

$$\int_{\Sigma_t} \frac{2S\Omega^2\tanh(z/2)}{n-2} h'^{ik}h'^{jl}F_{ij}F_{kl}\mathrm{d}V' \leq 0. \tag{5.323}$$

Since the integrand on the LHS is non-negative, it must be that $h'^{ik}h'^{jl}F_{ij}F_{kl} = 0$. Since $h'$ is Riemannian, this is equivalent to $F_{ij} = 0$.
Likewise, when $q < 0$, I have $z < 0$, $\psi > 0$ and finally

$$\int_{\Sigma_t} \frac{2S\Omega^2\tanh(z/2)}{n-2} h'^{ik}h'^{jl}F_{ij}F_{kl}\mathrm{d}V' \geq 0. \tag{5.324}$$

Since the integrand is non-positive now, the result is again $F_{ij} = 0$. □

**Corollary 5.26.1.** *When the conditions of the theorem hold, the problem reduces to the one studied in the earlier sections of the paper and thus all the results there apply again.*

# Appendix A

# Proof of theorem 5.12

In this appendix I'll prove that equations 5.44 and 5.45 are equivalent to

$$\Box^{(h')}\varphi = \frac{1}{2}\tanh(z/2)\nabla^{(h')}_i(\varphi)\nabla^{(h')i}(z) + \frac{1}{2}\coth(\varphi/2)\nabla^{(h')}_i(z)\nabla^{(h')i}(z) \text{ and} \tag{A.1}$$

$$\Box^{(h')}z = \frac{1}{2}\coth(\varphi/2)\nabla^{(h')}_i(\varphi)\nabla^{(h')i}(z) + \frac{1}{2}\tanh(z/2)\nabla^{(h')}_i(z)\nabla^{(h')i}(z). \tag{A.2}$$

Start with equation 5.45, $\nabla^{(h)}_i\left(\frac{1}{S}\nabla^{(h)i}\psi\right) = \nabla^{(h)}_i\left(\frac{1}{S}\nabla^{(h)i}(C\psi)\right) = 0$.
I'll first need the transformation of the Christoffel symbols.

$$\Gamma^{(h)i}{}_{jk} = \frac{1}{2}h^{il}(\partial_j h_{kl} + \partial_k h_{lj} - \partial_l h_{jk}) \tag{A.3}$$

$$= \frac{1}{2}\Omega^2 h'^{il}\left(\partial_j\left(\frac{1}{\Omega^2}h'_{kl}\right) + \partial_k\left(\frac{1}{\Omega^2}h'_{lj}\right) - \partial_l\left(\frac{1}{\Omega^2}h'_{jk}\right)\right) \tag{A.4}$$

$$= \Gamma^{(h\prime)i}{}_{jk} - \frac{1}{\Omega}h'^{il}(h'_{kl}\partial_j\Omega + h'_{lj}\partial_k\Omega - h'_{jk}\partial_l\Omega) \tag{A.5}$$

$$= \Gamma^{(h\prime)i}{}_{jk} - \delta^i{}_k\nabla^{(h')}_j(\ln(\Omega)) - \delta^i{}_j\nabla^{(h')}_k(\ln(\Omega)) + h'_{jk}\nabla^{(h')i}(\ln(\Omega)). \tag{A.6}$$

Using this, equation 5.45 transforms as

$$0 = \nabla^{(h)}_i\left(\frac{1}{S}\nabla^{(h)i}(C\psi)\right) \tag{A.7}$$

$$= \partial_i\left(\frac{1}{S}h^{ij}\nabla^{(h)}_j(C\psi)\right) + \Gamma^{(h)i}{}_{ji}\frac{1}{S}h^{jk}\nabla^{(h)}_k(C\psi) \tag{A.8}$$

$$= \partial_i\left(\frac{1}{S}\Omega^2 h'^{ij}\nabla^{(h')}_j(C\psi)\right)$$
$$+ (\Gamma^{(h\prime)i}{}_{ji} - \delta^i{}_i\nabla^{(h')}_j(\ln(\Omega)) - \delta^i{}_j\nabla^{(h')}_i(\ln(\Omega)) + h'_{ji}\nabla^{(h')i}(\ln(\Omega)))\frac{1}{S}\Omega^2 h'^{jk}\nabla^{(h')}_k(C\psi) \tag{A.9}$$

$$= \nabla^{(h')}_i\left(\frac{1}{S}\Omega^2\nabla^{(h')i}(C\psi)\right) - \frac{n-1}{S}\Omega^2\nabla^{(h')}_i(\ln(\Omega))\nabla^{(h')i}(C\psi) \tag{A.10}$$

$$= \Omega^2\nabla^{(h')}_i\left(\frac{1}{S}\nabla^{(h')i}(C\psi)\right) - \frac{n-3}{S}\Omega^2\nabla^{(h')}_i(\ln(\Omega))\nabla^{(h')i}(C\psi). \tag{A.11}$$

$$\therefore 0 = \nabla^{(h')}_i\left(\frac{1}{S}\nabla^{(h')i}(C\psi)\right) - \frac{n-3}{S}\nabla^{(h')}_i(\ln(\Omega))\nabla^{(h')i}(C\psi). \tag{A.12}$$

Now I have to actually start computing derivatives. By lemma 5.9,

$$\begin{aligned}
\nabla_i^{(h')}(C\psi) &= \nabla_i^{(h')}\left(\frac{\sinh(z/2)}{\cosh(\varphi/2)+\cosh(z/2)}\right) && \text{(A.13)}\\
&= \frac{1}{(\cosh(\varphi/2)+\cosh(z/2))^2}\left(\frac{1}{2}\cosh(z/2)\nabla_i^{(h')}(z)(\cosh(\varphi/2)+\cosh(z/2))\right.\\
&\quad \left.-\frac{1}{2}\sinh(z/2)(\sinh(z/2)\nabla_i^{(h')}z+\sinh(\varphi/2)\nabla_i^{(h')}\varphi\right) && \text{(A.14)}\\
&= \frac{(1+\cosh(\varphi/2)\cosh(z/2))\nabla_i^{(h')}z-\sinh(z/2)\sinh(\varphi/2)\nabla_i^{(h')}\varphi}{2(\cosh(\varphi/2)+\cosh(z/2))^2}. && \text{(A.15)}
\end{aligned}$$

Lemma 5.9 says the expression for $S$ is the same as the one for $C\psi$, but with $\varphi$ and $z$ swapped, so I can immediately read off that

$$\nabla_i^{(h')}S = \frac{(1+\cosh(\varphi/2)\cosh(z/2))\nabla_i^{(h')}\varphi-\sinh(z/2)\sinh(\varphi/2)\nabla_i^{(h')}z}{2(\cosh(\varphi/2)+\cosh(z/2))^2}. \tag{A.16}$$

Next, using equation 5.84,

$$\begin{aligned}
&\nabla_i^{(h')}(\ln(\Omega))\\
&= \frac{1}{\Omega}\nabla_i^{(h')}\Omega && \text{(A.17)}\\
&= \frac{1}{n-3}\left(\frac{2\cosh(z/2)}{\cosh(\varphi/2)+\cosh(z/2)}\right)^{-1}\nabla_i^{(h')}\left(\frac{2\cosh(z/2)}{\cosh(\varphi/2)+\cosh(z/2)}\right) && \text{(A.18)}\\
&= \frac{1}{2(n-3)\cosh(z/2)(\cosh(\varphi/2)+\cosh(z/2))}(\sinh(z/2)\nabla_i^{(h')}(z)(\cosh(\varphi/2)+\cosh(z/2))\\
&\quad -\cosh(z/2)(\sinh(z/2)\nabla_i^{(h')}z+\sinh(\varphi/2)\nabla_i^{(h')}\varphi)) && \text{(A.19)}\\
&= \frac{\tanh(z/2)\cosh(\varphi/2)\nabla_i^{(h')}z-\sinh(\varphi/2)\nabla_i^{(h')}\varphi}{2(n-3)(\cosh(\varphi/2)+\cosh(z/2))}. && \text{(A.20)}
\end{aligned}$$

The quantities that appear in A.12 are $\frac{1}{S}\nabla_i^{(h')}(C\psi)$ and $\frac{n-3}{S}\nabla_i^{(h')}(\ln(\Omega))$. They are

$$\begin{aligned}
\frac{1}{S}\nabla_i^{(h')}(C\psi) &= \frac{\cosh(\varphi/2)+\cosh(z/2)}{\sinh(\varphi/2)}\\
&\quad\times\frac{(1+\cosh(\varphi/2)\cosh(z/2))\nabla_i^{(h')}z-\sinh(z/2)\sinh(\varphi/2)\nabla_i^{(h')}\varphi}{2(\cosh(\varphi/2)+\cosh(z/2))^2} && \text{(A.21)}\\
&= \frac{\left(\frac{1}{\sinh(\varphi/2)}+\cosh(\varphi/2)\coth(\varphi/2)\right)\nabla_i^{(h')}z-\sinh(z/2)\nabla_i^{(h')}\varphi}{2(\cosh(\varphi/2)+\cosh(z/2))} \quad\text{and} && \text{(A.22)}\\
\frac{n-3}{S}\nabla_i^{(h')}(\ln(\Omega)) &= \frac{(n-3)\cosh(\varphi/2)+\cosh(z/2)}{\sinh(\varphi/2)}\\
&\quad\times\frac{\tanh(z/2)\cosh(\varphi/2)\nabla_i^{(h')}z-\sinh(\varphi/2)\nabla_i^{(h')}\varphi}{2(n-3)(\cosh(\varphi/2)+\cosh(z/2))} && \text{(A.23)}\\
&= \frac{1}{2}\tanh(z/2)\coth(\varphi/2)\nabla_i^{(h')}z-\frac{1}{2}\nabla_i^{(h')}\varphi. && \text{(A.24)}
\end{aligned}$$

The first term in equation A.12 is then

$$
\begin{aligned}
&\nabla^{(h')}_i\left(\frac{1}{S}\nabla^{(h')i}(C\psi)\right)\\
&=\nabla^{(h')}_i\left(\frac{\left(\frac{1}{\sinh(\varphi/2)}+\cosh(z/2)\coth(\varphi/2)\right)\nabla^{(h')i}z-\sinh(z/2)\nabla^{(h')i}\varphi}{2(\cosh(\varphi/2)+\cosh(z/2))}\right) \qquad\qquad \text{(A.25)}\\
&=\frac{1}{2(\cosh(\varphi/2)+\cosh(z/2))^2}\Bigg[(\cosh(\varphi/2)+\cosh(z/2))\Bigg(-\frac{\cosh(\varphi/2)}{2\sinh^2(\varphi/2)}\nabla^{(h')}_i(\varphi)\nabla^{(h')i}(z)\\
&\quad+\frac{1}{\sinh(\varphi/2)}\Box^{(h')}z+\frac{1}{2}\sinh(z/2)\coth(\varphi/2)\nabla^{(h')}_i(z)\nabla^{(h')i}(z)-\frac{\cosh(z/2)}{2\sinh^2(\varphi/2)}\nabla^{(h')}_i(\varphi)\nabla^{(h')i}(z)\\
&\quad+\cosh(z/2)\coth(\varphi/2)\Box^{(h')}z-\frac{1}{2}\cosh(z/2)\nabla^{(h')}_i(\varphi)\nabla^{(h')i}(z)-\sinh(z/2)\Box^{(h')}\varphi\Bigg)\\
&\quad-\left(\frac{1}{\sinh(\varphi/2)}\nabla^{(h')i}z+\cosh(z/2)\coth(\varphi/2)\nabla^{(h')i}z-\sinh(z/2)\nabla^{(h')i}\varphi\right)\\
&\quad\times\left(\frac{1}{2}\sinh(z/2)\nabla^{(h')}_i z+\frac{1}{2}\sinh(\varphi/2)\nabla^{(h')}_i\varphi\right)\Bigg]. \qquad\qquad \text{(A.26)}
\end{aligned}
$$

