A primer on spinors

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The object of this treatise is to give a general mathematical overview of spinors "from first principles." In the perspective I'll take here, the "first principles" are representations of the Clifford algebra; all properties of spinors are derived from studying irreducible representations of the Clifford algebra. My presentation here is a collation of results in [1], [2], [3] and [4][1]. My only contribution is to fill in some proofs which [3] and [4] did not have the courtesy of providing.

1 Arbitrary spacetimes

The study of spinors is intimately connected with the representation theory of "Clifford algebras." A Clifford algebra is a set of D objects (which can be thought of as matrices as only their representations in finite dimensional vector spaces are relevant[2]), $\{\gamma_a\}_{a=0}^{D-1}$, such that

$$\{\gamma_a, \gamma_b\} = \gamma_a\gamma_b + \gamma_b\gamma_a = -2\eta_{ab}I \tag{1}$$

where $\eta_{ab} = \mathrm{diag}(-1, \cdots -1, 1, \cdots 1)$ with t minus ones, s plus ones and $s + t = D$.

The first task is to study finite dimensional, complex, irreducible representations of this algebra. As I'll show, for questions such as the existence, uniqueness and dimension of the irreducible representations, it suffices to study the algebra, $\{\gamma_a, \gamma_b\} = 2\delta_{ab}I$.
Let $\{\tilde{\gamma}_a, \tilde{\gamma}_b\} = -2\eta_{ab}I$, $\gamma_a = \mathrm{i}\tilde{\gamma}_a$ for $t \leq a \leq D-1$ and $\gamma_a = \tilde{\gamma}_a$ for $0 \leq a \leq t-1$.
Then, for $a, b \geq t$,

$$\gamma_a\gamma_b + \gamma_b\gamma_a = \mathrm{i}\tilde{\gamma}_a\mathrm{i}\tilde{\gamma}_b + \mathrm{i}\tilde{\gamma}_b\mathrm{i}\tilde{\gamma}_a = -(\tilde{\gamma}_a\tilde{\gamma}_b + \tilde{\gamma}_b\tilde{\gamma}_a) = 2\eta_{ab}I = 2\delta_{ab}I \tag{2}$$

Likewise, for $a, b < t$,

$$\gamma_a\gamma_b + \gamma_b\gamma_a = \tilde{\gamma}_a\tilde{\gamma}_b + \tilde{\gamma}_b\tilde{\gamma}_a = -2\eta_{ab}I = 2\delta_{ab}I \tag{3}$$

Finally, when one of a and b is less than t and the other is greater than or equal to t,

$$\gamma_a\gamma_b + \gamma_b\gamma_a = \mathrm{i}(\tilde{\gamma}_a\tilde{\gamma}_b + \tilde{\gamma}_b\tilde{\gamma}_a) = -2\mathrm{i}\eta_{ab}I = 0 = 2\delta_{ab}I \tag{4}$$

$\therefore$ The original Clifford algebra can be transformed to one where $-\eta_{ab} \to \delta_{ab}$. Conversely, if $\{\gamma_a, \gamma_b\} = 2\delta_{ab}I$, then letting $\tilde{\gamma}_a = \gamma_a$ for $0 \leq a \leq t-1$ and $\tilde{\gamma}_a = -\mathrm{i}\gamma_a$ for $t \leq a \leq D-1$ yields $\{\tilde{\gamma}_a, \tilde{\gamma}_b\} = -2\eta_{ab}I$.
$\therefore$ The two Clifford algebras are equivalent. For now consider $\{\gamma_a, \gamma_b\} = 2\delta_{ab}I$.

[1] I have taken some proofs almost exactly as presented in these references.
[2] The Clifford algebra is assumed to be associative so that a matrix representation is well defined.

Let $\{\gamma_a\}_{a=0}^{D-1}$ be a finite dimensional, complex, irreducible representation of the Clifford algebra, $\{\gamma_a, \gamma_b\} = 2\delta_{ab}I$. Denote the dimension of the representation space by N.
Let $\{\Gamma_A\}_{A=0}^{2^D-1} = \{I, \gamma_a, \gamma_a\gamma_b \text{ with } a < b, \gamma_a\gamma_b\gamma_c \text{ with } a < b < c, \cdots, \gamma_0 \cdots \gamma_{D-1}\}$. By definition, all the Γ_A are $N \times N$ matrices.
$\gamma_a\gamma_b + \gamma_b\gamma_a = 2\delta_{ab}I \implies (\gamma_a)^2 = I$ and $\gamma_a\gamma_b = -\gamma_b\gamma_a$ for $a \neq b$.

$$\begin{aligned}
\therefore (\Gamma_A)^2 &= \gamma_{a_1} \cdots \gamma_{a_n}\gamma_{a_1} \cdots \gamma_{a_n} \text{ for some } 0 \leq n \leq D-1 \text{ and } a_1 < \cdots < a_n && (5)\\
&= \gamma_{a_1}\gamma_{a_1}(-1)^{n-1}\gamma_{a_2} \cdots \gamma_{a_n}\gamma_{a_2} \cdots \gamma_{a_n} && (6)\\
&= (-1)^{n-1}\gamma_{a_2} \cdots \gamma_{a_n}\gamma_{a_2} \cdots \gamma_{a_n} && (7)\\
&= (-1)^{n-1+n-2+\cdots+1}I && (8)\\
&= (-1)^{n(n-1)/2}I && (9)
\end{aligned}$$

$\therefore$ All the Γ_A are invertible and $(\Gamma_A)^{-1} = (-1)^{n(n-1)/2}\Gamma_A$.

Lemma 1.1. *$G = \{\pm\Gamma_A\}_{A=0}^{2^D-1}$ is a finite group of order 2^{D+1} under multiplication.*

Proof. That G has 2^{D+1} elements follows directly from the definition.
Matrix multiplication is already associative.
The identity matrix, I, is Γ_0 by definition and hence in G.
$(\pm\Gamma_A)^{-1} = \pm(-1)^{n(n-1)/2}\Gamma_A \in G$.
$\therefore$ All that's left to show is that multiplication is a well defined binary operation on G.
Let $\Gamma_A = \gamma_{a_1} \cdots \gamma_{a_m}$ and $\Gamma_B = \gamma_{b_1} \cdots \gamma_{b_n} \implies \Gamma_A\Gamma_B = \gamma_{a_1} \cdots \gamma_{a_m}\gamma_{b_1} \cdots \gamma_{b_n}$.
If $a_i \neq b_j \forall i, j$, then changing the order of the γ_{a_i} and γ_{b_j} (at the expense of some -1 factors) to make the sequence in ascending order of indices means $\Gamma_A\Gamma_B \in G$. If $a_i = b_j$ for some i and j, then changing the order to make them adjacent means $\gamma_{a_i}\gamma_{b_j} = I$ and those two γs are removed. This can be done until no a_i and b_j are equal.
$\therefore \Gamma_A\Gamma_B \in G$ again $\implies$ The binary operation is well defined. □

$\{\gamma_a\}_{a=0}^{D-1}$ is irreducible $\iff$ there is no subspace of $\mathbb{C}^N$ invariant under all γ_a.
$\therefore$ As $\{\gamma_a\}_{a=0}^{D-1} \subset G$, the elements of G also have no common invariant subspace.
$\therefore$ The irreducible representation of the Clifford algebra has automatically lead to an irreducible representation of G in the same representation space.

