Spinorial quasilocal mass for spacetimes with negative cosmological constant - extended notes

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Abstract

I define a new notion of quasilocal mass applicable to generic, compact, two dimensional, spacelike surfaces in spacetimes with negative cosmological constant. The definition is spinorial and based on work by Penrose and Dougan & Mason in the $\Lambda = 0$ case. Furthermore, it is proven to be non-negative, have an appropriate limit at $\mathcal{I}$, have an appropriate expression in linearised gravity, equal the Misner-Sharp mass in spherical symmetry and equal zero for every generic surface in AdS. These notes are based on [1], but written in a more informal (but more opiniated) and pedagogical style.

Contents

1 Introduction

One of the triumphs of mathematical general relativity is the positive energy theorem - originally proven by Schoen & Yau [2] based on minimal surface techniques and soon after by Witten [3] based on spinorial methods. Witten's method was subsequently extended to prove global mass-charge inequalities in 4D Einstein-Maxwell theory [4], 5D Einstein-Maxwell-Chern-Simons theory [5], global positive energy theorems for spacetimes with AdS-type asymptotics [6, 7, 8, 9, 10] and mass-charge inequalities in this context [11, 12, 13].

Meanwhile, one of the outstanding problems in mathematical general relativity is to find a completely satisfactory definition of quasilocal mass, a notion of mass associated to a closed, compact, spacelike, 2D hypersurface, usually taken to be diffeomorphic to a sphere - see [14] for a review on the many attempts in the literature. At a very high level, Witten's method equates a combination of the ADM quantities [15] to a non-negative volume integral over a Cauchy surface. This raises the tantalising possibility of replacing the Cauchy surface with a compact, spacelike, 3D, hypersurface and thereby finding a notion of quasilocal mass on the hypersurface's boundary. Furthermore, such a quasilocal mass would likely automatically satisfy a notion of positivity.

This idea culminated in the spinorial definition of quasilocal mass by Dougan & Mason [16], relying heavily on the Newman-Penrose (NP) [17] and Geroch-Held-Penrose (GHP) [18] formalisms. Their definition proved to have a number of physically desirable properties [19] and simplified Penrose's twistorial attempt [20] at making Witten's method quasilocal[1]. Dougan & Mason's quasilocal positive energy theorem was recently generalised to a quasilocal mass-charge inequality by Reall [23] in much the same way Gibbons & Hull [4] extended Witten's original work. In parallel with the increasing sophistication of global positive energy theorems, Reall speculates his results could be generalised to include a negative cosmological constant. However, to find a quasilocal mass-charge inequality in spacetimes with negative cosmological constant - let alone apply it to the third law of black hole mechanics like Reall - one must first have a satisfactory notion of quasilocal mass for these spacetimes. While quasilocal masses do exist for spacetimes with negative cosmological constant - for example the Hawking mass [24] can be generalised [25] and [26] generalises the Brown-York and Kijowski masses [27, 28] - these are not naturally spinorial. Thus, one seeks a generalisation of the Dougan-Mason quasilocal mass accommodating a negative cosmological constant.

In this work I define such a generalisation, roughly stated as follows.

Definition 1.1 (Quasilocal mass - rough version)**.** *Given a generic, 2D, surface, S, within a spacetime, (M, g), satisfying the Einstein equation with negative cosmological constant and matter fields satisfying the dominant energy condition, make the following constructions*[2]*.*
Let $\{l, n, m, \overline{m}\}$ be a Newman-Penrose tetrad adapted to S.
Assume the null expansions of S satisfy $\theta_l > 0$, $\theta_n < 0$ and $\theta_l\theta_n < \frac{2\Lambda}{3}$.
Let $\Phi = [\varphi_\alpha, \bar{\xi}^{\dot{\alpha}}]^T$ be a Dirac spinor satisfying $\overline{m}^a\nabla_a\Phi = 0$ on S and let $\{\Phi^A\}$ be a basis of solutions, i.e. $\Phi = c_A\Phi^A$ for some constants, c_A.

[1] See also [21] for a more recent spinorial definition of quasilocal mass and [22] for an attempt at using spinor methods to study positivity of quasilocal masses that are not themselves spinorial.

[2] It is non-trivial to show that all these constructions are possible.

Then, define the matrices, Q^{AB} and T^{AB}, by

$$Q^{AB} = \int_S l_a n_b (\overline{\Phi}^A \gamma^{abc} \nabla_c \Phi^B - \overline{\nabla_c(\Phi^A)} \gamma^{abc} \Phi^B) \,\mathrm{d}A \tag{1}$$

$$\text{and } T^{AB} = (\Phi^A)^T C^{-1} \Phi^B, \tag{2}$$

$$\text{where } \nabla_a \Phi = D_a \Phi + \mathrm{i}\sqrt{-\frac{\Lambda}{12}} \gamma_a \Phi \text{ with } D_a = \text{Levi} - \text{Civita connection} \tag{3}$$

$$\text{and } C = \text{charge conjugation matrix.} \tag{4}$$

Then, the quasilocal mass is defined to be

$$m(S) = \frac{1}{16\pi} \sqrt{-\operatorname{tr}\big(QT^{-1}\overline{Q}\overline{T}^{-1}\big)}. \tag{5}$$

My definition is based both on Dougan & Mason's work, but also on Penrose's twistorial definition[3]. Like the Dougan-Mason mass, my definition applies to 2D surfaces, S, which are "generic" in a sense I'll make precise later. Also, like Penrose's definition, but unlike Dougan & Mason's definition, my definition cannot decompose the mass into its constituents - e.g. energy and linear momentum - except near $\mathcal{I}$. Most importantly though, a good quasilocal mass should satisfy several properties of physical significance. Although no unanimously agreed list exists[4], I will show my definition satisfies the following properties.

- $m(S) \geq 0$.
- $m(S) = 0$ for every surface in AdS.
- $m(S)$ coincides with the Misner-Sharp mass (including cosmological constant) for spherically symmetric spacetimes.
- For asymptotically AdS spacetimes, $m(S)$ agrees with a global notion of mass as S approaches a sphere on $\mathcal{I}$.
- For gravity linearised about AdS, $m(S)$ agrees with a reasonable notion of mass built from the energy-momentum tensor, T_{ab}.

I begin in section 2 by setting up the problem and establishing various foundational identities regarding spinors and the GHP formalism. This is supplemented in section 3 by analysis required to show a Dirac-type operator admits a Green's function as required for Witten's method. Finally, I'm ready to state my new definition of quasilocal mass is section 4. The first two properties in the list above are shown to follow somewhat immediately. Section 5 is devoted to studying examples with high symmetry - namely spherical symmmetry in section 5.1 and toroidal symmetry in section 5.2. Section 6 then establishes the asymptotic properties, while section 7 studies gravity linearised around AdS. Section 8 then concludes with a recapitulation and some speculation on future work. My conventions are listed in appendix A. Most saliently, I use conventions based on [30]. However, since the Penrose-Rindler [31, 32] conventions have become somewhat ingrained in the general relativity community - despite these conventions clashing with standard conventions used in work without spinors - I provide a comparision between my conventions and the Penrose-Rindler conventions in appendix A.1. Finally, appendix B collates some identities I use frequently when manipulating two-component spinors and NP coefficients.

[3] It appears there has been one previous attempt at including a negative cosmological constant in Penrose's work [29]. However, I don't consider the mass in [29] to be quasilocal because it is only ever evaluated at $\mathcal{I}$. My definition also differs in that no reference is made to twistors.

[4] Furthermore, most authors have a tendency of devising lists that exactly match the properties their definition satisfies.

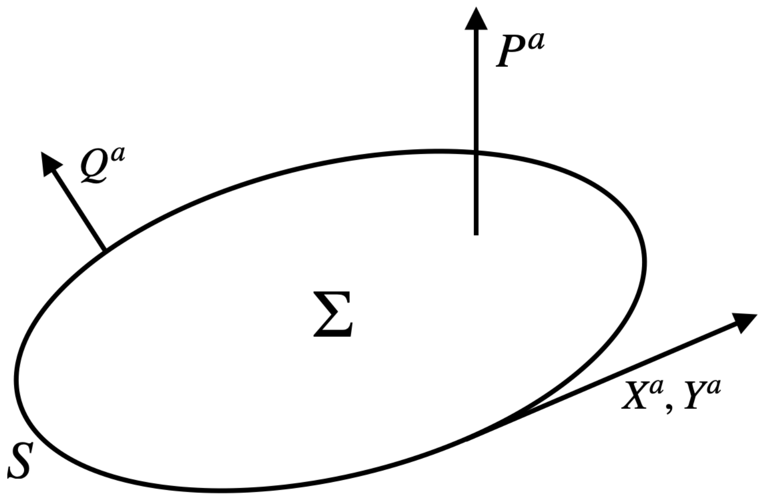

Figure 1: The set-up for defining quasilocal mass.

2 Set-up and the Lichnerowicz identity

Definition 2.1 (Σ, S, P, Q, X, Y)**.** *Let Σ be a three dimensional, compact manifold with boundary, S, within a spacetime, (M, g). Define $\{P, Q, X, Y\}$ to be a vielbein with X^a and Y^a tangent to S, Q^a an outward-pointing normal to S and P^a a timelike, future-directed normal to Σ.*

See figure 1 for a visual depiction of definition 2.1. A quasilocal mass is then a number, $m(S)$, for each applicable S.

Definition 2.2 (Newman-Penrose tetrad)**.** *Having chosen $\{P, Q, X, Y\}$ as described, define a Newman-Penrose (NP) tetrad [17] by*

$$l^a = \frac{1}{\sqrt{2}}\left(P^a + Q^a\right), \; n^a = \frac{1}{\sqrt{2}}\left(P^a - Q^a\right) \text{ and } m^a = \frac{1}{\sqrt{2}}\left(X^a + \mathrm{i}Y^a\right). \tag{6}$$

Equivalently, given an NP tetrad adapted to S and Σ, one can define

$$\begin{aligned} &P^a = \frac{1}{\sqrt{2}}\left(l^a + n^a\right), \; Q^a = \frac{1}{\sqrt{2}}\left(l^a - n^a\right), \; X^a = \frac{1}{\sqrt{2}}\left(m^a + \overline{m}^a\right) \\ &\text{and } Y^a = \frac{1}{\mathrm{i}\sqrt{2}}\left(m^a - \overline{m}^a\right). \end{aligned} \tag{7}$$

Lemma 2.3. *The NP coefficients, μ and ρ, are real. Furthermore, they are related to the expansions along the null normals by $\theta_l = -2\rho$ and $\theta_n = 2\mu$.*

Proof. For a spacelike, 2D surface with null normals, l^a and n^a, the generalised extrinsic curvature is defined to be

$$\mathbb{K}^a{}_{bc} = -\beta^d{}_b\beta^e{}_c(D_d(l_e)n^a + D_d(n_e)l^a). \tag{8}$$

However, $\mathbb{K}^a{}_{bc}$ is known to be symmetric in the lower two indices.

$$\begin{aligned} \therefore l_a\mathbb{K}^a{}_{bc}m^b\overline{m}^c = l_a\mathbb{K}^a{}_{bc}\overline{m}^b m^c &\iff \beta^d{}_b\beta^e{}_c D_d(l_e)m^b\overline{m}^c = \beta^d{}_b\beta^e{}_c D_d(l_e)\overline{m}^b m^c && (9)\\ &\iff m^d\overline{m}^e D_d(l_e) = \overline{m}^d m^e D_d(l_e) && (10)\\ &\iff -\bar{\rho} = -\rho. && (11)\\ \text{Similarly, } n_a\mathbb{K}^a{}_{bc}m^b\overline{m}^c = n_a\mathbb{K}^a{}_{bc}\overline{m}^b m^c &\iff m^d\overline{m}^e D_d(n_e) = \overline{m}^d m^e D_d(n_e) \iff \mu = \bar{\mu}. && (12) \end{aligned}$$

Thus μ and ρ are indeed real.
In the NP formalism, $g_{ab} = -l_a n_b - n_a l_b + m_a \overline{m}_b + \overline{m}_a m_b$.
$\therefore$ The indiced metric on S is $\beta_{ab} = g_{ab} + l_a n_b + n_a l_b = m_a \overline{m}_b + \overline{m}_a m_b$.
By definition, $\theta_l = \beta^{ab} D_a l_b$ and $\theta_n = \beta^{ab} D_a n_b$.
$\therefore \theta_l = (m^a \overline{m}^b + \overline{m}^a m^b) D_a l_b = -\bar{\rho} - \rho = -2\rho$ and $\theta_l = (m^a \overline{m}^b + \overline{m}^a m^b) D_a n_b = \mu + \bar{\mu} = 2\mu$. □

Given a pair of null normals to S, it will be very natural to use the Geroch-Held-Penrose (GHP) formalism [18] in what follows. The primary construction underpinning the GHP formalism is the spinor dyad.

Definition 2.4 (Spinor dyad, $A_\alpha, B_\alpha, a(\psi), b(\psi)$)**.** *When converted to two-components spinors, write the NP tetrad in terms of a spinor dyad, $\{A, B\}$, as*

$$l_{\alpha\dot{\alpha}} = A_\alpha \bar{A}_{\dot{\alpha}} \text{ and } n_{\alpha\dot{\alpha}} = B_\alpha \bar{B}_{\dot{\alpha}}. \tag{13}$$

with $B^\alpha A_\alpha = \sqrt{2}$. Subsequently, decompose any two-component spinor, ψ_α, as

$$\psi_\alpha = a(\psi) A_\alpha + b(\psi) B_\alpha \tag{14}$$

$$\iff a(\psi) = \frac{1}{\sqrt{2}} B^\alpha \psi_\alpha \text{ and } b(\psi) = -\frac{1}{\sqrt{2}} A^\alpha \psi_\alpha. \tag{15}$$

Finally, in terms of the spinor dyad,

$$m_{\alpha\dot{\alpha}} = B_\alpha \bar{A}_{\dot{\alpha}} \text{ and } \overline{m}_{\alpha\dot{\alpha}} = A_\alpha \bar{B}_{\dot{\alpha}}. \tag{16}$$

Proof. These constructions are from [18], but I'll explain in more detail why they're possible. By definition,

$$l_{\alpha\dot{\alpha}} = l^a (\sigma_a)_{\alpha\dot{\alpha}} \equiv \begin{bmatrix} l^0 + l^3 & l^1 - \mathrm{i} l^2 \\ l^1 + \mathrm{i} l^2 & l^0 - l^3 \end{bmatrix}. \tag{17}$$

$$\therefore \det(l_{\alpha\dot{\alpha}}) = (l^0)^2 - (l^1)^2 - (l^2)^2 - (l^3)^2 = 0. \tag{18}$$

$\therefore l_{\alpha\dot{\alpha}}$ is a 2×2, non-zero, rank-1 matrix.
$\therefore$ The columns of $l_{\alpha\dot{\alpha}}$ must be proportional to each other.
$\therefore \exists\, u_\alpha$ and v_α such that

$$l_{\alpha\dot{\alpha}} = u_\alpha \bar{v}_{\dot{\alpha}} \equiv \begin{bmatrix} u_1 \bar{v}_{\dot{1}} & u_1 \bar{v}_{\dot{2}} \\ u_2 \bar{v}_{\dot{1}} & u_2 \bar{v}_{\dot{2}} \end{bmatrix}. \tag{19}$$

Then, $l_{\alpha\dot{\alpha}}$ is hermitian $\implies u_1 \bar{v}_{\dot{1}}, u_2 \bar{v}_{\dot{2}} \in \mathbb{R}$ and $u_1 \bar{v}_{\dot{2}} = \bar{u}_{\dot{2}} v_1$.
$\therefore u_1 \bar{v}_{\dot{1}} = \bar{u}_{\dot{2}} |v_1|^2 / \bar{v}_{\dot{2}}$ and thus $\bar{u}_{\dot{2}} / \bar{v}_{\dot{2}} \in \mathbb{R}$, say c_2 (if v_1 or v_2 is zero then $l_{\alpha\dot{\alpha}} = A_\alpha \bar{A}_{\dot{\alpha}}$ holds immediately with one of A_1 or A_2 being zero).
Similarly, $u_2 \bar{v}_{\dot{2}} = \bar{u}_{\dot{1}} |v_2|^2 / \bar{v}_{\dot{1}} \implies \bar{u}_{\dot{1}} / \bar{v}_{\dot{1}} = c_1 \in \mathbb{R}$.

$$\therefore l_{\alpha\dot{\alpha}} \equiv \begin{bmatrix} c_1 |v_1|^2 & c_1 v_1 \bar{v}_{\dot{2}} \\ c_2 v_2 \bar{v}_{\dot{1}} & c_2 |v_2|^2 \end{bmatrix}. \tag{20}$$

Now, $c_2 v_2 \bar{v}_{\dot{1}} = \overline{(c_1 v_1 \bar{v}_{\dot{2}})} \implies c_1 = c_2$.
$\therefore l_{\alpha\dot{\alpha}} = c v_\alpha \overline{v}_{\dot{\alpha}}$ for some two-component spinor, v_α, and some real number, c.
$l_{\alpha\dot{\alpha}} \neq 0 \implies c \neq 0$. Then, l^a is causal and future directed $\implies l_{1\dot{1}} = l^0 + l^3 \geq 0 \implies c > 0$.
Let $A_\alpha = \sqrt{c} v_\alpha$ to finally get $l_{\alpha\dot{\alpha}} = A_\alpha \bar{A}_{\dot{\alpha}}$.
Similarly, $\exists B_\alpha$ such that $n_{\alpha\dot{\alpha}} = B_\alpha \bar{B}_{\dot{\alpha}}$.
$\therefore -1 = l^a n_a = -\frac{1}{2} l^{\alpha\dot{\alpha}} n_{\alpha\dot{\alpha}} = -\frac{1}{2} A^\alpha \bar{A}^{\dot{\alpha}} B_\alpha \bar{B}_{\dot{\alpha}} = -\frac{1}{2} |A^\alpha B_\alpha|^2 \implies |A^\alpha B_\alpha| = \sqrt{2}$.

In these definitions I still have the freedom to change A_α or B_α by a phase. I'll use this to fix $B^\alpha A_\alpha = \sqrt{2}$.
Furthermore, it follows that A_α and B_α are pointwise linearly independent.
$\therefore$ A_α and B_α form a pointwise basis for two-component spinors.
$\therefore$ Any two-component spinor, ψ, can be decomposed as $\psi = a(\psi)A_\alpha + b(\psi)B_\alpha$ for some functions, $a(\psi)$ and $b(\psi)$. These functions are determined by

$$A^\alpha \psi_\alpha = A^\alpha \left(a(\psi)A_\alpha + b(\psi)B_\alpha\right) = 0 - \sqrt{2}b(\psi) \tag{21}$$

$$\text{and } B^\alpha \psi_\alpha = B^\alpha \left(a(\psi)A_\alpha + b(\psi)B_\alpha\right) = \sqrt{2}a(\psi) + 0. \tag{22}$$

Once l^a and n^a are chosen, the choice of m^a is fixed uniquely up to an $SO(2)$ rotation. This freedom matches with the remaining phase freedom left after choosing $B^\alpha A_\alpha = \sqrt{2}$.
$\therefore$ Any choice/guess that works for $m_{\alpha\dot\alpha}$ in terms of A_α and B_α is good enough.
Choose $m_{\alpha\dot\alpha} = B_\alpha \bar{A}_{\dot\alpha}$. Then, $m^a m_a = -\frac{1}{2} m^{\alpha\dot\alpha} m_{\alpha\dot\alpha} = -\frac{1}{2} B^\alpha \bar{A}^{\dot\alpha} B_\alpha \bar{A}_{\dot\alpha} = 0$ and
$m^a \overline{m}_a = -\frac{1}{2} m^{\alpha\dot\alpha} \overline{m}_{\alpha\dot\alpha} = -\frac{1}{2} B^\alpha \bar{A}^{\dot\alpha} \overline{B}_{\dot\alpha} A_\alpha = 1$ as required. □

Definition 2.5 (Modified connection). *When acting on any Dirac spinor, Ψ, define the modified connection, ∇, by*

$$\nabla_a \Psi = D_a \Psi + \mathrm{i}k\gamma_a \Psi \quad \text{and} \tag{23}$$

$$\nabla_a \overline{\Psi} = D_a \overline{\Psi} - \mathrm{i}k \overline{\Psi}\gamma_a = (\nabla_a \Psi)^\dagger \gamma^0, \tag{24}$$

where $k = \sqrt{-\frac{\Lambda}{12}}$ and D_a is the Levi-Civita connection.

Definition 2.6 ($E^{ab}(\Psi), E^{ab}(\Psi_1, \Psi_2)$). *For a Dirac spinor, Ψ, let*

$$E^{ab}(\Psi) = \overline{\Psi}\gamma^{abc}\nabla_c \Psi + \text{c.c} = \overline{\Psi}\gamma^{abc}\nabla_c\Psi - \nabla_c(\overline{\Psi})\gamma^{abc}\Psi. \tag{25}$$

Similarly, define $E^{ab}(\Psi_1, \Psi_2)$ by

$$E^{ab}(\Psi_1, \Psi_2) = \overline{\Psi}_1 \gamma^{abc}\nabla_c \Psi_2 - \nabla_c(\overline{\Psi}_1)\gamma^{abc}\Psi_2. \tag{26}$$

$E^{ab}(\Psi)$ is the Hodge dual of what is usually called the Witten-Nester 2-form [33].

Theorem 2.7 (Lichnerowicz identity).

$$P_a D_b (E^{ba}(\Psi)) = 2\left(\nabla_I(\Psi)^\dagger \nabla^I \Psi - 4\pi T^{0a}\overline{\Psi}\gamma_a\Psi - (\gamma^I\nabla_I\Psi)^\dagger \gamma^J \nabla_J \Psi\right). \tag{27}$$

A variant of the Lichnerowicz identity is always the key result underpinning any Witten-style positive energy theorem. Note the RHS can be written in a more covariant looking way by replacing T^{0a} with $-P_b T^{ba}$ and replacing all ∇_I with $h^b{}_a \nabla_b$, where $h_{ab} = g_{ab} + P_a P_b$.

Proof.

$$\begin{aligned}
&D_b E^{ba}(\Psi) \\
&= D_b\left(\overline{\Psi}\gamma^{bac}\nabla_c\Psi - \nabla_c(\overline{\Psi})\gamma^{bac}\Psi\right) && (28)\\
&= D_b(\overline{\Psi})\gamma^{bac}\nabla_c\Psi + \overline{\Psi}\gamma^{bac}D_b(\nabla_c\Psi) - D_b(\nabla_c\overline{\Psi})\gamma^{bac}\Psi - \nabla_c(\overline{\Psi})\gamma^{bac}D_b\Psi && (29)\\
&= \nabla_b(\overline{\Psi})\gamma^{bac}\nabla_c\Psi + \mathrm{i}k\overline{\Psi}\gamma_b\gamma^{bac}\nabla_c\Psi + \overline{\Psi}\gamma^{bac}D_b(\nabla_c\Psi) - D_b(\nabla_c\overline{\Psi})\gamma^{bac}\Psi - \nabla_c(\overline{\Psi})\gamma^{bac}\nabla_b\Psi \\
&\quad + \mathrm{i}k\nabla_c(\overline{\Psi})\gamma^{bac}\gamma_b\Psi && (30)\\
&= 2\nabla_b(\overline{\Psi})\gamma^{bac}\nabla_c\Psi - 2\mathrm{i}k\overline{\Psi}\gamma^{ab}\nabla_b\Psi + \overline{\Psi}\gamma^{bac}D_b(\nabla_c\Psi) - D_b(\nabla_c\overline{\Psi})\gamma^{bac}\Psi - 2\mathrm{i}k\nabla_b(\overline{\Psi})\gamma^{ab}\Psi && (31)\\
&= 2\nabla_b(\overline{\Psi})\gamma^{bac}\nabla_c\Psi - 2\mathrm{i}k\overline{\Psi}\gamma^{ab}D_b\Psi + 2k^2\overline{\Psi}\gamma^{ab}\gamma_b\Psi + \overline{\Psi}\gamma^{bac}D_bD_c\Psi + \mathrm{i}k\overline{\Psi}\gamma^{bac}\gamma_cD_b\Psi \\
&\quad - D_bD_c(\overline{\Psi})\gamma^{bac}\Psi + \mathrm{i}kD_b(\overline{\Psi})\gamma_c\gamma^{bac}\Psi - 2\mathrm{i}kD_b(\overline{\Psi})\gamma^{ab}\Psi - 2k^2\overline{\Psi}\gamma_b\gamma^{ab}\Psi && (32)\\
&= 2\nabla_b(\overline{\Psi})\gamma^{bac}\nabla_c\Psi - 2\mathrm{i}k\overline{\Psi}\gamma^{ab}D_b\Psi - 6k^2\overline{\Psi}\gamma^a\Psi + \overline{\Psi}\gamma^{bac}D_bD_c\Psi - 2\mathrm{i}k\overline{\Psi}\gamma^{ba}D_b\Psi \\
&\quad - D_bD_c(\overline{\Psi})\gamma^{bac}\Psi - 2\mathrm{i}kD_b(\overline{\Psi})\gamma^{ba}\Psi - 2\mathrm{i}kD_b(\overline{\Psi})\gamma^{ab}\Psi - 6k^2\overline{\Psi}\gamma^a\Psi && (33)\\
&= 2\nabla_b(\overline{\Psi})\gamma^{bac}\nabla_c\Psi - 12k^2\overline{\Psi}\gamma^a\Psi - \overline{\Psi}\gamma^{abc}D_bD_c\Psi - D_bD_c(\overline{\Psi})\gamma^{cba}\Psi. && (34)
\end{aligned}$$

For the second derivative terms, one applies the standard Licherowicz identity. In particular,

$$\gamma^{abc}D_bD_c\Psi = \frac{1}{2}\gamma^{abc}[D_b, D_c]\Psi \text{ by antisymmetry} \tag{35}$$
$$= -\frac{1}{8}R^{de}{}_{bc}\gamma^{abc}\gamma_{de}\Psi \tag{36}$$
$$= -\frac{1}{8}R^{de}{}_{bc}\left(\gamma^{abc}{}_{de} - 6\gamma^{[ab}{}_{[e}\delta^{c]}{}_{d]} + 6\gamma^{[a}\delta^{b}{}_{[e}\delta^{c]}{}_{d]}\right)\Psi \tag{37}$$
$$= \frac{1}{8}R^{de}{}_{bc}\left(6\gamma^{[ab}{}_{[e}\delta^{c]}{}_{d]} - 6\gamma^{[a}\delta^{b}{}_{[e}\delta^{c]}{}_{d]}\right)\Psi \text{ by the Bianchi identity} \tag{38}$$
$$= \frac{3}{4}R^{de}{}_{bc}\left(\gamma^{[ab}{}_{e}\delta^{c]}{}_{d} - \gamma^{[a}\delta^{b}{}_{e}\delta^{c]}{}_{d}\right)\Psi \text{ by antisymmetry} \tag{39}$$
$$= \frac{1}{4}R^{de}{}_{bc}\left(\gamma^{ab}{}_{e}\delta^{c}{}_{d} + \gamma^{bc}{}_{e}\delta^{a}{}_{d} + \gamma^{ca}{}_{e}\delta^{b}{}_{d}\right)\Psi$$
$$- \frac{1}{4}R^{de}{}_{bc}\left(\gamma^{a}\delta^{b}{}_{e}\delta^{c}{}_{d} + \gamma^{b}\delta^{c}{}_{e}\delta^{a}{}_{d} + \gamma^{c}\delta^{a}{}_{e}\delta^{b}{}_{d}\right)\Psi \tag{40}$$
$$= \frac{1}{4}\left(-R_{eb}\gamma^{abe} + R^{a}{}_{ebc}\gamma^{bce} + R_{ec}\gamma^{cae} + R\gamma^{a} - R^{ab}\gamma_{b} - R^{ac}\gamma_{c}\right)\Psi \tag{41}$$
$$= \frac{1}{4}(0 + 0 + 0 + R\gamma^{a} - 2R^{ab}\gamma_{b})\Psi \text{ by Bianchi identity and } R_{ab} = R_{ba} \tag{42}$$
$$= -\frac{1}{2}\left(R^{ab} - \frac{1}{2}\eta^{ab}R\right)\gamma_b\Psi. \tag{43}$$

Then, for the other second derivative term,

$$(\gamma^{abc}D_bD_c\Psi)^\dagger = D_bD_c(\Psi)^\dagger\gamma^0\gamma^{cba}\gamma^0 = D_bD_c(\overline{\Psi})\gamma^{cba}\gamma^0. \tag{44}$$
$$\therefore D_bD_c(\overline{\Psi})\gamma^{cba} = (\gamma^{abc}D_bD_c\Psi)^\dagger\gamma^0 \tag{45}$$
$$= \left(-\frac{1}{2}\left(R^{ab} - \frac{1}{2}\eta^{ab}R\right)\gamma_b\Psi\right)^\dagger\gamma^0 \tag{46}$$
$$= -\frac{1}{2}\left(R^{ab} - \frac{1}{2}\eta^{ab}R\right)\overline{\Psi}\gamma_b. \tag{47}$$

Substituting back,

$$D_bE^{ba}(\Psi) = 2\nabla_b(\overline{\Psi})\gamma^{bac}\nabla_c\Psi - 12k^2\overline{\Psi}\gamma^a\Psi + \left(R^{ab} - \frac{1}{2}\eta^{ab}R\right)\overline{\Psi}\gamma_b\Psi \tag{48}$$
$$= 2\nabla_b(\overline{\Psi})\gamma^{bac}\nabla_c\Psi + 8\pi T^{ab}\overline{\Psi}\gamma_b\Psi \text{ by the Einstein equation.} \tag{49}$$

I'm working in a vielbein where $P_a \equiv -\delta_{a0}$. Hence,

$$P_aD_bE^{ba}(\Psi) = -2\left(\nabla_b(\Psi)^\dagger\gamma^0\gamma^{b0c}\nabla_c\Psi + 4\pi T^{0b}\overline{\Psi}\gamma_b\Psi\right) \tag{50}$$
$$= 2\left(\nabla_I(\Psi)^\dagger\gamma^{IJ}\nabla_J\Psi - 4\pi T^{0a}\overline{\Psi}\gamma_a\Psi\right) \tag{51}$$
$$= 2\left(\nabla_I(\Psi)^\dagger(\gamma^I\gamma^J + \delta^{IJ}I)\nabla_J\Psi - 4\pi T^{0a}\overline{\Psi}\gamma_a\Psi\right) \tag{52}$$
$$= 2\left(\nabla_I(\Psi)^\dagger\nabla^I\Psi - (\gamma^I\nabla_I\Psi)^\dagger\gamma^J\nabla_J\Psi - 4\pi T^{0a}\overline{\Psi}\gamma_a\Psi\right), \tag{53}$$

which is the form of the Lichnerowicz identity I will need in this work. □

Lemma 2.8. *For any antisymmetric tensor, M^{ab},*

$$P_aD_bM^{ba} = \widetilde{D}_b(P_aM^{ba}), \tag{54}$$

where $\widetilde{D}$ is the induced covariant derivative on Σ.