This expands to the gargantuan mess[1] of

$$
\begin{aligned}
&\nabla^{(h')}_i\left(\frac{1}{S}\nabla^{(h')i}(C\psi)\right)\\
&=\frac{1}{4(\cosh(\varphi/2)+\cosh(z/2))^2}\Bigg[-\coth^2(\varphi/2)\nabla^{(h')}_i(\varphi)\nabla^{(h')i}(z)\\
&\quad-\frac{\cosh(z/2)\cosh(\varphi/2)}{\sinh^2(\varphi/2)}\nabla^{(h')}_i(\varphi)\nabla^{(h')i}(z)+2\coth(\varphi/2)\Box^{(h')}z+\frac{2\cosh(z/2)}{\sinh(\varphi/2)}\Box^{(h')}z\\
&\quad+\frac{\sinh(z/2)\cosh^2(\varphi/2)}{\sinh(\varphi/2)}\nabla^{(h')}_i(z)\nabla^{(h')i}(z)+\cosh(z/2)\sinh(z/2)\coth(\varphi/2)\nabla^{(h')}_i(z)\nabla^{(h')i}(z)\\
&\quad-\frac{\cosh(z/2)\cosh(\varphi/2)}{\sinh^2(\varphi/2)}\nabla^{(h')}_i(\varphi)\nabla^{(h')i}(z)-\frac{\cosh^2(z/2)}{\sinh^2(\varphi/2)}\nabla^{(h')}_i(\varphi)\nabla^{(h')i}(z)\\
&\quad+\frac{2\cosh(z/2)\cosh^2(\varphi/2)}{\sinh(\varphi/2)}\Box^{(h')}z+2\cosh^2(z/2)\coth(\varphi/2)\Box^{(h')}z\\
&\quad-\cosh(z/2)\cosh(\varphi/2)\nabla^{(h')}_i(\varphi)\nabla^{(h')i}(z)-\cosh^2(z/2)\nabla^{(h')}_i(\varphi)\nabla^{(h')i}(z)\\
&\quad-2\sinh(z/2)\cosh(\varphi/2)\Box^{(h')}\varphi-2\sinh(z/2)\cosh(z/2)\Box^{(h')}\varphi-\frac{\sinh(z/2)}{\sinh(\varphi/2)}\nabla^{(h')}_i(z)\nabla^{(h')i}(z)\\
&\quad-\nabla^{(h')}_i(\varphi)\nabla^{(h')i}(z)-\sinh(z/2)\cosh(z/2)\coth(\varphi/2)\nabla^{(h')}_i(z)\nabla^{(h')i}(z)\\
&\quad-\cosh(z/2)\cosh(\varphi/2)\nabla^{(h')}_i(\varphi)\nabla^{(h')i}(z)+\sinh^2(z/2)\nabla^{(h')}_i(\varphi)\nabla^{(h')i}(z)\\
&\quad+\sinh(z/2)\sinh(\varphi/2)\nabla^{(h')}_i(\varphi)\nabla^{(h')i}(\varphi)\Bigg]. \qquad\qquad \text{(A.27)}
\end{aligned}
$$

[1] Although, as messes go, this expression pales in comparison to some of the equations involved in calculating $R^{(h')}_{ij}$ in terms of $\varphi$, $z$ and $h'$. Luckily, I didn't end up actually needing that full expression in this work.

The other term in equation A.12 is

$$\frac{n-3}{S}\nabla_i^{(h')}(\ln(\Omega))\nabla^{(h')i}(C\psi) \tag{A.28}$$

$$= \frac{1}{2}\left(\tanh(z/2)\coth(\varphi/2)\nabla_i^{(h')}z - \nabla_i^{(h')}\varphi\right)$$
$$\times \frac{(1+\cosh(\varphi/2)\cosh(z/2))\nabla^{(h')i}z - \sinh(z/2)\sinh(\varphi/2)\nabla^{(h')i}\varphi}{2(\cosh(\varphi/2)+\cosh(z/2))^2} \tag{A.29}$$

$$= \frac{1}{4(\cosh(\varphi/2)+\cosh(z/2))^2}\Bigg[\sinh(\varphi/2)\sinh(z/2)\nabla_i^{(h')}(\varphi)\nabla^{(h')i}(\varphi)$$
$$+ \nabla_i^{(h')}(z)\nabla^{(h')i}(z)\left(\tanh(z/2)\coth(\varphi/2) + \frac{\sinh(z/2)\cosh^2(\varphi/2)}{\sinh(\varphi/2)}\right)$$
$$- \nabla_i^{(h')}(\varphi)\nabla^{(h')i}(z)\left(1+\cosh(z/2)\cosh(\varphi/2) + \frac{\cosh(\varphi/2)\sinh^2(z/2)}{\cosh(z/2)}\right)\Bigg]. \tag{A.30}$$

Mercifully, putting these expressions together in equation A.12 leads to some simplification. In particular,

$$0 = 2\Box^{(h')}(z)\left(\coth(\varphi/2) + \frac{\cosh(z/2)}{\sinh(\varphi/2)} + \frac{\cosh(z/2)\cosh^2(\varphi/2)}{\sinh(\varphi/2)} + \cosh^2(z/2)\coth(\varphi/2)\right)$$
$$- 2\Box^{(h')}(\varphi)(\sinh(z/2)\cosh(\varphi/2) + \sinh(z/2)\cosh(z/2))$$
$$+ \nabla_i^{(h')}(z)\nabla^{(h')i}(z)\Bigg(\frac{\sinh(z/2)\cosh^2(\varphi/2)}{\sinh(\varphi/2)} + \cosh(z/2)\sinh(z/2)\coth(\varphi/2) - \frac{\sinh(z/2)}{\sinh(\varphi/2)}$$
$$- \sinh(z/2)\cosh(z/2)\coth(\varphi/2) - \tanh(z/2)\coth(\varphi/2) - \frac{\sinh(z/2)\cosh^2(\varphi/2)}{\sinh(\varphi/2)}\Bigg)$$
$$+ \nabla_i^{(h')}(\varphi)\nabla^{(h')i}(\varphi)(\sinh(z/2)\sinh(\varphi/2) - \sinh(z/2)\sinh(\varphi/2))$$
$$+ \nabla_i^{(h')}(\varphi)\nabla^{(h')i}(z)\Bigg(-\coth^2(\varphi/2) - \frac{\cosh(\varphi/2)\cosh(z/2)}{\sinh^2(\varphi/2)} - \frac{\cosh(\varphi/2)\cosh(z/2)}{\sinh^2(\varphi/2)}$$
$$- \frac{\cosh^2(z/2)}{\sinh^2(\varphi/2)} - \cosh(z/2)\cosh(\varphi/2) - \cosh^2(z/2) - 1 - \cosh(z/2)\cosh(\varphi/2) + \sinh^2(z/2)$$
$$+ 1 + \cosh(z/2)\cosh(\varphi/2) + \frac{\cosh(\varphi/2)\sinh^2(z/2)}{\cosh(z/2)}\Bigg) \tag{A.31}$$

$$= 2\Box^{(h')}(z)\left(\coth(\varphi/2) + \frac{\cosh(z/2)}{\sinh(\varphi/2)} + \frac{\cosh(z/2)\cosh^2(\varphi/2)}{\sinh(\varphi/2)} + \cosh^2(z/2)\coth(\varphi/2)\right)$$
$$- 2\Box^{(h')}(\varphi)\sinh(z/2)(\cosh(\varphi/2) + \cosh(z/2))$$
$$+ \nabla_i^{(h')}(z)\nabla^{(h')i}(z)\left(-\frac{\sinh(z/2)}{\sinh(\varphi/2)} - \tanh(z/2)\coth(\varphi/2)\right)$$
$$+ \nabla_i^{(h')}(\varphi)\nabla^{(h')i}(z)\Bigg(-\coth^2(\varphi/2) - \frac{2\cosh(\varphi/2)\cosh(z/2)}{\sinh^2(\varphi/2)} - \frac{\cosh^2(z/2)}{\sinh^2(\varphi/2)}$$
$$- \cosh(z/2)\cosh(\varphi/2) - \cosh^2(z/2) + \sinh^2(z/2) + \frac{\cosh(\varphi/2)\sinh^2(z/2)}{\cosh(z/2)}\Bigg). \tag{A.32}$$

The coefficient of each derivative combination simplifies as follows.

$$\coth(\varphi/2) + \frac{\cosh(z/2)}{\sinh(\varphi/2)} + \frac{\cosh(z/2)\cosh^2(\varphi/2)}{\sinh(\varphi/2)} + \cosh^2(z/2)\coth(\varphi/2) \tag{A.33}$$

$$= \frac{1}{\sinh(\varphi/2)}\big(\cosh(\varphi/2) + \cosh(z/2) + \cosh(z/2)\cosh^2(\varphi/2) + \cosh^2(z/2)\cosh(\varphi/2)\big) \tag{A.34}$$

$$= \frac{1+\cosh(\varphi/2)\cosh(z/2)}{\sinh(\varphi/2)}(\cosh(\varphi/2) + \cosh(z/2)) \text{ and} \tag{A.35}$$

$$-\frac{\sinh(z/2)}{\sinh(\varphi/2)} - \tanh(z/2)\coth(\varphi/2) = -\frac{\tanh(z/2)}{\sinh(\varphi/2)}(\cosh(z/2) + \cosh(\varphi/2)). \tag{A.36}$$

The last one is more obscure. Observe that

$$-\frac{\cosh(\varphi/2)+\cosh(z/2)}{\sinh^2(\varphi/2)\cosh(z/2)}(\cosh^2(z/2) + \sinh^2(\varphi/2) + \cosh(\varphi/2)\cosh(z/2)) \tag{A.37}$$

$$= -\left(\frac{\cosh(\varphi/2)}{\cosh(z/2)\sinh^2(\varphi/2)} + \frac{1}{\sinh^2(\varphi/2)}\right) \times (\cosh^2(z/2) + \sinh^2(\varphi/2) + \cosh(\varphi/2)\cosh(z/2)) \tag{A.38}$$

$$= -\frac{\cosh(\varphi/2)\cosh(z/2)}{\sinh^2(\varphi/2)} - \frac{\cosh^2(z/2)}{\sinh^2(\varphi/2)} - \frac{\cosh(\varphi/2)}{\cosh(z/2)} - 1 - \coth^2(\varphi/2) - \frac{\cosh(\varphi/2)\cosh(z/2)}{\sinh^2(\varphi/2)} \tag{A.39}$$

$$= -\frac{2\cosh(\varphi/2)\cosh(z/2)}{\sinh^2(\varphi/2)} - \frac{\cosh^2(z/2)}{\sinh^2(\varphi/2)} - \frac{\cosh(\varphi/2)}{\cosh(z/2)} - 1 - \coth^2(\varphi/2) \tag{A.40}$$

$$= -\frac{2\cosh(\varphi/2)\cosh(z/2)}{\sinh^2(\varphi/2)} - \frac{\cosh^2(z/2)}{\sinh^2(\varphi/2)} - \cosh(\varphi/2)\cosh(z/2) + \frac{\cosh(\varphi/2)\sinh^2(z/2)}{\cosh(z/2)} - \cosh^2(z/2) + \sinh^2(z/2) - \coth^2(\varphi/2). \tag{A.41}$$