Theorem 1.2. *The dimension of an irreducible representation's representation space N, can only be $2^{\lfloor D/2 \rfloor}$.*

Proof. Let Y be an arbitrary $N \times N$ matrix and let

$$S = \sum_{A=0}^{2^D-1} (\Gamma_A)^{-1}Y\Gamma_A \tag{10}$$

where I've adopted the convention of explicitly showing all summations on the $A, B, \ldots$ indices.

$$\therefore (\Gamma_B)^{-1}S\Gamma_B = \sum_{A=0}^{2^D-1} (\Gamma_B)^{-1}(\Gamma_A)^{-1}Y\Gamma_A\Gamma_B = \sum_{A=0}^{2^D-1} (\Gamma_A\Gamma_B)^{-1}Y\Gamma_A\Gamma_B \tag{11}$$

$\Gamma_B\Gamma_A \in G$ and as Γ_B is invertible, $\Gamma_{A_1}\Gamma_B = \pm\Gamma_{A_2}\Gamma_B \implies \Gamma_{A_1} = \pm\Gamma_{A_2}$.
$\therefore \{\Gamma_A\Gamma_B\}_{A=0}^{2^D-1} = \{\pm\Gamma_C\}_{C=0}^{2^D-1}$ where on the RHS, a $+$ or $-$ is chosen for each C depending on

whether $\Gamma_A\Gamma_B = \Gamma_C$ or $\Gamma_A\Gamma_B = -\Gamma_C$ (hence $\{\pm\Gamma_C\}_{C=0}^{2^D-1}$ has only half as many elements as the group, G).

$$\therefore (\Gamma_B)^{-1}S\Gamma_B = \sum_{C=0}^{2^D-1}(\pm\Gamma_C)^{-1}Y(\pm\Gamma_C) = \sum_{C=0}^{2^D-1}(\Gamma_C)^{-1}Y\Gamma_C = S \tag{12}$$

$\therefore S\Gamma_B = \Gamma_B S \ \ \forall B.$
$\therefore S = \lambda I$ for some $\lambda \in \mathbb{C}$ by Schur's lemma.

$$\therefore \lambda I = \sum_{A=0}^{2^D-1}(\Gamma_A)^{-1}Y\Gamma_A \tag{13}$$

$$\therefore \mathrm{tr}(\lambda I) = \mathrm{tr}\left(\sum_{A=0}^{2^D-1}(\Gamma_A)^{-1}Y\Gamma_A\right) \tag{14}$$

$$\therefore \lambda N = \sum_{A=0}^{2^D-1}\mathrm{tr}((\Gamma_A)^{-1}Y\Gamma_A) = \sum_{A=0}^{2^D-1}\mathrm{tr}(\Gamma_A(\Gamma_A)^{-1}Y) = 2^D\mathrm{tr}(Y) \tag{15}$$

$$\therefore \lambda = \frac{2^D\mathrm{tr}(Y)}{N} \implies \frac{2^D\mathrm{tr}(Y)}{N}I = \sum_{A=0}^{2^D-1}(\Gamma_A)^{-1}Y\Gamma_A \tag{16}$$

In the last equation,

$$\mathrm{LHS} = \frac{2^D Y_{kk}}{N}\delta_{ij} = \frac{2^D}{N}\delta_{kl}\delta_{ij}Y_{kl} \tag{17}$$

$$\mathrm{RHS} = \sum_{A=0}^{2^D-1}(\Gamma_A^{-1})_{ik}Y_{kl}(\Gamma_A)_{lj} \tag{18}$$

Then, since Y_{kl} is arbitrary,

$$\mathrm{LHS} = \mathrm{RHS} \implies \frac{2^D}{N}\delta_{kl}\delta_{ij} = \sum_{A=0}^{2^D-1}(\Gamma_A^{-1})_{ik}(\Gamma_A)_{lj} \tag{19}$$

$$\therefore \frac{2^D}{N}\delta_{ij}\delta_{ij} = \sum_{A=0}^{2^D-1}(\Gamma_A^{-1})_{ii}(\Gamma_A)_{jj} \tag{20}$$

$$\iff 2^D = \sum_{A=0}^{2^D-1}\mathrm{tr}(\Gamma_A)\mathrm{tr}((\Gamma_A)^{-1}) \tag{21}$$

Let $\Gamma_A = \gamma_{a_1}\cdots\gamma_{a_n}$ for some $1 \le n \le D-1$ (any Γ_A other than $\Gamma_0 = I$ and $\Gamma_{2^D-1} = \gamma_0\cdots\gamma_{D-1}$ can be written in such a form by definition).
$\therefore \exists b \in \{0, 1, \cdots, D-1\}$ such that $b \neq a_i \ \ \forall i$. Then, if n is odd,

$$\begin{aligned}(\gamma_b)^{-1}\Gamma_A\gamma_b &= \gamma_b\gamma_{a_1}\cdots\gamma_{a_n}\gamma_b &(22)\\ &= (\gamma_b)^2(-1)^n\gamma_{a_1}\cdots\gamma_{a_n} &(23)\\ &= (-1)^n\Gamma_A &(24)\\ &= -\Gamma_A \quad \text{as } n \text{ is odd} &(25)\end{aligned}$$

$$\therefore \mathrm{tr}((\gamma_b)^{-1}\Gamma_A\gamma_b) = \mathrm{tr}(-\Gamma_A) \iff \mathrm{tr}(\Gamma_A) = \mathrm{tr}(-\Gamma_A) \implies \mathrm{tr}(\Gamma_A) = 0 \tag{26}$$

On the other hand, if n is even,

$$\begin{aligned}(\gamma_{a_1})^{-1}\Gamma_A\gamma_{a_1} &= \gamma_{a_1}\gamma_{a_1}\cdots\gamma_{a_n}\gamma_{a_1} && (27)\\ &= \gamma_{a_1}\gamma_{a_1}\gamma_{a_1}(-1)^{n-1}\gamma_{a_2}\cdots\gamma_{a_n} && (28)\\ &= (-1)^{n-1}\gamma_{a_1}\cdots\gamma_{a_n} && (29)\\ &= -\Gamma_A \quad \text{as } n \text{ is even} && (30)\end{aligned}$$

$\therefore \mathrm{tr}(\Gamma_A) = 0$ by the same logic as before.
Hence, in equation 21, the only non-traceless matrices in the sum are when $A = 0$ and when $A = 2^D - 1$.

$$\begin{aligned}\therefore 2^D &= \mathrm{tr}(I)\mathrm{tr}(I^{-1}) + \mathrm{tr}(\gamma_0\cdots\gamma_{D-1})\mathrm{tr}((\gamma_0\cdots\gamma_{D-1})^{-1}) && (31)\\ &= N^2 + \mathrm{tr}(\gamma_0\cdots\gamma_{D-1})\mathrm{tr}((\gamma_0\cdots\gamma_{D-1})^{-1}) && (32)\end{aligned}$$