Proof. Let h_{ab} be the induced metric on Σ, i.e. $h_{ab} = g_{ab} + P_a P_b$.
Observe that $P_b M^{ba}$ is invariant under projection, i.e. because of M^{ab}'s antisymmetry, $h^a{}_c P_b M^{cb} = \delta^a{}_c P_b M^{cb} + P^a P_c P_b M^{bc} = P_b M^{ab}$.
∴ The induced covariant derivative acts as

$$
\begin{aligned}
\widetilde{D}_b(P_a M^{ba}) &= h^c{}_b h^b{}_d D_c(P_a M^{da}) && (55)\\
&= h^c{}_b D_c(P_a M^{ba}) && (56)\\
&= h^c{}_b D_c(P_a) M^{ba} + h^c{}_b P_a D_c M^{ba} && (57)\\
&= K_{ba} M^{ba} + \delta^c{}_b P_a D_c M^{ba} + P^c P_b P_a D_c M^{ba} \text{ where } K_{ab} = \text{extrinsic curvature} && (58)\\
&= P_a D_b M^{ba} \text{ by } M^{ba}\text{'s antisymmetry,} && (59)
\end{aligned}
$$

which is the claimed result. □

Definition 2.9 ($Q(\Psi), Q(\Psi_1, \Psi_2)$)**.** *For a Dirac spinor, Ψ, define $Q(\Psi)$ by*

$$Q(\Psi) = \int_\Sigma P_a D_b(E^{ba}(\Psi))\mathrm{d}V. \tag{60}$$

By lemma 2.8,

$$Q(\Psi) = \int_S P_a Q_b E^{ba}(\Psi)\mathrm{d}A. \tag{61}$$

Meanwhile, by theorem 2.7,

$$Q(\Psi) = 2\int_\Sigma \left(\nabla_I(\Psi)^\dagger \nabla^I \Psi - 4\pi T^{0a}\overline{\Psi}\gamma_a\Psi - (\gamma^I\nabla_I\Psi)^\dagger\gamma^J\nabla_J\Psi\right)\mathrm{d}V. \tag{62}$$

Similarly, define $Q(\Psi_1, \Psi_2)$ by

$$Q(\Psi_1, \Psi_2) = \int_\Sigma P_a D_b(E^{ba}(\Psi_1, \Psi_2))\mathrm{d}V. \tag{63}$$

Although Dirac spinors are more convenient on Σ, the positive energy theorem associated to the Dougan-Mason construction requires two-component spinors on S.

Lemma 2.10. *If $\Psi = [\psi_\alpha, \bar{\chi}^{\dot{\alpha}}]^T$, then*

$$
\begin{aligned}
Q(\Psi) = 4\int_S \Big(& b(\psi)\eth\bar{a}(\psi) + \bar{b}(\psi)\bar{\eth}a(\psi) - \bar{a}(\chi)\eth b(\chi) - a(\chi)\bar{\eth}\bar{b}(\chi)\\
&+ \rho|a(\psi)|^2 + \mu|b(\psi)|^2 + \rho|a(\chi)|^2 + \mu|b(\chi)|^2\\
&+ \mathrm{i}k\sqrt{2}\big(a(\psi)b(\chi) + b(\psi)a(\chi) - \bar{a}(\psi)\bar{b}(\chi) - \bar{b}(\psi)\bar{a}(\chi)\big)\Big)\mathrm{d}A,
\end{aligned} \tag{64}
$$

where $\eth$ and $\bar{\eth}$ are the edth and edth-bar operators defined in [18].

Proof. By definitions 2.2 and 2.9,

$$
\begin{aligned}
Q(\Psi) &= \int_S P_a Q_b E^{ba}(\Psi)\mathrm{d}A && (65)\\
&= \frac{1}{2}\int_S (l_a + n_a)(l_b - n_b)E^{ba}(\Psi)\mathrm{d}A && (66)\\
&= \int_S l_a n_b E^{ab}(\Psi)\mathrm{d}A \text{ by } E^{ab\prime}\text{s antisymmetry} && (67)\\
&= \frac{1}{4}\int_S l_{\alpha\dot{\alpha}} n_{\beta\dot{\beta}} E^{\alpha\dot{\alpha}\beta\dot{\beta}}(\Psi)\mathrm{d}A. && (68)
\end{aligned}
$$

Finding $E_{\alpha\dot\alpha\beta\dot\beta}(\Psi)$ is a long, tedious calculation.

$$
\begin{aligned}
E_{\alpha\dot\alpha\beta\dot\beta}(\Psi) &= (\sigma_a)_{\alpha\dot\alpha}(\sigma_b)_{\beta\dot\beta}E^{ab}(\Psi) &(69)\\
&= (\sigma_a)_{\alpha\dot\alpha}(\sigma_b)_{\beta\dot\beta}(\overline{\Psi}\gamma^{abc}\nabla_c\Psi - \nabla_c(\overline{\Psi})\gamma^{abc}\Psi) &(70)
\end{aligned}
$$

I'll evaluate the first term on the RHS and then just take the complex conjugate to get the second term.

$$
\begin{aligned}
&(\sigma_a)_{\alpha\dot\alpha}(\sigma_b)_{\beta\dot\beta}\overline{\Psi}\gamma^{abc}\nabla_c\Psi\\
&= (\sigma_a)_{\alpha\dot\alpha}(\sigma_b)_{\beta\dot\beta}\overline{\Psi}\gamma^{abc}(D_c\Psi + \mathrm{i}k\gamma_c\Psi) &(71)\\
&= (\sigma_a)_{\alpha\dot\alpha}(\sigma_b)_{\beta\dot\beta}\overline{\Psi}\gamma^{abc}D_c\Psi - 2\mathrm{i}k(\sigma_a)_{\alpha\dot\alpha}(\sigma_b)_{\beta\dot\beta}\overline{\Psi}\gamma^{ab}\Psi &(72)\\
&= (\sigma_a)_{\alpha\dot\alpha}(\sigma_b)_{\beta\dot\beta}\left[-\chi^\gamma, -\overline{\psi}_{\dot\gamma}\right]\begin{bmatrix}0 & (\sigma^{[a}\tilde\sigma^b\sigma^{c]})_{\gamma\dot\gamma}\\ (\tilde\sigma^{[a}\sigma^b\tilde\sigma^{c]})^{\dot\gamma\gamma} & 0\end{bmatrix}\begin{bmatrix}D_c\psi_\gamma\\ D_c\bar\chi^{\dot\gamma}\end{bmatrix}\\
&\quad - 2\mathrm{i}k(\sigma_a)_{\alpha\dot\alpha}(\sigma_b)_{\beta\dot\beta}\left[-\chi^\gamma, -\overline{\psi}_{\dot\gamma}\right]\begin{bmatrix}(\sigma^{[a}\tilde\sigma^{b]})_\gamma{}^\delta & 0\\ 0 & (\tilde\sigma^{[a}\sigma^{b]})^{\dot\gamma}{}_{\dot\delta}\end{bmatrix}\begin{bmatrix}\psi_\delta\\ \bar\chi^{\dot\delta}\end{bmatrix} &(73)\\
&= -(\sigma_a)_{\alpha\dot\alpha}(\sigma_b)_{\beta\dot\beta}\chi^\gamma(\sigma^{[a}\tilde\sigma^b\sigma^{c]})_{\gamma\dot\gamma}D_c\bar\chi^{\dot\gamma} - (\sigma_a)_{\alpha\dot\alpha}(\sigma_b)_{\beta\dot\beta}\overline{\psi}_{\dot\gamma}(\tilde\sigma^{[a}\sigma^b\tilde\sigma^{c]})^{\dot\gamma\gamma}D_c\psi_\gamma\\
&\quad + 2\mathrm{i}k(\sigma_a)_{\alpha\dot\alpha}(\sigma_b)_{\beta\dot\beta}\chi^\gamma(\sigma^{[a}\tilde\sigma^{b]})_\gamma{}^\delta\psi_\delta + 2\mathrm{i}k(\sigma_a)_{\alpha\dot\alpha}(\sigma_b)_{\beta\dot\beta}\overline{\psi}_{\dot\gamma}(\tilde\sigma^{[a}\sigma^{b]})^{\dot\gamma}{}_{\dot\delta}\bar\chi^{\dot\delta} &(74)
\end{aligned}
$$

Consider this expression term by term.
From the identity,

$$
(\sigma_a)_{\alpha\dot\beta}(\tilde\sigma_b)^{\dot\beta\beta}(\sigma_c)_{\beta\dot\alpha} = \eta_{ca}(\sigma_b)_{\alpha\dot\alpha} - \eta_{bc}(\sigma_a)_{\alpha\dot\alpha} - \eta_{ab}(\sigma_c)_{\alpha\dot\alpha} + \mathrm{i}\varepsilon_{abcd}(\sigma^d)_{\alpha\dot\alpha}, \qquad (75)
$$

it follows that

$$
\begin{aligned}
(\sigma_{[a}\tilde\sigma_b\sigma_{c]})_{\alpha\dot\alpha} &= \mathrm{i}\varepsilon_{abcd}(\sigma^d)_{\alpha\dot\alpha} &(76)\\
&= (\sigma_a)_{\alpha\dot\beta}(\tilde\sigma_b)^{\dot\beta\beta}(\sigma_c)_{\beta\dot\alpha} - \eta_{ca}(\sigma_b)_{\alpha\dot\alpha} + \eta_{bc}(\sigma_a)_{\alpha\dot\alpha} + \eta_{ab}(\sigma_c)_{\alpha\dot\alpha}. &(77)
\end{aligned}
$$

$$
\begin{aligned}
&\therefore -(\sigma_a)_{\alpha\dot\alpha}(\sigma_b)_{\beta\dot\beta}\chi^\gamma(\sigma^{[a}\tilde\sigma^b\sigma^{c]})_{\gamma\dot\gamma}D_c\bar\chi^{\dot\gamma}\\
&= -(\sigma_a)_{\alpha\dot\alpha}(\sigma_b)_{\beta\dot\beta}\left((\sigma^a)_{\gamma\dot\delta}(\tilde\sigma^b)^{\dot\delta\delta}(\sigma^c)_{\delta\dot\gamma} - \eta^{ca}(\sigma^b)_{\gamma\dot\gamma} + \eta^{bc}(\sigma^a)_{\gamma\dot\gamma} + \eta^{ab}(\sigma^c)_{\gamma\dot\gamma}\right)\chi^\gamma D_c\bar\chi^{\dot\gamma} &(78)\\
&= \left(-4\varepsilon_{\alpha\gamma}\varepsilon_{\dot\alpha\dot\delta}\delta^\delta{}_\beta\delta^{\dot\delta}{}_{\dot\beta}(\sigma^c)_{\delta\dot\gamma} - 2\varepsilon_{\beta\gamma}\varepsilon_{\dot\beta\dot\gamma}(\sigma^c)_{\alpha\dot\alpha} + 2\varepsilon_{\alpha\gamma}\varepsilon_{\dot\alpha\dot\gamma}(\sigma^c)_{\beta\dot\beta} + 2\varepsilon_{\alpha\beta}\varepsilon_{\dot\alpha\dot\beta}(\sigma^c)_{\gamma\dot\gamma}\right)\chi^\gamma D_c\bar\chi^{\dot\gamma} &(79)\\
&= \left(-4\varepsilon_{\alpha\gamma}\varepsilon_{\dot\alpha\dot\beta}(\sigma^c)_{\beta\dot\gamma} - 2\varepsilon_{\beta\gamma}\varepsilon_{\dot\beta\dot\gamma}(\sigma^c)_{\alpha\dot\alpha} + 2\varepsilon_{\alpha\gamma}\varepsilon_{\dot\alpha\dot\gamma}(\sigma^c)_{\beta\dot\beta} + 2\varepsilon_{\alpha\beta}\varepsilon_{\dot\alpha\dot\beta}(\sigma^c)_{\gamma\dot\gamma}\right)\chi^\gamma D_c\bar\chi^{\dot\gamma} &(80)\\
&= -4\varepsilon_{\dot\alpha\dot\beta}\chi_\alpha D_{\beta\dot\gamma}\bar\chi^{\dot\gamma} - 2\chi_\beta D_{\alpha\dot\alpha}\bar\chi_{\dot\beta} + 2\chi_\alpha D_{\beta\dot\beta}\bar\chi_{\dot\alpha} + 2\varepsilon_{\alpha\beta}\varepsilon_{\dot\alpha\dot\beta}\chi^\gamma D_{\gamma\dot\gamma}\bar\chi^{\dot\gamma} &(81)\\
&= -4\varepsilon_{\dot\alpha\dot\beta}\varepsilon^{\dot\gamma\dot\delta}\chi_\alpha D_{\beta\dot\gamma}\bar\chi_{\dot\delta} - 2\chi_\beta D_{\alpha\dot\alpha}\bar\chi_{\dot\beta} + 2\chi_\alpha D_{\beta\dot\beta}\bar\chi_{\dot\alpha} + 2\varepsilon_{\alpha\beta}\varepsilon_{\dot\alpha\dot\beta}\varepsilon^{\gamma\delta}\varepsilon^{\dot\gamma\dot\delta}\chi_\delta D_{\gamma\dot\gamma}\bar\chi_{\dot\delta} &(82)\\
&= 4(\delta^{\dot\gamma}{}_{\dot\alpha}\delta^{\dot\delta}{}_{\dot\beta} - \delta^{\dot\gamma}{}_{\dot\beta}\delta^{\dot\delta}{}_{\dot\alpha})\chi_\alpha D_{\beta\dot\gamma}\bar\chi_{\dot\delta} - 2\chi_\beta D_{\alpha\dot\alpha}\bar\chi_{\dot\beta} + 2\chi_\alpha D_{\beta\dot\beta}\bar\chi_{\dot\alpha}\\
&\quad + 2(\delta^\gamma{}_\alpha\delta^\delta{}_\beta - \delta^\gamma{}_\beta\delta^\delta{}_\alpha)(\delta^{\dot\gamma}{}_{\dot\alpha}\delta^{\dot\delta}{}_{\dot\beta} - \delta^{\dot\gamma}{}_{\dot\beta}\delta^{\dot\delta}{}_{\dot\alpha})\chi_\delta D_{\gamma\dot\gamma}\bar\chi_{\dot\delta} &(83)\\
&= 4\chi_\alpha D_{\beta\dot\alpha}\bar\chi_{\dot\beta} - 4\chi_\alpha D_{\beta\dot\beta}\bar\chi_{\dot\alpha} - 2\chi_\beta D_{\alpha\dot\alpha}\bar\chi_{\dot\beta} + 2\chi_\alpha D_{\beta\dot\beta}\bar\chi_{\dot\alpha} + 2\chi_\beta D_{\alpha\dot\alpha}\bar\chi_{\dot\beta} - 2\chi_\alpha D_{\beta\dot\alpha}\bar\chi_{\dot\beta}\\
&\quad - 2\chi_\beta D_{\alpha\dot\beta}\bar\chi_{\dot\alpha} + 2\chi_\alpha D_{\beta\dot\beta}\bar\chi_{\dot\alpha} &(84)\\
&= 2\chi_\alpha D_{\beta\dot\alpha}\bar\chi_{\dot\beta} - 2\chi_\beta D_{\alpha\dot\beta}\bar\chi_{\dot\alpha}. &(85)
\end{aligned}
$$

The second term in equation 74 is handled similarly.

$$
\begin{aligned}
(\tilde\sigma_a)^{\dot\alpha\beta}(\sigma_b)_{\beta\dot\beta}(\tilde\sigma_c)^{\dot\beta\alpha} &= \eta_{ca}(\tilde\sigma_b)^{\dot\alpha\alpha} - \eta_{bc}(\tilde\sigma_a)^{\dot\alpha\alpha} - \eta_{ab}(\tilde\sigma_c)^{\dot\alpha\alpha} - \mathrm{i}\varepsilon_{abcd}(\tilde\sigma^d)^{\dot\alpha\alpha} &(86)\\
\implies (\tilde\sigma_{[a}\sigma_b\tilde\sigma_{c]})^{\dot\alpha\alpha} &= -\mathrm{i}\varepsilon_{abcd}(\tilde\sigma^d)^{\dot\alpha\alpha} &(87)\\
&= (\tilde\sigma_a)^{\dot\alpha\beta}(\sigma_b)_{\beta\dot\beta}(\tilde\sigma_c)^{\dot\beta\alpha} - \eta_{ca}(\tilde\sigma_b)^{\dot\alpha\alpha} + \eta_{bc}(\tilde\sigma_a)^{\dot\alpha\alpha} + \eta_{ab}(\tilde\sigma_c)^{\dot\alpha\alpha}. &(88)
\end{aligned}
$$

$$\begin{aligned}
&\therefore -(\sigma_a)_{\alpha\dot\alpha}(\sigma_b)_{\beta\dot\beta}\overline{\psi}_{\dot\gamma}(\tilde\sigma^{[a}\sigma^b\tilde\sigma^{c]})^{\dot\gamma\gamma}D_c\psi_\gamma \\
&\quad= -(\sigma_a)_{\alpha\dot\alpha}(\sigma_b)_{\beta\dot\beta}\Big((\tilde\sigma^a)^{\dot\gamma\delta}(\sigma^b)_{\delta\dot\delta}(\tilde\sigma^c)^{\dot\delta\gamma} - \eta^{ca}(\tilde\sigma^b)^{\dot\gamma\gamma} + \eta^{bc}(\tilde\sigma^a)^{\dot\gamma\gamma} + \eta^{ab}(\tilde\sigma^c)^{\dot\gamma\gamma}\Big)\overline{\psi}_{\dot\gamma}D_c\psi_\gamma && (89)\\
&\quad= \Big(-4\delta^\delta{}_\alpha\delta^{\dot\gamma}{}_{\dot\alpha}\varepsilon_{\beta\delta}\varepsilon_{\dot\beta\dot\delta}(\tilde\sigma^c)^{\dot\delta\gamma} - 2\delta^\gamma{}_\beta\delta^{\dot\gamma}{}_{\dot\beta}(\sigma^c)_{\alpha\dot\alpha} + 2\delta^\gamma{}_\alpha\delta^{\dot\gamma}{}_{\dot\alpha}(\sigma^c)_{\beta\dot\beta} + 2\varepsilon_{\alpha\beta}\varepsilon_{\dot\alpha\dot\beta}(\tilde\sigma^c)^{\dot\gamma\gamma}\Big)\overline{\psi}_{\dot\gamma}D_c\psi_\gamma && (90)\\
&\quad= -4\varepsilon_{\beta\alpha}\overline{\psi}_{\dot\alpha}D^\gamma{}_{\dot\beta}\psi_\gamma - 2\overline{\psi}_{\dot\beta}D_{\alpha\dot\alpha}\psi_\beta + 2\overline{\psi}_{\dot\alpha}D_{\beta\dot\beta}\psi_\alpha + 2\varepsilon_{\alpha\beta}\varepsilon_{\dot\alpha\dot\beta}\overline{\psi}_{\dot\gamma}D^{\gamma\dot\gamma}\psi_\gamma && (91)\\
&\quad= -4\varepsilon_{\beta\alpha}\varepsilon^{\gamma\delta}\overline{\psi}_{\dot\alpha}D_{\delta\dot\beta}\psi_\gamma - 2\overline{\psi}_{\dot\beta}D_{\alpha\dot\alpha}\psi_\beta + 2\overline{\psi}_{\dot\alpha}D_{\beta\dot\beta}\psi_\alpha + 2\varepsilon_{\alpha\beta}\varepsilon_{\dot\alpha\dot\beta}\varepsilon^{\gamma\delta}\varepsilon^{\dot\gamma\dot\delta}\overline{\psi}_{\dot\gamma}D_{\delta\dot\delta}\psi_\gamma && (92)\\
&\quad= 4(\delta^\gamma{}_\beta\delta^\delta{}_\alpha - \delta^\gamma{}_\alpha\delta^\delta{}_\beta)\overline{\psi}_{\dot\alpha}D_{\delta\dot\beta}\psi_\gamma - 2\overline{\psi}_{\dot\beta}D_{\alpha\dot\alpha}\psi_\beta + 2\overline{\psi}_{\dot\alpha}D_{\beta\dot\beta}\psi_\alpha \\
&\qquad + 2(\delta^\gamma{}_\alpha\delta^\delta{}_\beta - \delta^\gamma{}_\beta\delta^\delta{}_\alpha)(\delta^{\dot\gamma}{}_{\dot\alpha}\delta^{\dot\delta}{}_{\dot\beta} - \delta^{\dot\gamma}{}_{\dot\beta}\delta^{\dot\delta}{}_{\dot\alpha})\overline{\psi}_{\dot\gamma}D_{\delta\dot\delta}\psi_\gamma && (93)\\
&\quad= 4\overline{\psi}_{\dot\alpha}D_{\alpha\dot\beta}\psi_\beta - 4\overline{\psi}_{\dot\alpha}D_{\beta\dot\beta}\psi_\alpha - 2\overline{\psi}_{\dot\beta}D_{\alpha\dot\alpha}\psi_\beta + 2\overline{\psi}_{\dot\alpha}D_{\beta\dot\beta}\psi_\alpha + 2\overline{\psi}_{\dot\alpha}D_{\beta\dot\beta}\psi_\alpha - 2\overline{\psi}_{\dot\alpha}D_{\alpha\dot\beta}\psi_\beta \\
&\qquad - 2\overline{\psi}_{\dot\beta}D_{\beta\dot\alpha}\psi_\alpha + 2\overline{\psi}_{\dot\beta}D_{\alpha\dot\alpha}\psi_\beta && (94)\\
&\quad= 2\overline{\psi}_{\dot\alpha}D_{\alpha\dot\beta}\psi_\beta - 2\overline{\psi}_{\dot\beta}D_{\beta\dot\alpha}\psi_\alpha. && (95)
\end{aligned}$$

Finally, the last two terms of equation 74 simplify to

$$\begin{aligned}
&2\mathrm{i}k(\sigma_a)_{\alpha\dot\alpha}(\sigma_b)_{\beta\dot\beta}\chi^\gamma(\sigma^{[a}\tilde\sigma^{b]})_\gamma{}^\delta\psi_\delta + 2\mathrm{i}k(\sigma_a)_{\alpha\dot\alpha}(\sigma_b)_{\beta\dot\beta}\overline{\psi}_{\dot\gamma}(\tilde\sigma^{[a}\sigma^{b]})^{\dot\gamma}{}_{\dot\delta}\bar\chi^{\dot\delta} \\
&\quad= \mathrm{i}k(\sigma_a)_{\alpha\dot\alpha}(\sigma_b)_{\beta\dot\beta}\big((\sigma^a)_{\gamma\dot\delta}(\tilde\sigma^b)^{\dot\delta\delta}\chi^\gamma\psi_\delta - (\sigma^b)_{\gamma\dot\delta}(\tilde\sigma^a)^{\dot\delta\delta}\chi^\gamma\psi_\delta + (\tilde\sigma^a)^{\dot\gamma\delta}(\sigma^b)_{\delta\dot\delta}\overline{\psi}_{\dot\gamma}\bar\chi^{\dot\delta} \\
&\qquad - (\tilde\sigma^b)^{\dot\gamma\delta}(\sigma^a)_{\delta\dot\delta}\overline{\psi}_{\dot\gamma}\bar\chi^{\dot\delta}\big) && (96)\\
&\quad= 4\mathrm{i}k\big(\varepsilon_{\alpha\gamma}\varepsilon_{\dot\alpha\dot\delta}\delta^\delta{}_\beta\delta^{\dot\delta}{}_{\dot\beta}\chi^\gamma\psi_\delta - \varepsilon_{\beta\gamma}\varepsilon_{\dot\beta\dot\delta}\delta^\delta{}_\alpha\delta^{\dot\delta}{}_{\dot\alpha}\chi^\gamma\psi_\delta + \varepsilon_{\beta\delta}\varepsilon_{\dot\beta\dot\delta}\delta^\delta{}_\alpha\delta^{\dot\gamma}{}_{\dot\alpha}\overline{\psi}_{\dot\gamma}\bar\chi^{\dot\delta} - \varepsilon_{\alpha\delta}\varepsilon_{\dot\alpha\dot\delta}\delta^\delta{}_\beta\delta^{\dot\gamma}{}_{\dot\beta}\overline{\psi}_{\dot\gamma}\bar\chi^{\dot\delta}\big) && (97)\\
&\quad= 4\mathrm{i}k\big(\varepsilon_{\dot\alpha\dot\beta}\chi_\alpha\psi_\beta - \varepsilon_{\dot\beta\dot\alpha}\chi_\beta\psi_\alpha + \varepsilon_{\beta\alpha}\overline{\psi}_{\dot\alpha}\bar\chi_{\dot\beta} - \varepsilon_{\alpha\beta}\overline{\psi}_{\dot\beta}\bar\chi_{\dot\alpha}\big) && (98)\\
&\quad= -4\mathrm{i}k\varepsilon_{\alpha\beta}(\overline{\psi}_{\dot\alpha}\bar\chi_{\dot\beta} + \overline{\psi}_{\dot\beta}\bar\chi_{\dot\alpha}) + 4\mathrm{i}k\varepsilon_{\dot\alpha\dot\beta}(\chi_\alpha\psi_\beta + \chi_\beta\psi_\alpha) && (99)
\end{aligned}$$

Putting it all together, equation 74 reduces to

$$\begin{aligned}
(\sigma_a)_{\alpha\dot\alpha}(\sigma_b)_{\beta\dot\beta}\overline{\Psi}\gamma^{abc}\nabla_c\Psi = {}& 2\chi_\alpha D_{\beta\dot\alpha}\bar\chi_{\dot\beta} - 2\chi_\beta D_{\alpha\dot\beta}\bar\chi_{\dot\alpha} + 2\overline{\psi}_{\dot\alpha}D_{\alpha\dot\beta}\psi_\beta - 2\overline{\psi}_{\dot\beta}D_{\beta\dot\alpha}\psi_\alpha \\
&- 4\mathrm{i}k\varepsilon_{\alpha\beta}(\overline{\psi}_{\dot\alpha}\bar\chi_{\dot\beta} + \overline{\psi}_{\dot\beta}\bar\chi_{\dot\alpha}) + 4\mathrm{i}k\varepsilon_{\dot\alpha\dot\beta}(\chi_\alpha\psi_\beta + \chi_\beta\psi_\alpha). \qquad (100)
\end{aligned}$$