Hence, equation A.32 simplifies to

$$0 = \frac{2(1+\cosh(\varphi/2)\cosh(z/2))}{\sinh(\varphi/2)}\Box^{(h')}(z) - 2\sinh(z/2)\Box^{(h')}(\varphi) - \frac{\tanh(z/2)}{\sinh(\varphi/2)}\nabla_i^{(h')}(z)\nabla^{(h')i}(z) - \left(\frac{\cosh(z/2)}{\sinh^2(\varphi/2)} + \frac{1}{\cosh(z/2)} + \frac{\cosh(\varphi/2)}{\sinh^2(\varphi/2)}\right)\nabla_i^{(h')}(\varphi)\nabla^{(h')i}(z). \tag{A.42}$$

This is as far as I can go with equation 5.45 for now. To complete the proof of theorem 5.12, I'll need to consider equation 5.44 as well.
First observe that for any function, $f$, using equation A.6,

$$\nabla_i^{(h)}\nabla_j^{(h)} f$$

$$= \partial_i\partial_j f - \Gamma^{(h)k}{}_{ji}\nabla_k^{(h)} f \tag{A.43}$$

$$= \partial_i\partial_j f - (\Gamma^{(h\prime)k}{}_{ji} - \delta^k{}_i\nabla_j^{(h')}(\ln(\Omega)) - \delta^k{}_j\nabla_i^{(h')}(\ln(\Omega)) + h'_{ji}\nabla^{(h')k}(\ln(\Omega)))\nabla_k^{(h')} f \tag{A.44}$$

$$= \nabla_i^{(h')}\nabla_j^{(h')} f + \nabla_i^{(h')}(f)\nabla_j^{(h')}(\ln(\Omega)) + \nabla_j^{(h')}(f)\nabla_i^{(h')}(\ln(\Omega)) - h'_{ij}\nabla_k^{(h')}(f)\nabla^{(h')k}(\ln(\Omega)). \tag{A.45}$$

$$\therefore \Box^{(h)} f = \Omega^2 h'^{ij}\nabla_i^{(h)}\nabla_j^{(h)} f = \Omega^2(\Box^{(h')} f - (n-3)\nabla_i^{(h')}(f)\nabla^{(h')i}(\ln(\Omega))). \tag{A.46}$$

Hence, equation 5.44, transforms as

$$\Box^{(h)} S = \frac{1}{S}\nabla_i^{(h)}(C\psi)\nabla^{(h)i}(C\psi) \tag{A.47}$$

$$\iff \Omega^2(\Box^{(h')} S - (n-3)\nabla_i^{(h')}(S)\nabla^{(h')i}(\ln(\Omega))) = \frac{1}{S}\Omega^2\nabla_i^{(h')}(C\psi)\nabla^{(h')i}(C\psi) \tag{A.48}$$

$$\iff \Box^{(h')} S = (n-3)\nabla_i^{(h')}(S)\nabla^{(h')i}(\ln(\Omega)) + \frac{1}{S}\nabla_i^{(h')}(C\psi)\nabla^{(h')i}(C\psi). \tag{A.49}$$

I'll evaluate each of these terms in terms of $\varphi$ and $z$ next. Using equation A.16 and A.20,

$$\begin{aligned}
&(n-3)\nabla_i^{(h')}(S)\nabla^{(h')i}(\ln(\Omega)) \\
&= (n-3)\frac{(1+\cosh(\varphi/2)\cosh(z/2))\nabla_i^{(h')}\varphi - \sinh(z/2)\sinh(\varphi/2)\nabla_i^{(h')}z}{2(\cosh(\varphi/2)+\cosh(z/2))^2} \\
&\quad \times \frac{\tanh(z/2)\cosh(\varphi/2)\nabla^{(h')i}z - \sinh(\varphi/2)\nabla^{(h')i}\varphi}{2(n-3)(\cosh(\varphi/2)+\cosh(z/2))}
\end{aligned} \tag{A.50}$$

$$\begin{aligned}
&= \frac{1}{4(\cosh(\varphi/2)+\cosh(z/2))^3}\Bigg[-\frac{\sinh^2(z/2)\sinh(\varphi/2)\cosh(\varphi/2)}{\cosh(z/2)}\nabla_i^{(h')}(z)\nabla^{(h')i}(z) \\
&\quad - \nabla_i^{(h')}(\varphi)\nabla^{(h')i}(\varphi)(\sinh(\varphi/2)+\cosh(z/2)\cosh(\varphi/2)\sinh(\varphi/2)) \\
&\quad + \nabla_i^{(h')}(\varphi)\nabla^{(h')i}(z)(\tanh(z/2)\cosh(\varphi/2)+\sinh(z/2)\cosh^2(\varphi/2) \\
&\quad + \sinh(z/2)\sinh^2(\varphi/2))\Bigg].
\end{aligned} \tag{A.51}$$

Using equation A.15,

$$\begin{aligned}
&\frac{1}{S}\nabla_i^{(h')}(C\psi)\nabla^{(h')i}(C\psi) \\
&= \frac{(1+\cosh(\varphi/2)\cosh(z/2))\nabla_i^{(h')}z - \sinh(z/2)\sinh(\varphi/2)\nabla_i^{(h')}\varphi}{4\sinh(\varphi/2)(\cosh(\varphi/2)+\cosh(z/2))^3} \\
&\quad \times ((1+\cosh(\varphi/2)\cosh(z/2))\nabla^{(h')i}z - \sinh(z/2)\sinh(\varphi/2)\nabla^{(h')i}\varphi)
\end{aligned} \tag{A.52}$$

$$\begin{aligned}
&= \frac{1}{4(\cosh(\varphi/2)+\cosh(z/2))^3}\Bigg[\sinh^2(z/2)\sinh(\varphi/2)\nabla_i^{(h')}(\varphi)\nabla^{(h')i}(\varphi) \\
&\quad + \nabla_i^{(h')}(z)\nabla^{(h')i}(z)\left(\frac{1}{\sinh(\varphi/2)} + 2\cosh(z/2)\coth(\varphi/2) + \frac{\cosh^2(z/2)\cosh^2(\varphi/2)}{\sinh(\varphi/2)}\right) \\
&\quad - 2\nabla_i^{(h')}(\varphi)\nabla^{(h')i}(z)(\sinh(z/2)+\sinh(z/2)\cosh(z/2)\cosh(\varphi/2))\Bigg].
\end{aligned} \tag{A.53}$$

Lastly, using equation A.16,

$$\Box^{(h')} S = \nabla^{(h')i}\left(\frac{(1+\cosh(\varphi/2)\cosh(z/2))\nabla_i^{(h')}\varphi - \sinh(z/2)\sinh(\varphi/2)\nabla_i^{(h')}z}{2(\cosh(\varphi/2)+\cosh(z/2))^2}\right) \tag{A.54}$$

$$\begin{aligned}
&= -\frac{\sinh(\varphi/2)\nabla^{(h')i}(\varphi) + \sinh(z/2)\nabla^{(h')i}(z)}{2(\cosh(\varphi/2)+\cosh(z/2))^3} \\
&\quad \times (\nabla_i^{(h')}(\varphi) + \cosh(\varphi/2)\cosh(z/2)\nabla_i^{(h')}(\varphi) - \sinh(z/2)\sinh(\varphi/2)\nabla_i^{(h')}(z)) \\
&\quad + \frac{1}{4(\cosh(\varphi/2)+\cosh(z/2))^2}\Big[2\Box^{(h')}\varphi + \sinh(\varphi/2)\cosh(z/2)\nabla_i^{(h')}(\varphi)\nabla^{(h')i}(\varphi) \\
&\quad + \cosh(\varphi/2)\sinh(z/2)\nabla_i^{(h')}(\varphi)\nabla^{(h')i}(z) + 2\cosh(\varphi/2)\cosh(z/2)\Box^{(h')}\varphi \\
&\quad - \cosh(\varphi/2)\sinh(z/2)\nabla_i^{(h')}(\varphi)\nabla^{(h')i}(z) - \sinh(\varphi/2)\cosh(z/2)\nabla_i^{(h')}(z)\nabla^{(h')i}(z) \\
&\quad - 2\sinh(\varphi/2)\sinh(z/2)\Box^{(h')}z\Big]
\end{aligned} \tag{A.55}$$

$$\begin{aligned}
&\therefore \Box^{(h')} S \\
&= -\frac{1}{2(\cosh(\varphi/2)+\cosh(z/2))^3}\Big(\sinh^2(z/2)\sinh(\varphi/2)\nabla_i^{(h')}(z)\nabla^{(h')i}(z) \\
&\quad + \nabla_i^{(h')}(\varphi)\nabla^{(h')i}(z)(\sinh(z/2) + \sinh(z/2)\cosh(z/2)\cosh(\varphi/2) - \sinh^2(\varphi/2)\sinh(z/2)) \\
&\quad + \nabla_i^{(h')}(\varphi)\nabla^{(h')i}(\varphi)(\sinh(\varphi/2) + \sinh(\varphi/2)\cosh(\varphi/2)\cosh(z/2))\Big) \\
&\quad + \frac{1}{4(\cosh(\varphi/2)+\cosh(z/2))^2}\Big[2\Box^{(h')}(\varphi)(1+\cosh(\varphi/2)\cosh(z/2)) \\
&\quad - 2\sinh(\varphi/2)\sinh(z/2)\Box^{(h')}(z) - \sinh(\varphi/2)\cosh(z/2)\nabla_i^{(h')}(z)\nabla^{(h')i}(z) \\
&\quad + \sinh(\varphi/2)\cosh(z/2)\nabla_i^{(h')}(\varphi)\nabla^{(h')i}(\varphi)\Big].
\end{aligned} \tag{A.56}$$

Putting these three expressions together, equation A.49 says

$$\begin{aligned}
0 &= 2\Box^{(h')}(\varphi)(1+\cosh(\varphi/2)\cosh(z/2)) - 2\sinh(\varphi/2)\sinh(z/2)\Box^{(h')}(z) \\
&\quad - \sinh(\varphi/2)\cosh(z/2)\nabla_i^{(h')}(z)\nabla^{(h')i}(z) + \sinh(\varphi/2)\cosh(z/2)\nabla_i^{(h')}(\varphi)\nabla^{(h')i}(\varphi) \\
&\quad + \frac{1}{\cosh(\varphi/2)+\cosh(z/2)}\Big[-\nabla_i^{(h')}(\varphi)\nabla^{(h')i}(z)\Big(2\sinh(z/2) \\
&\quad + 2\sinh(z/2)\cosh(z/2)\cosh(\varphi/2) - 2\sinh^2(\varphi/2)\sinh(z/2) + \tanh(z/2)\cosh(\varphi/2) \\
&\quad + \sinh(z/2)\cosh^2(\varphi/2) + \sinh(z/2)\sinh^2(\varphi/2) - 2\sinh(z/2) - 2\sinh(z/2)\cosh(z/2)\cosh(\varphi/2)\Big) \\
&\quad - \nabla_i^{(h')}(\varphi)\nabla^{(h')i}(\varphi)\Big(2\sinh(\varphi/2) + 2\sinh(\varphi/2)\cosh(\varphi/2)\cosh(z/2) - \sinh(\varphi/2) \\
&\quad - \cosh(z/2)\sinh(\varphi/2)\cosh(\varphi/2) + \sinh^2(z/2)\sinh(\varphi/2)\Big) \\
&\quad + \nabla_i^{(h')}(z)\nabla^{(h')i}(z)\Big(2\sinh^2(z/2)\sinh(\varphi/2) + \frac{\sinh^2(z/2)\sinh(\varphi/2)\cosh(\varphi/2)}{\cosh(z/2)} - \frac{1}{\sinh(\varphi/2)} \\
&\quad - 2\cosh(z/2)\coth(\varphi/2) - \frac{\cosh^2(z/2)\cosh^2(\varphi/2)}{\sinh(\varphi/2)}\Big)\Big].
\end{aligned} \tag{A.57}$$

$$
\begin{aligned}
\therefore 0 = {} & 2\Box^{(h')}(\varphi)(1+\cosh(\varphi/2)\cosh(z/2)) - 2\sinh(\varphi/2)\sinh(z/2)\Box^{(h')}(z) \\
& - \sinh(\varphi/2)\cosh(z/2)\nabla_i^{(h')}(z)\nabla^{(h')i}(z) + \sinh(\varphi/2)\cosh(z/2)\nabla_i^{(h')}(\varphi)\nabla^{(h')i}(\varphi) \\
& + \frac{1}{\cosh(\varphi/2)+\cosh(z/2)}\Bigg[ -\nabla_i^{(h')}(\varphi)\nabla^{(h')i}(z)\Big( -\sinh^2(\varphi/2)\sinh(z/2) \\
& + \tanh(z/2)\cosh(\varphi/2) + \sinh(z/2)\cosh^2(\varphi/2)\Big) \\
& - \nabla_i^{(h')}(\varphi)\nabla^{(h')i}(\varphi)\Big( \sinh(\varphi/2) + \sinh(\varphi/2)\cosh(\varphi/2)\cosh(z/2) + \sinh^2(z/2)\sinh(\varphi/2)\Big) \\
& + \nabla_i^{(h')}(z)\nabla^{(h')i}(z)\Bigg( 2\sinh^2(z/2)\sinh(\varphi/2) + \frac{\sinh^2(z/2)\sinh(\varphi/2)\cosh(\varphi/2)}{\cosh(z/2)} - \frac{1}{\sinh(\varphi/2)} \\
& - 2\cosh(z/2)\coth(\varphi/2) - \frac{\cosh^2(z/2)\cosh^2(\varphi/2)}{\sinh(\varphi/2)}\Bigg)\Bigg] \qquad \text{(A.58)}
\end{aligned}
$$