It will now be necessary to consider D even and odd separately; I'll start with the former.

$$\begin{aligned}\therefore \mathrm{tr}(\gamma_0\cdots\gamma_{D-1}) &= \mathrm{tr}(\gamma_{D-1}\gamma_0\cdots\gamma_{D-2}) && (33)\\ &= \mathrm{tr}(\gamma_0\cdots\gamma_{D-1}(-1)^{D-1}) && (34)\\ &= \mathrm{tr}(-\gamma_0\cdots\gamma_{D-1}) \quad \text{as } D \text{ is even} && (35)\\ \therefore \mathrm{tr}(\gamma_0\cdots\gamma_{D-1}) &= 0 && (36)\\ \therefore 2^D &= N^2 && (37)\\ \therefore N &= 2^{D/2} = 2^{\lfloor D/2\rfloor} && (38)\end{aligned}$$

However, when D is odd,

$$\begin{aligned}\gamma_a\gamma_0\cdots\gamma_{D-1} &= \gamma_a\gamma_0\cdots\gamma_{a-1}\gamma_a\gamma_{a+1}\cdots\gamma_{D-1} && (39)\\ &= \gamma_0\cdots\gamma_{a-1}\gamma_a(-1)^a\gamma_a\gamma_{a+1}\cdots\gamma_{D-1} && (40)\\ &= \gamma_0\cdots\gamma_{a-1}\gamma_a(-1)^a\gamma_{a+1}\cdots\gamma_{D-1}\gamma_a(-1)^{D-a-1} && (41)\\ &= (-1)^{D-1}\gamma_0\cdots\gamma_{D-1}\gamma_a && (42)\\ &= \gamma_0\cdots\gamma_{D-1}\gamma_a \quad \text{as } D \text{ is odd} && (43)\end{aligned}$$

Then, since all elements of G are products of the γs and possibly a factor of -1,
$g\gamma_0\cdots\gamma_{D-1} = \gamma_0\cdots\gamma_{D-1}g \quad \forall g \in G$.
$\therefore \gamma_0\cdots\gamma_{D-1} = \lambda I$ for some $\lambda \in \mathbb{C}\setminus\{0\}$ by Schur's lemma (not the same λ as before).

$$\begin{aligned}\therefore 2^D &= N^2 + \mathrm{tr}(\lambda I)\mathrm{tr}((\lambda I)^{-1}) && (44)\\ &= N^2 + (N\lambda)\left(\frac{N}{\lambda}\right) && (45)\\ &= 2N^2 && (46)\\ \therefore N &= 2^{(D-1)/2} = 2^{\lfloor D/2\rfloor} && (47)\end{aligned}$$

Hence, for any dimension, D, N is uniquely determined to be $2^{\lfloor D/2\rfloor}$. □

The previous theorem uniquely determines the representation space's dimension, but as yet I've said nothing about the number of inequivalent representations in $\mathbb{C}^N$.

Theorem 1.3. *For even dimensions, a finite dimensional, complex, irreducible representation of the Clifford algebra is unique up to equivalence, where as in odd dimensions, there are two inequivalent representations related by a factor of* -1*.*

Proof. Let $\{\gamma_a\}_{a=0}^{D-1}$ and $\{\tilde{\gamma}_a\}_{a=0}^{D-1}$ be two inequivalent, finite dimensional, complex irreducible representations of the Clifford algebra. Let G and $\tilde{G}$ be the two corresponding finite groups generated as before. For an arbitrary $N \times N$ matrix, Y, this time let

$$S = \sum_{A=0}^{2^D-1} (\Gamma_A)^{-1} Y \tilde{\Gamma}_A \tag{48}$$

$$\therefore (\Gamma_B)^{-1} S \tilde{\Gamma}_B = \sum_{A=0}^{2^D-1} (\Gamma_B)^{-1} (\Gamma_A)^{-1} Y \tilde{\Gamma}_A \tilde{\Gamma}_B \tag{49}$$

$$= \sum_{A=0}^{2^D-1} (\Gamma_A \Gamma_B)^{-1} Y \tilde{\Gamma}_A \tilde{\Gamma}_B \tag{50}$$

$$= \sum_{C=0}^{2^D-1} (\Gamma_C)^{-1} Y \tilde{\Gamma}_C \tag{51}$$

$$= S \tag{52}$$

$$\therefore S\tilde{\Gamma}_B = \Gamma_B S \quad \forall B \tag{53}$$

with the 3rd last line following by the same reasoning as equation 12. Now, since the representations of G & $\tilde{G}$ are inequivalent, $S\tilde{\Gamma}_B = \Gamma_B S \implies S = 0$ by Schur's 2nd lemma.

$$\therefore \sum_{A=0}^{2^D-1} (\Gamma_A^{-1})_{ik} Y_{kl} (\tilde{\Gamma}_A)_{lj} = 0 \tag{54}$$

However, since Y_{kl} is arbitrary, it must be that

$$0 = \sum_{A=0}^{2^D-1} (\Gamma_A^{-1})_{ik} (\tilde{\Gamma}_A)_{lj} \tag{55}$$

$$\therefore 0 = \sum_{A=0}^{2^D-1} (\Gamma_A^{-1})_{ii} (\tilde{\Gamma}_A)_{jj} \tag{56}$$

$$= \sum_{A=0}^{2^D-1} \mathrm{tr}((\Gamma_A)^{-1}) \mathrm{tr}(\tilde{\Gamma}_A) \tag{57}$$

For even D, it was shown in the proof of theorem 1.2 that $\tilde{\Gamma}_0 = I$ is the only one of the $\tilde{\Gamma}_A$s that is not traceless.
$\therefore 0 = \mathrm{tr}(I^{-1})\mathrm{tr}(I) = N^2 \implies N = 0$, contradicting theorem 1.2.
$\therefore$ For even dimensions, there could not have been two inequivalent representations to begin with, thereby proving the 1st half of the theorem.

Meanwhile for odd D, it was shown in the proof of theorem 1.2 that $\tilde{\Gamma}_0 = I$ and $\tilde{\Gamma}_{D-1} = \tilde{\lambda} I$ are the only non-traceless Γ_As.

$$\therefore 0 = \mathrm{tr}(I^{-1})\mathrm{tr}(I) + \mathrm{tr}((\lambda I)^{-1})\mathrm{tr}(\tilde{\lambda} I) \tag{58}$$

$$= N^2 + \frac{\tilde{\lambda}}{\lambda} N^2 \tag{59}$$

$$\therefore \tilde{\lambda} = -\lambda \tag{60}$$

Because of this result, there cannot be a 3rd inequivalent representation as follows.
Let $\{\gamma'_a\}_{a=0}^{D-1}$ be a 3rd inequivalent representation. Then, considering the three representations

pairwise, $\lambda = -\tilde{\lambda}$, $\lambda' = -\tilde{\lambda}$ and $\lambda' = -\lambda$. The 1st and 3rd of these equations together imply $\lambda' = \tilde{\lambda}$, which contradicts the 2nd equation.