Then, adding the complex conjugate gives

$$\begin{aligned}
E_{\alpha\dot\alpha\beta\dot\beta} = {}& 2\big(\chi_\alpha D_{\beta\dot\alpha}\bar\chi_{\dot\beta} - \chi_\beta D_{\alpha\dot\beta}\bar\chi_{\dot\alpha} + \overline{\psi}_{\dot\alpha}D_{\alpha\dot\beta}\psi_\beta - \overline{\psi}_{\dot\beta}D_{\beta\dot\alpha}\psi_\alpha + \bar\chi_{\dot\alpha}D_{\alpha\dot\beta}\chi_\beta - \bar\chi_{\dot\beta}D_{\beta\dot\alpha}\chi_\alpha \\
&+ \psi_\alpha D_{\beta\dot\alpha}\overline{\psi}_{\dot\beta} - \psi_\beta D_{\alpha\dot\beta}\overline{\psi}_{\dot\alpha}\big) + 8\mathrm{i}k\big(-\varepsilon_{\alpha\beta}(\overline{\psi}_{\dot\alpha}\bar\chi_{\dot\beta} + \overline{\psi}_{\dot\beta}\bar\chi_{\dot\alpha}) + \varepsilon_{\dot\alpha\dot\beta}(\psi_\alpha\chi_\beta + \psi_\beta\chi_\alpha)\big). \quad (101)
\end{aligned}$$

Thus, by definition 2.4, the required integrand is

$$\begin{aligned}
l^{\alpha\dot\alpha}n^{\beta\dot\beta}E_{\alpha\dot\alpha\beta\dot\beta}(\Psi) &= A^\alpha\bar A^{\dot\alpha}B^\beta\bar B^{\dot\beta}E_{\alpha\dot\alpha\beta\dot\beta} && (102)\\
&= 2A^\alpha\bar A^{\dot\alpha}B^\beta\bar B^{\dot\beta}\big(\chi_\alpha D_{\beta\dot\alpha}\bar\chi_{\dot\beta} - \chi_\beta D_{\alpha\dot\beta}\bar\chi_{\dot\alpha} + \overline{\psi}_{\dot\alpha}D_{\alpha\dot\beta}\psi_\beta - \overline{\psi}_{\dot\beta}D_{\beta\dot\alpha}\psi_\alpha + \bar\chi_{\dot\alpha}D_{\alpha\dot\beta}\chi_\beta \\
&\qquad - \bar\chi_{\dot\beta}D_{\beta\dot\alpha}\chi_\alpha + \psi_\alpha D_{\beta\dot\alpha}\overline{\psi}_{\dot\beta} - \psi_\beta D_{\alpha\dot\beta}\overline{\psi}_{\dot\alpha}\big) \\
&\qquad + 8\mathrm{i}kA^\alpha\bar A^{\dot\alpha}B^\beta\bar B^{\dot\beta}\big(-\varepsilon_{\alpha\beta}(\overline{\psi}_{\dot\alpha}\bar\chi_{\dot\beta} + \overline{\psi}_{\dot\beta}\bar\chi_{\dot\alpha}) + \varepsilon_{\dot\alpha\dot\beta}(\psi_\alpha\chi_\beta + \psi_\beta\chi_\alpha)\big) && (103)\\
&= 4\sqrt{2}\big(b(\chi)\bar B^{\dot\beta}\delta\bar\chi_{\dot\beta} + a(\chi)\bar A^{\dot\alpha}\bar\delta\bar\chi_{\dot\alpha} + \bar b(\psi)B^\beta\bar\delta\psi_\beta + \bar a(\psi)A^\alpha\delta\psi_\alpha + \bar b(\chi)B^\beta\bar\delta\chi_\beta \\
&\qquad + \bar a(\chi)A^\alpha\delta\chi_\alpha + b(\psi)\bar B^{\dot\beta}\delta\overline{\psi}_{\dot\beta} + a(\psi)\bar A^{\dot\alpha}\bar\delta\overline{\psi}_{\dot\alpha}\big) \\
&\qquad + 16\sqrt{2}\mathrm{i}k\big(-\bar b(\psi)\bar a(\chi) - \bar a(\psi)\bar b(\chi) + b(\psi)a(\chi) + a(\psi)b(\chi)\big). && (104)
\end{aligned}$$

All the derivative terms can be re-written in terms of the GHP ð and $\bar{\text{ð}}$ operators [18]. In particular, A_α and B_α are GHP type-(0, 1) and type-(0, -1) respectively by definition. Since

ψ_α (and likewise for χ_α etc.) is invariant under choice of spinor dyad, it must be that $a(\psi)$ and $b(\psi)$ are type-(0, -1) and type-(0, 1) respectively. For a type-(p, q) object, $f_{p,q}$, ð and $\bar{ð}$ are defined to act as

$$ðf_{p,q} = \delta f_{p,q} - p\beta f_{p,q} - q\bar{\alpha} f_{p,q} \tag{105}$$

$$\text{and } \bar{ð} f_{p,q} = \bar{\delta} f_{p,q} - p\alpha f_{p,q} - q\bar{\beta} f_{p,q}. \tag{106}$$

Consider the derivative terms one by one. It suffices to calculate half of them and take complex conjugates for the other half (noting that a type-(p, q) object becomes a type-(q, p) object under complex conjugation).

$$\begin{aligned}
\bar{B}^{\dot{\beta}} \delta \bar{\chi}_{\dot{\beta}} &= \bar{B}^{\dot{\alpha}} \delta \left(\bar{a}(\chi)\bar{A}_{\dot{\alpha}} + \bar{b}(\chi)\bar{B}_{\dot{\alpha}}\right) && (107)\\
&= \sqrt{2}\delta\bar{a}(\chi) + \bar{a}(\chi)\bar{B}^{\dot{\alpha}}\delta\bar{A}_{\dot{\alpha}} + 0 + \bar{b}(\chi)\bar{B}^{\dot{\alpha}}\delta\bar{B}_{\dot{\alpha}} && (108)\\
&= \sqrt{2}\left(\delta\bar{a}(\chi) + \beta\bar{a}(\chi) + \mu\bar{b}(\chi)\right) && (109)\\
&= \sqrt{2}\left(ð\bar{a}(\chi) + \mu\bar{b}(\chi)\right). && (110)
\end{aligned}$$

$$\begin{aligned}
\bar{A}^{\dot{\alpha}} \bar{\delta} \bar{\chi}_{\dot{\alpha}} &= \bar{A}^{\dot{\alpha}} \bar{\delta} \left(\bar{a}(\chi)\bar{A}_{\dot{\alpha}} + \bar{b}(\chi)\bar{B}_{\dot{\alpha}}\right) && (111)\\
&= 0 + \bar{a}(\chi)\bar{A}^{\dot{\alpha}}\bar{\delta}\bar{A}_{\dot{\alpha}} - \sqrt{2}\bar{\delta}\bar{b}(\chi) + \bar{b}(\chi)\bar{A}^{\dot{\alpha}}\bar{\delta}\bar{B}_{\dot{\alpha}} && (112)\\
&= -\sqrt{2}\left(\bar{\delta}\bar{b}(\chi) - \alpha\bar{b}(\chi) - \rho\bar{a}(\chi)\right) && (113)\\
&= -\sqrt{2}\left(\bar{ð}\bar{b}(\chi) - \rho\bar{a}(\chi)\right). && (114)
\end{aligned}$$

$$\begin{aligned}
B^{\beta} \bar{\delta} \psi_{\beta} &= B^{\alpha} \bar{\delta} \left(a(\psi)A_{\alpha} + b(\psi)B_{\alpha}\right) && (115)\\
&= \sqrt{2}\bar{\delta}a(\psi) + a(\psi)B^{\alpha}\bar{\delta}A_{\alpha} + 0 + b(\psi)B^{\alpha}\bar{\delta}B_{\alpha} && (116)\\
&= \sqrt{2}\left(\bar{\delta}a(\psi) + \bar{\beta}a(\psi) + \mu b(\psi)\right) && (117)\\
&= \sqrt{2}\left(\bar{ð}a(\psi) + \mu b(\psi)\right). && (118)
\end{aligned}$$

$$\begin{aligned}
A^{\alpha} \delta \psi_{\alpha} &= A^{\alpha} \delta \left(a(\psi)A_{\alpha} + b(\psi)B_{\alpha}\right) && (119)\\
&= 0 + a(\psi)A^{\alpha}\delta A_{\alpha} - \sqrt{2}\delta b(\psi) + b(\psi)A^{\alpha}\delta B_{\alpha} && (120)\\
&= -\sqrt{2}\left(\delta b(\psi) - \bar{\alpha}b(\psi) - \rho a(\psi)\right) && (121)\\
&= -\sqrt{2}\left(ðb(\psi) - \rho a(\psi)\right). && (122)
\end{aligned}$$

Substituting back, I get

$$\begin{aligned}
l^{\alpha\dot{\alpha}} n^{\beta\dot{\beta}} E_{\alpha\dot{\alpha}\beta\dot{\beta}}(\Psi) = 8\big(& b(\chi)(ð\bar{a}(\chi) + \mu\bar{b}(\chi)) - a(\chi)(\bar{ð}\bar{b}(\chi) - \rho\bar{a}(\chi)) + \bar{b}(\psi)(\bar{ð}a(\psi) + \mu b(\psi))\\
&- \bar{a}(\psi)(ðb(\psi) - \rho a(\psi)) + \bar{b}(\chi)(\bar{ð}a(\chi) + \mu b(\chi)) - \bar{a}(\chi)(ðb(\chi) - \rho a(\chi))\\
&+ b(\psi)(ð\bar{a}(\psi) + \mu\bar{b}(\psi)) - a(\psi)(\bar{ð}\bar{b}(\psi) - \rho\bar{a}(\psi))\big)\\
&+ 16\sqrt{2}\mathrm{i}k\big(-\bar{b}(\psi)\bar{a}(\chi) - \bar{a}(\psi)\bar{b}(\chi) + b(\psi)a(\chi) + a(\psi)b(\chi)\big) \qquad (123)\\
= 8\big(& b(\chi)ð\bar{a}(\chi) - a(\chi)\bar{ð}\bar{b}(\chi) + \bar{b}(\psi)\bar{ð}a(\psi) - \bar{a}(\psi)ðb(\psi)\\
&+ \bar{b}(\chi)\bar{ð}a(\chi) - \bar{a}(\chi)ðb(\chi) + b(\psi)ð\bar{a}(\psi) - a(\psi)\bar{ð}\bar{b}(\psi)\big)\\
&+ 16\big(\mu|b(\chi)|^2 + \rho|a(\chi)|^2 + \mu|b(\psi)|^2 + \rho|a(\psi)|^2\big)\\
&+ 16\sqrt{2}\mathrm{i}k\big(-\bar{b}(\psi)\bar{a}(\chi) - \bar{a}(\psi)\bar{b}(\chi) + b(\psi)a(\chi) + a(\psi)b(\chi)\big). \qquad (124)
\end{aligned}$$

The ð and $\bar{ð}$ operators were constructed by GHP [18] such that integration by parts is valid on S, e.g.

$$\int_S \bar{a}(\psi)ð(b(\psi))\mathrm{d}A = -\int_S b(\psi)ð(\bar{a}(\psi))\mathrm{d}A. \tag{125}$$

Substituting equation 124 into equation 68 and integrating by parts proves the lemma. □

3 Elements of analysis

A key idea of Witten's method is applying the Lichnerowicz identity with a spinor, Ψ, solving $\gamma^I \nabla_I \Psi = 0$ on Σ. This section is dedicated to proving this is always possible given appropriate boundary conditions on S and given an appropriate functional space for Ψ. My presentation is heavily based on [34, 35, 36].

Definition 3.1 (C_b^∞)**.** *Let C_b^∞ be the space of Dirac spinors, $\Psi = [\psi_\alpha, \bar{\chi}^{\dot{\alpha}}]^T$, which are smooth on Σ and subject to the boundary conditions, $a(\psi) = b(\chi) = 0$ on S.*

Definition 3.2 ($\langle \cdot, \cdot \rangle_{C_b^\infty}$)**.** *Assume the dominant energy condition holds on Σ and the null expansions on S satisfy $\theta_l > 0$, $\theta_n < 0$ & $\theta_l \theta_n < -8k^2$. Then, define an inner product by*

$$\langle \Psi_1, \Psi_2 \rangle_{C_b^\infty} = \int_\Sigma \Big((\nabla_I \Psi_1)^\dagger \nabla^I \Psi_2 + 4\pi T^{0a} \Psi_1^\dagger \gamma_0 \gamma_a \Psi_2 \Big) \mathrm{d}V - Q(\Psi_1, \Psi_2). \tag{126}$$

Proof. It much be checked that $\langle \cdot, \cdot \rangle_{C_b^\infty}$ is a well defined inner product. Conjugate symmetry and linearity in the second argument are manifest, so only positive definiteness remains.

$$\langle \Psi, \Psi \rangle_{C_b^\infty} = \int_\Sigma \big((\nabla_I \Psi)^\dagger \nabla^I \Psi + 4\pi T^{0a} \Psi^\dagger \gamma_0 \gamma_a \Psi \big) \mathrm{d}V - Q(\Psi). \tag{127}$$

$\Psi \in C_b^\infty \implies a(\psi) = b(\chi) = 0$ on S. Then, by lemma 2.10,

$$Q(\Psi) = 4 \int_S \Big(\mu |b(\psi)|^2 + \rho |a(\chi)|^2 + \mathrm{i}k\sqrt{2} \big(b(\psi) a(\chi) - \bar{b}(\psi) \bar{a}(\chi) \big) \Big) \mathrm{d}A \tag{128}$$

For any nowhere-vanishing, complex function, z, on S, I can re-write $Q(\Psi)$ as

$$Q(\Psi) = 4 \int_S \left(\frac{\mu}{|z|^2} |z b(\psi)|^2 + \rho |z|^2 \left| \frac{1}{z} a(\chi) \right|^2 + \mathrm{i}k\sqrt{2} \left(z b(\psi) \frac{a(\chi)}{z} - \bar{z} \bar{b}(\psi) \frac{\bar{a}(\chi)}{\bar{z}} \right) \right) \mathrm{d}A \tag{129}$$

Let $\mu' = \mu / |z|^2$, $\rho' = |z|^2 \rho$, $b'(\psi) = z b(\psi)$ and $a'(\chi) = a(\chi)/z$.

$$\begin{aligned} \therefore Q(\Psi) &= 4 \int_S \Big(\mu' |b'(\psi)|^2 + \rho' |a'(\chi)|^2 + \mathrm{i}k\sqrt{2} \big(b'(\psi) a'(\chi) - \bar{b}'(\psi) \bar{a}'(\chi) \big) \Big) \mathrm{d}A & (130) \\ &= 4 \int_S \Big((\mu' + k\sqrt{2}) |b'(\psi)|^2 + (\rho' + k\sqrt{2}) |a'(\chi)|^2 - k\sqrt{2} |b'(\psi) + \mathrm{i}\bar{a}'(\chi)|^2 \Big) \mathrm{d}A & (131) \\ &\le 4 \int_S \Big((\mu' + k\sqrt{2}) |b'(\psi)|^2 + (\rho' + k\sqrt{2}) |a'(\chi)|^2 \Big) \mathrm{d}A & (132) \end{aligned}$$

Choose $z = \sqrt[4]{\mu/\rho}$ so that $\mu' = \rho' = -\sqrt{\mu\rho} = -\frac{1}{2}\sqrt{-\theta_l \theta_n} < -k\sqrt{2}$.
$\therefore Q(\Psi) \le 0$.
Next, consider $T^{0a} \Psi^\dagger \gamma_0 \gamma_a \Psi = \Psi^\dagger (T^{00} I + T^{0I} \gamma_0 \gamma_I) \Psi$.
The eigenvalues of $T^{0I} \gamma_0 \gamma_I$ are[5] $\pm\sqrt{T^{0I} T^0{}_I}$, so $T^{0a} \gamma_0 \gamma_a$ is non-negative definite if and only if $T^{00} \ge \sqrt{T^{0I} T^0{}_I}$.
The dominant energy condition says $-T^a{}_b V^b$ is future directed and causal for any future directed, causal vector, V^a.
Choose $Va = \delta^{a0}$.
$\therefore -T^a{}_0 = T^{0a}$ is future directed and causal.
$\therefore T^{00} \ge 0$ and $0 \ge \eta_{ab} T^{0a} T^{0b} \iff (T^{00})^2 \ge T^{0I} T^0{}_I$, which is the condition above.

[5] This can be seen by supposing $T^{0I} \gamma_0 \gamma_I v = \lambda v$. Then, $\lambda^2 v = T^{0I} T^0{}_I v$ by the Clifford algebra. Both $\pm\sqrt{T^{0I} T^0{}_I}$ must be eigenvalues because if v is in one eigenspace, then $\gamma_0 v$ is in the other eigenspace.

$\therefore T^{0a}\Psi^\dagger\gamma_0\gamma_a\Psi \geq 0$.
In summary, all three terms in equation 127 are non-negative.
$\therefore \langle\Psi,\Psi\rangle_{C_b^\infty} \geq 0$.
Finally, suppose $\langle\Psi,\Psi\rangle_{C_b^\infty} = 0$.
Then, by equation 127, $\nabla_I\Psi = 0$ on Σ and $Q(\Psi) = 0$.
The boundary conditions already imply $a(\psi) = b(\chi) = 0$ on S. Equation 132 then implies $a(\chi) = b(\psi) = 0$ on S too.
$\therefore \Psi = 0$ on S.
Choose an arbitrary point, $p \in \Sigma$, and a smooth curve from any point on S to p. Let t^I be the tangent to the curve.
Then, $t^I\nabla_I\Psi = 0$ along the curve and $\Psi = 0$ at the initial point.
Since $t^I\nabla_I\Psi = 0$ is just a linear, homogeneous, 1st order ODE along the curve and Ψ is smooth, initial value problems will have unique solution.
Since $\Psi = 0$ is manifestly a solution for $t^I\nabla_I\Psi = 0$ and $\Psi|_S = 0$, this must be the only solution along the curve.
Since p is arbitrary, this applies for any point on Σ, meaning $\Psi = 0$ everywhere on Σ.
$\therefore \langle\cdot,\cdot\rangle_{C_b^\infty}$ is positive definite. □

Definition 3.3 (G). *Define a linear operator, $G : C_b^\infty \to L^2$, by $G : \Psi \mapsto \gamma^I\nabla_I\Psi$.*

Corollary 3.3.1. $\langle\Psi_1,\Psi_2\rangle_{C_b^\infty} = \langle G(\Psi_1), G(\Psi_2)\rangle_{L^2}$.

Proof. Apply theorem 2.7, definition 2.9 and the polarisation identity for relating norms and inner products. □

Definition 3.4 ($\mathcal{H}$). *Define $\mathcal{H}$ to be the completion of C_b^∞ under $\langle\cdot,\cdot\rangle_{C_b^\infty}$.*

Lemma 3.5. *G extends to a continuous (i.e. bounded) linear operator from $\mathcal{H}$ to L^2 such that $\langle\Psi_1,\Psi_2\rangle_{\mathcal{H}} = \langle G(\Psi_1), G(\Psi_2)\rangle_{L^2}$.*

Proof. G is already defined for $\Psi \in C_b^\infty$. The points in $\mathcal{H}\backslash C_b^\infty$ are equivalence classes of Cauchy sequences.
Let $\{\Psi_A\}_{A=0}^\infty$ be a Cauchy sequence in C_b^∞ with limit in $\mathcal{H}\backslash C_b^\infty$.
Observe that by corollary 3.3.1, $||G(\Psi_A) - G(\Psi_B)||_{L^2} = ||G(\Psi_A - \Psi_B)||_{L^2} = ||\Psi_A - \Psi_B||_{C_b^\infty}$.
$\therefore \{G(\Psi_A)\}_{A=0}^\infty$ is a Cauchy sequence in L^2.
$\therefore$ Since L^2 is complete, $\exists \lim_{A\to\infty} G(\Psi_A) \in L^2$.
Extend the definition of G to $\mathcal{H}\backslash C_b^\infty$ by defining $G(\lim_{A\to\infty}\Psi_A) = \lim_{A\to\infty} G(\Psi_A)$.
This definition is independent of my original choice of Cauchy sequence, $\{\Psi_A\}_{A=0}^\infty$, because if I'd chosen a different Cauchy sequence with the same "limit," $\{\Psi'_A\}_{A=0}^\infty$, then $\{G(\Psi_A), G(\Psi'_B)\}$ would be a Cauchy sequence in L^2 by a similar computation to above. Hence, they would have the same limit in L^2.
Next, bserve that this definition implies corollary 3.3.1 extends to $\mathcal{H}$. In particular, suppose $\Psi = \lim_{A\to\infty}\Psi_A$ and $\Psi' = \lim_{A\to\infty}\Psi'_A$ for Cauchy sequences[6], $\{\Psi_A\}_{A=0}^\infty, \{\Psi'_A\}_{A=0}^\infty \in C_b^\infty$. Then,

$$\langle\Psi,\Psi'\rangle_{\mathcal{H}} = \lim_{A\to\infty}\lim_{B\to\infty}\langle\Psi_A,\Psi'_B\rangle_{C_b^\infty} \text{ by the definition of } \langle\cdot,\cdot\rangle_{\mathcal{H}} \tag{133}$$
$$= \lim_{A\to\infty}\lim_{B\to\infty}\langle G(\Psi_A), G(\Psi'_B)\rangle_{L^2} \text{ by corollary 3.3.1} \tag{134}$$
$$= \left\langle \lim_{A\to\infty} G(\Psi_A), \lim_{B\to\infty} G(\Psi'_B)\right\rangle_{L^2} \text{ by } \langle\cdot,\cdot\rangle'_{L^2}s \text{ continuity} \tag{135}$$
$$= \langle G(\Psi), G(\Psi')\rangle_{L^2} \text{ by } G's \text{ definition.} \tag{136}$$

[6]Strictly speaking, Ψ and Ψ' are equivalence classes of Cauchy sequences, but I'm going to abuse notation by denoting them as if they were ordinary spinors themselves.

As an immediate consequence, I get

$$||G(\Psi)||_{L^2} = ||\Psi||_{\mathcal{H}}, \tag{137}$$

which implies that G is a continuous/bounded linear operator. □

Theorem 3.6. *G is a continuous, linear isomorphism between $\mathcal{H}$ and L^2.*

Most saliently, the theorem implies $(\gamma^I \nabla_I)^{-1} : L^2 \to \mathcal{H}$ exists.

Proof. Linearity is by construction and continuity has already been shown by lemma 3.5. Next, suppose $G(\Psi) = 0$. Then, by lemma 3.5,

$$0 = ||G(\Psi)||_{L^2} = ||\Psi||_{\mathcal{H}} \implies \Psi = 0. \tag{138}$$

$\therefore G$ is injective.
It remains to prove surjectivity.
Let θ be an arbitrary element of L^2.
Define $F_\theta : \mathcal{H} \to \mathbb{C}$ by

$$F_\theta(\Psi) = \langle \theta, G(\Psi) \rangle_{L^2}. \tag{139}$$

F_θ is manifestly linear. It is also continuous/bounded because the Cauchy-Schwarz inequality and lemma 3.5 imply $|F_\theta(\Psi)| = |\langle \theta, G(\Psi)\rangle_{L^2}| \leq ||\theta||_{L^2} ||G(\Psi)||_{L^2} = ||\theta||_{L^2} ||\Psi||_{\mathcal{H}}$.
$\therefore$ By the Riesz representation theorem, $\exists \mathcal{Z} \in \mathcal{H}$ such that $F_\theta(\Psi) = \langle \mathcal{Z}, \Psi \rangle_{\mathcal{H}}$.
$\therefore F_\theta(\Psi) = \langle G(\mathcal{Z}), G(\Psi) \rangle_{L^2}$ by lemma 3.5.
By equation 139, if follows that

$$\langle W, G(\Psi) \rangle_{L^2} = 0 \ \forall \Psi \in \mathcal{H}, \text{ where } W = \theta - G(\mathcal{Z}). \tag{140}$$

Consider a formal integration by parts on this equation.

$$\therefore 0 = \int_\Sigma W^\dagger G(\Psi) \mathrm{d}V \tag{141}$$

$$= \int_\Sigma W^\dagger (\gamma^I D_I \Psi - 3\mathrm{i}k\Psi) \mathrm{d}V \tag{142}$$

$$= \int_\Sigma \left(-P_a \overline{W} \gamma^{ab} D_b \Psi - 3\mathrm{i}k W^\dagger \Psi \right) \mathrm{d}V \tag{143}$$

$$= \int_\Sigma \left(-P_a D_b(\overline{W} \gamma^{ab} \Psi) + P_a D_b(\overline{W}) \gamma^{ab} \Psi - 3\mathrm{i}k W^\dagger \Psi \right) \mathrm{d}V \tag{144}$$

$$= -\int_S P_a Q_b \overline{W} \gamma^{ab} \Psi \mathrm{d}A - \int_\Sigma \left(D_I(W)^\dagger \gamma^I \Psi - 3\mathrm{i}k W^\dagger \Psi \right) \mathrm{d}V \ \text{ by lemma 2.8} \tag{145}$$

$$= \int_S l_a n_b \overline{W} \gamma^{ab} \Psi \mathrm{d}A + \int_\Sigma \left(\gamma^I D_I W + 3\mathrm{i}k W \right)^\dagger \Psi \mathrm{d}V. \tag{146}$$

Let $W = [\phi_\alpha, \bar{\zeta}^{\dot{\alpha}}]^T$ and $\Psi = [\psi_\alpha, \bar{\chi}^{\dot{\alpha}}]^T$ in terms of two-component spinors.

$$\therefore l_a n_b \overline{W} \gamma^{ab} \Psi = l_a n_b \begin{bmatrix} -\zeta^\alpha & -\bar{\phi}_{\dot{\alpha}} \end{bmatrix} \begin{bmatrix} (\sigma^{[a}\tilde{\sigma}^{b]})_\alpha{}^\beta & 0 \\ 0 & (\tilde{\sigma}^{[a}\sigma^{b]})^{\dot{\alpha}}{}_{\dot{\beta}} \end{bmatrix} \begin{bmatrix} \psi_\beta \\ \bar{\chi}^{\dot{\beta}} \end{bmatrix} \tag{147}$$

$$= -\frac{1}{2} l_a n_b \big((\sigma^a)_{\alpha\dot{\alpha}} (\tilde{\sigma}^b)^{\dot{\alpha}\beta} \zeta^\alpha \psi_\beta - (\sigma^b)_{\alpha\dot{\alpha}} (\tilde{\sigma}^a)^{\dot{\alpha}\beta} \zeta^\alpha \psi_\beta + (\tilde{\sigma}^a)^{\dot{\alpha}\alpha} (\sigma^b)_{\alpha\dot{\beta}} \bar{\phi}_{\dot{\alpha}} \bar{\chi}^{\dot{\beta}} - (\tilde{\sigma}^b)^{\dot{\alpha}\alpha} (\sigma^a)_{\alpha\dot{\beta}} \bar{\phi}_{\dot{\alpha}} \bar{\chi}^{\dot{\beta}} \big) \tag{148}$$

$$= \frac{1}{2} \big(n_{\alpha\dot{\alpha}} l^{\beta\dot{\alpha}} \zeta^\alpha \psi_\beta - l_{\alpha\dot{\alpha}} n^{\beta\dot{\alpha}} \zeta^\alpha \psi_\beta + n^{\alpha\dot{\alpha}} l_{\alpha\dot{\beta}} \bar{\phi}_{\dot{\alpha}} \bar{\chi}^{\dot{\beta}} - l^{\alpha\dot{\alpha}} n_{\alpha\dot{\beta}} \bar{\phi}_{\dot{\alpha}} \bar{\chi}^{\dot{\beta}} \big) \tag{149}$$

$$= \frac{1}{2} \big(B_\alpha \bar{B}_{\dot{\alpha}} A^\beta \bar{A}^{\dot{\alpha}} \zeta^\alpha \psi_\beta - A_\alpha \bar{A}_{\dot{\alpha}} B^\beta \bar{B}^{\dot{\alpha}} \zeta^\alpha \psi_\beta + B^\alpha \bar{B}^{\dot{\alpha}} A_\alpha \bar{A}_{\dot{\beta}} \bar{\phi}_{\dot{\alpha}} \bar{\chi}^{\dot{\beta}} - A^\alpha \bar{A}^{\dot{\alpha}} B_\alpha \bar{B}_{\dot{\beta}} \bar{\phi}_{\dot{\alpha}} \bar{\chi}^{\dot{\beta}} \big) \tag{150}$$

$$= \sqrt{2} \big(-a(\zeta) b(\psi) - b(\zeta) a(\psi) + \bar{a}(\phi) \bar{b}(\chi) + \bar{b}(\phi) \bar{a}(\chi) \big). \tag{151}$$

Since $a(\psi) = b(\chi) = 0$ on S $\forall \Psi \in \mathcal{H}$, the formal integration by parts says

$$0 = \sqrt{2} \int_S \big(\bar{b}(\phi) \bar{a}(\chi) - a(\zeta) b(\psi) \big) \, \mathrm{d}A + \int_\Sigma \big(\gamma^I D_I W + 3\mathrm{i}kW \big)^\dagger \Psi \mathrm{d}V. \tag{152}$$