Again, the coefficient of each derivative combination simplifies.

$$
\begin{aligned}
& -\sinh^2(\varphi/2)\sinh(z/2) + \tanh(z/2)\cosh(\varphi/2) + \sinh(z/2)\cosh^2(\varphi/2) && \text{(A.59)} \\
& = \sinh(z/2) + \tanh(z/2)\cosh(\varphi/2) && \text{(A.60)} \\
& = \tanh(z/2)(\cosh(\varphi/2)+\cosh(z/2)) \quad \text{and} && \text{(A.61)} \\
& \sinh(\varphi/2) + \sinh(\varphi/2)\cosh(\varphi/2)\cosh(z/2) + \sinh^2(z/2)\sinh(\varphi/2) && \text{(A.62)} \\
& = \cosh^2(z/2)\sinh(\varphi/2) + \sinh(\varphi/2)\cosh(\varphi/2)\cosh(z/2) && \text{(A.63)} \\
& = \sinh(\varphi/2)\cosh(z/2)(\cosh(\varphi/2)+\cosh(z/2)). && \text{(A.64)}
\end{aligned}
$$

The third one is again more obscure. Observe that

$$
\begin{aligned}
& (\cosh(\varphi/2)+\cosh(z/2))\left(\frac{\sinh(\varphi/2)\sinh^2(z/2)}{\cosh(z/2)} - \frac{\cosh(z/2)}{\sinh(\varphi/2)} - \coth(\varphi/2)\right) \\
& = \frac{\cosh(\varphi/2)\sinh(\varphi/2)\sinh^2(z/2)}{\cosh(z/2)} + \sinh(\varphi/2)\sinh^2(z/2) - \cosh(z/2)\coth(\varphi/2) - \frac{\cosh^2(z/2)}{\sinh(\varphi/2)} \\
& \quad - \frac{\cosh^2(\varphi/2)}{\sinh(\varphi/2)} - \cosh(z/2)\coth(\varphi/2) && \text{(A.65)} \\
& = \frac{\cosh(\varphi/2)\sinh(\varphi/2)\sinh^2(z/2)}{\cosh(z/2)} + \sinh(\varphi/2)\sinh^2(z/2) - 2\cosh(z/2)\coth(\varphi/2) \\
& \quad - \frac{1+\sinh^2(z/2)}{\sinh(\varphi/2)} - \frac{\cosh^2(\varphi/2)}{\sinh(\varphi/2)}(\cosh^2(z/2)-\sinh^2(z/2)) && \text{(A.66)} \\
& = \frac{\cosh(\varphi/2)\sinh(\varphi/2)\sinh^2(z/2)}{\cosh(z/2)} - 2\cosh(z/2)\coth(\varphi/2) - \frac{1}{\sinh(\varphi/2)} \\
& \quad - \frac{\cosh^2(\varphi/2)\cosh^2(z/2)}{\sinh(\varphi/2)} + \sinh(\varphi/2)\sinh^2(z/2) - \frac{\sinh^2(z/2)}{\sinh(\varphi/2)}(1-\cosh^2(\varphi/2)) && \text{(A.67)} \\
& = \frac{\cosh(\varphi/2)\sinh(\varphi/2)\sinh^2(z/2)}{\cosh(z/2)} - 2\cosh(z/2)\coth(\varphi/2) - \frac{1}{\sinh(\varphi/2)} \\
& \quad - \frac{\cosh^2(\varphi/2)\cosh^2(z/2)}{\sinh(\varphi/2)} + 2\sinh(\varphi/2)\sinh^2(z/2) && \text{(A.68)}
\end{aligned}
$$

which is the coefficient of $\nabla_i^{(h')}(z)\nabla^{(h')i}(z)$ in equation A.58. Hence, equation A.58 simplifies to

$$\begin{aligned}
0 &= 2\Box^{(h')}(\varphi)(1+\cosh(\varphi/2)\cosh(z/2)) - 2\sinh(\varphi/2)\sinh(z/2)\Box^{(h')}(z) \\
&\quad - \sinh(\varphi/2)\cosh(z/2)\nabla_i^{(h')}(z)\nabla^{(h')i}(z) + \sinh(\varphi/2)\cosh(z/2)\nabla_i^{(h')}(\varphi)\nabla^{(h')i}(\varphi) \\
&\quad - \tanh(z/2)\nabla_i^{(h')}(\varphi)\nabla^{(h')i}(z) - \sinh(\varphi/2)\cosh(z/2)\nabla_i^{(h')}(\varphi)\nabla^{(h')i}(\varphi) \\
&\quad + \left(\frac{\sinh(\varphi/2)\sinh^2(z/2)}{\cosh(z/2)} - \frac{\cosh(z/2)}{\sinh(\varphi/2)} - \coth(\varphi/2)\right)\nabla_i^{(h')}(z)\nabla^{(h')i}(z) \qquad \text{(A.69)} \\
&= 2\Box^{(h')}(\varphi)(1+\cosh(\varphi/2)\cosh(z/2)) - 2\sinh(\varphi/2)\sinh(z/2)\Box^{(h')}(z) \\
&\quad - \tanh(z/2)\nabla_i^{(h')}(\varphi)\nabla^{(h')i}(z) \\
&\quad + \nabla_i^{(h')}(z)\nabla^{(h')i}(z)\Bigg(\frac{\sinh(\varphi/2)\sinh^2(z/2)}{\cosh(z/2)} - \frac{\cosh(z/2)}{\sinh(\varphi/2)} - \coth(\varphi/2) \\
&\quad - \sinh(\varphi/2)\cosh(z/2)\Bigg). \qquad \text{(A.70)}
\end{aligned}$$

Hence, I finally get that equation 5.44 is equivalent to

$$\begin{aligned}
0 &= 2\Box^{(h')}(\varphi)(1+\cosh(\varphi/2)\cosh(z/2)) - 2\sinh(\varphi/2)\sinh(z/2)\Box^{(h')}(z) \\
&\quad - \tanh(z/2)\nabla_i^{(h')}(\varphi)\nabla^{(h')i}(z) \\
&\quad + \nabla_i^{(h')}(z)\nabla^{(h')i}(z)\left(\frac{\sinh(\varphi/2)\sinh^2(z/2)}{\cosh(z/2)} - \frac{\cosh(z/2)\cosh^2(\varphi/2)}{\sinh(\varphi/2)} - \coth(\varphi/2)\right). \qquad \text{(A.71)}
\end{aligned}$$

Re-arranging for $\Box^{(h')}z$, I get

$$\begin{aligned}
\Box^{(h')}z &= \left(\frac{1}{\sinh(\varphi/2)\sinh(z/2)} + \coth(\varphi/2)\coth(z/2)\right)\Box^{(h')}\varphi \\
&\quad + \frac{1}{2}\nabla_i^{(h')}(z)\nabla^{(h')i}(z)\left(\tanh(z/2) - \coth(z/2)\coth^2(\varphi/2) - \frac{\cosh(\varphi/2)}{\sinh(z/2)\sinh^2(\varphi/2)}\right) \\
&\quad - \frac{1}{2\sinh(\varphi/2)\cosh(z/2)}\nabla_i^{(h')}(\varphi)\nabla^{(h')i}(z). \qquad \text{(A.72)}
\end{aligned}$$

Substituting this back into equation A.42 gives

$$\begin{aligned}
0 &= \frac{2(1+\cosh(\varphi/2)\cosh(z/2))}{\sinh(\varphi/2)}\left(\frac{1}{\sinh(\varphi/2)\sinh(z/2)} + \coth(\varphi/2)\coth(z/2)\right)\Box^{(h')}\varphi \\
&\quad + \frac{1}{2}\nabla_i^{(h')}(z)\nabla^{(h')i}(z)\left(\tanh(z/2) - \coth(z/2)\coth^2(\varphi/2) - \frac{\cosh(\varphi/2)}{\sinh(z/2)\sinh^2(\varphi/2)}\right) \\
&\quad \times \frac{2(1+\cosh(\varphi/2)\cosh(z/2))}{\sinh(\varphi/2)} \\
&\quad - \frac{1}{2\sinh(\varphi/2)\cosh(z/2)}\nabla_i^{(h')}(\varphi)\nabla^{(h')i}(z)\frac{2(1+\cosh(\varphi/2)\cosh(z/2))}{\sinh(\varphi/2)} \\
&\quad - 2\sinh(z/2)\Box^{(h')}(\varphi) - \frac{\tanh(z/2)}{\sinh(\varphi/2)}\nabla_i^{(h')}(z)\nabla^{(h')i}(z) \\
&\quad - \left(\frac{\cosh(z/2)}{\sinh^2(\varphi/2)} + \frac{1}{\cosh(z/2)} + \frac{\cosh(\varphi/2)}{\sinh^2(\varphi/2)}\right)\nabla_i^{(h')}(\varphi)\nabla^{(h')i}(z). \qquad \text{(A.73)}
\end{aligned}$$

$$
\begin{aligned}
\therefore 0 = 2\Box^{(h')}(\varphi)\Bigg(&\frac{1}{\sinh^2(\varphi/2)\sinh(z/2)} + \frac{2\coth(z/2)\cosh(\varphi/2)}{\sinh^2(\varphi/2)} + \frac{\coth^2(\varphi/2)\cosh^2(z/2)}{\sinh(z/2)} \\
&- \sinh(z/2)\Bigg) \\
&- \nabla_i^{(h')}(\varphi)\nabla^{(h')i}(z)\Bigg(\frac{1}{\sinh^2(\varphi/2)\cosh(z/2)} + \frac{2\cosh(\varphi/2)}{\sinh^2(\varphi/2)} + \frac{\cosh(z/2)}{\sinh^2(\varphi/2)} + \frac{1}{\cosh(z/2)}\Bigg) \\
&+ \nabla_i^{(h')}(z)\nabla^{(h')i}(z)\Bigg(\coth(\varphi/2)\sinh(z/2) - \frac{2\coth(z/2)\cosh^2(\varphi/2)}{\sinh^3(\varphi/2)} \\
&- \frac{\cosh^2(z/2)\coth^3(\varphi/2)}{\sinh(z/2)} - \frac{\cosh(\varphi/2)}{\sinh(z/2)\sinh^3(\varphi/2)}\Bigg). \qquad \text{(A.74)}
\end{aligned}
$$