There could yet be two inequivalent representations though. Let $\tilde{\gamma}_a = -\gamma_a$. Then,

$$\tilde{\gamma}_a\tilde{\gamma}_b + \tilde{\gamma}_b\tilde{\gamma}_a = (-\gamma_a)(-\gamma_b) + (-\gamma_b)(-\gamma_a) = \gamma_a\gamma_b + \gamma_b\gamma_a = 2\delta_{ab}I \tag{61}$$

$\therefore \{\tilde{\gamma}_a\}_{a=0}^{D-1} = \{-\gamma_a\}_{a=0}^{D-1}$ also satisfies the Clifford algebra.
Assume $\exists$ an $N \times N$ matrix, A, such that $\tilde{\gamma}_a = A^{-1}\gamma_a A$ for a contradiction.

$$\begin{aligned}
\therefore \tilde{\gamma}_0 \cdots \tilde{\gamma}_{D-1} &= C^{-1}\gamma_0 C \cdots C^{-1}\gamma_{D-1}C && (62)\\
&= C^{-1}\gamma_0 \cdots \gamma_{D-1}C && (63)\\
&= C^{-1}\lambda I C && (64)\\
&= \lambda I && (65)
\end{aligned}$$

However, $\tilde{\gamma}_0 \cdots \tilde{\gamma}_{D-1} = (-1)^D\gamma_0 \cdots \gamma_{D-1} = -\lambda I$.
$\therefore \lambda I = -\lambda I$, which contradicts $\lambda \neq 0$.
$\therefore$ In odd dimensions, $\{\gamma_a\}_{a=0}^{D-1}$ and $\{-\gamma_a\}_{a=0}^{D-1}$ are indeed inequivalent representations, hence completing the proof of all parts of the theorem. $\square$

Having established these properties, it's time to return to the general Clifford algebra, $\{\gamma_a, \gamma_b\} = -2\eta_{ab}I$, where the previous two theorems will continue to hold via the reasons outlined earlier. Spinors can now be defined as the N components of $\mathbb{C}^N$, the representation space of the Clifford algebra. As I'll outline, these spinors will allow representations of the spin groups (the universal covering groups of $\mathrm{SO}^{\uparrow}(s,t)$).
From hereon, let $\gamma_0 \cdots \gamma_{D-1}$ be denoted by γ_{D+1}.

Let $\Lambda^a{}_b \in \mathrm{SO}^{\uparrow}(s,t)$ and let $\gamma'_a = (\Lambda^{-1})^b{}_a\gamma_b$, i.e. as if γ^a was a Lorentz vector.

$$\begin{aligned}
\therefore \gamma'_a\gamma'_b + \gamma'_b\gamma'_a &= (\Lambda^{-1})^c{}_a(\Lambda^{-1})^d{}_b(\gamma_c\gamma_d + \gamma_d\gamma_c) && (66)\\
&= -2\eta_{cd}(\Lambda^{-1})^c{}_a(\Lambda^{-1})^d{}_b I && (67)\\
&= -2\eta_{ab}I \quad \text{by the defining properties of } \mathrm{SO}^{\uparrow}(s,t) && (68)
\end{aligned}$$

$\therefore \{\gamma'_a\}_{a=0}^{D-1}$ also satisfy the Clifford algebra.
$\therefore$ In even dimensions, since the irreducible representation is unique, $\exists S(\Lambda)$ such that $\gamma'_a = S(\Lambda)^{-1}\gamma_a S(\Lambda)$. However, in odd dimensions, both $\gamma'_a = S(\Lambda)^{-1}\gamma_a S(\Lambda)$ and $\gamma'_a = S(\Lambda)^{-1}(-\gamma_a)S(\Lambda)$ could be possible by the previous theorem. Consider the latter case.
$\gamma_{D+1} = \gamma_0 \cdots \gamma_{D-1} = \frac{1}{N!}\varepsilon^{a_1 \cdots a_D}\gamma_{a_1} \cdots \gamma_{a_D}$ by anticommutativity.

$$\begin{aligned}
\therefore S(\Lambda)^{-1}\gamma_{D+1}S(\Lambda) &= \frac{1}{N!}\varepsilon^{a_1\cdots a_D}S(\Lambda)^{-1}\gamma_{a_1}S(\Lambda)\cdots S(\Lambda)^{-1}\gamma_{a_D}S(\Lambda) && (69)\\
&= \frac{(-1)^D}{N!}\varepsilon^{a_1\cdots a_D}\gamma'_{a_1}\cdots\gamma'_{a_D} && (70)\\
&= \frac{(-1)^D}{N!}\varepsilon^{a_1\cdots a_D}(\Lambda^{-1})^{b_1}{}_{a_1}\cdots(\Lambda^{-1})^{b_D}{}_{a_D}\gamma_{b_1}\cdots\gamma_{b_D} && (71)\\
&= \frac{(-1)^D}{N!}\det(\Lambda^{-1})\varepsilon^{b_1\cdots b_D}\gamma'_{b_1}\cdots\gamma'_{b_D} && (72)\\
&= -\gamma_{D+1} \quad \text{as } D \text{ is odd and } \det(\Lambda^{-1}) = 1 && (73)
\end{aligned}$$

However, I showed earlier that in odd dimensions, $\gamma_{D+1} = \lambda I$ for some complex $\lambda \neq 0$.
$\therefore$ The last equation says $S(\Lambda)^{-1}\lambda I S(\Lambda) = -\lambda I \iff \lambda I = -\lambda I \iff \lambda = 0$, which

contradicts $\lambda \neq 0$.
$\therefore$ Even in odd dimensions, $\gamma'_a = S(\Lambda)^{-1}\gamma_a S(\Lambda)$. Hence, in any dimension,

$$\begin{aligned}
S(\Lambda_1)^{-1}S(\Lambda_2)^{-1}\gamma_a S(\Lambda_2)S(\Lambda_1) &= S(\Lambda_1)^{-1}(\Lambda_2^{-1})^b{}_a\gamma_b S(\Lambda_1) && (74)\\
&= (\Lambda_1^{-1})^c{}_b(\Lambda_2^{-1})^b{}_a\gamma_c && (75)\\
&= ((\Lambda_2\Lambda_1)^{-1})^b{}_a\gamma_b && (76)\\
&= S(\Lambda_2\Lambda_1)^{-1}\gamma_a S(\Lambda_2\Lambda_1) && (77)\\
\therefore \gamma_a S(\Lambda_2)S(\Lambda_1)S(\Lambda_2\Lambda_1)^{-1} &= S(\Lambda_2)S(\Lambda_1)S(\Lambda_2\Lambda_1)^{-1}\gamma_a && (78)
\end{aligned}$$