As $\Psi \in \mathcal{H} \supset C_b^\infty$ is arbitrary, it must be that W is a weak solution to $\gamma^I D_I W + 3\mathrm{i}kW = 0$ on Σ subject to the boundary conditions, $b(\phi) = a(\zeta) = 0$ on S.
It is then a technical mathematical problem to ascertain whether weak solutions lift to strong solutions in this context. This question was studied in depth by [34, 35] and I will appeal especially to their theorem 6.4 to conclude this is indeed the case.
Now, I can apply/establish a modified Lichnerowicz identity for W, as follows.
Let $\widetilde{\nabla}_a W = D_a W - \mathrm{i}k\gamma_a W$, i.e. $k \mapsto -k$ compared with the original connection, ∇.
$\therefore \gamma^I D_I W + 3\mathrm{i}kW = 0 \iff \gamma^I \widetilde{\nabla}_I W = 0$.
The sign of k was never essential in the proof of the Lichnerowicz identity; it merely mattered that $k^2 = -\Lambda/12$.
$\therefore$ From the proofs of theorem 2.7 and lemma 2.10, it immediately follows that

$$0 = \int_\Sigma (\gamma^I \widetilde{\nabla}_I W)^\dagger \gamma^J \widetilde{\nabla}_J (W) \mathrm{d}V \tag{153}$$

$$= \int_\Sigma \Big((\widetilde{\nabla}_I W)^\dagger \widetilde{\nabla}^I W - 4\pi T^{0a} \overline{W} \gamma_a W \Big) \, \mathrm{d}V - \widetilde{Q}(W), \tag{154}$$

$$\text{where } \widetilde{Q}(W) = 2 \int_S \Big(b(\phi) \eth \bar{a}(\phi) + \bar{b}(\phi) \bar{\eth} a(\phi) - \bar{a}(\zeta) \eth b(\zeta) - a(\zeta) \bar{\eth} \bar{b}(\zeta) + \rho |a(\phi)|^2 + \mu |b(\phi)|^2 + \rho |a(\zeta)|^2 + \mu |b(\zeta)|^2 - \mathrm{i}k\sqrt{2} \big(a(\phi) b(\zeta) + b(\phi) a(\zeta) - \bar{a}(\phi) \bar{b}(\zeta) - \bar{b}(\phi) \bar{a}(\zeta) \big) \Big) \mathrm{d}A. \tag{155}$$

However, $b(\phi) = a(\zeta) = 0$ on S from earlier.

$$\therefore \widetilde{Q}(W) = 2 \int_S \Big(\rho |a(\phi)|^2 + \mu |b(\zeta)|^2 - \mathrm{i}k\sqrt{2} \big(a(\phi) b(\zeta) - \bar{a}(\phi) \bar{b}(\zeta) \big) \Big) \, \mathrm{d}A \tag{156}$$

As in the proof of definition 3.2, let $\mu' = \mu/|z|^2$, $\rho' = |z|^2 \rho$, $a'(\phi) = a(\phi)/z$ and $b'(\zeta) = zb(\zeta)$. Again, choose $z = \sqrt[4]{\mu/\rho}$ so that $\mu' = \rho' = -\sqrt{\mu\rho} = -\frac{1}{2}\sqrt{-\theta_l \theta_n} < -k\sqrt{2}$.

$$\therefore \widetilde{Q}(W) = 2 \int_S \Big(\rho' |a'(\phi)|^2 + \mu' |b'(\zeta)|^2 - \mathrm{i}k\sqrt{2} \big(a'(\phi) b'(\zeta) - \bar{a}'(\phi) \bar{b}'(\zeta) \big) \Big) \, \mathrm{d}A \tag{157}$$

$$= 2 \int_S \Big((\rho' + k\sqrt{2}) |a'(\phi)|^2 + (\mu' + k\sqrt{2}) |b'(\zeta)|^2 - k\sqrt{2} |\bar{a}'(\phi) + \mathrm{i}b'(\zeta)|^2 \Big) \, \mathrm{d}A \tag{158}$$

$$\leq 0. \tag{159}$$

Thus, combined with the dominant energy condition as used in the proof of definition 3.2, every term on the RHS of equation 154 is non-negative.
$\therefore \widetilde{\nabla}_I W = 0$ and $\widetilde{Q}(W) = 0$.
The latter implies $a(\phi) = b(\zeta) = 0$ on S by equation 158.
$\therefore W = 0$ on S since $b(\phi) = a(\zeta) = 0$ on S already.
In the proof of definition 3.2 I showed $\nabla_I \Psi = 0$ on Σ with $\Psi = 0$ on S implies $\Psi = 0$ on Σ. By the same logic used there, it now follows that $W = 0$ on Σ.
$\therefore \theta = G(\mathcal{Z})$.
Since $\theta \in L^2$ is arbitrary, it must be that G is surjective. □

4 New quasilocal mass and its positivity

Definition 4.1 (Φ)**.** *Define $\Phi = [\varphi_\alpha, \bar{\xi}^{\dot{\alpha}}]^T$ to be a Dirac spinor satisfying $\overline{m}^a \nabla_a \Phi = 0$ on S.*

Definition 4.2 (Φ^A)**.** *Let $\{\Phi^A = [\varphi^A_\alpha, \bar{\xi}^{A\dot{\alpha}}]^T\}$ denote a basis for the space of solutions to $\overline{m}^a \nabla_a \Phi = 0$ on S. Use $A, B, ...$ as indices on this space[7].*

Lemma 4.3. *In the GHP formalism, $\overline{m}^a \nabla_a \Phi = 0$ is equivalent to*

$$0 = \bar{\eth} a(\varphi) + \mu b(\varphi) - \mathrm{i}k\sqrt{2}\bar{a}(\xi), \tag{160}$$
$$0 = \bar{\eth}\bar{b}(\xi) - \rho\bar{a}(\xi) - \mathrm{i}k\sqrt{2}b(\varphi), \tag{161}$$
$$0 = \bar{\eth} b(\varphi) - \bar{\sigma} a(\varphi) \text{ and} \tag{162}$$
$$0 = \bar{\eth}\bar{a}(\xi) + \lambda\bar{b}(\xi). \tag{163}$$

Proof. In terms of two component spinors,

$$\overline{m}^a \nabla_a \Phi = \overline{m}^a D_a \Phi + \mathrm{i}k\overline{m}^a \gamma_a \Phi \tag{164}$$
$$= \begin{bmatrix} \overline{m}^a D_a \varphi_\alpha \\ \overline{m}^a D_a \bar{\xi}^{\dot{\alpha}} \end{bmatrix} + \mathrm{i}k\overline{m}^{\alpha\dot{\alpha}} \begin{bmatrix} 0 & (\sigma_a)_{\alpha\dot{\alpha}} \\ (\tilde{\sigma}_a)^{\dot{\alpha}\alpha} & 0 \end{bmatrix} \begin{bmatrix} \varphi_\alpha \\ \bar{\xi}^{\dot{\alpha}} \end{bmatrix} \tag{165}$$
$$= \begin{bmatrix} \bar{\delta}\varphi_\alpha + \mathrm{i}k\overline{m}_{\alpha\dot{\alpha}}\bar{\xi}^{\dot{\alpha}} \\ \bar{\delta}\bar{\xi}^{\dot{\alpha}} + \mathrm{i}k\overline{m}^{\alpha\dot{\alpha}}\varphi_\alpha \end{bmatrix} \tag{166}$$
$$= \begin{bmatrix} \bar{\delta}\varphi_\alpha + \mathrm{i}kA_\alpha\bar{B}_{\dot{\alpha}}\bar{\xi}^{\dot{\alpha}} \\ \bar{\delta}\bar{\xi}^{\dot{\alpha}} + \mathrm{i}kA^\alpha\bar{B}^{\dot{\alpha}}\varphi_\alpha \end{bmatrix} \tag{167}$$
$$= \begin{bmatrix} \bar{\delta}\varphi_\alpha - \mathrm{i}k\sqrt{2}\bar{a}(\xi)A_\alpha \\ \bar{\delta}\bar{\xi}^{\dot{\alpha}} - \mathrm{i}k\sqrt{2}b(\varphi)\bar{B}^{\dot{\alpha}} \end{bmatrix}. \tag{168}$$

Contract the first equation with A^α.

$$\therefore 0 = A^\alpha \left(\bar{\delta}\varphi_\alpha - \mathrm{i}k\sqrt{2}\bar{a}(\xi)A_\alpha\right) \tag{169}$$
$$= A^\alpha \left(\bar{\delta}\left(a(\varphi)A_\alpha + b(\varphi)B_\alpha\right) - \mathrm{i}k\sqrt{2}\bar{a}(\xi)A_\alpha\right) \tag{170}$$
$$= a(\varphi)A^\alpha\bar{\delta}A_\alpha + 0 + b(\varphi)A^\alpha\bar{\delta}B_\alpha - \sqrt{2}\bar{\delta}b(\varphi) - 0 \tag{171}$$
$$= \sqrt{2}\bar{\sigma}a(\varphi) + \sqrt{2}\bar{\beta}b(\varphi) - \sqrt{2}\bar{\delta}b(\varphi) \tag{172}$$
$$= \sqrt{2}\left(\bar{\sigma}a(\varphi) - \bar{\eth}b(\varphi)\right), \tag{173}$$

[7] This implicitly assumes the solution space has countable dimension. As I'll explain later, this assumption is acceptable because for generic S, I expect the solution space to be just four dimensional.

which proves equation 162.
Similarly, one finds the remaining three equations as follows.

$$
\begin{aligned}
0 &= B^\alpha \left(\bar{\delta}\varphi_\alpha - \mathrm{i}k\sqrt{2}\bar{a}(\xi)A_\alpha\right) &(174)\\
&= B^\alpha \left(\bar{\delta}\left(a(\varphi)A_\alpha + b(\varphi)B_\alpha\right) - \mathrm{i}k\sqrt{2}\bar{a}(\xi)A_\alpha\right) &(175)\\
&= a(\varphi)B^a\bar{\delta}A_\alpha + \sqrt{2}\bar{\delta}a(\varphi) + b(\varphi)B^\alpha\bar{\delta}B_\alpha + 0 - 2\mathrm{i}k\bar{a}(\xi) &(176)\\
&= \sqrt{2}\bar{\beta}a(\varphi) + \sqrt{2}\bar{\delta}a(\varphi) + \sqrt{2}\mu b(\varphi) - 2\mathrm{i}k\bar{a}(\xi) &(177)\\
&= \sqrt{2}\left(\bar{\eth}a(\varphi) + \mu b(\varphi) - \mathrm{i}k\sqrt{2}\bar{a}(\xi)\right). &(178)
\end{aligned}
$$

$$
\begin{aligned}
0 &= \bar{A}_{\dot\alpha}\left(\bar{\delta}\bar{\xi}^{\dot\alpha} - \mathrm{i}k\sqrt{2}b(\varphi)\bar{B}^{\dot\alpha}\right) &(179)\\
&= \bar{A}_{\dot\alpha}\left(\bar{\delta}(\bar{a}(\xi)\bar{A}^{\dot\alpha} + \bar{b}(\xi)\bar{B}^{\dot\alpha}) - \mathrm{i}k\sqrt{2}b(\varphi)\bar{B}^{\dot\alpha}\right) &(180)\\
&= \bar{a}(\xi)\bar{A}_{\dot\alpha}\bar{\delta}\bar{A}^{\dot\alpha} + 0 + \bar{b}(\xi)\bar{A}_{\dot\alpha}\bar{\delta}\bar{B}^{\dot\alpha} + \sqrt{2}\bar{\delta}\bar{b}(\xi) - 2\mathrm{i}kb(\varphi) &(181)\\
&= -\sqrt{2}\rho\bar{a}(\xi) - \sqrt{2}\alpha\bar{b}(\xi) + \sqrt{2}\bar{\delta}\bar{b}(\xi) - 2\mathrm{i}kb(\varphi) &(182)\\
&= \sqrt{2}\left(\bar{\eth}\bar{b}(\xi) - \rho\bar{a}(\xi) - \mathrm{i}k\sqrt{2}b(\varphi)\right). &(183)
\end{aligned}
$$

$$
\begin{aligned}
0 &= \bar{B}_{\dot\alpha}\left(\bar{\delta}\bar{\xi}^{\dot\alpha} - \mathrm{i}k\sqrt{2}b(\varphi)\bar{B}^{\dot\alpha}\right) &(184)\\
&= \bar{B}_{\dot\alpha}\left(\bar{\delta}(\bar{a}(\xi)\bar{A}^{\dot\alpha} + \bar{b}(\xi)\bar{B}^{\dot\alpha}) - \mathrm{i}k\sqrt{2}b(\varphi)\bar{B}^{\dot\alpha}\right) &(185)\\
&= \bar{a}(\xi)\bar{B}_{\dot\alpha}\bar{\delta}\bar{A}^{\dot\alpha} - \sqrt{2}\bar{\delta}\bar{a}(\xi) + \bar{b}(\xi)\bar{B}_{\dot\alpha}\bar{\delta}\bar{B}^{\dot\alpha} + 0 - 0 &(186)\\
&= -\sqrt{2}\alpha\bar{a}(\xi) - \sqrt{2}\bar{\delta}\bar{a}(\xi) - \sqrt{2}\lambda\bar{b}(\xi) &(187)\\
&= -\sqrt{2}\left(\bar{\eth}\bar{a}(\xi) + \lambda\bar{b}(\xi)\right). &(188)
\end{aligned}
$$

□

Definition 4.4 (Q^{AB})**.** *Define the hermitian matrix, Q^{AB}, by*

$$
\begin{aligned}
Q^{AB} = 4\int_S &\Big(\rho\bar{a}(\varphi^A)a(\varphi^B) + \mu b(\xi^A)\bar{b}(\xi^B) - \rho a(\xi^A)\bar{a}(\xi^B) - \mu\bar{b}(\varphi^A)b(\varphi^B)\\
&+ \mathrm{i}k\sqrt{2}\big(b(\xi^A)a(\varphi^B) - \bar{a}(\varphi^A)\bar{b}(\xi^B) - a(\xi^A)b(\varphi^B) + \bar{b}(\varphi^A)\bar{a}(\xi^B)\big)\Big)\mathrm{d}A. \qquad (189)
\end{aligned}
$$

Theorem 4.5. *If the dominant energy condition holds on Σ and the null expansions on S satisfy $\theta_l > 0$, $\theta_n < 0$ & $\theta_l\theta_n < -8k^2$, then Q^{AB} is a non-negative definite matrix.*

Proof. From lemma 2.10,

$$
\begin{aligned}
Q(\Phi) = 4\int_S &\Big(b(\varphi)\eth\bar{a}(\varphi) + \bar{b}(\varphi)\bar{\eth}a(\varphi) - \bar{a}(\xi)\eth b(\xi) - a(\xi)\bar{\eth}\bar{b}(\xi)\\
&+ \rho|a(\varphi)|^2 + \mu|b(\varphi)|^2 + \rho|a(\xi)|^2 + \mu|b(\xi)|^2\\
&+ \mathrm{i}k\sqrt{2}\big(a(\varphi)b(\xi) + b(\varphi)a(\xi) - \bar{a}(\varphi)\bar{b}(\xi) - \bar{b}(\varphi)\bar{a}(\xi)\big)\Big)\mathrm{d}A. \qquad (190)
\end{aligned}
$$

From equations 160 and 161,

$$
\begin{aligned}
\bar{b}(\varphi)\bar{\eth}a(\varphi) &= -\mu|b(\varphi)|^2 + \mathrm{i}k\sqrt{2}\bar{b}(\varphi)\bar{a}(\xi) \text{ and} &(191)\\
a(\xi)\bar{\eth}\bar{b}(\xi) &= \rho|a(\xi)|^2 + \mathrm{i}k\sqrt{2}a(\xi)b(\varphi). &(192)
\end{aligned}
$$

$$
\begin{aligned}
\therefore Q(\Phi) = 4\int_S &\Big(\rho|a(\varphi)|^2 - \mu|b(\varphi)|^2 - \rho|a(\xi)|^2 + \mu|b(\xi)|^2\\
&+ \mathrm{i}k\sqrt{2}\big(a(\varphi)b(\xi) - a(\xi)b(\varphi) - \bar{a}(\varphi)\bar{b}(\xi) + \bar{a}(\xi)\bar{b}(\varphi)\big)\Big)\mathrm{d}A. \qquad (193)
\end{aligned}
$$

Let $\mathcal{Z} = [\phi_\alpha, \bar{\zeta}^{\dot{\alpha}}]^T$ be any element of $H^1(\Sigma)$ such that on S, $a(\phi) = a(\varphi)$ and $b(\zeta) = b(\xi)$.
$\mathcal{Z} \in H^1(\Sigma) \implies \gamma^I \nabla_I \mathcal{Z} \in L^2(\Sigma)$.
$\therefore$ By theorem 3.6, $\exists \Psi' \in \mathcal{H}$ such that $\gamma^I \nabla_I \Psi' = G(\Psi') = -\gamma^I \nabla_I \mathcal{Z}$.
$\therefore \Psi = \Psi' + \mathcal{Z}$ satisfies $\gamma^I \nabla_I \Psi = 0$.
$\therefore$ By definition 2.9,

$$Q(\Psi) = \int_\Sigma \big(\nabla_I(\Psi)^\dagger \nabla^I \Psi - 4\pi T^{0a}\overline{\Psi}\gamma^a \Psi\big)\,\mathrm{d}V \geq 0 \tag{194}$$

where the first term is manifestly non-negative and the second term is non-negative by the dominant energy condition (same reasoning as in the proof that definition 3.2 is well-defined). Furthermore, since every element, $\Psi' \in \mathcal{H}$, has $a(\psi') = b(\chi') = 0$ on S by construction, it follows that Ψ has $a(\psi) = a(\varphi)$ and $b(\chi) = b(\xi)$ on S.
$\therefore$ By definition 2.9, lemma 2.10 and the fact all the derivatives in lemma 2.10 are tangent to S, $Q(\Psi)$ can also be written as

$$\begin{aligned} Q(\Psi) = 4\int_S \Big(& b(\psi)\eth\bar{a}(\varphi) + \bar{b}(\psi)\bar{\eth}a(\varphi) - \bar{a}(\chi)\eth b(\xi) - a(\chi)\bar{\eth}\bar{b}(\xi) \\ & + \rho|a(\varphi)|^2 + \mu|b(\psi)|^2 + \rho|a(\chi)|^2 + \mu|b(\xi)|^2 \\ & + \mathrm{i}k\sqrt{2}\big(a(\varphi)b(\xi) + b(\psi)a(\chi) - \bar{a}(\varphi)\bar{b}(\xi) - \bar{b}(\psi)\bar{a}(\chi)\big)\Big)\mathrm{d}A. \end{aligned} \tag{195}$$

Then, from equations 160 and 161,

$$\begin{aligned} Q(\Psi) = 4\int_S \Big(& - b(\psi)\big(\mu\bar{b}(\varphi) + \mathrm{i}k\sqrt{2}a(\xi)\big) - \bar{b}(\psi)\big(\mu b(\varphi) - \mathrm{i}k\sqrt{2}\bar{a}(\xi)\big) \\ & - \bar{a}(\chi)\big(\rho a(\xi) - \mathrm{i}k\sqrt{2}\bar{b}(\varphi)\big) - a(\chi)\big(\rho\bar{a}(\xi) + \mathrm{i}k\sqrt{2}b(\varphi)\big) \\ & + \rho|a(\varphi)|^2 + \mu|b(\psi)|^2 + \rho|a(\chi)|^2 + \mu|b(\xi)|^2 \\ & + \mathrm{i}k\sqrt{2}\big(a(\varphi)b(\xi) + b(\psi)a(\chi) - \bar{a}(\varphi)\bar{b}(\xi) - \bar{b}(\psi)\bar{a}(\chi)\big)\Big)\mathrm{d}A \end{aligned} \tag{196}$$

$$\begin{aligned} = 4\int_S \Big(& \mu\big(- b(\psi)\bar{b}(\varphi) - \bar{b}(\psi)b(\varphi) + |b(\psi)|^2 + |b(\xi)|^2\big) \\ & + \rho\big(- \bar{a}(\chi)a(\xi) - a(\chi)\bar{a}(\xi) + |a(\varphi)|^2 + |a(\chi)|^2\big) \\ & + \mathrm{i}k\sqrt{2}\big(- b(\psi)a(\xi) + \bar{b}(\psi)\bar{a}(\xi) + \bar{a}(\chi)\bar{b}(\varphi) - a(\chi)b(\varphi) \\ & \quad + a(\varphi)b(\xi) + b(\psi)a(\chi) - \bar{a}(\varphi)\bar{b}(\xi) - \bar{b}(\psi)\bar{a}(\chi)\big)\Big)\mathrm{d}A. \end{aligned} \tag{197}$$

$\therefore$ Equation 193 can be re-written as

$$\begin{aligned} Q(\Phi) = 4\int_S \Big(& - \mu\big(|b(\varphi)|^2 - b(\psi)\bar{b}(\varphi) - \bar{b}(\psi)b(\varphi) + |b(\psi)|^2\big) \\ & - \rho\big(|a(\xi)|^2 - \bar{a}(\chi)a(\xi) - a(\chi)\bar{a}(\xi) + |a(\chi)|^2\big) \\ & - \mathrm{i}k\sqrt{2}\big(- b(\psi)a(\xi) + \bar{b}(\psi)\bar{a}(\xi) + \bar{a}(\chi)\bar{b}(\varphi) - a(\chi)b(\varphi) \\ & \quad + b(\psi)a(\chi) - \bar{b}(\psi)\bar{a}(\chi) + a(\xi)b(\varphi) - \bar{a}(\xi)\bar{b}(\varphi)\big)\Big)\mathrm{d}A + Q(\Psi) \end{aligned} \tag{198}$$

$$\begin{aligned} = 4\int_S \Big(& - \mathrm{i}k\sqrt{2}\big((a(\xi) - a(\chi))(b(\varphi) - b(\psi)) - (\bar{a}(\xi) - \bar{a}(\chi))(\bar{b}(\varphi) - \bar{b}(\psi))\big) \\ & - \mu|b(\varphi) - b(\psi)|^2 - \rho|a(\xi) - a(\chi)|^2\Big)\mathrm{d}A + Q(\Psi). \end{aligned} \tag{199}$$

As done previously, let $\mu' = \mu/|z|^2$, $\rho' = |z|^2\rho$, $a'(\xi) = a(\xi)/z$, $a'(\chi) = a(\chi)/z$, $b'(\varphi) = zb(\varphi)$

and $b'(\psi) = zb(\psi)$. Again, choose $z = \sqrt[4]{\mu/\rho}$ so that $\mu' = \rho' = -\sqrt{\mu\rho} = -\frac{1}{2}\sqrt{-\theta_l\theta_n} < -k\sqrt{2}$.

$$\therefore Q(\Phi) = 4\int_S \Big(-\mathrm{i}k\sqrt{2}\big((a'(\xi) - a'(\chi))(b'(\varphi) - b'(\psi)) - (\bar{a}'(\xi) - \bar{a}'(\chi))(\bar{b}'(\varphi) - \bar{b}'(\psi))\big) - \mu'|b'(\varphi) - b'(\psi)|^2 - \rho'|a'(\xi) - a'(\chi)|^2\Big)\mathrm{d}A + Q(\Psi) \tag{200}$$

$$= 4\int_S \Big(\sqrt{2}k|a'(\xi) - a'(\chi) + \mathrm{i}\bar{b}'(\varphi) - \mathrm{i}\bar{b}'(\psi)|^2 - (\mu' + \sqrt{2}k)|b'(\varphi) - b'(\psi)|^2 - (\rho' + \sqrt{2}k)|a'(\xi) - a'(\chi)|^2\Big)\mathrm{d}A + Q(\Psi) \tag{201}$$

$$\geq 0. \tag{202}$$

Since $\{\Phi^A\}$ is a basis for the solution space to $\overline{m}^a\nabla_a\Phi = 0$, I can let $\Phi = c_A\Phi^A$ for any constants, c_A.
$\therefore a(\varphi) = c_A a(\varphi^A)$, $b(\varphi) = c_A b(\varphi^A)$, $\bar{a}(\xi) = c_A\bar{a}(\xi^A)$ and $\bar{b}(\xi) = c_A\bar{b}(\xi^A)$.
$\therefore$ By definition 4.4 and equation 193,

$$0 \leq Q(\Phi) = \bar{c}_A Q^{AB} c_B. \tag{203}$$

Since c_A are arbitrary, it must be that Q^{AB} is non-negative definite. □

While this theorem achieves a manifestly non-negative object, it's not possible to extract a mass from the full matrix, Q^{AB}, without some auxiliary constructions.

Definition 4.6 (T^{AB})**.** *Define the matrix, T^{AB}, by*

$$T^{AB} = \varepsilon^{\alpha\beta}\varphi^A_\alpha\varphi^B_\beta - \varepsilon^{\dot{\alpha}\dot{\beta}}\bar{\xi}^A_{\dot{\alpha}}\bar{\xi}^B_{\dot{\beta}} \tag{204}$$

$$= \sqrt{2}\left(a(\varphi^A)b(\varphi^B) - a(\varphi^B)b(\varphi^A) + \bar{a}(\xi^B)\bar{b}(\xi^A) - \bar{a}(\xi^A)\bar{b}(\xi^B)\right). \tag{205}$$

The notion of a surface, S, being "generic" can now finally be stated precisely.

Definition 4.7 (Generic - T^{AB} form)**.** *The surface, S, is called generic if and only if T^{AB} is invertible.*

Definition 4.8 (Generic - Φ^A form)**.** *The surface, S, is said to be generic if and only if the solution space to $\overline{m}^a\nabla_a\Phi = 0$ on S is four dimensional and the basis, $\{\Phi^A\}_{A=1}^4$, is pointwise linearly independent at least at one point of S.*

I will show later that the Φ^A version of generic implies the T^{AB} version. In any case, I will only use the property of S being generic once - albeit in a rather essential way in definition 4.11, where I rely on T^{-1} existing. More importantly though, surfaces generic in name should be generic in practice too. For the T^{AB} form, it could be argued that since the set of singular $n\times n$ matrices are measure zero in the set of all $n\times n$ matrices, this is indeed a valid notion of generic. However, it's not obvious the solution space is finite dimensional and this argument doesn't consider the possibility there is something specific to this situation precluding T^{AB}'s invertibility. Furthermore, the examples considered in sections 5 and 6 either satisfy both notions of generic or neither notion. Hence, it's unclear whether $m(S)$ constructed on a surface satisfying the T^{AB} form, but not the Φ^A form, of generic has physical meaning beyond simple mathematical validity. Finally, from a practical point of view, one would like to know what size of matrix to expect for T^{AB} - and for that matter, Q^{AB}. As defined so far, they could be of arbitrarily large size, maybe even infinitely large. Luckily, when S has spherical topology, there are reasons to believe the Φ^A form is also a valid notion of generic, implying T^{AB} is only a 4×4 matrix.

It is known - e.g. from section 8.2.2 of [14] - that $\bar{\delta}$ is an elliptic operator and the compactness of S then guarantees $\bar{\delta}$ has finite dimensional kernel. Then, it is also known [14] that $\bar{\delta}$'s index (dimension of kernel minus dimension of cokernel) is $4(1-g)$ when S has genus, g.
The difference between $\overline{m}^a\nabla_a$ - the operator I'm actually interested in - and $\bar{\delta}$ is $\mathrm{i}k\overline{m}^a\gamma_a$, which is a compact operator since S is compact and $\mathrm{i}k\overline{m}^a\gamma_a$ is just a 4×4 matrix.
$\therefore$ By Fredholm theory, $\mathrm{index}(\overline{m}^a\nabla_a) = \mathrm{index}(\bar{\delta}) = 4(1-g)$.
$\therefore$ If S is diffeomorphic to a sphere, then $\overline{m}^a\nabla_a\Phi = 0$ must have at least four linearly independent solutions.
In the spherical examples of sections 5 and 6, there happen to be precisely four linearly independent solutions. Faced with a similar situation, Penrose then argues [20] as long as S is not too far from canonical situations - such as the examples to be considered - there would still remain precisely four linearly independent solutions.
At least for spherical S, this justifies the first half of the generic definition in Φ^A form. For non-spherical S, the situation far less constrained and I cannot say whether either definition of generic is actually realistic. In section 5.2 I study examples with toroidal S. In one example $\overline{m}^a\nabla_a\Phi = 0$ will have only two linearly independent solutions and the corresponding T^{AB} will just zero, while in the other example $\overline{m}^a\nabla_a\Phi = 0$ won't have any non-trivial solutions to begin with. Hence both definitions of generic fail - however the wider implications are unclear.
The second half of definition 4.8 is motivated by a possibility that occurs in the Dougan-Mason definition, where one needs to solve the analogous equation, $\bar{\delta}\varphi_\alpha = 0$. It turns out there exist "exceptional" surfaces - e.g. a bifurcate Killing surface - where there are two solutions to $\bar{\delta}\varphi_\alpha = 0$ (the maximum expected or desired) which are linearly independent as functions despite being pointwise linearly dependent at every point of S. The Dougan-Mason mass cannot be defined on such surfaces because the analogue of T^{AB} just becomes zero. However, based on considerations of holomorphic spin bundles, Dougan and Mason argue such surfaces really are indeed exceptional and not generic.
Similarly, I will insist $\{\Phi^A\}_{A=1}^4$ are pointwise linearly independent at least at one point of S for S to be called generic in the Φ^A sense.