As has become custom by now, each of these daunting coefficients simplifies significantly.

$$
\begin{aligned}
&\frac{1}{\sinh^2(\varphi/2)\sinh(z/2)} + \frac{2\coth(z/2)\cosh(\varphi/2)}{\sinh^2(\varphi/2)} + \frac{\coth^2(\varphi/2)\cosh^2(z/2)}{\sinh(z/2)} - \sinh(z/2) \\
&= \frac{1 + 2\cosh(z/2)\cosh(\varphi/2) + \cosh^2(\varphi/2)\cosh^2(z/2) - \sinh^2(\varphi/2)\sinh^2(z/2)}{\sinh^2(\varphi/2)\sinh(z/2)} \qquad \text{(A.75)} \\
&= \frac{1 + 2\cosh(z/2)\cosh(\varphi/2) + \cosh^2(\varphi/2)\cosh^2(z/2) - (\cosh^2(\varphi/2) - 1)(\cosh^2(z/2) - 1)}{\sinh^2(\varphi/2)\sinh(z/2)} \\
&\qquad \text{(A.76)} \\
&= \frac{(\cosh(\varphi/2) + \cosh(z/2))^2}{\sinh^2(\varphi/2)\sinh(z/2)}, \qquad \text{(A.77)} \\
&\frac{1}{\sinh^2(\varphi/2)\cosh(z/2)} + \frac{2\cosh(\varphi/2)}{\sinh^2(\varphi/2)} + \frac{\cosh(z/2)}{\sinh^2(\varphi/2)} + \frac{1}{\cosh(z/2)} \\
&= \frac{1 + 2\cosh(z/2)\cosh(z/2) + \cosh^2(z/2) + \sinh^2(\varphi/2)}{\sinh^2(\varphi/2)\cosh(z/2)} \qquad \text{(A.78)} \\
&= \frac{(\cosh(\varphi/2) + \cosh(z/2))^2}{\sinh^2(\varphi/2)\cosh(z/2)} \quad \text{and} \qquad \text{(A.79)} \\
&\coth(\varphi/2)\sinh(z/2) - \frac{2\coth(z/2)\cosh^2(\varphi/2)}{\sinh^3(\varphi/2)} - \frac{\cosh^2(z/2)\coth^3(\varphi/2)}{\sinh(z/2)} \\
&- \frac{\cosh(\varphi/2)}{\sinh(z/2)\sinh^3(\varphi/2)} \\
&= -\frac{\cosh(\varphi/2)}{\sinh^3(\varphi/2)\sinh(z/2)}\Big( - \sinh^2(\varphi/2)\sinh^2(z/2) + 2\cosh(\varphi/2)\cosh(z/2) \\
&\quad + \cosh^2(z/2)\cosh^2(\varphi/2) + 1\Big) \qquad \text{(A.80)} \\
&= -\frac{\cosh(\varphi/2)}{\sinh^3(\varphi/2)\sinh(z/2)}\Big( - (\cosh^2(\varphi/2) - 1)(\cosh^2(z/2) - 1) + 2\cosh(\varphi/2)\cosh(z/2) \\
&\quad + \cosh^2(z/2)\cosh^2(\varphi/2) + 1\Big) \qquad \text{(A.81)} \\
&= -\frac{\cosh(\varphi/2)(\cosh(\varphi/2) + \cosh(z/2))^2}{\sinh^3(\varphi/2)\sinh(z/2)}. \qquad \text{(A.82)}
\end{aligned}
$$

Hence equation A.74 says

$$
\begin{aligned}
0 &= 2\frac{(\cosh(\varphi/2)+\cosh(z/2))^2}{\sinh^2(\varphi/2)\sinh(z/2)}\Box^{(h')}\varphi - \frac{(\cosh(\varphi/2)+\cosh(z/2))^2}{\sinh^2(\varphi/2)\cosh(z/2)}\nabla_i^{(h')}(\varphi)\nabla^{(h')i}(z) \\
&\quad - \frac{\cosh(\varphi/2)(\cosh(\varphi/2)+\cosh(z/2))^2}{\sinh^3(\varphi/2)\sinh(z/2)}\nabla_i^{(h')}(z)\nabla^{(h')i}(z) \qquad \text{(A.83)}\\
\iff \Box^{(h')}\varphi &= \frac{1}{2}\tanh(z/2)\nabla_i^{(h')}(\varphi)\nabla^{(h')i}(z) + \frac{1}{2}\coth(\varphi/2)\nabla_i^{(h')}(z)\nabla^{(h')i}(z), \qquad \text{(A.84)}
\end{aligned}
$$

which completes the proof of equation A.1. Substituting this result back into equation A.72 gives

$$
\begin{aligned}
\Box^{(h')}z &= \frac{1}{2}\left(\frac{1}{\sinh(\varphi/2)\sinh(z/2)} + \coth(\varphi/2)\coth(z/2)\right) \\
&\quad \times \left(\tanh(z/2)\nabla_i^{(h')}(\varphi)\nabla^{(h')i}(z) + \coth(\varphi/2)\nabla_i^{(h')}(z)\nabla^{(h')i}(z)\right) \\
&\quad + \frac{1}{2}\nabla_i^{(h')}(z)\nabla^{(h')i}(z)\left(\tanh(z/2) - \coth(z/2)\coth^2(\varphi/2) - \frac{\cosh(\varphi/2)}{\sinh(z/2)\sinh^2(\varphi/2)}\right) \\
&\quad - \frac{1}{2\sinh(\varphi/2)\cosh(z/2)}\nabla_i^{(h')}(\varphi)\nabla^{(h')i}(z) \qquad \text{(A.85)}\\
&= \frac{1}{2}\nabla_i^{(h')}(\varphi)\nabla^{(h')i}(z)\left(\frac{1}{\sinh(\varphi/2)\cosh(z/2)} + \coth(\varphi/2) - \frac{1}{\sinh(\varphi/2)\cosh(z/2)}\right) \\
&\quad + \frac{1}{2}\nabla_i^{(h')}(z)\nabla^{(h')i}(z)\left(\frac{\cosh(\varphi/2)}{\sinh^2(\varphi/2)\sinh(z/2)} + \coth^2(\varphi/2)\coth(z/2) + \tanh(z/2)\right. \\
&\quad \left. - \coth(z/2)\coth^2(\varphi/2) - \frac{\cosh(\varphi/2)}{\sinh(z/2)\sinh^2(\varphi/2)}\right) \qquad \text{(A.86)}\\
&= \frac{1}{2}\coth(\varphi/2)\nabla_i^{(h')}(\varphi)\nabla^{(h')i}(z) + \frac{1}{2}\tanh(z/2)\nabla_i^{(h')}(z)\nabla^{(h')i}(z), \qquad \text{(A.87)}
\end{aligned}
$$

which proves equation A.2.

# Appendix B

# Proof of theorem 5.14

In this appendix I'll prove that equation 5.46,

$$R^{(h)}_{ij} = \frac{1}{S}\nabla^{(h)}_i\nabla^{(h)}_j S + \frac{C^2}{(n-3)S^2}h_{ij}\nabla^{(h)}_k(\psi)\nabla^{(h)k}(\psi) - \frac{(n-2)C^2}{(n-3)S^2}\nabla^{(h)}_i(\psi)\nabla^{(h)}_j(\psi), \tag{B.1}$$

is equivalent to

$$\begin{aligned} R^{(h')}_{ij} &= \frac{1}{2}\coth(\varphi/2)\nabla^{(h')}_i\nabla^{(h')}_j(\varphi) - \frac{1}{4(n-3)}\nabla^{(h')}_i(\varphi)\nabla^{(h')}_j(\varphi) \\ &\quad + \frac{1}{4(n-3)}h'_{ij}\nabla^{(h')}_k(\varphi)\nabla^{(h')k}(\varphi). \end{aligned} \tag{B.2}$$

I can assume $z$ is a constant, because this is already proven in the main text before $R^{(h)}_{ij}$ is ever required in terms of $h'$, $\varphi$ and $z$.
From page 42 of [16],

$$R^{(h')}_{ij} = R^{(h)}_{ij} + \frac{n-3}{4}\Omega^2\Omega_{ij} + \frac{1}{4}\Omega^2 h_{ij}\Omega^k{}_k, \text{ where} \tag{B.3}$$

$$\Omega_{ij} = \frac{4}{\Omega}\nabla^{(h)}_i\nabla^{(h)}_j\left(\frac{1}{\Omega}\right) - 2h_{ij}\nabla^{(h)}_k\left(\frac{1}{\Omega}\right)\nabla^{(h)k}\left(\frac{1}{\Omega}\right) \text{ and } \Omega^i{}_i = h^{ij}\Omega_{ij}. \tag{B.4}$$

By equation A.45,

$$\begin{aligned} \nabla^{(h)}_i\nabla^{(h)}_j\left(\frac{1}{\Omega}\right) &= \nabla^{(h')}_i\nabla^{(h')}_j\left(\frac{1}{\Omega}\right) + \nabla^{(h')}_i\left(\frac{1}{\Omega}\right)\nabla^{(h')}_j(\ln(\Omega)) + \nabla^{(h')}_j\left(\frac{1}{\Omega}\right)\nabla^{(h')}_i(\ln(\Omega)) \\ &\quad - h'_{ij}\nabla^{(h')}_k\left(\frac{1}{\Omega}\right)\nabla^{(h')k}(\ln(\Omega)) \end{aligned} \tag{B.5}$$

$$\begin{aligned} &= \nabla^{(h')}_i\nabla^{(h')}_j\left(\frac{1}{\Omega}\right) - \frac{1}{\Omega^2}\nabla^{(h')}_i(\Omega)\nabla^{(h')}_j(\ln(\Omega)) - \frac{1}{\Omega^2}\nabla^{(h')}_j(\Omega)\nabla^{(h')}_i(\ln(\Omega)) \\ &\quad + h'_{ij}\frac{1}{\Omega^2}\nabla^{(h')}_k(\Omega)\nabla^{(h')k}(\ln(\Omega)). \end{aligned} \tag{B.6}$$

Substituting this back into the $\Omega_{ij}$ definition, I get

$$\begin{aligned}
&\Omega^2\Omega_{ij} \\
&= 4\Omega\nabla_i^{(h')}\nabla_j^{(h')}\left(\frac{1}{\Omega}\right) - \frac{4}{\Omega}\nabla_i^{(h')}(\Omega)\nabla_j^{(h')}(\ln(\Omega)) - \frac{4}{\Omega}\nabla_j^{(h')}(\Omega)\nabla_i^{(h')}(\ln(\Omega)) \\
&\quad + h'_{ij}\frac{4}{\Omega}\nabla_k^{(h')}(\Omega)\nabla^{(h')k}(\ln(\Omega)) - 2h_{ij}\frac{1}{\Omega^2}\nabla_k^{(h')}(\Omega)\nabla_l^{(h')}(\Omega)h^{kl} \qquad \text{(B.7)} \\
&= 4\Omega\nabla_i^{(h')}\nabla_j^{(h')}\left(\frac{1}{\Omega}\right) - 8\nabla_i^{(h')}(\ln(\Omega))\nabla_j^{(h')}(\ln(\Omega)) + 2h'_{ij}\nabla_k^{(h')}(\ln(\Omega))\nabla^{(h')k}(\ln(\Omega)). \qquad \text{(B.8)}
\end{aligned}$$

$$\begin{aligned}
&\therefore \Omega^2 h_{ij}\Omega^k{}_k \\
&= \Omega^2 h_{ij}\Omega_{kl}h^{kl} \qquad \text{(B.9)} \\
&= \Omega^2 h'_{ij}\Omega_{kl}h'^{kl} \qquad \text{(B.10)} \\
&= 4\Omega h'_{ij}\Box^{(h')}\left(\frac{1}{\Omega}\right) + 2(n-5)h'_{ij}\nabla_k^{(h')}(\ln(\Omega))\nabla^{(h')k}(\ln(\Omega)). \qquad \text{(B.11)}
\end{aligned}$$

$$\begin{aligned}
&\therefore \frac{n-3}{4}\Omega^2\Omega_{ij} + \frac{1}{4}\Omega^2 h_{ij}\Omega^k{}_k \\
&= (n-3)\Omega\nabla_i^{(h')}\nabla_j^{(h')}\left(\frac{1}{\Omega}\right) - 2(n-3)\nabla_i^{(h')}(\ln(\Omega))\nabla_j^{(h')}(\ln(\Omega)) + \Omega h'_{ij}\Box^{(h')}\left(\frac{1}{\Omega}\right) \\
&\quad + (n-4)h'_{ij}\nabla_k^{(h')}(\ln(\Omega))\nabla^{(h')k}(\ln(\Omega)). \qquad \text{(B.12)}
\end{aligned}$$