Since the last equation holds $\forall a$, $gS(\Lambda_2)S(\Lambda_1)S(\Lambda_2\Lambda_1)^{-1} = S(\Lambda_2)S(\Lambda_1)S(\Lambda_2\Lambda_1)^{-1}g \ \ \forall g \in G$.
$\therefore$ By Schur's lemma, $S(\Lambda_2)S(\Lambda_1)S(\Lambda_2\Lambda_1)^{-1} = f(\Lambda_2,\Lambda_1)I \iff S(\Lambda_1)S(\Lambda_2) = f(\Lambda_1,\Lambda_2)S(\Lambda_1\Lambda_2)$ for some $f(\Lambda_1,\Lambda_2) \in \mathbb{C}$.
$\therefore$ S is a projective representation of $\mathrm{SO}^{\uparrow}(s,t)$.
In general, this is the best that can be done for $\mathrm{SO}^{\uparrow}(s,t)$. However, since $S(\Lambda)^{-1}\gamma_a S(\Lambda) = (\Lambda^{-1})^b{}_a\gamma_b$ is invariant under $S(\Lambda) \to \beta S(\Lambda)$ for any $\beta \in \mathbb{C}\setminus\{0\}$, S can be extended to a representation of $\mathrm{Spin}(s,t)$, the universal covering group of $\mathrm{SO}^{\uparrow}(s,t)$. In this case, it can be shown[3] S can be made into a linear representation, rather than only a projective representation. This property distinguishes the spinor representation from other tensor representations; spinors facilitate a representation of $\mathrm{Spin}(s,t)$, not $\mathrm{SO}^{\uparrow}(s,t)$.
$\therefore$ From henceforth, let $S(\Lambda) \equiv S(N)$ where N is a pre-image of Λ under the covering map.

A natural way to generate a representation of $\mathrm{Spin}(s,t)$, is to exponentiate[4] elements of $\mathfrak{spin}(s,t)$. Since a group and its universal cover are locally isomorphic, $\mathfrak{spin}(s,t) \cong \mathfrak{so}^{\uparrow}(\mathfrak{s},\mathfrak{t})$.
$\therefore$ One must study the connection between Lorentz groups and Clifford algebras at the level of Lie algebras. To do so, let $M_{ab} = -\frac{1}{4}[\gamma_a,\gamma_b]$.

$$\begin{aligned}
\therefore [M_{ab},M_{cd}] &= \frac{1}{16}[[\gamma_a,\gamma_b],[\gamma_c,\gamma_d]] && (79)\\
&= \frac{1}{16}[\gamma_a\gamma_b-\gamma_b\gamma_a,\gamma_c\gamma_d-\gamma_d\gamma_c] && (80)\\
&= \frac{1}{16}(\gamma_a\gamma_b-\gamma_b\gamma_a)(\gamma_c\gamma_d-\gamma_d\gamma_c) - \frac{1}{16}(\gamma_c\gamma_d-\gamma_d\gamma_c)(\gamma_a\gamma_b-\gamma_b\gamma_a)v && (81)\\
&= \frac{1}{16}(\gamma_a\gamma_b\gamma_c\gamma_d - \gamma_a\gamma_b\gamma_d\gamma_c - \gamma_b\gamma_a\gamma_c\gamma_d + \gamma_b\gamma_a\gamma_d\gamma_c - \gamma_c\gamma_d\gamma_a\gamma_b + \gamma_c\gamma_d\gamma_b\gamma_a\\
&\qquad + \gamma_d\gamma_c\gamma_a\gamma_b - \gamma_d\gamma_c\gamma_b\gamma_a) && (82)
\end{aligned}$$

Using the Clifford algebra,

$$\begin{aligned}
\gamma_c\gamma_d\gamma_a\gamma_b &= -\gamma_c\gamma_a\gamma_d\gamma_b - 2\eta_{ad}\gamma_c\gamma_b && (83)\\
&= \gamma_a\gamma_c\gamma_d\gamma_b + 2\eta_{ac}\gamma_d\gamma_b - 2\eta_{ad}\gamma_c\gamma_b && (84)\\
&= -\gamma_a\gamma_c\gamma_b\gamma_d - 2\eta_{bd}\gamma_a\gamma_c + 2\eta_{ac}\gamma_d\gamma_b - 2\eta_{ad}\gamma_c\gamma_b && (85)\\
&= \gamma_a\gamma_b\gamma_c\gamma_d + 2\eta_{bc}\gamma_a\gamma_d - 2\eta_{bd}\gamma_a\gamma_c + 2\eta_{ac}\gamma_d\gamma_b - 2\eta_{ad}\gamma_c\gamma_b && (86)\\
\therefore \gamma_a\gamma_b\gamma_c\gamma_d - \gamma_c\gamma_d\gamma_a\gamma_b &= 2(\eta_{ad}\gamma_c\gamma_b - \eta_{ac}\gamma_d\gamma_b + \eta_{bd}\gamma_a\gamma_c - \eta_{bc}\gamma_a\gamma_d) && (87)
\end{aligned}$$

[3] I will sketch how this can be done below and in the next section
[4] I will have an example in the next section

$\gamma_c\gamma_d\gamma_b\gamma_a - \gamma_b\gamma_a\gamma_c\gamma_d$, $\gamma_d\gamma_c\gamma_a\gamma_b - \gamma_a\gamma_b\gamma_d\gamma_c$ and $\gamma_b\gamma_a\gamma_d\gamma_c - \gamma_d\gamma_c\gamma_b\gamma_a$ follow by relabelling indices.

$$\begin{aligned}
\therefore [M_{ab}, M_{cd}] = \frac{1}{8}(&\eta_{ad}\gamma_c\gamma_b - \eta_{ac}\gamma_d\gamma_b + \eta_{bd}\gamma_a\gamma_c - \eta_{bc}\gamma_a\gamma_d \\
&+ \eta_{ca}\gamma_b\gamma_d - \eta_{cb}\gamma_a\gamma_d + \eta_{da}\gamma_c\gamma_b - \eta_{db}\gamma_c\gamma_a \\
&+ \eta_{db}\gamma_a\gamma_c - \eta_{da}\gamma_b\gamma_c + \eta_{cb}\gamma_d\gamma_a - \eta_{ca}\gamma_d\gamma_b \\
&+ \eta_{bc}\gamma_d\gamma_a - \eta_{bd}\gamma_c\gamma_a + \eta_{ac}\gamma_b\gamma_d - \eta_{ad}\gamma_b\gamma_c) && (88) \\
= \frac{1}{4}(&\eta_{ad}[\gamma_c, \gamma_b] + \eta_{ac}[\gamma_b, \gamma_d] + \eta_{bd}[\gamma_a, \gamma_c] + \eta_{bc}[\gamma_d, \gamma_a]) && (89) \\
= \ &\eta_{ad}M_{bc} - \eta_{ac}M_{bd} + \eta_{bc}M_{ad} - \eta_{bd}M_{ac} && (90)
\end{aligned}$$

$\therefore M_{ab} = -\frac{1}{4}[\gamma_a, \gamma_b]$ satisfy the Lie algebra of $\mathfrak{so}^{\uparrow}(3,1)$, i.e. M_{ab} are Lorentz generators.