Lemma 4.9. *T^{AB} is antisymmetric and constant on S. Furthermore, the notion of generic in definition 4.8 implies the notion of generic in definition 4.7.*

Proof. Antisymmetry follows directly from the definition.
Next, observe that

$$\begin{aligned}
\bar{\delta}T^{AB} &= \bar{\delta}\left(\varepsilon^{\alpha\beta}\varphi_\alpha^A\varphi_\beta^B - \varepsilon^{\dot{\alpha}\dot{\beta}}\bar{\xi}_{\dot{\alpha}}^A\bar{\xi}_{\dot{\beta}}^B\right) && (206)\\
&= \varepsilon^{\alpha\beta}\bar{\delta}(\varphi_\alpha^A)\varphi_\beta^B + \varepsilon^{\alpha\beta}\varphi_\alpha^A\bar{\delta}\varphi_\beta^B - \varepsilon^{\dot{\alpha}\dot{\beta}}\bar{\delta}(\bar{\xi}_{\dot{\alpha}}^A)\bar{\xi}_{\dot{\beta}}^B - \varepsilon^{\dot{\alpha}\dot{\beta}}\bar{\xi}_{\dot{\alpha}}^A\bar{\delta}\bar{\xi}_{\dot{\beta}}^B && (207)\\
&= \varepsilon^{\alpha\beta}\mathrm{i}k\sqrt{2}\bar{a}(\xi^A)A_\alpha\varphi_\beta^B + \varepsilon^{\alpha\beta}\varphi_\alpha^A\mathrm{i}k\sqrt{2}\bar{a}(\xi^B)A_\beta \\
&\quad - \varepsilon^{\dot{\alpha}\dot{\beta}}\mathrm{i}k\sqrt{2}b(\varphi^A)\bar{B}_{\dot{\alpha}}\bar{\xi}_{\dot{\beta}}^B - \varepsilon^{\dot{\alpha}\dot{\beta}}\bar{\xi}_{\dot{\alpha}}^A\mathrm{i}k\sqrt{2}b(\varphi^B)\bar{B}_{\dot{\beta}} \quad \text{by equation 168} && (208)\\
&= 2\mathrm{i}k\left(\bar{a}(\xi^A)b(\varphi^B) - \bar{a}(\xi^B)b(\varphi^A) + b(\varphi^A)\bar{a}(\xi^B) - b(\varphi^B)\bar{a}(\xi^A)\right) && (209)\\
&= 0. && (210)
\end{aligned}$$

$\therefore$ For each A and B, T^{AB} is a holomorphic function on S.
$\therefore$ Since S is compact, Liouville's theorem implies T^{AB} is constant on S.
To prove invertibility, it's easier to work in Dirac spinor notation. In the conventions I'm using,

the charge conjugation matrix is

$$C = \begin{bmatrix} \varepsilon_{\alpha\beta} & 0 \\ 0 & \varepsilon^{\dot{\alpha}\dot{\beta}} \end{bmatrix}. \tag{211}$$

$$\therefore (\Phi^A)^T C^{-1} \Phi^B = \begin{bmatrix} \varphi^A_\alpha & \bar{\xi}^{A\dot{\alpha}} \end{bmatrix} \begin{bmatrix} \varepsilon^{\alpha\beta} & 0 \\ 0 & \varepsilon_{\dot{\alpha}\dot{\beta}} \end{bmatrix} \begin{bmatrix} \varphi^B_\beta \\ \bar{\xi}^{B\dot{\beta}} \end{bmatrix} \tag{212}$$

$$= \varphi^A_\alpha \varepsilon^{\alpha\beta} \varphi^B_\beta + \bar{\xi}^{A\dot{\alpha}} \varepsilon_{\dot{\alpha}\dot{\beta}} \bar{\xi}^{B\dot{\beta}} \tag{213}$$

$$= T^{AB}. \tag{214}$$

Let v_A be a vector in the nullspace of T^{AB}.
$\therefore T^{AB} v_B = 0$.
Let $\mathcal{Z} = v_A \Phi^A$. Then, $T^{AB} v_B = 0 \iff (\Phi^A)^T C^{-1} \mathcal{Z} = 0 \iff w_A (\Phi^A)^T C^{-1} \mathcal{Z} = 0$ for any vector, w_A.
Definition 4.8 says there are four different Φ^A and they are pointwise linearly independent at least at one point, say p, on S.
$\therefore$ Since Dirac spinors also have four components, $\{\Phi^A\}_{A=1}^4$ must form a pointwise basis at p.
$\therefore w_A \Phi^A$ can be any Dirac spinor at p.
$\therefore C^{-1} \mathcal{Z}|_p = 0$.
$\therefore \mathcal{Z}|_p = 0$ since C^{-1} is invertible.
$\therefore v_A = 0$ by the linear independence of $\{\Phi^A\}_{A=1}^4$ at p.
$\therefore T^{AB}$ has trivial nullspace. □

Lemma 4.10. *For any non-negative definite, hermitian matrix, H, and antisymmetric matrix, A, $\operatorname{tr}(HA\overline{H}\bar{A})$ is real and $\operatorname{tr}(HA\overline{H}\bar{A}) \leq 0$.*

Proof. In this proof I will write all indices downstairs and I will write all summations explicitly.
Every hermitian matrix is orthogonally diagonalisable and has real eigenvalues.
$\therefore \exists$ Vectors, $\{v_{(A)}\}$, such that $v^\dagger_{(A)} v_{(B)} = \delta_{AB}$ and $H v_{(A)} = \lambda_A$ for some $\lambda_A \in \mathbb{R}$.
H being non-negative definite implies $\lambda_A \geq 0 \ \forall A$.
Let $U_{AB} = v_{(B)A}$. Then, the orthogonal diagonalisation statement is that

$$U^\dagger H U \equiv \sum_{C,D} (U^\dagger)_{AC} H_{CD} U_{DB} \sum_{C,D} \bar{v}_{(A)C} H_{CD} v_{(B)D} = \delta_{AB} \lambda_B = D_{AB}. \tag{215}$$

$\therefore$ By U's unitarity,

$$H_{AB} = \sum_{C,D} U_{AC} D_{CD} (U^\dagger)_{DB} = \sum_{C,D} v_{(C)A} \delta_{CD} \lambda_D \bar{v}_{(D)B} = \sum_C \lambda_C v_{(C)A} \bar{v}_{(C)B}. \tag{216}$$

Then, the quantity of interest is

$$\operatorname{tr}(HA\overline{H}\bar{A}) = \sum_{A,B,C,D} H_{AB} A_{BC} \overline{H}_{CD} \bar{A}_{DA} \tag{217}$$

$$= \sum_{A,B,C,D,E,F} \lambda_E v_{(E)A} \bar{v}_{(E)B} A_{BC} \lambda_F \bar{v}_{(F)C} v_{(F)D} \bar{A}_{DA} \tag{218}$$

$$= -\sum_{A,B,C,D,E,F} \lambda_E v_{(E)A} \bar{v}_{(E)B} A_{BC} \lambda_F \bar{v}_{(F)C} v_{(F)D} \bar{A}_{AD} \tag{219}$$

$$= -\sum_{A,B,C,D,E,F} \lambda_E \lambda_F \bar{v}_{(E)B} A_{BC} \bar{v}_{(F)C} v_{(E)A} \bar{A}_{AD} v_{(F)D} \tag{220}$$

$$= -\sum_{E,F} \lambda_E \lambda_F |v^\dagger_{(E)} A \bar{v}_{(F)}|^2, \tag{221}$$

which is manifestly real and non-positive. □

Definition 4.11 (Quasilocal mass). *Suppose the dominant energy condition holds on Σ, the null expansions on S satisfy $\theta_l > 0$, $\theta_n < 0$ & $\theta_l\theta_n < -8k^2$ and S is generic (either definition). Then, construct Q^{AB} and T^{AB} by definitions 4.4 & 4.6 and define the quasilocal mass, $m(S)$, to be*

$$m(S) = \frac{1}{16\pi}\sqrt{-\operatorname{tr}\left(QT^{-1}\overline{Q}\overline{T}^{-1}\right)}. \tag{222}$$

Theorem 4.5, lemma 4.9 and lemma 4.10 ensure $m(S)$ is well-defined and manifestly non-negative. Furthermore, $m(S)$ is independent of the choice of basis, $\{\Phi^A\}$. For example, suppose I perform a change of basis, $\Phi'^A = B^A{}_B\Phi^B$. Then, by definitions 4.4 and 4.6,

$$Q'^{AB} = \bar{B}^A{}_C Q^{CD} B^B{}_D \iff Q' = \bar{B}QB^T \text{ and} \tag{223}$$

$$T'^{AB} = B^A{}_C T^{CD} B^B{}_D \iff T' = BTB^T. \tag{224}$$

$$\therefore \operatorname{tr}\left(Q'(T')^{-1}\overline{Q}'(\overline{T}')^{-1}\right) = \operatorname{tr}\left(\bar{B}QB^TB^{-T}T^{-1}B^{-1}B\overline{Q}\bar{B}^T\bar{B}^{-T}\overline{T}^{-1}\bar{B}^{-1}\right) \tag{225}$$

$$= \operatorname{tr}\left(QT^{-1}\overline{Q}\overline{T}^{-1}\right). \tag{226}$$

Corollary 4.11.1. *$m(S) = 0$ for every surface, S, in AdS that's generic in the Φ^A sense.*

Proof. For generic surfaces I have at most four linearly independent solutions to $\overline{m}^a\nabla_a\Phi = 0$. However, AdS already has a four dimensional space of Killing spinors, i.e. solutions to $\nabla_a\varepsilon_k = 0$. $\therefore$ Since $\nabla_a\varepsilon_k = 0$ is a stronger condition, I can use the Killing spinors of AdS as $\{\Phi^A\}_{A=1}^4$. Then, $\Phi = \varepsilon_k$ and $\nabla_a\varepsilon_k = 0 \implies E^{ab}(\Phi) = 0 \implies Q(\Phi) = 0 \implies m(S) = 0$. □

There is a weak converse to this property which can be proved somewhat immediately.

Corollary 4.11.2. *If $Q^{AB} = 0$ and the S is generic in the Φ^A sense, then the spacetime is maximally symmetric on Σ.*

Proof. From the proof of theorem 4.5, especially equation 194, $Q^{AB} = 0 \implies \exists$ 4 linearly independent spinors[8], Ψ^A, such that $\nabla_I\Psi^A = 0$.
First, $\nabla_I\Psi = 0$ implies the 'integrability condition,'

$$0 = [\nabla_I, \nabla_J]\Psi \tag{227}$$

$$= [D_I + \mathrm{i}k\gamma_I, D_J + \mathrm{i}k\gamma_J]\Psi \tag{228}$$

$$= [D_I, D_J]\Psi + \mathrm{i}k\gamma_J D_I\Psi - \mathrm{i}k\gamma_I D_J\Psi + \mathrm{i}k\gamma_I\nabla_J\Psi - \mathrm{i}k\gamma_J\nabla_I\Psi \tag{229}$$

$$= -\frac{1}{4}R_{abIJ}\gamma^{ab}\Psi + \mathrm{i}k\gamma_J(-\mathrm{i}k\gamma_I\Psi) - \mathrm{i}k\gamma_I(-\mathrm{i}k\gamma_J\Psi) + 0 - 0 \tag{230}$$

$$= -\frac{1}{4}R_{abIJ}\gamma^{ab}\Psi - 2k^2\gamma_{IJ}\Psi \tag{231}$$

$$= -\frac{1}{4}(R_{abIJ} + 8k^2\eta_{aI}\eta_{bJ})\gamma^{ab}\Psi. \tag{232}$$

The generic assumption implied Φ^A are pointwise linearly independent at least at one point, p. Thus, the Ψ^A are also linearly independent at p.
However, then the Ψ^A are pointwise linearly independent everywhere (on Σ), as follows.
Let $\Psi = c_A\Psi^A$ for some constants, c_A.
$\therefore \nabla_I\Psi^A = 0 \implies \nabla_I\Psi = 0$.
Suppose $\Psi = 0$ at some point, q.
Choose a curve from q to p with tangent vector, t^I. Then, $t^I\nabla_I\Psi = 0$ is a homogeneous, linear,

[8] The boundary conditions for the Ψ^A PDE only have half the degrees of freedom of each Φ^A, so perhaps this conclusion is more non-trivial than I'm making it sound.

1st order ODE problem with initial condition, $\Psi|_q = 0$.
Since ODEs have unique local solutions on smooth manifolds, it must be that $\Psi = 0$ everywhere on the curve.
However, $\Psi = c_A\Psi^A = 0$ at p implies $c_A = 0$ by $\{\Psi^A\}_{A=1}^4$'s linear independence at p.
Hence, the Ψ^A are indeed pointwise linearly independent everywhere.
$\therefore$ Equation 232 implies $(R_{abIJ} + 8k^2\eta_{aI}\eta_{bJ})\gamma^{ab} = 0$.
Since $\{\gamma^{ab}\}$ are also linearly independent, it must be that $R_{abIJ} = -4k^2(\eta_{aI}\eta_{bJ} - \eta_{aJ}\eta_{bI})$.
It remains to be seen what happens for R_{ab0I}.
$R_{JK0I} = R_{0IJK} = -(\eta_{0J}\eta_{IK} - \eta_{0K}\eta_{IJ}) = 0$.
That leaves $R_{0J0I} = -R_{00IJ} - R_{0IJ0} = R_{0I0J}$.
Since a basis of Ψ is allowed, equation 194 also implies $T^{0a}\gamma_0\gamma_a = 0$. But, the eigenvalues of $T^{0a}\gamma_0\gamma_a = 0$ are $T^{00} \pm \sqrt{T^{0I}T^{0}{}_{I}}$, so it must be that T^{00} and T^{0I} are both zero, i.e. $T^{a0} = 0$.
By the dominant energy condition, $-T^a{}_bV^b$ is future directed and causal whenever V^a is future directed and causal.
Choose $V^a = \delta^{a0} + \delta^{aI}$ for some value of I.
$\therefore -T^a{}_bV^b = -T^a{}_0 - T^a{}_I = 0 - \delta^{aJ}T^{JI}$.
However, this can only be causal if $T^{IJ} = 0$.
$\therefore$ Ultimately, $T_{ab} = 0$.
$\therefore R_{ab} = \Lambda\eta_{ab} = -12k^2\eta_{ab}$.
$\therefore -12k^2\delta_{IJ} = R^a{}_{IaJ} = -R_{0I0J} + R^K{}_{IKJ} = -R_{0I0J} - 4k^2(\delta^K{}_K\delta_{IJ} - \delta^K{}_J\delta_{KI}) = -R_{0I0J} - 8k^2\delta_{IJ}$.
$\therefore R_{0I0J} = 4k^2\delta_{IJ}$.
$\therefore$ Putting all the components together, $R_{abcd} = -4k^2(\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc})$. □

More generally, one may wonder whether this converse applies if $m(S) = 0$, which could happen if Q^{AB} has just one non-trivial vector in its nullspace, rather than every vector being in the nullspace. This problem is considerably harder even in the asymptotically flat or asymptotically hyperbolic context - see [37, 38] for recent progress in those cases - and I will not consider it in this work.

My definition of quasilocal mass is closest in spirit to Penrose's definition, albeit I can make-do with spinors instead of twistors[9]. In particular, my Q^{AB} is analogous to his "kinematical twistor" - see the material around equation 23 in [20] - while my T^{AB} is analogous to his surface "infinity twistor" - see the discussion between equations 25 and 26 in [20]. Meanwhile, my definition is also closely related to the Dougan-Mason mass. When $\Lambda = 0$, the left-handed and right-handed sectors of all the equations decouple, meaning it suffices to simply set the right-handed sector to zero. Then, $A, B, \cdots$ only run $1, 2$. Thus, my T^{AB} can be normalised to ε^{AB} and one can use it to manipulate two-component spinors with Q^{AB} now viewed as $P^{\dot{A}A}$, a 4-momentum converted to two-component spinors. Then, my definition can be recast as

$$-256\pi^2 m(S)^2 = \operatorname{tr}\left(QT^{-1}\overline{Q}\overline{T}^{-1}\right) = Q^{AB}T_{BC}\overline{Q}^{CD}\overline{T}_{DA} \equiv P^{\dot{A}A}\varepsilon_{AB}P^{B\dot{B}}\varepsilon_{\dot{B}\dot{A}}, \qquad (233)$$

which is the Dougan-Mason mass (up to normalisation). However, since Dougan and Mason have a full energy-momentum vector, $P^{\dot{A}A}$, they are able to further decompose $m(S)$ into a quasilocal energy and quasilocal linear momentum. This decomposition is lost in my definition - as it is in Penrose's definition when S is away from $\mathcal{I}$. While the technical reason is simply that $A, B, \cdots$ run over four indices, instead of two, it remains to be seen whether there is a deeper physical reason for this discrepancy.

[9] In fact, as foreshadowed in [39], it is desirable to not use twistors if possible and this was one of the motivations behind constructing the Dougan-Mason mass.

5 Highly symmetric examples

For an arbitrary surface, S, the quasilocal mass of definition 4.11 will likely be very difficult, if not impossible, to calculate analytically. However, if the surface has a high degree of symmetry, then more progress can be made. In section 5.1 I'll study spherically symmetric spacetimes and show my definition reduces to the Misner-Sharp mass[10] [40] of such spacetimes[11]. Likewise, in section 5.2, I'll study toroidal symmetry, where it will turn out that a number of assumptions required for definition 4.11 don't hold. The canonical examples of spacetimes with such high symmetry are the Schwarzschild spacetime and its variations, described by the metric,

$$g = -\left(c - \frac{2M}{r} + 4k^2\right) \mathrm{d}t \otimes \mathrm{d}t + \frac{\mathrm{d}r \otimes \mathrm{d}r}{c - 2M/r + 4k^2} + r^2 g^{(c)}, \tag{234}$$

where $c = 1$, 0 or -1 and $g^{(c)}$ is the standard metric on the round 2-sphere, the 2-torus or a compactified 2D hyperbolic space respectively.

5.1 Spherical symmetry

Definition 5.1 (Spherical symmetry in double null coordinates)**.** *For any spherically symmetric spacetime, let r be the area-radius function and let u & v be null coordinates normal to the symmetry spheres, S_r^2. Then, in such "double null" coordinates, spherical symmetry dictates the metric is*

$$g = -\Omega(u,v)^2(\mathrm{d}u \otimes \mathrm{d}v + \mathrm{d}v \otimes \mathrm{d}u) + r(u,v)^2 g_{S^2}. \tag{235}$$

for some function, $\Omega(u,v)$. Without loss of generality assume u is outgoing and v is ingoing, i.e. $\partial_u r > 0$ and $\partial_v r < 0$.

Definition 5.2 (NP tetrad in spherical symmetry)**.** *For any S_r^2 in a spherically symmetric spacetime, choose the NP tetrad,*

$$l = \frac{1}{\Omega}\frac{\partial}{\partial u}, \;\; n = \frac{1}{\Omega}\frac{\partial}{\partial v} \;\text{ and }\; m = \frac{1}{r\sqrt{2}}\left(\frac{\partial}{\partial \theta} + \frac{\mathrm{i}}{\sin(\theta)}\frac{\partial}{\partial \phi}\right). \tag{236}$$

Lemma 5.3. *For the tetrad chosen in definition 5.2,*

$$\sigma = \lambda = 0, \;\; \rho = -\frac{\partial_u r}{\Omega r}, \;\; \mu = \frac{\partial_v r}{\Omega r} \;\text{ and }\; \beta = -\alpha = \frac{1}{2\sqrt{2}r}\cot(\theta). \tag{237}$$

Proof. The proof is to simply calculate each NP coefficient directly.

$$\sigma = -m^a \delta l_a \tag{238}$$
$$= -m^\mu m^\nu \partial_\nu l_\mu + m^\mu m^\nu \Gamma^\rho{}_{\mu\nu} l_\rho \tag{239}$$
$$= 0 + \frac{1}{2} m^\mu m^\nu l^\rho (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}) \;\text{ as } l_\mu \text{ doesn't depend on } \theta \text{ or } \phi \tag{240}$$
$$= -\frac{1}{2} l^\rho m^\mu m^\nu \partial_\rho g_{\mu\nu} \;\text{ as } g \text{ has no cross terms between } \mathrm{d}u \,\&\, \mathrm{d}v \text{ and } \mathrm{d}\theta \,\&\, \mathrm{d}\phi \tag{241}$$
$$= -\frac{1}{2\Omega}\left(\frac{1}{2r^2}\partial_u(r^2) + \frac{\mathrm{i}^2}{2r^2\sin^2(\theta)}\partial_u(r^2\sin^2(\theta))\right) \tag{242}$$
$$= 0. \tag{243}$$

[10] The Misner-Sharp mass is usually taken as the standard mass for spherically symmetric spacetimes [14].

[11] While this could appear to be merely a sanity check, in fact it is non-trivial. For example, the Brown-York mass [27] does not agree with the Misner-Sharp mass and in fact produces $m(S_r^2) = r(1 - \sqrt{1 - 2M/r})$ in the Schwarzschild spacetime (with $\Lambda = 0$) despite being physically very well-motivated.

Similarly, $\lambda = m^a \delta n_a = 0$ too. Also analogously,

$$\rho = -m^a \bar{\delta} l_a \tag{244}$$
$$= -\frac{1}{2} l^\rho m^\mu \overline{m}^\nu \partial_\rho g_{\mu\nu} \tag{245}$$
$$= -\frac{1}{2\Omega}\left(\frac{1}{2r^2}\partial_u(r^2) + \frac{\mathrm{i}(-\mathrm{i})}{2r^2\sin^2(\theta)}\partial_u(r^2\sin^2(\theta))\right) \tag{246}$$
$$= -\frac{\partial_u r}{\Omega r} \quad \text{and} \tag{247}$$
$$\mu = \overline{m}^a \delta n_a = \frac{1}{2} n^\rho \overline{m}^\mu m^\nu \partial_\rho g_{\mu\nu} = \frac{1}{2\Omega}\left(\frac{\partial_v(r^2)}{2r^2} + \frac{(-\mathrm{i})\mathrm{i}\partial_v(r^2\sin^2(\theta))}{2r^2\sin^2(\theta)}\right) = \frac{\partial_v r}{\Omega r}. \tag{248}$$

It remains to find α and β.

$$\beta = \frac{1}{2}(\overline{m}^a \delta m_a - n^a \delta l_a) \tag{249}$$
$$= \frac{1}{2}\left(\overline{m}^\mu m^\nu \partial_\nu m_\mu - \overline{m}^\mu m^\nu \Gamma^\rho{}_{\mu\nu} m_\rho - n^\mu m^\nu \partial_\nu l_\mu + n^\mu m^\nu \Gamma^\rho{}_{\mu\nu} l_\rho\right) \tag{250}$$
$$= \frac{1}{2}\left(\overline{m}^\mu m^\nu \partial_\nu m_\mu - \overline{m}^\mu m^\nu \Gamma^\rho{}_{\mu\nu} m_\rho - 0 + n^\mu m^\nu \Gamma^\rho{}_{\mu\nu} l_\rho\right) \quad \text{as } m^\nu \partial_\nu l_\mu = 0. \tag{251}$$

Consider each of these terms separately.

$$n^\mu m^\nu \Gamma^\rho{}_{\mu\nu} l_\rho = \frac{1}{2} l^\rho n^\mu m^\nu (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}) \tag{252}$$
$$= \frac{1}{2} l^\rho n^\mu m^\nu \partial_\nu g_{\rho\mu} \quad \text{as } g \text{ has no cross terms between } \mathrm{d}u \,\&\, \mathrm{d}v \text{ and } \mathrm{d}\theta \,\&\, \mathrm{d}\phi \tag{253}$$
$$= 0 \quad \text{as } g_{uv} \text{ doesn't depend on } \theta \text{ or } \phi. \tag{254}$$

$$\overline{m}^\mu m^\nu \partial_\nu m_\mu = \frac{1}{r\sqrt{2}} \overline{m}^\mu \partial_\theta m_\mu = \frac{1}{r\sqrt{2}} \frac{-\mathrm{i}}{r\sin(\theta)\sqrt{2}} \partial_\theta \left(\frac{\mathrm{i}}{r\sin(\theta)\sqrt{2}} r^2 \sin^2(\theta)\right) = \frac{\cot(\theta)}{2\sqrt{2}r}. \tag{255}$$

$$\overline{m}^\mu m^\nu \Gamma^\rho{}_{\mu\nu} m_\rho = \frac{1}{2} m^\rho \overline{m}^\mu m^\nu (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}) \tag{256}$$
$$= \frac{1}{2} m^\rho \overline{m}^\mu m^\nu \partial_\mu g_{\nu\rho} \tag{257}$$
$$= \frac{1}{2} m^\rho \frac{1}{r\sqrt{2}} m^\nu \partial_\theta g_{\nu\rho} \tag{258}$$
$$= \frac{1}{2} \frac{\mathrm{i}}{r\sin(\theta)\sqrt{2}} \frac{1}{r\sqrt{2}} \frac{\mathrm{i}}{r\sin(\theta)\sqrt{2}} \partial_\theta (r^2\sin^2(\theta)) \tag{259}$$
$$= -\frac{\cot(\theta)}{2r\sqrt{2}}. \tag{260}$$

Putting the different terms together, I get $\beta = \frac{1}{2r\sqrt{2}}\cot(\theta)$. Finally,

$$\alpha = \frac{1}{2}(\overline{m}^a \bar{\delta} m_a - n^a \bar{\delta} l_a) = \frac{1}{2}\overline{(m^a \delta \overline{m}_a - n^a \delta l_a)} = \frac{1}{2}\overline{(-\overline{m}^a \delta m_a - n^a \delta l_a)} = -\frac{\cot(\theta)}{2r\sqrt{2}} \tag{261}$$

using the calculations above for $\overline{m}^a \delta m_a$ and $n^a \delta l_a$. □

Corollary 5.3.1. *For Schwarzschild-AdS, the conditions imposed on θ_l and θ_n in definition 4.11 are equivalent to $r > 2M$.*

This result is somewhat mysterious because $r = 2M$ is no longer a special radius once a cosmological constant is added to the Schwarzschild metric.

Proof. By lemma 2.3, the conditions to check are $\mu, \rho < 0$ and $\mu\rho = -\frac{1}{4}\theta_l\theta_n > 2k^2$. These conditions are all invariant of the actual NP tetrad chosen. For the purpose of this corollary, it will be easier to swap the l and n used above for

$$l = \frac{1}{\sqrt{2}}\left(\frac{1}{f}\frac{\partial}{\partial t} + f\frac{\partial}{\partial r}\right) \quad \text{and} \quad n = \frac{1}{\sqrt{2}}\left(\frac{1}{f}\frac{\partial}{\partial t} - f\frac{\partial}{\partial r}\right), \tag{262}$$

where $f = \sqrt{1 - 2M/r + 4k^2r^2}$. Then, proceeding as in the main lemma yields

$$\rho = \mu = -\frac{f}{r\sqrt{2}}. \tag{263}$$

$$\therefore \mu\rho = \frac{f^2}{2r^2} = \frac{1 - 2M/r + 4k^2r^2}{2r^2}. \tag{264}$$

$\therefore$ The required condition on $\mu\rho$ is equivalent to $r > 2M$.
$\mu, \rho < 0$ is automatically satisfied when $r > 2M$. □

Lemma 5.4. *The general solution to $\overline{m}^a\nabla_a\Phi$ on S^2_r has*

$$a(\xi) = \bar{c}_1\left({}_{1/2}Y_{1/2,-1/2}\right) + \bar{c}_2\left({}_{1/2}Y_{1/2,1/2}\right), \tag{265}$$

$$b(\varphi) = c_3\left({}_{-1/2}Y_{1/2,-1/2}\right) + c_4\left({}_{-1/2}Y_{1/2,1/2}\right), \tag{266}$$

$$a(\varphi) = -\left(\frac{\sqrt{2}}{\Omega}\partial_v(r)c_3 + 2\mathrm{i}krc_2\right)\left({}_{1/2}Y_{1/2,-1/2}\right) - \left(\frac{\sqrt{2}}{\Omega}\partial_v(r)c_4 - 2\mathrm{i}krc_1\right)\left({}_{1/2}Y_{1/2,1/2}\right), \tag{267}$$

$$b(\xi) = \left(\frac{\sqrt{2}}{\Omega}\partial_u(r)\bar{c}_1 + 2\mathrm{i}kr\bar{c}_4\right)\left({}_{-1/2}Y_{1/2,-1/2}\right) + \left(\frac{\sqrt{2}}{\Omega}\partial_u(r)\bar{c}_2 - 2\mathrm{i}kr\bar{c}_3\right)\left({}_{-1/2}Y_{1/2,1/2}\right), \tag{268}$$

where c_A are arbitrary constants and $({}_sY_{jm})$ are spin-weighted spherical harmonics[12].