Substituting this back into equations B.3 and B.1,

$$\begin{aligned}
R_{ij}^{(h')} &= \frac{1}{S}\nabla_i^{(h)}\nabla_j^{(h)}S + \frac{C^2}{(n-3)S^2}h_{ij}\nabla_k^{(h)}(\psi)\nabla^{(h)k}(\psi) - \frac{(n-2)C^2}{(n-3)S^2}\nabla_i^{(h)}(\psi)\nabla_j^{(h)}(\psi) \\
&\quad + (n-3)\Omega\nabla_i^{(h')}\nabla_j^{(h')}\left(\frac{1}{\Omega}\right) - 2(n-3)\nabla_i^{(h')}(\ln(\Omega))\nabla_j^{(h')}(\ln(\Omega)) + \Omega h'_{ij}\Box^{(h')}\left(\frac{1}{\Omega}\right) \\
&\quad + (n-4)h'_{ij}\nabla_k^{(h')}(\ln(\Omega))\nabla^{(h')k}(\ln(\Omega)). \qquad \text{(B.13)}
\end{aligned}$$

In the last expression, using equation A.45 again,

$$\begin{aligned}
\nabla_i^{(h)}\nabla_j^{(h)}S &= \nabla_i^{(h')}\nabla_j^{(h')}S + \nabla_i^{(h')}(S)\nabla_j^{(h')}(\ln(\Omega)) + \nabla_j^{(h')}(S)\nabla_i^{(h')}(\ln(\Omega)) \\
&\quad - h'_{ij}\nabla_k^{(h')}(S)\nabla^{(h')k}(\ln(\Omega)). \qquad \text{(B.14)}
\end{aligned}$$

Meanwhile, I also have,

$$\frac{C^2}{(n-3)S^2}h_{ij}\nabla_k^{(h)}(\psi)\nabla^{(h)k}(\psi) = \frac{C^2}{(n-3)S^2}h'_{ij}\nabla_k^{(h')}(\psi)\nabla^{(h')k}(\psi) \text{ and} \qquad \text{(B.15)}$$

$$\frac{(n-2)C^2}{(n-3)S^2}\nabla_i^{(h)}(\psi)\nabla_j^{(h)}(\psi) = \frac{(n-2)}{(n-3)S^2}\nabla_i^{(h')}(C\psi)\nabla_j^{(h')}(C\psi). \qquad \text{(B.16)}$$

Putting all these pieces together,

$$\begin{aligned}
R_{ij}^{(h')} &= \frac{1}{S}\nabla_i^{(h')}\nabla_j^{(h')}S + \frac{1}{S}\nabla_i^{(h')}(S)\nabla_j^{(h')}(\ln(\Omega)) + \frac{1}{S}\nabla_j^{(h')}(S)\nabla_i^{(h')}(\ln(\Omega)) \\
&\quad - \frac{1}{S}h'_{ij}\nabla_k^{(h')}(S)\nabla^{(h')k}(\ln(\Omega)) + \frac{1}{(n-3)S^2}h'_{ij}\nabla_k^{(h')}(C\psi)\nabla^{(h')k}(C\psi) \\
&\quad - \frac{(n-2)}{(n-3)S^2}\nabla_i^{(h')}(C\psi)\nabla_j^{(h')}(C\psi) + (n-3)\Omega\nabla_i^{(h')}\nabla_j^{(h')}\left(\frac{1}{\Omega}\right) \\
&\quad - 2(n-3)\nabla_i^{(h')}(\ln(\Omega))\nabla_j^{(h')}(\ln(\Omega)) + \Omega h'_{ij}\Box^{(h')}\left(\frac{1}{\Omega}\right) \\
&\quad + (n-4)h'_{ij}\nabla_k^{(h')}(\ln(\Omega))\nabla^{(h')k}(\ln(\Omega)). \qquad \text{(B.17)}
\end{aligned}$$

All that remains is the unenviable task of evaluating all these derivatives and simplifying the result. Some of these derivatives have already been calculated in appendix A. With $z$ constant, equations A.15, A.16 and A.20 say

$$\nabla_i^{(h')}(C\psi) = -\frac{\sinh(\varphi/2)\sinh(z/2)}{2(\cosh(\varphi/2)+\cosh(z/2))^2}, \tag{B.18}$$

$$\nabla_i^{(h')}(S) = \frac{(1+\cosh(\varphi/2)\cosh(z/2))\nabla_i^{(h')}\varphi}{2(\cosh(\varphi/2)+\cosh(z/2))^2} \quad \text{and} \tag{B.19}$$

$$\nabla_i^{(h')}(\ln(\Omega)) = -\frac{\sinh(\varphi/2)\nabla_i^{(h')}\varphi}{2(n-3)(\cosh(\varphi/2)+\cosh(z/2))}. \tag{B.20}$$

Putting these together with lemma 5.9,

$$\frac{1}{S}\nabla_i^{(h')}(S)\nabla_j^{(h')}(\ln(\Omega)) = -\frac{1+\cosh(\varphi/2)\cosh(z/2)}{4(n-3)(\cosh(\varphi/2)+\cosh(z/2))^2}\nabla_i^{(h')}(\varphi)\nabla_j^{(h')}(\varphi), \tag{B.21}$$

$$\frac{1}{S^2}\nabla_i^{(h')}(C\psi)\nabla_j^{(h')}(C\psi) = \frac{\sinh^2(z/2)}{4(\cosh(\varphi/2)+\cosh(z/2))^2}\nabla_i^{(h')}(\varphi)\nabla_j^{(h')}(\varphi) \quad \text{and} \tag{B.22}$$

$$(n-3)\nabla_i^{(h')}(\ln(\Omega))\nabla_j^{(h')}(\ln(\Omega)) = \frac{\sinh^2(\varphi/2)}{4(n-3)(\cosh(\varphi/2)+\cosh(z/2))^2}\nabla_i^{(h')}(\varphi)\nabla_j^{(h')}(\varphi). \tag{B.23}$$

Meanwhile, the 2nd derivative terms are as follows.

$$\begin{aligned}
&\frac{1}{S}\nabla_i^{(h')}\nabla_j^{(h')}S \\
&= \frac{\cosh(\varphi/2)+\cosh(z/2)}{\sinh(\varphi/2)}\nabla_i^{(h')}\left(\frac{(1+\cosh(\varphi/2)\cosh(z/2))\nabla_j^{(h')}\varphi}{2(\cosh(\varphi/2)+\cosh(z/2))^2}\right) \qquad \text{(B.24)}\\
&= -\frac{\cosh(\varphi/2)+\cosh(z/2)}{\sinh(\varphi/2)}\frac{\sinh(\varphi/2)\nabla_i^{(h')}\varphi}{2(\cosh(\varphi/2)+\cosh(z/2))^3}(1+\cosh(\varphi/2)\cosh(z/2))\nabla_j^{(h')}\varphi \\
&\quad + \frac{1}{4\sinh(\varphi/2)(\cosh(\varphi/2)+\cosh(z/2))}\sinh(\varphi/2)\cosh(z/2)\nabla_i^{(h')}(\varphi)\nabla_j^{(h')}(\varphi) \\
&\quad + \frac{1}{2\sinh(\varphi/2)(\cosh(\varphi/2)+\cosh(z/2))}(1+\cosh(\varphi/2)\cosh(z/2))\nabla_i^{(h')}\nabla_j^{(h')}\varphi \qquad \text{(B.25)}\\
&= \frac{\cosh^2(z/2)-\cosh(\varphi/2)\cosh(z/2)-2}{4(\cosh(\varphi/2)+\cosh(z/2))^2}\nabla_i^{(h')}(\varphi)\nabla_j^{(h')}(\varphi) \\
&\quad + \frac{\frac{1}{\sinh(\varphi/2)}+\cosh(z/2)\coth(\varphi/2)}{2(\cosh(\varphi/2)+\cosh(z/2))}\nabla_i^{(h')}\nabla_j^{(h')}\varphi. \qquad \text{(B.26)}
\end{aligned}$$

From equation 5.84,

$$
\begin{aligned}
&(n-3)\Omega\nabla_i^{(h')}\nabla_j^{(h')}\left(\frac{1}{\Omega}\right) \\
&= (n-3)\left(\frac{2\cosh(z/2)}{\cosh(\varphi/2)+\cosh(z/2)}\right)^{1/(n-3)}\nabla_i^{(h')}\nabla_j^{(h')}\left(\left(\frac{\cosh(\varphi/2)}{2\cosh(z/2)}+\frac{1}{2}\right)^{1/(n-3)}\right) \qquad \text{(B.27)} \\
&= \left(\frac{2\cosh(z/2)}{\cosh(\varphi/2)+\cosh(z/2)}\right)^{1/(n-3)} \\
&\quad \times \nabla_i^{(h')}\left(\left(\frac{\cosh(\varphi/2)}{2\cosh(z/2)}+\frac{1}{2}\right)^{(4-n)/(n-3)}\frac{\sinh(\varphi/2)}{4\cosh(z/2)}\nabla_j^{(h')}\varphi\right) \qquad \text{(B.28)} \\
&= \frac{2\cosh(z/2)}{\cosh(\varphi/2)+\cosh(z/2)}\left(\frac{\cosh(\varphi/2)}{8\cosh(z/2)}\nabla_i^{(h')}(\varphi)\nabla_j^{(h')}(\varphi)+\frac{\sinh(\varphi/2)}{4\cosh(z/2)}\nabla_i^{(h')}\nabla_j^{(h')}(\varphi)\right) \\
&\quad - \frac{n-4}{n-3}\left(\frac{2\cosh(z/2)}{\cosh(\varphi/2)+\cosh(z/2)}\right)^2\frac{\sinh(\varphi/2)}{4\cosh(z/2)}\nabla_i^{(h')}(\varphi)\frac{\sinh(\varphi/2)}{4\cosh(z/2)}\nabla_j^{(h')}(\varphi) \qquad \text{(B.29)} \\
&= \frac{\cosh(\varphi/2)\nabla_i^{(h')}(\varphi)\nabla_j^{(h')}(\varphi)+2\sinh(\varphi/2)\nabla_i^{(h')}\nabla_j^{(h')}\varphi}{4(\cosh(\varphi/2)+\cosh(z/2))} \\
&\quad -(n-4)(n-3)\nabla_i^{(h')}(\ln(\Omega))\nabla_j^{(h')}(\ln(\Omega)). \qquad \text{(B.30)}
\end{aligned}
$$