It's now time to study the effects of these transformation properties of the Clifford algebra on the properties of spinors themselves. Spinors were originally used most prominently in physics in the context of the Dirac equation,

$$(\mathrm{i}\gamma^a \nabla_a - q\gamma^a A_a(x) - m)\Psi(x) = 0 \tag{91}$$

where Ψ is a $2^{\lfloor D/2 \rfloor}$ component spinor. To be a well defined equation of motion, the Dirac equation must transform covariantly.
$\therefore$ Under a local Lorentz transformation, ${e'}_a{}^m(x) = (\Lambda^{-1})^b{}_a e_b{}^m(x)$, the Dirac equation must be $0 = (\mathrm{i}\gamma^a\nabla'_a - q\gamma^a A'_a(x) - m)\Psi'(x)$. This equation still has γ^a, not γ'^a, because despite appearances, γ^a are supposed to be a set of constant matrices; they cannot be different for different observers.
Since $\nabla_a = \Lambda^b{}_a \nabla'_b$ and $A_a = \Lambda^b{}_a A'_b$, the original Dirac equation can be re-written as

$$\begin{aligned}
0 &= (\mathrm{i}\gamma^a \nabla_a - q\gamma^a A_a(x) - m)\Psi(x) && (92) \\
&= (\Lambda^b{}_a \gamma^a(\mathrm{i}\nabla'_b - qA'_b(x)) - m)\Psi(x) && (93)
\end{aligned}$$

Earlier, I showed that $\gamma'_a = (\Lambda^{-1})^b{}_a \gamma_a \implies \gamma'_a = S(\Lambda)^{-1}\gamma_a S(\Lambda)$ for some group representation, $S(\Lambda)$. Let $T(\Lambda)$ be the corresponding representation for contravariant indices[5],
i.e. $\gamma'^a = \Lambda^a{}_b \gamma^b \implies \gamma'^a = T(\Lambda)^{-1}\gamma^a T(\Lambda)$.

$$\begin{aligned}
\therefore 0 &= (T(\Lambda)^{-1}\gamma^b T(\Lambda)(\mathrm{i}\nabla'_b - qA'_b(x)) - m)\Psi(x) && (94) \\
&= T(\Lambda)^{-1}(\gamma^a(\mathrm{i}\nabla'_a - qA'_a(x)) - m)T(\Lambda)\Psi(x) && (95) \\
\therefore 0 &= (\mathrm{i}\gamma^a\nabla'_a - q\gamma^a A'_a(x) - m)T(\Lambda)\Psi(x) && (96)
\end{aligned}$$

$\therefore$ It must be that $\Psi'(x) = T(\Lambda)\Psi(x)$. This defines the transformation property of spinors[6].
If one restricts attention to special relativity, then the transformation of interest is $x'^a = \Lambda^a{}_b x^b$.
Then, the Dirac equation is $0 = (\mathrm{i}\gamma^a\partial_a - q\gamma^a A_a(x) - m)\Psi(x)$ and the transformation property required of spinors is $\Psi'(x') = T(\Lambda)\Psi(x)$, or equivalently $\Psi'(x) = T(\Lambda)\Psi(\Lambda^{-1}x)$.

There are still many properties of spinors left to consider. For "calculation" purposes, it will be useful to choose a basis in the spinor/representation space of the Clifford algebra. As G is a finite group, $\exists$ an inner product (that's unique up to scaling) invariant under the action

[5] T and S might be equivalent representations, or even the same representation, but there's nothing to gain by being intelligent here. It will be fine to simply treat them separately.

[6] Rather than take the Dirac equation as fundamental and derive spinors' transformation properties from there, a more mathematical perspective would be to define spinors to transform as $\Psi'(x) = T(\Lambda)\Psi(x)$ and use that to prove the Dirac equation transforms covariantly.

of the representation. Since scaling is arbitrary, any scaling of this unique inner product can be chosen. Then, choose a basis that's orthonormal with respect to this inner product.
$\therefore$ In this basis, all γ_a are unitary, i.e. $\gamma_a^\dagger = (\gamma_a)^{-1}$.
However, $\gamma_a\gamma_b + \gamma_b\gamma_a = -2\eta_{ab}I \implies (\gamma_a)^2 = -\eta_{aa}I$ (no sum).
$\therefore (\gamma_a)^{-1} = \gamma_a$ for $0 \leq a \leq t-1$ and $(\gamma_a)^{-1} = -\gamma_a$ for $t \leq a \leq s+t-1$.
$\therefore \gamma_a^\dagger = \gamma_a$ for $0 \leq a \leq t-1$ and $\gamma_a^\dagger = -\gamma_a$ for $t \leq a \leq s+t-1$.

Theorem 1.4. *Let $A = \gamma_0\gamma_1\cdots\gamma_{t-1}$. Then, A is unitary and $\gamma_a^\dagger = (-1)^{t+1}A\gamma_a A^{-1}$.*

Proof. For $0 \leq a \leq t-1$, $\gamma_a^\dagger = (\gamma_a)^{-1} = \gamma_a$.

$$\begin{aligned}
\therefore A^\dagger A &= (\gamma_0\cdots\gamma_{t-1})^\dagger(\gamma_0\cdots\gamma_{t-1}) && (97)\\
&= \gamma_{t-1}^\dagger\cdots\gamma_0^\dagger\gamma_0\cdots\gamma_{t-1} && (98)\\
&= (\gamma_{t-1})^{-1}\cdots(\gamma_0)^{-1}\gamma_0\cdots\gamma_{t-1} && (99)\\
&= I \quad \implies A \text{ is unitary} && (100)
\end{aligned}$$

For $0 \leq b \leq t-1$, $(\gamma_b)^{-1} = \gamma_b$ and hence $A^{-1} = \gamma_{t-1}\cdots\gamma_0$.
For $t \leq a \leq s+t-1$,

$$\begin{aligned}
(-1)^{t+1}A\gamma_a A^{-1} &= (-1)^{t+1}\gamma_0\cdots\gamma_{t-1}\gamma_a\gamma_{t-1}\cdots\gamma_0 && (101)\\
&= (-1)^{t-1}\gamma_a(-1)^t\gamma_0\cdots\gamma_{t-1}\gamma_{t-1}\cdots\gamma_0 && (102)\\
&= (-1)^{2t+1}\gamma_a && (103)\\
&= -\gamma_a && (104)\\
&= \gamma_a^\dagger && (105)
\end{aligned}$$

For $0 \leq a \leq t-1$,

$$\begin{aligned}
(-1)^{t+1}A\gamma_a A^{-1} &= (-1)^{t+1}\gamma_0\cdots\gamma_{t-1}\gamma_a\gamma_{t-1}\cdots\gamma_0 && (106)\\
&= (-1)^{t-1}\gamma_0\cdots\gamma_{t-1}\gamma_a\gamma_{t-1}\cdots\gamma_a\cdots\gamma_0 \ \text{(no sum)} && (107)\\
&= (-1)^{t+1}\gamma_0\cdots\gamma_{t-1}\gamma_{t-1}\cdots\gamma_a(-1)^{t-a-1}\gamma_a\cdots\gamma_0 && (108)\\
&= (-1)^{2t-a}\gamma_0\cdots\gamma_a\gamma_a\gamma_a\gamma_{a-1}\cdots\gamma_0 && (109)\\
&= (-1)^a\gamma_0\cdots\gamma_{a-1}\gamma_a\gamma_{a-1}\cdots\gamma_0 && (110)\\
&= (-1)^a\gamma_0\cdots\gamma_{a-1}\gamma_{a-1}\cdots\gamma_0\gamma_a(-1)^a && (111)\\
&= (-1)^{2a}\gamma_a && (112)\\
&= \gamma_a && (113)\\
&= \gamma_a^\dagger && (114)
\end{aligned}$$