Proof. Since $a(\varphi)$ & $a(\xi)$ are type-$(0,-1)$ and $b(\varphi)$ & $b(\xi)$ are type-$(0,1)$ in the GHP formalism, lemma 5.3 implies the $\overline{m}^a\nabla_a\Phi = 0$ equations from lemma 4.3 reduce to

$$0 = \frac{1}{r\sqrt{2}}\left(\frac{\partial}{\partial\theta} - \frac{\mathrm{i}}{\sin(\theta)}\frac{\partial}{\partial\phi} + \frac{1}{2}\cot(\theta)\right)a(\varphi) + \frac{\partial_v r}{\Omega r}b(\varphi) - \mathrm{i}k\sqrt{2}\bar{a}(\xi), \tag{269}$$

$$0 = \frac{1}{r\sqrt{2}}\left(\frac{\partial}{\partial\theta} + \frac{\mathrm{i}}{\sin(\theta)}\frac{\partial}{\partial\phi} + \frac{1}{2}\cot(\theta)\right)b(\xi) + \frac{\partial_u r}{\Omega r}a(\xi) + \mathrm{i}k\sqrt{2}\bar{b}(\varphi), \tag{270}$$

$$0 = \frac{1}{r\sqrt{2}}\left(\frac{\partial}{\partial\theta} - \frac{\mathrm{i}}{\sin(\theta)}\frac{\partial}{\partial\phi} - \frac{1}{2}\cot(\theta)\right)b(\varphi) \quad \text{and} \tag{271}$$

$$0 = \frac{1}{r\sqrt{2}}\left(\frac{\partial}{\partial\theta} + \frac{\mathrm{i}}{\sin(\theta)}\frac{\partial}{\partial\phi} - \frac{1}{2}\cot(\theta)\right)a(\xi). \tag{272}$$

Let $\eth_s$ and $\bar{\eth}_s$ be differential operators that act on functions, F, by

$$\eth_s F = -(\sin(\theta))^s\left(\frac{\partial}{\partial\theta} + \frac{\mathrm{i}}{\sin(\theta)}\frac{\partial}{\partial\phi}\right)\left((\sin(\theta)^{-s}F\right) \tag{273}$$

$$= s\cot(\theta)F - \left(\frac{\partial}{\partial\theta} + \frac{\mathrm{i}}{\sin(\theta)}\frac{\partial}{\partial\phi}\right)F \quad \text{and} \tag{274}$$

$$\bar{\eth}_s F = -(\sin(\theta))^{-s}\left(\frac{\partial}{\partial\theta} - \frac{\mathrm{i}}{\sin(\theta)}\frac{\partial}{\partial\phi}\right)\left((\sin(\theta)^{s}F\right) \tag{275}$$

$$= -s\cot(\theta)F - \left(\frac{\partial}{\partial\theta} - \frac{\mathrm{i}}{\sin(\theta)}\frac{\partial}{\partial\phi}\right)F. \tag{276}$$

[12]The exact expressions for the four spin-weighted spherical harmonics I'll need are listed in appendix A.

∴ The $\overline{m}^a\nabla_a\Phi = 0$ equations above can be written as

$$
\begin{aligned}
0 &= \bar{\eth}_{1/2}a(\varphi) - \frac{\sqrt{2}}{\Omega}\partial_v(r)b(\varphi) + 2\mathrm{i}rk\bar{a}(\xi), && (277)\\
0 &= \eth_{-1/2}b(\xi) - \frac{\sqrt{2}}{\Omega}\partial_u(r)a(\xi) - 2\mathrm{i}rk\bar{b}(\varphi), && (278)\\
0 &= \bar{\eth}_{-1/2}b(\varphi) \text{ and} && (279)\\
0 &= \eth_{1/2}a(\xi). && (280)
\end{aligned}
$$

The spin-weighted spherical harmonics, $({}_sY_{jm})$, are known [31] to be eigenfunctions of $\eth_s$ and $\bar{\eth}_s$; in particular

$$
\begin{aligned}
\eth_s\left({}_sY_{jm}\right) &= \sqrt{(j-s)(j+s+1)}\left({}_{s+1}Y_{jm}\right), && (281)\\
\bar{\eth}_s\left({}_sY_{jm}\right) &= -\sqrt{(j+s)(j-s+1)}\left({}_{s-1}Y_{jm}\right) \text{ and} && (282)\\
\overline{\left({}_sY_{jm}\right)} &= (-1)^{s+m}\left({}_{-s}Y_{j(-m)}\right). && (283)
\end{aligned}
$$

Furthermore, they form a complete basis for expanding functions on the round sphere.
∴ It immediately follows that the solutions to equations 279 and 280 are

$$
\begin{aligned}
b(\varphi) &= c_3\left({}_{-1/2}Y_{1/2,-1/2}\right) + c_4\left({}_{-1/2}Y_{1/2,1/2}\right) \text{ and} && (284)\\
a(\xi) &= \bar{c}_1\left({}_{1/2}Y_{1/2,-1/2}\right) + \bar{c}_2\left({}_{1/2}Y_{1/2,1/2}\right) && (285)
\end{aligned}
$$

for some constants, c_1, c_2, c_3 and c_4.
Substituting these into equations 277 and 278 then says

$$
\begin{aligned}
\bar{\eth}_{1/2}a(\varphi) &= \left(\frac{\sqrt{2}}{\Omega}\partial_v(r)c_3 + 2\mathrm{i}krc_2\right)\left({}_{-1/2}Y_{1/2,-1/2}\right)\\
&\quad + \left(\frac{\sqrt{2}}{\Omega}\partial_v(r)c_4 - 2\mathrm{i}krc_1\right)\left({}_{-1/2}Y_{1/2,1/2}\right) \text{ and} && (286)\\
\eth_{-1/2}b(\xi) &= \left(\frac{\sqrt{2}}{\Omega}\partial_u(r)\bar{c}_1 + 2\mathrm{i}kr\bar{c}_4\right)\left({}_{1/2}Y_{1/2,-1/2}\right)\\
&\quad + \left(\frac{\sqrt{2}}{\Omega}\partial_u(r)\bar{c}_2 - 2\mathrm{i}kr\bar{c}_3\right)\left({}_{1/2}Y_{1/2,1/2}\right). && (287)
\end{aligned}
$$

The claimed expressions for $a(\varphi)$ and $b(\xi)$ then follow by once again applying the completeness and eigenfunction properties (under $\eth_s$ and $\bar{\eth}_s$) of spin-weighted spherical harmonics. □

Definition 5.5 (Misner-Sharp mass). *Including a cosmological constant, the Misner-Sharp mass for spherically symmetric spacetimes is defined to be [40]*

$$
m_{MS}(S_r^2) = \frac{r}{2}\left(1 + 4k^2r^2 - (\eta^{ab} - \beta^{ab})D_a(r)D_b(r)\right). \tag{288}
$$

Theorem 5.6. *$m(S_r^2)$ agrees with the Misner-Sharp mass (with cosmological constant) for spherically symmetric spacetimes.*

Proof. Taking the four c_A to be the coefficients multiplying the four linearly independent solutions, from computer algebra it follows that

$$Q^{AB} \equiv \frac{4r(2\partial_u(r)\partial_v(r)+\Omega^2(1+4k^2r^2))}{\Omega^3}\begin{bmatrix} \partial_u r & 0 & 0 & -\mathrm{i}k\Omega r\sqrt{2} \\ 0 & \partial_u r & \mathrm{i}k\Omega r\sqrt{2} & 0 \\ 0 & -\mathrm{i}k\Omega r\sqrt{2} & -\partial_v r & 0 \\ \mathrm{i}k\Omega r\sqrt{2} & 0 & 0 & -\partial_v r \end{bmatrix}, \tag{289}$$

$$T^{AB} \equiv \frac{1}{\pi\Omega}\begin{bmatrix} 0 & -\partial_u r & -\mathrm{i}k\Omega r\sqrt{2} & 0 \\ \partial_u r & 0 & 0 & -\mathrm{i}k\Omega r\sqrt{2} \\ \mathrm{i}k\Omega r\sqrt{2} & 0 & 0 & -\partial_v r \\ 0 & \mathrm{i}k\Omega r\sqrt{2} & \partial_v r & 0 \end{bmatrix} \text{ and hence} \tag{290}$$

$$T^{-1} = \frac{\pi\Omega}{\partial_u(r)\partial_v(r)+2k^2\Omega^2r^2}\begin{bmatrix} 0 & \partial_v r & -\mathrm{i}k\Omega r\sqrt{2} & 0 \\ -\partial_v r & 0 & 0 & -\mathrm{i}k\Omega r\sqrt{2} \\ \mathrm{i}k\Omega r\sqrt{2} & 0 & 0 & \partial_u r \\ 0 & \mathrm{i}k\Omega r\sqrt{2} & -\partial_u r & 0 \end{bmatrix}. \tag{291}$$

Then, again kneeling at the altar of the computer for matrix algebra yields

$$m(S^2_r) = \frac{1}{16\pi}\sqrt{-\operatorname{tr}\left(QT^{-1}\overline{Q}\overline{T}^{-1}\right)} = \frac{r}{2}\left(\frac{2}{\Omega^2}\partial_u(r)\partial_v(r)+1+4k^2r^2\right), \tag{292}$$

which is the Misner-Sharp mass in double null coordinates (note the Misner-Sharp mass is manifestly coordinate independent). ☐

Corollary 5.6.1. *For the Schwarzschild-AdS spacetime, $m(S^2_r)$ coincides with the mass parameter, M, in the metric.*

Proof. The Misner-Sharp mass for Schwarzschild-AdS is most easily calculated in the standard (t, r, θ, ϕ) coordinates instead of double null coordinates. Hence,

$$m(S^2_r) = \frac{r}{2}\left(1+4k^2+\frac{1}{1+4k^2-2M/r}\partial_t(r)^2-(1+4k^2-2M/r)\partial_r(r)^2\right) = M, \tag{293}$$

as expected. ☐

5.2 Toroidal symmetry

In this section I'll consider some toroidal examples. I'll start with the toroidal Kottler spacetime, i.e. the $c = 0$ case in equation 234 and then consider the AdS soliton [41]. It will turn out that my quasilocal mass construction is not possible in either example.

Definition 5.7 (Toroidal Kottler)**.** *The domain of outer communication of the toroidal Kottler spacetime is $\mathbb{R} \times [r_0, \infty) \times \mathbb{T}^2$ with the metric,*

$$g = -f(r)^2\mathrm{d}t\otimes\mathrm{d}t + \frac{\mathrm{d}r\otimes\mathrm{d}r}{f(r)^2} + r^2(\mathrm{d}\theta\otimes\mathrm{d}\theta+\mathrm{d}\phi\otimes\mathrm{d}\phi), \tag{294}$$

$$\text{where } f(r) = \sqrt{-\frac{2M}{r}+4k^2r^2} \tag{295}$$

and (θ, ϕ) are coordinates on each $\mathbb{T}^2 = S^1 \times S^1$.

In this subsection, I'll always choose S to be the "radius"-r surface, $\mathbb{T}^2_r$.

Definition 5.8 (NP tetrad for toroidal Kottler)**.** *For any $\mathbb{T}^2_r$ in toroidal Kottler, choose the NP tetrad,*

$$l = \frac{1}{\sqrt{2}}\left(\frac{1}{f}\frac{\partial}{\partial t} + f\frac{\partial}{\partial r}\right), \; n = \frac{1}{\sqrt{2}}\left(\frac{1}{f}\frac{\partial}{\partial t} - f\frac{\partial}{\partial r}\right) \text{ and } m = \frac{1}{r\sqrt{2}}\left(\frac{\partial}{\partial \theta} + \mathrm{i}\frac{\partial}{\partial \phi}\right). \tag{296}$$

Lemma 5.9. *For the tetrad chosen in definition 5.8,*

$$\sigma = \lambda = \alpha = \beta = 0 \text{ and } \rho = \mu = -\frac{f}{r\sqrt{2}}. \tag{297}$$

Proof. Follow exactly analogous steps to lemma 5.3. □

Corollary 5.9.1. *The $\theta_l\theta_n < -8k^2$ assumption never holds.*

Proof. From lemma 2.3, the $\theta_{l,n}$ conditions reduce to $f > 2rk$ (as they did for Schwarzschild-AdS). However, unlike Schwarzschild-AdS, because $f = \sqrt{-2M/r + 4k^2r^2} < \sqrt{4k^2r^2} = 2kr$ this condition never holds. □

Corollary 5.9.2. *The $\overline{m}^a\nabla_a\Phi = 0$ equations from lemma 4.3 reduce to*

$$0 = \frac{1}{r\sqrt{2}}\left(\frac{\partial}{\partial \theta} - \mathrm{i}\frac{\partial}{\partial \phi}\right)a(\varphi) - \frac{f}{r\sqrt{2}}b(\varphi) - \mathrm{i}k\sqrt{2}\bar{a}(\xi), \tag{298}$$

$$0 = \frac{1}{r\sqrt{2}}\left(\frac{\partial}{\partial \theta} + \mathrm{i}\frac{\partial}{\partial \phi}\right)b(\xi) + \frac{f}{r\sqrt{2}}a(\xi) + \mathrm{i}k\sqrt{2}\bar{b}(\varphi), \tag{299}$$

$$0 = \frac{1}{r\sqrt{2}}\left(\frac{\partial}{\partial \theta} - \mathrm{i}\frac{\partial}{\partial \phi}\right)b(\varphi) \text{ and} \tag{300}$$

$$0 = \frac{1}{r\sqrt{2}}\left(\frac{\partial}{\partial \theta} + \mathrm{i}\frac{\partial}{\partial \phi}\right)a(\xi). \tag{301}$$

Proof. Direct substitution, with $a(\varphi)$ & $a(\xi)$ being type-$(0,-1)$ and $b(\varphi)$ & $b(\xi)$ being type-$(0,1)$ in the GHP formalism. □

Theorem 5.10. *The general solution to $\overline{m}^a\nabla_a\Phi = 0$ on $\mathbb{T}^2_r$ has*

$$a(\varphi) = c_1, \; a(\xi) = 0, \; b(\varphi) = 0 \text{ and } b(\xi) = \bar{c}_2, \tag{302}$$

where c_1 and c_2 are arbitrary constants.

Note that the Φ^A form of generic immediately fails because there are only two linearly independent solutions, not four.

Proof. Let $z = \theta - \mathrm{i}\phi$ define the complex variable on the torus.
$\therefore \theta = \frac{1}{2}(z + \bar{z})$ and $\phi = \frac{1}{2\mathrm{i}}(\bar{z} - z)$.

$$\therefore \frac{\partial}{\partial z} = \frac{\partial \theta}{\partial z}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial z}\frac{\partial}{\partial \phi} = \frac{1}{2}\left(\frac{\partial}{\partial \theta} + \mathrm{i}\frac{\partial}{\partial \phi}\right) \text{ and} \tag{303}$$

$$\frac{\partial}{\partial \bar{z}} = \frac{\partial \theta}{\partial \bar{z}}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial \bar{z}}\frac{\partial}{\partial \phi} = \frac{1}{2}\left(\frac{\partial}{\partial \theta} - \mathrm{i}\frac{\partial}{\partial \phi}\right). \tag{304}$$

$\therefore$ The equations in corollary 5.9.2 can be re-written as

$$0 = 2\partial_{\bar{z}}a(\varphi) - fb(\varphi) - 2\mathrm{i}kr\bar{a}(\xi), \tag{305}$$

$$0 = 2\partial_z b(\xi) + fa(\xi) + 2\mathrm{i}kr\bar{b}(\varphi), \tag{306}$$

$$0 = \partial_{\bar{z}}b(\varphi) \text{ and} \tag{307}$$

$$0 = \partial_z a(\xi). \tag{308}$$

$\therefore$ By Liouville's theorem, $b(\varphi)$ and $a(\xi)$ must be constants, say c_3 and $\bar{c}_4$.
$\therefore$ The remaining two equations become

$$\partial_{\bar{z}}a(\varphi) = \frac{1}{2}(fc_3 + 2\mathrm{i}c_4kr) \text{ and } \partial_z b(\xi) = -\frac{1}{2}(f\bar{c}_4 + 2\mathrm{i}\bar{c}_3kr). \tag{309}$$

Since r is also just a constant on $\mathbb{T}^2_r$, the equations can be immediately integrated to

$$a(\varphi) = \frac{1}{2}(fc_3 + 2\mathrm{i}c_4kr)\bar{z} + c_1(z) \text{ and } b(\xi) = -\frac{1}{2}(f\bar{c}_4 + 2\mathrm{i}\bar{c}_3kr)z + \bar{c}_2(\bar{z}) \tag{310}$$

for some holomorphic functions, c_1 and c_2.
However, by Liouville's theorem, c_1 and c_2 must be constants.
Furthermore, $\mathbb{T}^2_r$ has 2π periodicity in the θ and ϕ coordinates which neither $(fc_3 + 2\mathrm{i}c_4kr)\bar{z}$ nor $(f\bar{c}_4 + 2\mathrm{i}\bar{c}_3kr)z$ do.
$\therefore fc_3 + 2\mathrm{i}c_4kr = 0$ and $f\bar{c}_4 + 2\mathrm{i}\bar{c}_3kr = 0$.
$\therefore c_3 = -\frac{2\mathrm{i}kr}{f}c_4$ and $\left(f - \frac{4k^2r^2}{f}\right)\bar{c}_4 = 0$.
Since $f^2 < 4k^2r^2$, the only solution is $c_3 = c_4 = 0$. ☐

Corollary 5.10.1. *Taking the two linearly independent solutions to be* $(a(\varphi), b(\varphi), \bar{a}(\xi), \bar{b}(\xi)) = (1, 0, 0, 0)$ *and* $(a(\varphi), b(\varphi), \bar{a}(\xi), \bar{b}(\xi)) = (0, 0, 0, 1)$*, it follows that*

$$Q^{AB} = 8\sqrt{2}\pi^2 r \begin{bmatrix} -f & -2\mathrm{i}kr \\ 2\mathrm{i}kr & -f \end{bmatrix} \text{ and } T^{AB} = 0. \tag{311}$$

Proof. By definition,

$$\begin{aligned} &Q^{AB} \\ &= 4\int_S \Big(\rho\bar{a}(\varphi^A)a(\varphi^B) + \mu b(\xi^A)\bar{b}(\xi^B) - \rho a(\xi^A)\bar{a}(\xi^B) - \mu\bar{b}(\varphi^A)b(\varphi^B) \\ &\qquad + \mathrm{i}k\sqrt{2}\big(b(\xi^A)a(\varphi^B) - \bar{a}(\varphi^A)\bar{b}(\xi^B) - a(\xi^A)b(\varphi^B) + \bar{b}(\varphi^A)\bar{a}(\xi^B)\big)\Big)\mathrm{d}A \qquad (312) \\ &= 4(2\pi r)^2\left(-\frac{f}{r\sqrt{2}}\left(\bar{a}(\varphi^A)a(\varphi^B) + b(\xi^A)\bar{b}(\xi^B)\right) + \mathrm{i}k\sqrt{2}\left(b(\xi^A)a(\varphi^B) - \bar{a}(\varphi^A)\bar{b}(\xi^B)\right)\right) \qquad (313) \\ &= 8\sqrt{2}\pi^2 r\left(-f\left(\bar{a}(\varphi^A)a(\varphi^B) + b(\xi^A)\bar{b}(\xi^B)\right) + 2\mathrm{i}kr\left(b(\xi^A)a(\varphi^B) - \bar{a}(\varphi^A)\bar{b}(\xi^B)\right)\right) \qquad (314) \\ &\equiv 8\sqrt{2}\pi^2 r\begin{bmatrix} -f & -2\mathrm{i}kr \\ 2\mathrm{i}kr & -f \end{bmatrix}. \qquad (315) \end{aligned}$$

Meanwhile, $T^{AB} = \sqrt{2}\left(a(\varphi^A)b(\varphi^B) - a(\varphi^B)b(\varphi^A) + \bar{a}(\xi^B)\bar{b}(\xi^A) - \bar{a}(\xi^A)\bar{b}(\xi^B)\right)$.
$\therefore T^{AB} = 0$ because both solutions have $b(\varphi) = a(\xi) = 0$. ☐

Since $T^{AB} = 0$, the T^{AB} form of generic fails too and $m(\mathbb{T}^2_r)$ cannot be defined using the prescription I've developed in this work. The wider implications of this example are as yet unclear; it is possible my definition simply doesn't work for most non-spherical surfaces.

The next example dampens optimism further.

Definition 5.11 (AdS soliton)**.** *The AdS soliton is defined to be the spacetime,* $\mathbb{R}\times[r_0,\infty)\times\mathbb{T}^2$*, with the metric,*

$$g = -r^2\mathrm{d}\tau\otimes\mathrm{d}\tau + \frac{\mathrm{d}r\otimes\mathrm{d}r}{f(r)^2} + f(r)^2\mathrm{d}\omega\otimes\mathrm{d}\omega + r^2\mathrm{d}\phi\otimes\mathrm{d}\phi, \tag{316}$$

$$\text{where } f(r) = \sqrt{-\frac{2M}{r} + 4k^2r^2}, \tag{317}$$

r_0 *is the solution to* $f(r_0) = 0$ *and* (ω, ϕ) *are coordinates on the* $\mathbb{T}^2$*.* ϕ *is taken to be* 2π *periodic while* $\omega \sim \omega + \pi/3k^2r_0$*.*

This spacetime is constructed as per the procedure in [41]. In particular, start with the toroidal Kottler metric and define new coordinates, $\tau = \mathrm{i}\theta$ and $\omega = \mathrm{i}t$. Analytically continue the coordinates so that τ & ω are real, unwrap the the τ coordinate so $\tau \in \mathbb{R}$ and compactify the ω coordinate so that (ω, ϕ) are coordinates on a torus. The periodicity of ω is chosen so as to avoid a conical singularity at $r = r_0$, although I won't need the actual periodicity for what follows. Since the metric is found just by analytic continuation, the vacuum Einstein equation continues to be satisfied. Note that there is no longer any black hole; this really is a soliton.

Definition 5.12 (NP tetrad for the AdS soliton)**.** *For any $\mathbb{T}^2_r$ in the AdS soliton, choose the NP tetrad,*

$$l = \frac{1}{\sqrt{2}}\left(\frac{1}{r}\frac{\partial}{\partial \tau} + f\frac{\partial}{\partial r}\right), \ n = \frac{1}{\sqrt{2}}\left(\frac{1}{r}\frac{\partial}{\partial \tau} - f\frac{\partial}{\partial r}\right) \text{ and } m = \frac{1}{\sqrt{2}}\left(\frac{1}{f}\frac{\partial}{\partial \omega} + \frac{\mathrm{i}}{r}\frac{\partial}{\partial \phi}\right). \tag{318}$$

Lemma 5.13. *For the tetrad chosen in definition 5.12,*

$$\alpha = \beta = 0, \ \sigma = \lambda = -\frac{3M}{2\sqrt{2}r^2 f} \text{ and } \rho = \mu = -\frac{8k^2r^3 - M}{2\sqrt{2}r^2 f} = -\frac{f^2 + 12k^2r^2}{4\sqrt{2}rf}. \tag{319}$$

Proof. Follow exactly analogous steps to lemma 5.3. □

Corollary 5.13.1. *The θ_l and θ_n assumptions always hold.*

Proof. Since $\mu = \rho < 0$, by lemma 2.3, all I need to show is $\mu^2 > 2k^2$. Observe that

$$0 < 64k^2rM + \frac{4M^2}{r^2}. \tag{320}$$

$$\therefore 160k^4r^4 - 80k^2rM < -16k^2rM + \frac{4M^2}{r^2} + 160k^4r^4 \tag{321}$$

$$\therefore 40k^2r^2\left(-\frac{2M}{r} + 4k^2r^2\right) < \left(-\frac{2M}{r} + 4k^2r^2\right)^2 + 144k^4r^4 \tag{322}$$

$$\therefore 40k^2r^2f^2 < f^4 + 144k^4r^4 \tag{323}$$

$$\therefore 64k^2r^2f^2 < f^4 + 144k^4r^4 + 24k^2r^2f^2 = (f^2 + 12k^2r^2)^2 \tag{324}$$

$$\therefore 2k^2 < \frac{(f^2 + 12k^2r^2)^2}{32r^2f^2} = \mu^2 \tag{325}$$

as required. □

Theorem 5.14. *The only solution to $\overline{m}^a\nabla_a\Phi = 0$ on $\mathbb{T}^2_r$ is $\Phi = 0$.*

Proof. Package the GHP components of Φ into a vector, $[a(\varphi), b(\varphi), \bar{a}(\xi), \bar{b}(\xi)]^T$. Then, with the NP coefficients calculated, the equations of lemma 4.3 become $\overline{m}^\mu\partial_\mu v = Av$, where

$$A = \begin{bmatrix} 0 & -\mu & \mathrm{i}k\sqrt{2} & 0 \\ \bar{\sigma} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\lambda \\ 0 & \mathrm{i}k\sqrt{2} & \rho & 0 \end{bmatrix} = \frac{1}{2\sqrt{2}r^2 f}\begin{bmatrix} 0 & 8k^2r^3 - M & 4\mathrm{i}kr^2f & 0 \\ -3M & 0 & 0 & 0 \\ 0 & 0 & 0 & 3M \\ 0 & 4\mathrm{i}kr^2f & -(8k^2r^3 - M) & 0 \end{bmatrix} \tag{326}$$

is effectively a constant matrix on $\mathbb{T}^2_r$.
$\{m, \overline{m}\}$ induces a complex structure on $\mathbb{T}^2_r$. Choose a corresponding complex coordinate, $z = \frac{1}{\sqrt{2}}(f\omega - \mathrm{i}r\phi)$, so that $m^\mu\partial_\mu = \partial_z$ and $\overline{m}^\mu\partial_\mu = \partial_{\bar{z}}$.
$\therefore$ The equation to solve is $\partial_{\bar{z}}v = Av$. Integrating immediately yields

$$v = \mathrm{e}^{\bar{z}A}c(z) \tag{327}$$

for some holomorphic vector, $c(z)$.
However, by Liouville's theorem, $c(z)$ must be a constant vector, c. But, then v would be a globally defined, non-constant, antiholomorphic vector on the compact space, $\mathbb{T}^2_r$, contradicting Liouville's theorem.
The only way around this is to have $c \in \text{nullspace}(A)$, so that the $\bar{z}$ dependence falls out[13] in $v = \mathrm{e}^{\bar{z}A}c$. However, from computer algebra,

$$\det(A) = (\mu^2 - 2k^2)\lambda^2 > 0. \tag{328}$$

$\therefore$ The only solution is $v = 0$ and $\overline{m}^a \nabla_a \Phi = 0$ has no non-trivial solutions. □

6 Asymptotic limit

The next criterion I'll check for a good quasilocal mass definition is the asymptotic limit at $\mathcal{I}$. In this section, it will be convenient to set the "AdS length scale," to one[14]. Equivalently, one would choose units such that $\Lambda = -3$ and $k = 1/2$. The length scales can always be restored on dimensional grounds.

Definition 6.1 (Asymptotically AdS)**.** *A spacetime, (M, g), is said to be asymptotically AdS if and only if only if $\exists$ coordinates, $(r, x^m) = (r, t, \theta^\alpha)$, in an open neighbourhood of the "boundary" at infinity[15] such that $\{r = \infty\}$ is the "boundary" itself, constant r and t surfaces are diffeomorphic to S^2 and g admits a Fefferman-Graham expansion [42],*

$$\begin{aligned} g = \mathrm{e}^{2r}\Bigg(- \left(1 + \frac{1}{4}\mathrm{e}^{-2r}\right)^2 \mathrm{d}t \otimes \mathrm{d}t + \left(1 - \frac{1}{4}\mathrm{e}^{-2r}\right)^2 g_{S^2} \\ + \mathrm{e}^{-3r} f_{(3)mn}\, \mathrm{d}x^m \otimes \mathrm{d}x^n + O(\mathrm{e}^{-4r})\Bigg) + \mathrm{d}r \otimes \mathrm{d}r. \end{aligned} \tag{329}$$

Lemma 6.2. *The metric on AdS can also be written as*[16]

$$\begin{aligned} g_{\text{AdS}} &= -\left(\frac{1+\rho^2}{1-\rho^2}\right)^2 \mathrm{d}t \otimes \mathrm{d}t + \frac{4}{(1-\rho^2)^2}\delta_{IJ}\mathrm{d}x^I \otimes \mathrm{d}x^J & (330) \\ &= -\left(\frac{1+\rho^2}{1-\rho^2}\right)^2 \mathrm{d}t \otimes \mathrm{d}t + \frac{4}{(1-\rho^2)^2}\mathrm{d}\rho \otimes \mathrm{d}\rho + \frac{4\rho^2}{(1-\rho^2)^2} g_{S^2}. & (331) \end{aligned}$$

Then, with the tetrad,

$$e_0 = \frac{1-\rho^2}{1+\rho^2}\partial_t \ \text{ and } \ e_I = \frac{1-\rho^2}{2}\partial_I, \tag{332}$$

the Killing spinors of AdS can be written as

$$\varepsilon_k = \frac{1}{\sqrt{1-\rho^2}}\left(I - \mathrm{i}x_I\gamma^I\right)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0, \tag{333}$$

with ε_0 an arbitrary spinor that's constant with respect to the chosen tetrad.