Corollary 5.13.1 says $\Box^{(h')}\varphi = 0$, so it follows that

$$
\Omega\Box^{(h')}\left(\frac{1}{\Omega}\right) = \frac{\cosh(\varphi/2)\nabla_k^{(h')}(\varphi)\nabla^{(h')k}(\varphi)}{4(n-3)(\cosh(\varphi/2)+\cosh(z/2))} - (n-4)\nabla_k^{(h')}(\ln(\Omega))\nabla^{(h')k}(\ln(\Omega)). \qquad \text{(B.31)}
$$

Substituting all of these expressions back into equation B.17 gives

$$
\begin{aligned}
R_{ij}^{(h')} &= \frac{1}{S}\nabla_i^{(h')}\nabla_j^{(h')}S + \frac{1}{S}\nabla_i^{(h')}(S)\nabla_j^{(h')}(\ln(\Omega)) + \frac{1}{S}\nabla_j^{(h')}(S)\nabla_i^{(h')}(\ln(\Omega)) \\
&\quad - \frac{1}{S}h'_{ij}\nabla_k^{(h')}(S)\nabla^{(h')k}(\ln(\Omega)) + \frac{1}{(n-3)S^2}h'_{ij}\nabla_k^{(h')}(C\psi)\nabla^{(h')k}(C\psi) \\
&\quad - \frac{(n-2)}{(n-3)S^2}\nabla_i^{(h')}(C\psi)\nabla_j^{(h')}(C\psi) - (n-3)(n-2)\nabla_i^{(h')}(\ln(\Omega))\nabla_j^{(h')}(\ln(\Omega)) \\
&\quad + \frac{\cosh(\varphi/2)\nabla_i^{(h')}(\varphi)\nabla_j^{(h')}(\varphi)+2\sinh(\varphi/2)\nabla_i^{(h')}\nabla_j^{(h')}\varphi}{4(\cosh(\varphi/2)+\cosh(z/2))} \\
&\quad + h'_{ij}\frac{\cosh(\varphi/2)\nabla_k^{(h')}(\varphi)\nabla^{(h')k}(\varphi)}{4(n-3)(\cosh(\varphi/2)+\cosh(z/2))} \qquad \text{(B.32)} \\
&= \frac{\nabla_i^{(h')}\nabla_j^{(h')}(\varphi)}{2(\cosh(\varphi/2)+\cosh(z/2))}\left(\frac{1}{\sinh(\varphi/2)}+\cosh(z/2)\coth(\varphi/2)+\sinh(\varphi/2)\right) \\
&\quad + \frac{\nabla_i^{(h')}(\varphi)\nabla_j^{(h')}(\varphi)}{4(n-3)(\cosh(\varphi/2)+\cosh(z/2))^2}\Big((n-3)\cosh^2(z/2)-(n-3)\cosh(\varphi/2)\cosh(z/2) \\
&\quad -2(n-3)-2(1+\cosh(\varphi/2)\cosh(z/2))-(n-2)\sinh^2(z/2)-(n-2)\sinh^2(\varphi/2) \\
&\quad +(n-3)\cosh(\varphi/2)(\cosh(\varphi/2)+\cosh(z/2))\Big) \\
&\quad + \frac{h'_{ij}\nabla_k^{(h')}(\varphi)\nabla^{(h')k}(\varphi)}{4(n-3)(\cosh(\varphi/2)+\cosh(z/2))^2}\Big(1+\cosh(\varphi/2)\cosh(z/2)+\sinh^2(z/2) \\
&\quad +\cosh(\varphi/2)(\cosh(\varphi/2)+\cosh(z/2))\Big). \qquad \text{(B.33)}
\end{aligned}
$$

As usual, each of these ghastly derivative coefficients simplifies.

$$\frac{1}{\sinh(\varphi/2)} + \cosh(z/2)\coth(\varphi/2) + \sinh(\varphi/2)$$

$$= \coth(\varphi/2)\left(\frac{1}{\cosh(\varphi/2)} + \cosh(z/2) + \frac{\sinh^2(\varphi/2)}{\cosh(\varphi/2)}\right) \tag{B.34}$$

$$= \coth(\varphi/2)(\cosh(\varphi/2) + \cosh(z/2)), \tag{B.35}$$

$$(n-3)\cosh^2(z/2) - (n-3)\cosh(\varphi/2)\cosh(z/2) - 2(n-3)$$
$$- 2(1+\cosh(\varphi/2)\cosh(z/2)) - (n-2)\sinh^2(z/2) - (n-2)\sinh^2(\varphi/2)$$
$$+ (n-3)\cosh(\varphi/2)(\cosh(\varphi/2)+\cosh(z/2))$$
$$= (n-2) - \cosh^2(z/2) - 2(n-3) - 2 - 2\cosh(\varphi/2)\cosh(z/2) + (n-2)$$
$$- \cosh^2(\varphi/2) \tag{B.36}$$

$$= -(\cosh(\varphi/2)+\cosh(z/2))^2 \quad \text{and} \tag{B.37}$$

$$1 + \cosh(\varphi/2)\cosh(z/2) + \sinh^2(z/2) + \cosh(\varphi/2)(\cosh(\varphi/2)+\cosh(z/2))$$
$$= \cosh^2(z/2) + 2\cosh(\varphi/2)\cosh(z/2) + \cosh^2(\varphi/2) \tag{B.38}$$
$$= (\cosh(\varphi/2)+\cosh(z/2))^2. \tag{B.39}$$

Hence, equation B.33 says

$$R^{(h')}_{ij} = \frac{1}{2}\coth(\varphi/2)\nabla^{(h')}_i\nabla^{(h')}_j(\varphi) - \frac{1}{4(n-3)}\nabla^{(h')}_i(\varphi)\nabla^{(h')}_j(\varphi)$$
$$+ \frac{1}{4(n-3)}h'_{ij}\nabla^{(h')}_k(\varphi)\nabla^{(h')k}(\varphi), \tag{B.40}$$

which is what I set out to prove.

# Appendix C

# Proof of lemma 5.25

In this appendix, I prove that in the presence of a magnetic field,

$$\begin{aligned}\Box^{(h')}(z) &= \frac{1}{2}\coth(\varphi/2)\nabla_i^{(h')}(\varphi)\nabla^{(h')i}(z) + \frac{1}{2}\tanh(z/2)\nabla_i^{(h')}(z)\nabla^{(h')i}(z)\\ &\quad + \frac{2C}{n-2}\Omega^2\sinh^2(\varphi/2)\psi h'^{ik}h'^{jl}F_{ij}F_{kl}.\end{aligned} \tag{C.1}$$

Equation 5.45 is unchanged upon introducing the magnetic field.
Hence, the derivation of equation A.42 in appendix A still holds. It says

$$\begin{aligned}0 &= \frac{2(1+\cosh(\varphi/2)\cosh(z/2))}{\sinh(\varphi/2)}\Box^{(h')}(z) - 2\sinh(z/2)\Box^{(h')}(\varphi) - \frac{\tanh(z/2)}{\sinh(\varphi/2)}\nabla_i^{(h')}(z)\nabla^{(h')i}(z)\\ &\quad - \left(\frac{\cosh(z/2)}{\sinh^2(\varphi/2)} + \frac{1}{\cosh(z/2)} + \frac{\cosh(\varphi/2)}{\sinh^2(\varphi/2)}\right)\nabla_i^{(h')}(\varphi)\nabla^{(h')i}(z).\end{aligned} \tag{C.2}$$

$$\begin{aligned}\therefore \Box^{(h')}\varphi &= \frac{1+\cosh(\varphi/2)\cosh(z/2)}{\sinh(\varphi/2)\sinh(z/2)}\Box^{(h')}z - \frac{1}{2\sinh(\varphi/2)\cosh(z/2)}\nabla_i^{(h')}(z)\nabla^{(h')i}(z)\\ &\quad - \frac{1}{2}\left(\frac{\coth(z/2)}{\sinh^2(\varphi/2)} + \frac{1}{\cosh(z/2)\sinh(z/2)} + \frac{\cosh(\varphi/2)}{\sinh(z/2)\sinh^2(\varphi/2)}\right)\nabla_i^{(h')}(\varphi)\nabla^{(h')i}(z).\end{aligned} \tag{C.3}$$

Also from appendix A,

$$\begin{aligned}&\Box^{(h)}S - \frac{C^2}{S}\nabla_i^{(h)}(\psi)\nabla^{(h)i}(\psi)\\ &= \frac{\Omega^2}{4(\cosh(\varphi/2)+\cosh(z/2))^2}\Big[2\Box^{(h')}(\varphi)(1+\cosh(\varphi/2)\cosh(z/2))\\ &\quad - 2\sinh(\varphi/2)\sinh(z/2)\Box^{(h')}(z) - \tanh(z/2)\nabla_i^{(h')}(\varphi)\nabla^{(h')i}(z)\\ &\quad + \nabla_i^{(h')}(z)\nabla^{(h')i}(z)\left(\frac{\sinh(\varphi/2)\sinh^2(z/2)}{\cosh(z/2)} - \frac{\cosh(z/2)\cosh^2(\varphi/2)}{\sinh(\varphi/2)} - \coth(\varphi/2)\right)\Big].\end{aligned} \tag{C.4}$$

Hence, equation 5.238 says[1]

$$
\begin{aligned}
0 &= \Box^{(h)} S - \frac{C^2}{S}\nabla^{(h)}_i(\psi)\nabla^{(h)i}(\psi) - \frac{S}{n-2}F^{ij}F_{ij} && \text{(C.5)}\\
&= \frac{\Omega^2}{4(\cosh(\varphi/2)+\cosh(z/2))^2}\Bigg[2\Box^{(h')}(\varphi)(1+\cosh(\varphi/2)\cosh(z/2))\\
&\quad - 2\sinh(\varphi/2)\sinh(z/2)\Box^{(h')}(z) - \tanh(z/2)\nabla^{(h')}_i(\varphi)\nabla^{(h')i}(z)\\
&\quad + \nabla^{(h')}_i(z)\nabla^{(h')i}(z)\left(\frac{\sinh(\varphi/2)\sinh^2(z/2)}{\cosh(z/2)} - \frac{\cosh(z/2)\cosh^2(\varphi/2)}{\sinh(\varphi/2)} - \coth(\varphi/2)\right)\Bigg]\\
&\quad - \frac{S}{n-2}F_{ij}F^{ij} && \text{(C.6)}\\
\therefore 0 &= 2\Box^{(h')}(\varphi)(1+\cosh(\varphi/2)\cosh(z/2)) - \frac{4\sinh(\varphi/2)(\cosh(\varphi/2)+\cosh(z/2))}{(n-2)\Omega^2}F^{ij}F_{ij}\\
&\quad - 2\sinh(\varphi/2)\sinh(z/2)\Box^{(h')}(z) - \tanh(z/2)\nabla^{(h')}_i(\varphi)\nabla^{(h')i}(z)\\
&\quad + \nabla^{(h')}_i(z)\nabla^{(h')i}(z)\left(\frac{\sinh(\varphi/2)\sinh^2(z/2)}{\cosh(z/2)} - \frac{\cosh(z/2)\cosh^2(\varphi/2)}{\sinh(\varphi/2)} - \coth(\varphi/2)\right) && \text{(C.7)}
\end{aligned}
$$