$\therefore \gamma_a^\dagger = (-1)^{t+1}A\gamma_a A^{-1}$ in general. □

To derive the next few results, restrict attention to the case of D being even.

$$\begin{aligned}
(\pm\gamma_a)^*(\pm\gamma_b)^* + (\pm\gamma_b)^*(\pm\gamma_a)^* &= (\gamma_a\gamma_b + \gamma_b\gamma_a)^* && (115)\\
&= (-2\eta_{ab}I)^* && (116)\\
&= -2\eta_{ab}I && (117)
\end{aligned}$$

$\therefore \{\pm\gamma_a^*\}_{a=0}^{D-1}$ also satisfy the Clifford algebra.
Since the irreducible representation of the Clifford algebra is unique in even dimensions, $\exists$ matrices, B_1 and B_2, such that $\gamma_a^* = B_1\gamma_a(B_1)^{-1}$ and $-\gamma_a^* = B_2\gamma_a(B_2)^{-1}$. These two equations

can be wrapped together by saying $\gamma_a^* = \mu B\gamma_a B^{-1}$ where $\mu = \pm 1$. Here μ and B are taken to be interdependent, e.g. if $\mu = 1$, then $B = B_1$ while if $\mu = -1$, then $B = B_2$.

$$\begin{aligned}
\gamma_a^* = \mu B\gamma_a B^{-1} \implies \gamma_a &= \mu B^*\gamma_a^* B^{-*} && (118)\\
&= \mu B^*\mu B\gamma_a B^{-1}B^{-*} && (119)\\
&= B^*B\gamma_a(B^*B)^{-1} && (120)\\
\therefore \gamma_a B^*B &= B^*B\gamma_a \quad \forall a && (121)
\end{aligned}$$

$\therefore$ By Schur's lemma, $B^*B = \nu I$ for some $\nu \in \mathbb{C} \setminus \{0\}$.
$\therefore BB^* = \nu I$ as well since a matrix and its inverse commute.
$\therefore (BB^*)^* = \nu^* I \implies B^*B = \nu^* I \implies \nu I = \nu^* I \implies \nu \in \mathbb{R} \setminus \{0\}$. Then,

$$\begin{aligned}
BB^* = \nu I \implies \det(BB^*) &= \det(\nu I) && (122)\\
\therefore \det(B)\det(B^*) &= \nu^{2^{D/2}}\det(I) && (123)\\
\therefore \nu^{2^{D/2}} &= |\det(B)|^2 && (124)
\end{aligned}$$

For any $k \in \mathbb{C} \setminus \{0\}$, $(kB)\gamma_a(kB)^{-1} = B\gamma_a B^{-1} = \gamma_a^*$.
$\therefore B$ can be scaled without loss of generality as its definition only relies on $\mu B\gamma_a B^{-1} = \gamma_a^*$.
$\therefore$ Scale B so that $\det(B) = 1$.
$\therefore \nu^{2^{D/2}} = 1$ and hence $\nu = \pm 1$.
Since γ_a are unitary,

$$\begin{aligned}
I = \gamma_a\gamma_a^\dagger = \gamma_a(\gamma_a^*)^T &= \gamma_a\mu B^{-T}\gamma_a^T B^T && (125)\\
\therefore I^* &= (\mu\gamma_a B^{-T}\gamma_a^T B^T)^* && (126)\\
\therefore I &= \mu(\gamma_a B^{-T}\gamma_a^T B^T)^* && (127)\\
&= \mu\gamma_a^* B^{-\dagger}\gamma_a^\dagger B^\dagger && (128)\\
&= \mu^2 B\gamma_a B^{-1}B^{-\dagger}\gamma_a^\dagger B^\dagger && (129)\\
&= B\gamma_a B^{-1}B^{-\dagger}\gamma_a^\dagger B^\dagger && (130)\\
\therefore B^{-\dagger}\gamma_a &= B\gamma_a B^{-1}B^{-\dagger} && (131)\\
\therefore \gamma_a B^\dagger B &= B^\dagger B\gamma_a \quad \forall a && (132)
\end{aligned}$$

$\therefore$ By Schur's lemma, $B^\dagger B = \rho I$ for some $\rho \in \mathbb{C} \setminus \{0\}$. Hence, $\rho = \pm 1$ by the exact same reasoning by which ν was constrained to be ± 1.
$\therefore$ For any vector, $v \in \mathbb{C}^{2^{D/2}}$, $v^\dagger B^\dagger Bv = v^\dagger \rho Iv \implies ||Bv||^2 = \rho||v||^2$. Then, as $||Bv||^2 \geq 0$ and $||v||^2 \geq$, it must be that $\rho \geq 0$.
$\therefore \rho = 1$, thereby making B unitary.

Theorem 1.5. *Let $C = B^T A$. Then, C is unitary and $\gamma_a^T = (-1)^{t+1}\mu C\gamma_a C^{-1}$.*

Proof. $C^\dagger C = (B^T A)^\dagger B^T A = A^\dagger B^* B^T A = A^\dagger(BB^\dagger)^* A = A^\dagger A = I \implies C$ is unitary.
For the other part of the proof, applying theorem 1.4 along the way,

$$\begin{aligned}
\gamma_a^T &= (\gamma_a^\dagger)^* && (133)\\
&= ((-1)^{t+1}A\gamma_a A^{-1})^* && (134)\\
&= (-1)^{t+1}A^*\gamma_a^* A^{-*} && (135)
\end{aligned}$$

$$
\begin{aligned}
A^* &= (\gamma_0 \cdots \gamma_{t-1})^* && (136)\\
&= \gamma_0^* \cdots \gamma_{t-1}^* && (137)\\
&= \mu B \gamma_0 B^{-1} \cdots \mu B \gamma_{t-1} B^{-1} && (138)\\
&= \mu^t B A B^{-1} \quad \implies A^{-*} = \frac{1}{\mu^t} B A^{-1} B^{-1} && (139)\\
\therefore \gamma_a^T &= (-1)^{t+1} (\mu^t B A B^{-1})(\mu B \gamma_a B^{-1}) \left(\frac{1}{\mu^t} B A^{-1} B^{-1} \right) && (140)\\
&= (-1)^{t+1} \mu B A \gamma_a A^{-1} B^{-1} && (141)
\end{aligned}
$$

$B^* B = \nu I$ and $B^\dagger B = I \implies B^* = \nu B^\dagger \implies B = \nu B^T$.

$$
\begin{aligned}
\therefore \gamma_a^T &= (-1)^{t+1} \mu \nu B^T A \gamma_a A^{-1} B^{-T} / \nu && (142)\\
&= (-1)^{t+1} C \gamma_a C^{-1} && (143)
\end{aligned}
$$

□

Consider the effect of B and C on spinors in the context of the Dirac equation.