[13] In fact, the toroidal Schwarzschild-AdS example earlier can be analysed in exactly this way. Since $\sigma = \lambda = 0$ in that example, the analogue of A has two rows of zeroes, which then yield a 2D nullspace and the two constant solutions in equation 302.

[14] The real reason of course being that this is what is usually done in the asymptotically locally AdS literature and I can't be bothered tracking where all the factors of k should go.

[15] First of all, such a notion of "boundary" at infinity should exist on (M, g).

[16] Note that in this context ρ equals $\sqrt{x_I x^I}$ and not the ρ in the NP formalism, which will not appear explicitly.

Proof. See [13]. □

Definition 6.3 ("Conserved" quantities)**.** *In an asymptotically AdS spacetime, define the energy, linear momentum, angular momentum and centre of mass position as*

$$E = \frac{3}{16\pi}\int_{S^2_\infty} s^{\alpha\beta} f_{(3)\alpha\beta}\mathrm{d}(g_{S^2}), \tag{334}$$

$$P_I = \frac{3}{16\pi}\int_{S^2_\infty} s^{\alpha\beta} f_{(3)\alpha\beta}\hat{x}_I\mathrm{d}(g_{S^2}), \tag{335}$$

$$J_{IJ} = \frac{3}{16\pi}\int_{S^2_\infty} f_{(3)0\alpha}\left(\hat{x}_I\frac{\partial\theta^\alpha}{\partial x^J}\bigg|_{\rho=1} - \hat{x}_J\frac{\partial\theta^\alpha}{\partial x^I}\bigg|_{\rho=1}\right)\mathrm{d}(g_{S^2}) \text{ and} \tag{336}$$

$$K_I = \frac{3}{16\pi}\int_{S^2_\infty} f_{(3)0\alpha}\frac{\partial\theta^\alpha}{\partial x^J}\bigg|_{\rho=1}\left(\delta^J{}_I - \hat{x}^J\hat{x}_I\right)\mathrm{d}(g_{S^2}) \tag{337}$$

respectively. In these expressions, θ^α denote local coordinates on S^2, $s^{\alpha\beta}$ is the inverse of the round metric on S^2, S^2_∞ is the sphere at infinity, $\hat{x}^I$ denote unit vector Cartesian coordinates and $\rho = \sqrt{x_I x^I}$, i.e. $x^I = \rho\hat{x}^I$.

These definitions are explained in greater detail in [13].

Theorem 6.4. *When $S = S^2_\infty$, i.e. the sphere at infinity in an asymptotically AdS spacetime, $Q(\Phi) = Q(\varepsilon_k)$, where ε_k is a Killing spinor of AdS.*

Proof. AdS itself has four linearly independent solutions to $\nabla_a\Phi = 0$, namely the the 4D space of Killing spinors, ε_k.
$\therefore$ In AdS, the 4D space of solutions to $\overline{m}^a\nabla_a\Phi = 0$ can be spanned by the Killing vectors themselves.
By definition 6.1, the difference between g and g_{AdS} is $O(\mathrm{e}^{-3r})$.
$\therefore$ In the asymptotic region of (M, g), $\Phi = \varepsilon_k + \mathcal{Z}$ for some $\mathcal{Z}$ that's $O(\mathrm{e}^{-3r})$ below leading order. Equating $\frac{4}{(1-\rho^2)^2}\mathrm{d}\rho\otimes\mathrm{d}\rho$ with $\mathrm{d}r\otimes\mathrm{d}r$ in lemma 6.2 shows ε_k is $O(\mathrm{e}^{r/2})$ and thus $\mathcal{Z}$ must be $O(\mathrm{e}^{-5r/2})$.
In the context of Fefferman-Graham expansions, I'll work in a vielbein where $P_a = -\delta_{a0}$ and $Q_a = \delta_{a1} \equiv \mathrm{d}r$.

$$\therefore Q(\Phi) = \int_{S^2_\infty} P_a Q_b E^{ba}(\Phi)\mathrm{d}A \tag{338}$$

$$= \int_{S^2_\infty} E^{01}(\Phi)\mathrm{d}A \tag{339}$$

$$= \int_{S^2_\infty} \left(\Phi^\dagger\gamma^1\gamma^A\nabla_A\Phi + \nabla_A(\Phi)^\dagger\gamma^A\gamma^1\Phi\right)\mathrm{d}A \tag{340}$$

$$= Q(\varepsilon_k) + \int_{S^2_\infty}\big(\mathcal{Z}^\dagger\gamma^1\gamma^A\nabla_A\varepsilon_k + \varepsilon_k^\dagger\gamma^1\gamma^A\nabla_A\mathcal{Z} + \mathcal{Z}^\dagger\gamma^1\gamma^A\nabla_A\mathcal{Z} + \nabla_A(\varepsilon_k)^\dagger\gamma^A\gamma^1\mathcal{Z}$$
$$+ \nabla_A(\mathcal{Z})^\dagger\gamma^A\gamma^1\varepsilon_k + \nabla_A(\mathcal{Z})^\dagger\gamma^A\gamma^1\mathcal{Z}\big)\mathrm{d}A. \tag{341}$$

From equation 329, $\mathrm{d}A$ is $O(\mathrm{e}^{2r})$.
$\therefore \mathcal{Z}^\dagger\gamma^1\gamma^A\nabla_A\mathcal{Z}\mathrm{d}A$ and $\nabla_A(\mathcal{Z})^\dagger\gamma^A\gamma^1\mathcal{Z}\mathrm{d}A$ are both $O(\mathrm{e}^{-3r})$ and go to zero as $r\to\infty$.
From [13], $\nabla_A\varepsilon_k$ is also $O(\mathrm{e}^{-5r/2})$ in the asymptotic region.
$\therefore \mathcal{Z}^\dagger\gamma^1\gamma^A\nabla_A\varepsilon_k\mathrm{d}A$ and $\nabla_A(\varepsilon_k)^\dagger\gamma^A\gamma^1\mathcal{Z}\mathrm{d}A$ similarly contribute nothing.

$$\therefore Q(\Phi) = Q(\varepsilon_k) + \int_{S^2_\infty}\left(\varepsilon_k^\dagger\gamma^1\gamma^A\nabla_A\mathcal{Z} + \nabla_A(\mathcal{Z})^\dagger\gamma^A\gamma^1\varepsilon_k\right)\mathrm{d}A. \tag{342}$$

The 2nd term is the complex conjugate of the first so it suffices to prove the 1st term integrates to zero. Begin by re-writing the integrand as follows.

$$\varepsilon_k^\dagger\gamma^1\gamma^A\nabla_A\mathcal{Z} = \varepsilon_k^\dagger\gamma^1\gamma^A D_A\mathcal{Z} + \frac{\mathrm{i}}{2}\varepsilon_k^\dagger\gamma^1\gamma^A\gamma_A\mathcal{Z} \tag{343}$$

$$= D_A(\varepsilon_k^\dagger\gamma^1\gamma^A\mathcal{Z}) - D_A(\varepsilon_k)^\dagger\gamma^1\gamma^A\mathcal{Z} - \mathrm{i}\varepsilon_k^\dagger\gamma^1\mathcal{Z} \tag{344}$$

$$= D_A(\varepsilon_k^\dagger\gamma^1\gamma^A\mathcal{Z}) - \nabla_A(\varepsilon_k)^\dagger\gamma^1\gamma^A\mathcal{Z} - \left(-\frac{\mathrm{i}}{2}\gamma_A\varepsilon_k\right)^\dagger\gamma^1\gamma^A\mathcal{Z} - \mathrm{i}\varepsilon_k^\dagger\gamma^1\mathcal{Z} \tag{345}$$

$$= D_A(\varepsilon_k^\dagger\gamma^1\gamma^A\mathcal{Z}) - \nabla_A(\varepsilon_k)^\dagger\gamma^1\gamma^A\mathcal{Z} + \frac{\mathrm{i}}{2}\varepsilon_k^\dagger\gamma_A\gamma^1\gamma^A\mathcal{Z} - \mathrm{i}\varepsilon_k^\dagger\gamma^1\mathcal{Z} \tag{346}$$

$$= D_A(\varepsilon_k^\dagger\gamma^1\gamma^A\mathcal{Z}) - \nabla_A(\varepsilon_k)^\dagger\gamma^1\gamma^A\mathcal{Z}. \tag{347}$$

I've already found above that $\nabla_A(\varepsilon_k)^\dagger\gamma^1\gamma^A\mathcal{Z}$ contributes nothing to the integral.

$$\therefore \int_{S^2_\infty}\varepsilon_k^\dagger\gamma^1\gamma^A\nabla_A(Z)\mathrm{d}A = \int_{S^2_\infty} D_A(\varepsilon_k^\dagger\gamma^1\gamma^A\mathcal{Z})\mathrm{d}A. \tag{348}$$

Let $D_A^{(S)}$ be the intrinsic Levi-Civita connection of S, let K_{IJ} be the extrinsic curvature of Σ in M and let c_{AB} be the extrinsic curvature of S in Σ. Then,

$$D_A(\varepsilon_k^\dagger\gamma^1\gamma^A\mathcal{Z}) = D_A(\varepsilon_k)^\dagger\gamma^1\gamma^A\mathcal{Z} + \varepsilon_k^\dagger\gamma^1\gamma^A D_A(\mathcal{Z}) \tag{349}$$

$$\begin{aligned} &= \left(D_A^{(S)}\varepsilon_k - \frac{1}{2}K_{AI}\gamma^I\gamma^0\varepsilon_k - \frac{1}{2}c_{AB}\gamma^B\gamma^1\varepsilon_k\right)^\dagger\gamma^1\gamma^A\mathcal{Z} \\ &\quad + \varepsilon_k^\dagger\gamma^1\gamma^A\left(D_A^{(S)}\mathcal{Z} - \frac{1}{2}K_{AI}\gamma^I\gamma^0\mathcal{Z} - \frac{1}{2}c_{AB}\gamma^B\gamma^1\mathcal{Z}\right) \end{aligned} \tag{350}$$

$$\begin{aligned} &= D_A^{(S)}(\varepsilon_k^\dagger\gamma^1\gamma^A\mathcal{Z}) + \frac{1}{2}K_{AI}(\varepsilon_k^\dagger\gamma^0\gamma^I\gamma^1\gamma^A\mathcal{Z} - \varepsilon_k^\dagger\gamma^1\gamma^A\gamma^I\gamma^0\mathcal{Z}) \\ &\quad - \frac{1}{2}c_{AB}(\varepsilon_k^\dagger\gamma^1\gamma^B\gamma^1\gamma^A\mathcal{Z} + \varepsilon_k^\dagger\gamma^1\gamma^A\gamma^B\gamma^1\mathcal{Z}). \end{aligned} \tag{351}$$

The measure, $\mathrm{d}A$, is $O(\mathrm{e}^{2r})$ while the ε_k-$\mathcal{Z}$ products are already $O(\mathrm{e}^{-2r})$.
$\therefore$ To get a non-zero integral as $r \to \infty$ I only need to take the extrinsic curvatures to leading order, which is nothing but their values in AdS.
AdS is time symmetric, so $K_{IJ} = 0$ to leading order.
Meanwhile, $c_{AB}(\gamma^1\gamma^B\gamma^1\gamma^A + \gamma^1\gamma^A\gamma^B\gamma^1) = c_{AB}(-(\gamma^1)^2\gamma^B\gamma^A + (\gamma^1)^2\gamma^A\gamma^B) = 0$ as extrinsic curvatures are symmetric. That leaves

$$\int_{S^2_\infty}\varepsilon_k^\dagger\gamma^1\gamma^A\nabla_A(Z)\mathrm{d}A = \int_{S^2_\infty} D_A^{(S)}(\varepsilon_k^\dagger\gamma^1\gamma^A\mathcal{Z})\mathrm{d}A = 0 \tag{352}$$

by Stokes' theorem.
Equation 342 then implies $Q(\Phi) = Q(\varepsilon_k)$. □

Corollary 6.4.1. *When* $S = S^2_\infty$, $m(S) = \sqrt{E^2 - ||P||^2 + ||J||^2 - ||K||^2}$.

Proof. From [13],

$$Q(\varepsilon_k) = 8\pi\varepsilon_0^\dagger\mathrm{e}^{-\mathrm{i}\gamma^0 t/2}\left(EI - \mathrm{i}P_I\gamma^I + K_I\gamma^0\gamma^I + \frac{\mathrm{i}}{2}J_{IJ}\gamma^0\gamma^{IJ}\right)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0, \tag{353}$$

$\therefore$ The four components of the constant spinor, ε_0, parameterise the four linearly independent solutions, Φ^A.

$$\therefore Q^{AB} \equiv 8\pi\mathrm{e}^{-\mathrm{i}\gamma^0 t/2}\left(EI - \mathrm{i}P_I\gamma^I + K_I\gamma^0\gamma^I + \frac{\mathrm{i}}{2}J_{IJ}\gamma^0\gamma^{IJ}\right)\mathrm{e}^{\mathrm{i}\gamma^0 t/2} \tag{354}$$

To calculate $m(S^2_\infty)$, I also need to find T^{AB} in this context. Given I'm using Dirac spinors here, T^{AB} is most easily calculated using the alternative expression, $T^{AB} = (\Phi^A)^T C^{-1}\Phi^B$, of equation 214. In the conventions chosen, $(\gamma_a)^T = -C^{-1}\gamma_a C$.

$$\begin{aligned}
\therefore T^{AB} &= (\varepsilon_k^A)^T C^{-1}\varepsilon_k^B && (355)\\
&= \frac{1}{1-\rho^2}(\varepsilon_0^A)^T \left(\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\right)^T \left(I - \mathrm{i}x_I(\gamma^I)^T\right) C^{-1}\left(I - \mathrm{i}x_J\gamma^J\right)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0^B && (356)\\
&= \frac{1}{1-\rho^2}(\varepsilon_0^A)^T C^{-1}\mathrm{e}^{-\mathrm{i}\gamma^0 t/2} C\left(I + \mathrm{i}x_I C^{-1}\gamma^I C\right) C^{-1}\left(I - \mathrm{i}x_J\gamma^J\right)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0^B && (357)\\
&= \frac{1}{1-\rho^2}(\varepsilon_0^A)^T C^{-1}\mathrm{e}^{-\mathrm{i}\gamma^0 t/2}\left(I + \mathrm{i}x_I\gamma^I\right)\left(I - \mathrm{i}x_J\gamma^J\right)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0^B && (358)\\
&= \frac{1}{1-\rho^2}(\varepsilon_0^A)^T C^{-1}\mathrm{e}^{-\mathrm{i}\gamma^0 t/2}\left(I + x_I x_J\gamma^I\gamma^J\right)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0^B && (359)\\
&= \frac{1}{1-\rho^2}(\varepsilon_0^A)^T C^{-1}\mathrm{e}^{-\mathrm{i}\gamma^0 t/2}(1-\rho^2) I\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0^B && (360)\\
&= (C^{-1})^{AB}. && (361)
\end{aligned}$$

Finally, outsourcing matrix algebra to the computer,

$$m(S^2_\infty) = \frac{1}{16\pi}\sqrt{-\operatorname{tr}\left(QT^{-1}\overline{Q}\overline{T}^{-1}\right)} = \sqrt{E^2 - P_I P^I + J_I J^I - K_I K^I}, \tag{362}$$

where $J_I = \frac{1}{2}\varepsilon_{IJK}J^{JK}$. □

The question naturally arises whether $\sqrt{E^2 - ||P||^2 + ||J||^2 - ||K||^2}$ is an appropriate notion of mass in asymptotically AdS spacetimes. For example, from special relativity, one thinks of mass as just $\sqrt{E^2 - ||P||^2}$, without any contributions from angular momenta, J_{IJ}, or boost charges, K_I. However, it can be argued this is an artefact of Minkowski space's symmetry group, namely the Poincaré group. As in QFT, one could define m^2 to be proportional to a quadratic Casimir operator for (the Lie algebra of) the symmetry group [30]. Therefore, in the AdS context, I have to first find a quadratic Casimir for $\mathfrak{o}(3,2)$.

Definition 6.5 (J_{MN})**.** *Choose generators, $\{J_{MN} = -J_{NM}\}_{M,N=1}^5$, for $\mathfrak{o}(3,2)$ such that the defining Lie bracket is*[17]

$$[J^{MN}, J^{PQ}] = \mathrm{i}\left(\eta^{MP}J^{NQ} - \eta^{MQ}J^{NP} - \eta^{NP}J^{MQ} + \eta^{NQ}J^{MP}\right), \tag{363}$$

where $\eta_{MN} \equiv \mathrm{diag}(-1,1,1,1,-1)$ and all $M, N, \cdots$ indices are raised/lowered by η^{-1}/η.

Lemma 6.6. *$C = \frac{1}{2}J^{MN}J_{MN}$ is a quadratic Casimir*[18] *for $\mathfrak{o}(3,2)$.*

Proof. By definition, a Casimir operator is one that commutes with all elements of the Lie

[17] The fact such a basis exists can be seen immediately by following the analogous steps in [43] for $\mathfrak{o}(3,1)$.

[18] Assume I have a faithful matrix representation of the Lie algebra so that multiplying two Lie algebra elements is well-defined.

algebra.

$$[C, J^{MN}] = \frac{1}{2}[J^{PQ}J_{PQ}, J^{MN}] \tag{364}$$
$$= \frac{1}{2}[J^{PQ}, J^{MN}]J_{PQ} + \frac{1}{2}J_{PQ}[J^{PQ}, J^{MN}] \tag{365}$$
$$= -\frac{\mathrm{i}}{2}\left(\eta^{MP}J^{NQ} - \eta^{MQ}J^{NP} - \eta^{NP}J^{MQ} + \eta^{NQ}J^{MP}\right)J_{PQ}$$
$$- \frac{\mathrm{i}}{2}J_{PQ}\left(\eta^{MP}J^{NQ} - \eta^{MQ}J^{NP} - \eta^{NP}J^{MQ} + \eta^{NQ}J^{MP}\right) \tag{366}$$
$$= \frac{\mathrm{i}}{2}(-J^{NQ}J^{M}{}_{Q} + J^{NP}J_{P}{}^{M} + J^{MQ}J^{N}{}_{Q} - J^{MP}J_{P}{}^{N} - J^{M}{}_{Q}J^{NQ} + J_{P}{}^{M}J^{NP}$$
$$+ J^{N}{}_{Q}J^{MQ} - J_{P}{}^{N}J^{MP}) \tag{367}$$
$$= 0. \tag{368}$$

$\therefore C$ is indeed a Casimir operator. □

Interpret J^{5a} as a 4-momentum generator, $\mathbb{P}^a$, J^{0I} as boost generators, $\mathbb{K}^I$, and J^{IJ} as angular momentum generators, $\mathbb{J}_I = \frac{1}{2}\varepsilon_{IJK}J^{JK}$, in line with [10] and the logic used in [13] for definition 6.3. Then,

$$C = J^{5a}J_{5a} + \frac{1}{2}J^{IJ}J_{IJ} + J^{0I}J_{0I} \tag{369}$$
$$= \mathbb{P}^0\mathbb{P}^0 - \mathbb{P}^I\mathbb{P}_I + \mathbb{J}^I\mathbb{J}_I - \mathbb{K}^I\mathbb{K}_I \tag{370}$$

suggesting that the limit in corollary 6.4.1 is physically reasonable.

7 Linearised gravity

In this section, I'll consider perturbations of AdS sourced by a matter field. In particular, the metric is assumed to be

$$g_{ab} = B_{ab} + \eta h_{ab}, \tag{371}$$

where $B = g_{\text{AdS}}$ is the background metric, h is the perturbation and η is assumed to be an infinitesimal parameter. Furthermore, the energy-momentum tensor, T_{ab}, is assumed to be $O(\eta)$. The aim is to show that definition 4.11 captures the mass in T_{ab}. Throughout this section I'll use the same coordinates and tetrad as in lemma 6.2. Furthermore, it will once again be convenient to set the AdS length scale to one, i.e. choose units where $\Lambda = -3$ and $k = 1/2$.

Lemma 7.1. *The Killing vectors of AdS can be spanned by*

$$t = \partial_t, \tag{372}$$
$$j_{IJ} = x_I\partial_J - x_J\partial_I, \tag{373}$$
$$p_I = \frac{2x_I}{1+\rho^2}\cos(t)\partial_t + \frac{1}{2}\left((1+\rho^2)\delta^J{}_I - 2x^Jx_I\right)\sin(t)\partial_J \quad \text{and} \tag{374}$$
$$k_I = -\frac{2x_I}{1+\rho^2}\sin(t)\partial_t + \frac{1}{2}\left((1+\rho^2)\delta^J{}_I - 2x^Jx_I\right)\cos(t)\partial_J. \tag{375}$$

Proof. The vectors listed are manifestly linearly independent and AdS is known to have a 10D space of Killing vectors because it is maximallly symmetric.

$\therefore$ It suffices to check that the 10 vectors listed are indeed Killing vectors.
$\partial_t g_{\mu\nu} = 0 \implies t$ is Killing.
To better distinguish between coordinate and vielbein indices in this calculation, I will relabel the coordinates as x_i. However, I will still have $x^i = x^I$ and $x_i = x_I$, i.e. unlike tensors, the components of the coordinates will not change when swapping vielbein and coordinate indices.

$$(\mathcal{L}_{j_{ij}}g)_{\mu\nu} = j^\rho_{ij}\partial_\rho g_{\mu\nu} + g_{\rho\nu}\partial_\mu j^\rho_{ij} + g_{\mu\rho}\partial_\nu j^\rho_{ij} \tag{376}$$
$$= x_i\partial_j g_{\mu\nu} - x_j\partial_i g_{\mu\nu} + g_{j\nu}\partial_\mu x_i - g_{i\nu}\partial_\mu x_j + g_{\mu j}\partial_\nu x_i - g_{\mu i}\partial_\nu x_j. \tag{377}$$
$$\therefore (\mathcal{L}_{j_{ij}}g)_{00} = x_i\partial_j\left(-\left(\frac{1+\rho^2}{1-\rho^2}\right)^2\right) - x_j\partial_i\left(-\left(\frac{1+\rho^2}{1-\rho^2}\right)^2\right) + 0 - 0 + 0 - 0 \tag{378}$$
$$= -\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\left(\frac{1+\rho^2}{1-\rho^2}\right)^2\right)\left(x_i\frac{x_j}{\rho} - x_j\frac{x_i}{\rho}\right) \tag{379}$$
$$= 0. \tag{380}$$
$$(\mathcal{L}_{j_{ij}}g)_{0k} = 0 - 0 + 0 - 0 + 0 - 0 = 0. \tag{381}$$
$$(\mathcal{L}_{j_{ij}}g)_{kl} = x_i\partial_j\left(\frac{4\delta_{kl}}{(1-\rho^2)^2}\right) - x_j\partial_i\left(\frac{4\delta_{kl}}{(1-\rho^2)^2}\right) + \frac{4\delta_{jl}}{(1-\rho^2)^2}\partial_k x_i - \frac{4\delta_{il}}{(1-\rho^2)^2}\partial_k x_j + \frac{4\delta_{kj}}{(1-\rho^2)^2}\partial_l x_i - \frac{4\delta_{ki}}{(1-\rho^2)^2}\partial_l x_j \tag{382}$$
$$= \frac{16\rho\delta_{kl}}{(1-\rho^2)^3}\left(x_i\frac{x_j}{\rho} - x_j\frac{x_i}{\rho}\right) + \frac{4}{(1-\rho^2)^2}(\delta_{jl}\delta_{ki} - \delta_{il}\delta_{kj} + \delta_{kj}\delta_{li} - \delta_{ki}\delta_{lj}) \tag{383}$$
$$= 0. \tag{384}$$

$\therefore j_{ij}$ is indeed Killing.

$$(\mathcal{L}_{p_i}g)_{\mu\nu} = \frac{1}{2}\sin(t)(1+\rho^2)\partial_i g_{\mu\nu} - \sin(t)x_i x^j\partial_j g_{\mu\nu} + 2g_{0\nu}\partial_\mu\left(\frac{x_i\cos(t)}{1+\rho^2}\right) + \frac{1}{2}g_{i\nu}\partial_\mu((1+\rho^2)\sin(t)) - g_{j\nu}\partial_\mu(x^j x_i\sin(t)) + 2g_{\mu 0}\partial_\nu\left(\frac{x_i\cos(t)}{1+\rho^2}\right) + \frac{1}{2}g_{\mu i}\partial_\nu((1+\rho^2)\sin(t)) - g_{\mu j}\partial_\nu(x^j x_i\sin(t)). \tag{385}$$
$$\therefore (\mathcal{L}_{p_i}g)_{00} = \frac{1}{2}\sin(t)(1+\rho^2)\partial_i\left(-\left(\frac{1+\rho^2}{1-\rho^2}\right)^2\right) - \sin(t)x_i x^j\partial_j\left(-\left(\frac{1+\rho^2}{1-\rho^2}\right)^2\right) - 4\left(\frac{1+\rho^2}{1-\rho^2}\right)^2\partial_t\left(\frac{x_i\cos(t)}{1+\rho^2}\right) \tag{386}$$
$$= \frac{4\sin(t)(1+\rho^2)}{(1-\rho^2)^3}\left(-(1+\rho^2)\rho\frac{x_i}{\rho} + 2\rho x_i x^j\frac{x_j}{\rho} + x_i(1-\rho^2)\right) \tag{387}$$
$$= 0. \tag{388}$$

$$(\mathcal{L}_{p_i}g)_{0j}$$
$$= \frac{2\delta_{ij}}{(1-\rho^2)^2}\partial_t((1+\rho^2)\sin(t)) - \frac{4\delta_{kj}}{(1-\rho^2)^2}\partial_t(x^k x_i\sin(t)) - 2\left(\frac{1+\rho^2}{1-\rho^2}\right)^2\partial_j\left(\frac{x_i\cos(t)}{1+\rho^2}\right) \tag{389}$$
$$= \frac{2\delta_{ij}}{(1-\rho^2)^2}(1+\rho^2)\cos(t) - \frac{4x_i x_j}{(1-\rho^2)^2}\cos(t) - 2\frac{\cos(t)}{(1-\rho^2)^2}\left((1+\rho^2)\delta_{ij} - 2\rho x_i\frac{x_j}{\rho}\right) \tag{390}$$
$$= 0. \tag{391}$$

$$
\begin{aligned}
&(\mathcal{L}_{p_i} g)_{jk} \\
&= \frac{1}{2}\sin(t)(1+\rho^2)\partial_i\left(\frac{4\delta_{jk}}{(1-\rho^2)^2}\right) - \sin(t)x_i x^l \partial_l\left(\frac{4\delta_{jk}}{(1-\rho^2)^2}\right) + \frac{2\delta_{ik}}{(1-\rho^2)^2}\partial_j((1+\rho^2)\sin(t)) \\
&\quad - \frac{4\delta_{lk}}{(1-\rho^2)^2}\partial_j(x_i x^l \sin(t)) + \frac{2\delta_{ji}}{(1-\rho^2)^2}\partial_k((1+\rho^2)\sin(t)) - \frac{4\delta_{jl}}{(1-\rho^2)^2}\partial_k(x_i x^l\sin(t)) \qquad (392)\\
&= \frac{8\rho\sin(t)(1+\rho^2)}{(1-\rho^2)^3}\delta_{jk}\frac{x_i}{\rho} - \frac{16\rho\delta_{jk}x_i x^l\sin(t)}{(1-\rho^2)^3}\frac{x_l}{\rho} + \frac{4\rho\delta_{ik}\sin(t)}{(1-\rho^2)^2}\frac{x_j}{\rho} - \frac{4\sin(t)}{(1-\rho^2)^2}(x_k\delta_{ij}+x_i\delta_{kj}) \\
&\quad + \frac{4\rho\sin(t)\delta_{ji}}{(1-\rho^2)^2}\frac{x_k}{\rho} - \frac{4\sin(t)}{(1-\rho^2)^2}(x_j\delta_{ki}+x_i\delta_{kj}) \qquad (393)\\
&= \frac{4\sin(t)}{(1-\rho^2)^3}(2x_i\delta_{jk}(1+\rho^2) - 4x_i\delta_{jk}\rho^2 + x_j\delta_{ik}(1-\rho^2) - x_k\delta_{ij}(1-\rho^2) - x_i\delta_{kj}(1-\rho^2) \\
&\quad + x_k\delta_{ji}(1-\rho^2) - x_j\delta_{ki}(1-\rho^2) - x_i\delta_{kj}(1-\rho^2)) \qquad (394)\\
&= 0. \qquad (395)
\end{aligned}
$$

$\therefore p_i$ is Killing.
The differences between p_i and k_i is are only signs and swapping sin & cos. Hence, by following the same calculation it also follows that k_i is also Killing. □

In analogy with definition 6.3, I will define the following "matter charges."