Then, by equation C.3,

$$
\begin{aligned}
0 &= 2\Box^{(h')}(z)\Bigg(\frac{1}{\sinh(\varphi/2)\sinh(z/2)} + 2\coth(\varphi/2)\coth(z/2) + \frac{\cosh^2(\varphi/2)\cosh^2(z/2)}{\sinh(\varphi/2)\sinh(z/2)}\\
&\quad - \sinh(\varphi/2)\sinh(z/2)\Bigg) - \frac{4\sinh(\varphi/2)(\cosh(\varphi/2)+\cosh(z/2))}{(n-2)\Omega^2}F^{ij}F_{ij}\\
&\quad - \nabla^{(h')}_i(\varphi)\nabla^{(h')i}(z)\Bigg(\frac{\coth(z/2)}{\sinh^2(\varphi/2)} + \frac{1}{\cosh(z/2)\sinh(z/2)} + \frac{\cosh(\varphi/2)}{\sinh(z/2)\sinh^2(\varphi/2)}\\
&\quad + \frac{\cosh(\varphi/2)\cosh^2(z/2)}{\sinh(z/2)\sinh^2(\varphi/2)} + \frac{\cosh(\varphi/2)}{\sinh(z/2)} + \coth(z/2)\coth^2(\varphi/2) + \tanh(z/2)\Bigg)\\
&\quad + \nabla^{(h')}_i(z)\nabla^{(h')i}(z)\Bigg(-\frac{1}{\sinh(\varphi/2)\cosh(z/2)} - 2\coth(\varphi/2) + \frac{\sinh(\varphi/2)\sinh^2(z/2)}{\cosh(z/2)}\\
&\quad - \frac{\cosh(z/2)\cosh^2(\varphi/2)}{\sinh(\varphi/2)}\Bigg). && \text{(C.8)}
\end{aligned}
$$

The coefficients of each type of derivative term simplify as follows.

$$
\begin{aligned}
&\frac{1}{\sinh(\varphi/2)\sinh(z/2)} + 2\coth(\varphi/2)\coth(z/2) + \frac{\cosh^2(\varphi/2)\cosh^2(z/2)}{\sinh(\varphi/2)\sinh(z/2)}\\
&- \sinh(\varphi/2)\sinh(z/2)\\
&= \frac{1+2\cosh(\varphi/2)\cosh(z/2)+\cosh^2(\varphi/2)\cosh^2(z/2)-\sinh^2(\varphi/2)\sinh^2(z/2)}{\sinh(\varphi/2)\sinh(z/2)} && \text{(C.9)}\\
&= \frac{1}{\sinh(\varphi/2)\sinh(z/2)}(1+2\cosh(\varphi/2)\cosh(z/2)+\cosh^2(\varphi/2)\cosh^2(z/2)\\
&\quad - (\cosh^2(\varphi/2)-1)(\cosh^2(z/2)-1)) && \text{(C.10)}\\
&= \frac{(\cosh(\varphi/2)+\cosh(z/2))^2}{\sinh(\varphi/2)\sinh(z/2)}. && \text{(C.11)}
\end{aligned}
$$

[1] To clarify, throughout this section, $F^{ij}$ will mean $h^{ik}h^{jl}F_{kl}$ even though other $i, j, k$ ... indices are often manifestly raised using the $h'$ metric.

$$
\begin{aligned}
&\frac{\coth(z/2)}{\sinh^2(\varphi/2)} + \frac{1}{\cosh(z/2)\sinh(z/2)} + \frac{\cosh(\varphi/2)}{\sinh(z/2)\sinh^2(\varphi/2)} + \frac{\cosh(\varphi/2)\cosh^2(z/2)}{\sinh(z/2)\sinh^2(\varphi/2)} + \frac{\cosh(\varphi/2)}{\sinh(z/2)} \\
&+ \coth(z/2)\coth^2(\varphi/2) + \tanh(z/2) \\
&= \frac{\cosh(\varphi/2)}{\sinh(z/2)\sinh^2(\varphi/2)}\bigg(\frac{\cosh(z/2)}{\cosh(\varphi/2)} + \frac{\sinh^2(\varphi/2)}{\cosh(\varphi/2)\cosh(z/2)} + 1 + \cosh^2(z/2) + \sinh^2(\varphi/2) \\
&\quad + \cosh(\varphi/2)\cosh(z/2) + \frac{\sinh^2(z/2)\sinh^2(\varphi/2)}{\cosh(z/2)\cosh(\varphi/2)}\bigg) \qquad \text{(C.12)} \\
&= \frac{\cosh(\varphi/2)}{\sinh(z/2)\sinh^2(\varphi/2)}\bigg(\frac{\cosh(z/2)}{\cosh(\varphi/2)} + \frac{\cosh^2(\varphi/2)-1}{\cosh(\varphi/2)\cosh(z/2)} + \cosh^2(\varphi/2) + \cosh^2(z/2) \\
&\quad + \cosh(\varphi/2)\cosh(z/2) + \frac{(\cosh^2(z/2)-1)(\cosh^2(\varphi/2)-1)}{\cosh(z/2)\cosh(\varphi/2)}\bigg) \qquad \text{(C.13)} \\
&= \frac{\cosh(\varphi/2)}{\sinh(z/2)\sinh^2(\varphi/2)}\bigg(\frac{\cosh(z/2)}{\cosh(\varphi/2)} + \frac{\cosh(z/2)(\cosh^2(\varphi/2)-1)}{\cosh(\varphi/2)} + \cosh^2(\varphi/2) + \cosh^2(z/2) \\
&\quad + \cosh(\varphi/2)\cosh(z/2)\bigg) \qquad \text{(C.14)} \\
&= \frac{\cosh(\varphi/2)(\cosh(\varphi/2)+\cosh(z/2))^2}{\sinh(z/2)\sinh^2(\varphi/2)}. \qquad \text{(C.15)}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{\sinh(\varphi/2)\cosh(z/2)} - 2\coth(\varphi/2) + \frac{\sinh(\varphi/2)\sinh^2(z/2)}{\cosh(z/2)} - \frac{\cosh(z/2)\cosh^2(\varphi/2)}{\sinh(\varphi/2)} \\
&= \frac{-1 - 2\cosh(\varphi/2)\cosh(z/2) + \sinh^2(\varphi/2)\sinh^2(z/2) - \cosh^2(z/2)\cosh^2(\varphi/2)}{\sinh(\varphi/2)\cosh(z/2)} \qquad \text{(C.16)} \\
&= -\frac{(\cosh(\varphi/2)+\cosh(z/2))^2}{\sinh(\varphi/2)\cosh(z/2)}. \qquad \text{(C.17)}
\end{aligned}
$$

Substituting these results back up, I get

$$
\begin{aligned}
0 &= 2\Box^{(h')}(z)\frac{(\cosh(\varphi/2)+\cosh(z/2))^2}{\sinh(\varphi/2)\sinh(z/2)} - \frac{4\sinh(\varphi/2)(\cosh(\varphi/2)+\cosh(z/2))}{(n-2)\Omega^2}F^{ij}F_{ij} \\
&\quad - \nabla_i^{(h')}(\varphi)\nabla^{(h')i}(z)\frac{\cosh(\varphi/2)(\cosh(\varphi/2)+\cosh(z/2))^2}{\sinh(z/2)\sinh^2(\varphi/2)} \\
&\quad - \nabla_i^{(h')}(z)\nabla^{(h')i}(z)\frac{(\cosh(\varphi/2)+\cosh(z/2))^2}{\sinh(\varphi/2)\cosh(z/2)}. \qquad \text{(C.18)}
\end{aligned}
$$

$$
\begin{aligned}
\therefore \Box^{(h')}(z) &= \frac{2\sinh(z/2)\sinh^2(\varphi/2)}{(n-2)\Omega^2(\cosh(\varphi/2)+\cosh(z/2))}F^{ij}F_{ij} + \frac{1}{2}\coth(\varphi/2)\nabla_i^{(h')}(\varphi)\nabla^{(h')i}(z) \\
&\quad + \frac{1}{2}\tanh(z/2)\nabla_i^{(h')}(z)\nabla^{(h')i}(z) \qquad \text{(C.19)} \\
&= \frac{2C\psi\sinh^2(\varphi/2)\Omega^2}{(n-2)}h'^{ik}h'^{jl}F_{ij}F_{kl} + \frac{1}{2}\coth(\varphi/2)\nabla_i^{(h')}(\varphi)\nabla^{(h')i}(z) \\
&\quad + \frac{1}{2}\tanh(z/2)\nabla_i^{(h')}(z)\nabla^{(h')i}(z). \qquad \text{(C.20)}
\end{aligned}
$$

# Appendix D

# Notational conventions

The following symbols typically have the meanings given below.

- $M$: the spacetime manifold
- $n$: the dimension of $M$
- $g$: the Lorentzian metric on the full spacetime
- $k^a$: static Killing vector field
- $t$: a coordinate in a coordinate system where $k^a = \frac{\partial}{\partial t}$
- $\Sigma_t$: a surface of constant $t$
- $h$: the metric on a spacelike hypersurface within the full spacetime, typically $\Sigma_t$
- $\tilde{h}$ or $\beta$: the induced metric on an $(n-2)$-dimensional spacelike submanifold
- $H$ or $H^+$: the event horizon
- $\kappa$: the surface gravity of the event horizon
- $\mathcal{H}$: a spacelike cross-section - typically a constant $t$ slice - of the event horizon
- $C$: the domain of outer communication or the constant, $\sqrt{\frac{2(n-3)}{n-2}}$, based on context
- $\nabla$: the covariant derivative associated with $g$
- $\nabla^{(a)}$: the covariant derivative association with some given metric, $a$
- $\varepsilon$: the Levi-Civita tensor corresponding to $g$
- $\varepsilon^{(a)}$: the Levi-Civita tensor corresponding to some given metric, $a$
- $\therefore$: a symbol to denote "therefore"
- $\mathcal{I}^{\pm}$: future and past null infinity
- $S^{n-2}_{\infty}$: the surface at infinity on an $(n-1)$-dimensional asymptotically flat end
- $\omega_{n-2}$: the area of a unit radius $S^{n-2}$

The metric signature of $g$ will always be mostly pluses[1].

$n$ is always assumed to be at least 4.

$a$, $b$, ... will be abstract indices for tensors on $M$. They will be raised/lowered by $g^{ab}/g_{ab}$.

$\mu$, $\nu$, ... will run from 0 to $n-1$ and be indices for tensors on $M$ in a specific basis. They will be raised/lowered by $g^{\mu\nu}/g_{\mu\nu}$.

$i$, $j$, ... will run 1 to $n-1$ and be indices for tensors on some spacelike hypersurface, typically $\Sigma_t$. They will typically be raised/lowered by $h^{ij}/h_{ij}$ or a conformally equivalent metric.

$A$, $B$, ... will run 2 to $n-1$ and be indices for tensors on some spacelike $(n-2)$-dimensional surface. They will typically be raised/lowered by $\tilde{h}^{AB}/\tilde{h}_{AB}$.

The standard bilinear form on $p$-forms, $\alpha$ and $\beta$, is $(\alpha|\beta) = \frac{1}{p!}\alpha_{a_1\cdots a_p}\beta^{a_1\cdots a_p}$.

The Hodge dual of a $p$-form, $\alpha$, is defined to be $(\star\alpha)_{a_1\cdots a_{n-p}} = \frac{1}{p!}\varepsilon_{b_1\cdots b_p a_1\cdots a_{n-p}}\alpha^{b_1\cdots b_p}$.

The Riemann tensor is defined so that $[\nabla_a, \nabla_b]V^c = R^c{}_{dab}V^d$.

Newton's constant, $G$, and the speed of light, $c$, are both set to 1.

A series of derivatives acts on all terms enclosed in brackets, e.g. $\nabla_{a_1}\cdots\nabla_{a_n}(AB)$ means there are $n$ derivatives, $\nabla_{a_1}$, ..., $\nabla_{a_n}$, acting on the product, $AB$, with $\nabla_{a_n}$ acting first and $\nabla_{a_1}$ acting last. I've tried to never write an expression such as $\nabla^a A\nabla_a B$, which in principle could mean $\nabla^a(A)\nabla_a(B)$ or $\nabla^a(A\nabla_a B)$.

[1] This is the only sensible convention.

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