Definition 7.2 (Matter charges)**.** *Let matter charges on Σ be defined as*

$$
E = \int_\Sigma T_{0a}t^a \mathrm{d}V,\ \ P_I = \int_\Sigma T_{0a}p_I^a\mathrm{d}V,\ \ J_{IJ} = \int_\Sigma T_{0a}j_{IJ}^a\mathrm{d}V \ \text{ and } \ K_I = \int_\Sigma T_{0a}k_I^a\mathrm{d}V. \qquad (396)
$$

Theorem 7.3. *For gravity linearised about AdS, if S is generic in the Φ^A sense, then*

$$
m(S) = \sqrt{E^2 - ||P||^2 + ||J||^2 - ||K||^2}. \qquad (397)
$$

This result is formally identical to section 6 and therefore the result can once again be thought of as a Casimir mass, but this time for T_{ab}.

Proof. In AdS, the solutions to $\overline{m}^a\nabla_a\Phi = 0$ can be spanned on any surface generic in the Φ^A sense by the Killing spinors, ε_k, restricted to S.
$\therefore$ Since $g_{ab} = B_{ab} + \eta h_{ab}$, I can let $\Phi = \varepsilon_k + \eta\mathcal{Z}$ for some Dirac spinor, $\mathcal{Z}$.
Extend $\mathcal{Z}$'s definition off S in an arbitrary, but sufficiently regular, way so that $\Phi = \varepsilon_k + \eta\mathcal{Z}$ is defined on all of Σ.
$\therefore$ By definition 2.9,

$$
Q(\Phi) = 2\int_\Sigma \left(\nabla_I(\Phi)^\dagger\nabla^I\Phi - 4\pi T^{0a}\overline{\Phi}\gamma_a\Phi - (\gamma^I\nabla_I\Phi)^\dagger\gamma^J\nabla_J\Phi\right)\mathrm{d}V. \qquad (398)
$$

$\nabla_a^{(B)}\varepsilon_k = 0 \implies \nabla_a\Phi = O(\eta)$.
$\therefore$ The first and third terms in equation 398 are both $O(\eta^2)$.
Meanwhile, since T_{ab} is assumed to be $O(\eta)$, the second term is $-4\pi T^{0a}\bar{\varepsilon}_k\gamma_a\varepsilon_k + O(\eta^2)$.
$\therefore$ In the linearised limit,

$$
Q(\Phi) = 8\pi\int_\Sigma T_{0a}\bar{\varepsilon}_k\gamma^a\varepsilon_k\mathrm{d}V. \qquad (399)
$$

From lemma 6.2,

$$\bar{\varepsilon}_k\gamma^0\varepsilon_k = \varepsilon_k^\dagger\varepsilon_k \tag{400}$$
$$= \frac{1}{1-\rho^2}\varepsilon_0^\dagger \mathrm{e}^{-\mathrm{i}\gamma^0 t/2}(I - \mathrm{i}x_I\gamma^I)(I - \mathrm{i}x_J\gamma^J)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0 \tag{401}$$
$$= \frac{1}{1-\rho^2}\varepsilon_0^\dagger \mathrm{e}^{-\mathrm{i}\gamma^0 t/2}(I - 2\mathrm{i}x_I\gamma^I - x_Ix_J\gamma^I\gamma^J)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0 \tag{402}$$
$$= \frac{1}{1-\rho^2}\varepsilon_0^\dagger \mathrm{e}^{-\mathrm{i}\gamma^0 t/2}((1+\rho^2)I - 2\mathrm{i}x_I\gamma^I)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0 \tag{403}$$

and likewise

$$\bar{\varepsilon}_k\gamma^I\varepsilon_k = \varepsilon_k^\dagger\gamma^0\gamma^I\varepsilon_k \tag{404}$$
$$= \frac{1}{1-\rho^2}\varepsilon_0^\dagger \mathrm{e}^{-\mathrm{i}\gamma^0 t/2}(I - \mathrm{i}x_J\gamma^J)\gamma^0\gamma^I(I - \mathrm{i}x_K\gamma^K)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0 \tag{405}$$
$$= \frac{1}{1-\rho^2}\varepsilon_0^\dagger \mathrm{e}^{-\mathrm{i}\gamma^0 t/2}(\gamma^0 - \mathrm{i}x_J\gamma^J\gamma^0)(\gamma^I - \mathrm{i}x_K\gamma^I\gamma^K)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0 \tag{406}$$
$$= \frac{1}{1-\rho^2}\varepsilon_0^\dagger \mathrm{e}^{-\mathrm{i}\gamma^0 t/2}(\gamma^0\gamma^I - \mathrm{i}x_J\gamma^J\gamma^0\gamma^I - \mathrm{i}x_J\gamma^0\gamma^I\gamma^J - x_Jx_K\gamma^J\gamma^0\gamma^I\gamma^K)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0 \tag{407}$$
$$= \frac{1}{1-\rho^2}\varepsilon_0^\dagger \mathrm{e}^{-\mathrm{i}\gamma^0 t/2}(\gamma^0\gamma^I - 2\mathrm{i}x_J\gamma^0\gamma^{IJ} - 2x^Ix_J\gamma^0\gamma^J + \rho^2\gamma^0\gamma^I)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0 \tag{408}$$
$$= \frac{1}{1-\rho^2}\varepsilon_0^\dagger \mathrm{e}^{-\mathrm{i}\gamma^0 t/2}((1+\rho^2)\gamma^0\gamma^I - 2\mathrm{i}x_J\gamma^0\gamma^{IJ} - 2x^Ix_J\gamma^0\gamma^J)\mathrm{e}^{\mathrm{i}\gamma^0 t/2}\varepsilon_0. \tag{409}$$

Substituting back into equation 399, applying definition 7.2 and converting to vielbein indices where required using the tetrad in lemma 6.2, then gives

$$Q(\Phi) = 8\pi\varepsilon_0^\dagger\Bigg(\int_\Sigma \frac{1+\rho^2}{1-\rho^2}T_{00}\mathrm{d}V\, I - 2\mathrm{i}\int_\Sigma \frac{x_I}{1-\rho^2}T_{00}\mathrm{e}^{-\mathrm{i}\gamma^0 t}\mathrm{d}V\,\gamma^I - 2\mathrm{i}\int_\Sigma \frac{x_J}{1-\rho^2}T_{0I}\mathrm{d}V\,\gamma^0\gamma^{IJ}$$
$$+ \int_\Sigma \frac{1}{1-\rho^2}T_{0I}\left((1+\rho^2)\delta^I{}_J - 2x^Ix_J\right)\mathrm{e}^{-\mathrm{i}\gamma^0 t}\mathrm{d}V\,\gamma^0\gamma^J\Bigg)\varepsilon_0 \tag{410}$$
$$= 8\pi\varepsilon_0^\dagger\Bigg(\int_\Sigma \frac{1+\rho^2}{1-\rho^2}T_{00}\mathrm{d}V\, I + \mathrm{i}\int_\Sigma \frac{1}{1-\rho^2}(x_IT_{0J} - x_JT_{0I})\mathrm{d}V\,\gamma^0\gamma^{IJ}$$
$$- \mathrm{i}\int_\Sigma\left(\frac{2x_I\cos(t)}{1-\rho^2}T_{00} + \frac{\sin(t)}{1-\rho^2}T_{0I}\left((1+\rho^2)\delta^I{}_J - 2x^Ix_J\right)\right)\mathrm{d}V\,\gamma^I$$
$$+ \int_\Sigma\left(-\frac{2x_I\sin(t)}{1-\rho^2}T_{00} + \frac{\cos(t)}{1-\rho^2}T_{0I}\left((1+\rho^2)\delta^I{}_J - 2x^Ix_J\right)\right)\mathrm{d}V\,\gamma^0\gamma^I\Bigg)\varepsilon_0 \tag{411}$$
$$= 8\pi\varepsilon_0^\dagger\left(EI + \frac{\mathrm{i}}{2}J_{IJ}\gamma^0\gamma^{IJ} - \mathrm{i}P_I\gamma^I + K_I\gamma^0\gamma^I\right)\varepsilon_0. \tag{412}$$

Take the four components of the constant spinor, ε_0, to parameterise the four linearly independent solutions, Φ^A.

$$\therefore Q^{AB} = 8\pi\left(EI + \frac{\mathrm{i}}{2}J_{IJ}\gamma^0\gamma^{IJ} - \mathrm{i}P_I\gamma^I + K_I\gamma^0\gamma^I\right). \tag{413}$$

Since T_{ab}, and hence Q^{AB}, are already $O(\eta)$, for the linearised limit it suffices to take T^{AB} to $O(1)$ in definition 4.11.
$\therefore T^{AB} = (\varepsilon_k^A)^T C^{-1}\varepsilon_k = (C^{-1})^{AB}$, borrowing the calculation from the proof of corollary 6.4.1.
Finally, evaluating $m(S)$ for this Q^{AB} and T^{AB} using computer algebra gives

$$m(S) = \frac{1}{16\pi}\sqrt{-\operatorname{tr}\left(QT^{-1}\overline{Q}\overline{T}^{-1}\right)} = \sqrt{E^2 + J_IJ^I - P_IP^I - K_IK^I}, \tag{414}$$

where $J_I = \frac{1}{2}\varepsilon_{IJK}J^{JK}$. □

8 Conclusion

In this work I've defined a new quasilocal mass for spacetimes with negative cosmological constant. The new definition is spinorial and based on definitions by Penrose and Dougan & Mason - which are themselves inspired by Witten's proof of the positive energy theorem. I've shown my definition satisfies a number of physically desirable properties - namely that $m(S) \geq 0$, $m(S) = 0$ for every surface in AdS, $m(S_r^2)$ agrees with the Misner-Sharp mass in spherical symmetry and $m(S)$ has an appropriate limit, $\sqrt{E^2 - ||P||^2 + ||J||^2 - ||K||^2}$, in linearised gravity or when S approaches a sphere on $\mathcal{I}$ in an asymptotically AdS spacetime.

Some avenues of further research are immediately apparent at this juncture. This work was originally inspired by Reall's suggestion that a quasilocal mass-charge inequality could be established for spacetimes with $\Lambda < 0$ and such an inequality could be used to prove the 3rd law of black hole mechanics for supersymmetric horizons in this context. Having now established a workable quasilocal mass, a logical next step would be tackling this conjecture. Note that the 3rd law part of the conjecture might not be immediately accessible though because the $\theta_l > 2\sqrt{2}k$ requirement prevents taking S arbitrarily close to the event horizon (where $\theta_l = 0$).

Even in the field of quasilocal mass itself, some improvements could be made. I've given two different definitions of generic and it may be interesting to study further how the two definitions relate. It would be particularly desirable to find an example of a toroidal S where my construction can actually be completed in full - unlike the examples in section 5.2. Then, perhaps a more concrete conclusion can be made about whether either definition is generic in practice or physically relevant for higher genus surfaces.

Elsewhere, in terms of physical properties, one property I did not mention in this work is the "small sphere" limit. In particular, one hopes that given a point, $p \in M$, and a future direction, t^a, if S_r is a sphere reached by flowing an affine parameter distance, r, along the generators of p's future lightcone, then a quasilocal 4-momentum for S_r would be $P^a = -\frac{4\pi}{3}r^3 T^a{}_b t^b$, to leading order in r. Then, $m(S_r)^2 = -P^a P_a$. This happens to be true for both the Dougan-Mason and Penrose masses [16, 44]. For a vacuum spacetime a similar result holds at 5th order in r based on the Bel-Robinson tensor. It would be interesting to see if the same - or something analogous - also holds for my definition. Unfortunately, this was a calculation I did not make much progess on.

Furthermore, while my definition has good quantitative properties, it does share the qualitative failings of many other quasilocal masses. Unlike the Hamilton-Jacobi masses or the Hawking mass, the physical motivation for my definition is not clear, beyond some supergravity considerations [45] underpinning Witten's method to prove the positive energy theorem. More practically, like the Hamilton-Jacobi or Bartnik masses, my quasilocal mass is likely to be quite difficult to calculate for most metrics. Not only does one have to find a NP tetrad adapted to S, one then has to find all solutions to $\overline{m}^a \nabla_a \Phi = 0$ on S. The quest to find a truly satisfactory quasilocal mass goes on.

Another possible extension would be to consider spacetimes with $\Lambda > 0$ instead. Not only is the $\Lambda > 0$ case potentially most relevant to the real world, it is arguably also a pressing need for mathematical general relativity. Many familiar properties of conformal infinity break down when $\Lambda > 0$ [46] and this precludes defining anything directly analogous to the ADM [15] or Wang [6] masses. Nonetheless, a number of energy-momentum definitions have been devised in this context - see [47] for a review. Particularly relevant to the present work are extensions based on Witten's method [48, 49]. Ultimately though, these successes still have to work around the global challenges imposed by $\Lambda > 0$ - e.g. compact Cauchy surfaces, spacelike $\mathcal{I}^+$ or cosmological horizons. It may be that quasilocal mass is a viable alternative for avoiding these issues. In fact, an analogue of Penrose's quasilocal mass can be defined for asymptotically de Sitter spacetimes, albeit it no longer retains some key properties, such as positivity [49, 50]. Likewise, it would be interesting to see if the definition I've developed here can be adjusted for $\Lambda > 0$ and if so, which of its physical properties remains intact.

A Conventions

My conventions are based off [30]. However, since there are slight differences and most people in the GR community are unlikely to be familiar with [30], I list the main points below.

I use a $(-1, +1, +1, +1)$ metric signature[19].

The following symbols are frequently used.

- M: The full spacetime
- g: The (Lorentzian) metric on M
- Σ: 3D, compact, spacelike hypersurface with boundary
- S: The boundary of Σ
- Λ: A negative cosmological constant
- $k = \sqrt{-\Lambda/12}$
- C_b^∞: The space of smooth Dirac spinors on Σ subject to the boundary conditions given in definition 3.1
- $\mathcal{H}$: The completion of C_b^∞ under the inner product in definition 3.2
- $\overline{\Psi} = \Psi^\dagger \gamma^0$ for a Dirac spinor, Ψ
- D_a: The Levi-Civita connection of g
- $\nabla_a \Psi = D_a \psi + \mathrm{i} k \gamma_a \Psi$ for a Dirac spinor, Ψ
- $\nabla_a \overline{\Psi} = D_a \overline{\Psi} - \mathrm{i} k \overline{\Psi} \gamma_a = (\nabla_a \Psi)^\dagger \gamma^0$ for a Dirac spinor, Ψ
- I: The identity matrix
- $\{A_\alpha, B_\alpha\}$: A GHP spinor dyad
- $\delta = m^a D_a$ in the context of the NP formalism
- $\bar{\delta} = \overline{m}^a D_a$ in the context of the NP formalism

I use many different types of indices, as given below.

- $a, b, c, \cdots$ are vielbein indices running $0, 1, 2, 3$. However, in most equations it will be apparent that these could equally well denote abstract indices.

[19] This is the only sensible convention.

- $\mu, \nu, \rho, \cdots$ are coordinate indices running $0, 1, 2, 3$.
- $I, J, K, \cdots$ are vielbein indices running $1, 2, 3$.
- $\alpha, \beta, \gamma, \cdots$ are two-component spinor indices for the $(1/2, 0)$ representation, i.e. left-handed Weyl spinors, and run 1, 2.
- $\dot{\alpha}, \dot{\beta}, \dot{\gamma}, \cdots$ are two-component spinor indices for the $(0, 1/2)$ representation, i.e. right-handed Weyl spinors, and run $\dot{1}$, $\dot{2}$.
- $A, B, C, \cdots$ run $1, 2, 3, 4$ and index the linearly independent solutions to $\overline{m}^a \nabla_a \Phi = 0$.

The Riemann tensor is defined such that $[D_a, D_b]V^c = R^c{}_{dab} V^d$.

Complex conjugation of an object - unless it is a Dirac spinor - will be denoted by a bar over the object, e.g. $\bar{z}$.

Levi-Civita symbols are normalised by $\varepsilon_{12} = -1$, $\varepsilon^{12} = 1$, $\varepsilon_{\dot{1}\dot{2}} = -1$, $\varepsilon^{\dot{1}\dot{2}} = 1$, $\varepsilon_{0123} = -1$ and $\varepsilon^{0123} = 1$. Then, $\varepsilon^{\alpha\gamma}\varepsilon_{\gamma\beta} = \delta^\alpha{}_\beta$ and likewise for the dotted indices.

Two-component spinors are raised and lowered from the left, i.e. $\psi_\alpha = \varepsilon_{\alpha\beta}\psi^\beta$ and $\psi^\alpha = \varepsilon^{\alpha\beta}\psi_\beta$.

The extended Pauli matrices are

$$(\sigma_a)_{\alpha\dot{\alpha}} \equiv (I, \sigma_1, \sigma_2, \sigma_3) \text{ and} \tag{415}$$

$$(\tilde{\sigma}_a)^{\dot{\alpha}\alpha} = \varepsilon^{\alpha\beta}\varepsilon^{\dot{\alpha}\dot{\beta}}(\sigma_a)_{\beta\dot{\beta}} \equiv (I, -\sigma_1, -\sigma_2, -\sigma_3) \tag{416}$$

with $\sigma_{1,2,3}$ being the standard Pauli matrices.

I will convert between vielbein indices and two-component spinor indices by $V_{\alpha\dot{\alpha}} = (\sigma_a)_{\alpha\dot{\alpha}} V^a$ and $V_a = -\frac{1}{2}(\tilde{\sigma}_a)^{\dot{\alpha}\alpha} V_{\alpha\dot{\alpha}}$.

Dirac spinors are decomposed into two-component spinors by $\Psi = [\psi_\alpha, \bar{\chi}^{\dot{\alpha}}]^T$.

I will use the Weyl representation of gamma matrices, i.e.

$$\gamma_a = \begin{bmatrix} 0 & (\sigma_a)_{\alpha\dot{\alpha}} \\ (\tilde{\sigma}_a)^{\dot{\alpha}\alpha} & 0 \end{bmatrix}. \tag{417}$$

Hence, the gamma matrices are unitary and satisfy $\gamma^a\gamma^b + \gamma^b\gamma^a = -2\eta^{ab} I$. Furthermore, in terms of two-component spinors, $\overline{\Psi} = \Psi^\dagger \gamma^0 = [-\chi^\alpha, -\overline{\psi}_{\dot{\alpha}}]$.

The charge conjugation matrix is

$$C = \begin{bmatrix} \varepsilon_{\alpha\beta} & 0 \\ 0 & \varepsilon^{\dot{\alpha}\dot{\beta}} \end{bmatrix} \iff C^{-1} = \begin{bmatrix} \varepsilon^{\alpha\beta} & 0 \\ 0 & \varepsilon_{\dot{\alpha}\dot{\beta}} \end{bmatrix}. \tag{418}$$

The spin-weighted spherical harmonics I'll use in section 5 are

$$\left({}_{1/2}Y_{1/2,1/2}\right) = \frac{\mathrm{i}}{\sqrt{2\pi}} \sin\left(\frac{\theta}{2}\right) \mathrm{e}^{\mathrm{i}\phi/2}, \quad \left({}_{1/2}Y_{1/2,-1/2}\right) = -\frac{\mathrm{i}}{\sqrt{2\pi}} \cos\left(\frac{\theta}{2}\right) \mathrm{e}^{-\mathrm{i}\phi/2},$$
$$\left({}_{-1/2}Y_{1/2,1/2}\right) = \frac{\mathrm{i}}{\sqrt{2\pi}} \cos\left(\frac{\theta}{2}\right) \mathrm{e}^{\mathrm{i}\phi/2} \text{ and } \left({}_{-1/2}Y_{1/2,-1/2}\right) = \frac{\mathrm{i}}{\sqrt{2\pi}} \sin\left(\frac{\theta}{2}\right) \mathrm{e}^{-\mathrm{i}\phi/2}. \tag{419}$$

Section 6 features some additional or modified conventions as listed below.

- Based on context, $\alpha, \beta, \gamma, \cdots$ also denote coordinate indices running $2, 3$.
- Based on context, $A, B, C, \cdots$ also denote vielbein indices running $2, 3$.
- $m, n, p, \cdots$ are coordinate indices running $0, 2, 3$.
- $M, N, P, \cdots$ run $0, 1, 2, 3, 4$ and index the embedding Cartesian coordinates when AdS is viewed as a surface in $\mathbb{R}^{3,2}$.
- The cosmological constant is set to $\Lambda = -3$.

A.1 Comparison to Penrose-Rindler conventions

The monogrpahs of Penrose and Rindler [31, 32] have become the standard references for two-component spinors in the GR community. Unfortunately their conventions differ significantly at times from mine; I list the key differences below.

- I use a mostly plus metric while they use a mostly minus metric. This is the primary reason I've chosen not to follow their conventions.
- I use lowercase letters from the start of the Greek alphabet for two-component spinor indices while they use uppercase Latin letters.
- My undotted spinor indices correspond to their primed spinor indices and my dotted indices correspond to their unprimed indices.
- I take the two-component spinor indices to run over the values 1 and 2, whereas they take them to run over 0 and 1.
- I convert to spinor indices by $V_{\alpha\dot{\alpha}} = (\sigma_a)_{\alpha\dot{\alpha}} V^a$, while they have $V_{\alpha\dot{\alpha}} = \frac{1}{\sqrt{2}}(\sigma_a)_{\alpha\dot{\alpha}} V^a$. This $\sqrt{2}$ discrepancy appears in a number of equations when comparing the two conventions.
- The elements of my spinor dyad, A_α and B_α, are denoted as $o_{A'}$ and $\iota_{A'}$ respectively in their notation. Unfortunately, ι, and especially o, are not great letters to use when writing mathematics, especially by hand, hence why I've chosen A and B instead.
- The $\sqrt{2}$ difference when converting to spinor indices means I require $B^\alpha A_\alpha = \bar{B}^{\dot{\alpha}} \bar{A}_{\dot{\alpha}} = \sqrt{2}$ while they have $\iota^A o_A = \iota^{A'} o_{A'} = 1$.
- For any spinor, ψ_α, my $a(\psi)$ and $b(\psi)$ would be called $\frac{1}{\sqrt{2}}\psi_{1'}$ and $-\frac{1}{\sqrt{2}}\psi_{0'}$ respectively in their notation.
- I write Dirac spinors as $\Psi = [\psi_\alpha,\, \bar{\chi}^{\dot{\alpha}}]^T$, while they would write $[\bar{\chi}^A,\, \psi_{A'}]^T$, i.e. the left and right handed componenets are written in the opposite order.
- I raise and lower indices from the left, i.e. $\psi^\alpha = \varepsilon^{\alpha\beta}\psi_\beta$ and $\psi_\alpha = \varepsilon_{\alpha\beta}\psi^\beta$, while they raise from the left, but lower from the right, i.e. $\psi^A = \varepsilon^{AB}\psi_B$ but $\psi_A = \psi^B \varepsilon_{BA}$. This difference means I have $\varepsilon^{12} = 1$, but $\varepsilon_{12} = -1$. Furthermore, it means I have $\varepsilon^{\alpha\gamma}\varepsilon_{\gamma\beta} = \delta^\alpha{}_\beta$ while they have $\varepsilon^{AC}\varepsilon_{CB} = -\delta^A{}_B$. The asymmetry between raising and lowering indices is the 2nd biggest reason I've chosen not use Penrose and Rindler's conventions.

B Frequently used spinor identities

The following are some basic two-component spinor identities I'll use liberally without proof or explicit mention. Most of them are given in [30].

$$V^{\alpha\dot{\alpha}}W_{\alpha\dot{\alpha}} = -2V^aW_a \tag{420}$$

$$\overline{(\psi_\alpha)} = \overline{\psi}_{\dot{\alpha}} \tag{421}$$

$$\psi_\alpha\chi^\alpha = -\psi^\alpha\chi_\alpha \tag{422}$$

$$(\sigma_a)_{\alpha\dot{\alpha}}(\tilde{\sigma}_b)^{\dot{\alpha}\beta} + (\sigma_b)_{\alpha\dot{\alpha}}(\tilde{\sigma}_a)^{\dot{\alpha}\beta} = -2\eta_{ab}\delta_\alpha{}^\beta \tag{423}$$

$$(\tilde{\sigma}_a)^{\dot{\alpha}\alpha}(\sigma_b)_{\alpha\dot{\beta}} + (\tilde{\sigma}_b)^{\dot{\alpha}\alpha}(\sigma_a)_{\alpha\dot{\beta}} = -2\eta_{ab}\delta^{\dot{\alpha}}{}_{\dot{\beta}} \tag{424}$$

$$(\sigma_a)_{\alpha\dot{\alpha}}(\tilde{\sigma}_b)^{\dot{\alpha}\alpha} = -2\eta_{ab} \tag{425}$$

$$(\sigma^a)_{\alpha\dot{\alpha}}(\tilde{\sigma}_a)^{\dot{\beta}\beta} = -2\delta^\beta{}_\alpha\delta^{\dot{\beta}}{}_{\dot{\alpha}} \tag{426}$$

$$(\sigma_a)_{\alpha\dot{\beta}}(\tilde{\sigma}_b)^{\dot{\beta}\beta}(\sigma_c)_{\beta\dot{\alpha}} = \eta_{ca}(\sigma_b)_{\alpha\dot{\alpha}} - \eta_{bc}(\sigma_a)_{\alpha\dot{\alpha}} - \eta_{ab}(\sigma_c)_{\alpha\dot{\alpha}} + \mathrm{i}\varepsilon_{abcd}(\sigma^d)_{\alpha\dot{\alpha}} \tag{427}$$

$$(\tilde{\sigma}_a)^{\dot{\alpha}\beta}(\sigma_b)_{\beta\dot{\beta}}(\tilde{\sigma}_c)^{\dot{\beta}\alpha} = \eta_{ca}(\tilde{\sigma}_b)^{\dot{\alpha}\alpha} - \eta_{bc}(\tilde{\sigma}_a)^{\dot{\alpha}\alpha} - \eta_{ab}(\tilde{\sigma}_c)^{\dot{\alpha}\alpha} - \mathrm{i}\varepsilon_{abcd}(\tilde{\sigma}^d)^{\dot{\alpha}\alpha} \tag{428}$$

$$\varepsilon_{\alpha\beta}\varepsilon^{\gamma\delta} = -\left(\delta^\gamma{}_\alpha\delta^\delta{}_\beta - \delta^\delta{}_\alpha\delta^\gamma{}_\beta\right) \tag{429}$$

I'll need the following NP coefficients in terms of GHP variables.

$$\sqrt{2}\mu = \sqrt{2}\overline{m}^a\delta n_a = \bar{B}^{\dot{\alpha}}\delta\bar{B}_{\dot{\alpha}} \tag{430}$$

$$\sqrt{2}\rho = -\sqrt{2}m^a\bar{\delta}l_a = \bar{A}^{\dot{\alpha}}\bar{\delta}\bar{A}_{\dot{\alpha}} \tag{431}$$

$$\sqrt{2}\alpha = \frac{1}{\sqrt{2}}\left(\overline{m}^a\bar{\delta}m_a - n^a\bar{\delta}l_a\right) = \bar{B}^{\dot{\alpha}}\bar{\delta}\bar{A}_{\dot{\alpha}} \tag{432}$$

$$\sqrt{2}\beta = \frac{1}{\sqrt{2}}\left(\overline{m}^a\delta m_a - n^a\delta l_a\right) = \bar{B}^{\dot{\alpha}}\delta\bar{A}_{\dot{\alpha}} \tag{433}$$

$$\sqrt{2}\sigma = -\sqrt{2}m^a\delta l_a = \bar{A}^{\dot{\alpha}}\delta\bar{A}_{\dot{\alpha}} \tag{434}$$

$$\sqrt{2}\lambda = \sqrt{2}\overline{m}^a\bar{\delta}n_a = \bar{B}^{\dot{\alpha}}\bar{\delta}\bar{B}_{\dot{\alpha}} \tag{435}$$

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