$$
\begin{aligned}
0 &= (\mathrm{i}\gamma^a \nabla_a - q\gamma^a A_a - m)\Psi && (144)\\
\therefore 0 &= (-\mathrm{i}\gamma^{a*} \nabla_a - q\gamma^{a*} A_a - m)\Psi^* && (145)\\
&= (-\mathrm{i}\mu B \gamma^a B^{-1} \nabla_a - q\mu B \gamma^a B^{-1} A_a - m)\Psi^* && (146)\\
&= B(-\mathrm{i}\mu \gamma^a \nabla_a - q\mu \gamma^a A_a - m) B^{-1} \Psi^* && (147)\\
\therefore 0 &= (-\mathrm{i}\mu \gamma^a \nabla_a - q\mu \gamma^a A_a - m) B^{-1} \Psi^* && (148)
\end{aligned}
$$

$\therefore$ If $\mu = -1$, then $B^{-1}\Psi^*$ satisfies the same Dirac equation as Ψ but with $q \to -q$.
$\therefore$ If $\mu = -1$, then $B^{-1}\Psi^*$ describes the antiparticle of the particle described by Ψ.
On the other hand, if $\mu = 1$, then $B^{-1}\Psi^*$ satisfies the same Dirac equation as Ψ but with both $q \to -q$ and $m \to -m$.
When $\mu = -1$, a particle is its own antiparticle if and only if $B^{-1}\Psi^* = \Psi \iff \Psi^* = B\Psi$.
$\therefore \Psi = (B\Psi)^* = B^*\Psi^* = B^* B \Psi = \nu \Psi \implies \nu = 1$.

Definition 1.6. *If $\mu = -1$, $\nu = 1$ and $\Psi^* = B\Psi$, then Ψ is called a Majorana spinor. If $\mu = 1$, $\nu = 1$ and $\Psi^* = B\Psi$, then Ψ is called a pseudo-Majorana spinor.*
*If $\nu = -1$ and one has two spinors, $\Psi_i (i = 1, 2)$, then one can impose an "*SU(2) *reality condition," $\overline{\Psi}^i = (\Psi_i)^* = \varepsilon^{ij} B \Psi_j$. In this case, the $\mu = -1$ and $\mu = 1$ cases are called* SU(2) *Majorana and* SU(2) *pseudo-Majorana spinors respectively.*

The matrix, C, can also be related to antiparticles as follows. From the Dirac equation,

$$
\begin{aligned}
0 &= ((\mathrm{i}\gamma^a \nabla_a - q\gamma^a A_a - m)\Psi)^\dagger && (149)\\
&= -\mathrm{i}\nabla_a(\Psi^\dagger)(\gamma^a)^\dagger - q\Psi^\dagger(\gamma^a)^\dagger A_a - m\Psi^\dagger && (150)\\
&= -\mathrm{i}\nabla_a(\Psi^\dagger)(-1)^{t+1} A\gamma^a A^{-1} - (-1)^{t+1} q\Psi^\dagger A\gamma^a A^{-1} A_a - m\Psi^\dagger && (151)\\
&= (-\mathrm{i}\nabla_a(\Psi^\dagger A)(-1)^{t+1}\gamma^a - (-1)^{t+1} q\Psi^\dagger A\gamma^a A_a - m\Psi^\dagger) A^{-1} && (152)
\end{aligned}
$$

Let $\Psi^\dagger A = \overline{\Psi}$; $\overline{\Psi}$ is called the adjoint spinor.

$$
\begin{aligned}
\therefore 0 &= \overline{\Psi}((-1)^{t+1}\mathrm{i}\gamma^a \overleftarrow{\nabla}_a + (-1)^{t+1} q\gamma^a A_a + m) && (153)\\
\therefore 0 &= ((-1)^{t+1}\mathrm{i}(\gamma^a)^T \nabla_a + (-1)^{t+1} q(\gamma^a)^T A_a + m)\overline{\Psi}^T && (154)\\
&= ((-1)^{t+1}\mathrm{i}(-1)^{t+1}\mu C\gamma^a C^{-1}\nabla_a + (-1)^{t+1} q(-1)^{t+1}\mu C\gamma^a C^{-1} A_a + m)\overline{\Psi}^T && (155)\\
\therefore 0 &= (\mathrm{i}\mu\gamma^a \nabla_a + q\mu\gamma^a A_a + m) C^{-1}\overline{\Psi}^T && (156)
\end{aligned}
$$

$\therefore$ Again, if $\mu = -1$, then $C^{-1}\overline{\Psi}^T$ describes the antiparticle of the particle described by Ψ. For this reason, $C^{-1}\overline{\Psi}^T$ is denoted $\overline{\Psi}_C$ and C is called the charge conjugation matrix. For reasons unknown, B doesn't have a special name despite serving a similar function. It is however no coincidence that $B^{-1}\Psi^*$ and $C^{-1}\overline{\Psi}^T$ serve the same purpose.

Theorem 1.7. *$B^{-1}\Psi^*$ and $C^{-1}\overline{\Psi}^T$ are proportional to each other.*

Proof. $C^{-1}\overline{\Psi}^T = (B^TA)^{-1}(\Psi^\dagger A)^T = A^{-1}B^{-T}A^T\Psi^* = A^{-1}(AB^{-1})^T\Psi^*$
$A^* = \gamma_0^* \cdots \gamma_{t-1}^* = \mu B\gamma_0 B^{-1} \cdots \mu B\gamma_{t-1}B^{-1} = \mu^t BAB^{-1} \implies AB^{-1} = \mu^t B^{-1}A^*$
$\therefore C^{-1}\overline{\Psi}^T = \mu^t A^{-1}A^\dagger B^{-T}\Psi^*$.
However, I showed earlier that $B = \nu B^T$. Thus, $B^{-1} = \nu B^{-T} \iff B^{-T} = \nu B^{-1}$ since $\nu^2 = 1$. Meanwhile, for the other two matrices,

$$A^{-1}A^\dagger = (\gamma_0 \cdots \gamma_{t-1})^{-1}(\gamma_0 \cdots \gamma_{t-1})^\dagger \tag{157}$$
$$= \gamma_{t-1}^{-1} \cdots \gamma_0^{-1}\gamma_{t-1}^\dagger \cdots \gamma_0^\dagger \tag{158}$$
$$= \gamma_{t-1} \cdots \gamma_0\gamma_{t-1} \cdots \gamma_0 \tag{159}$$
$$= (-1)^{t-1+t-2+\cdots+1}I \tag{160}$$
$$= (-1)^{t(t-1)/2}I \tag{161}$$
$$\therefore C^{-1}\overline{\Psi}^T = \nu\mu^t(-1)^{t(t-1)/2}B^{-1}\Psi^* \tag{162}$$

□

As it happens, ν and μ are not independent.

Theorem 1.8. *ν is a function of μ, t and s by*

$$\nu = \cos\left(\frac{\pi}{4}(s-t)\right) - \mu\sin\left(\frac{\pi}{4}(s-t)\right) \tag{163}$$

Proof. I've already shown $B^T = \nu B$. Then, using theorems 1.4 and 1.5,

$$C^T = (B^TA)^T \tag{164}$$
$$= \gamma_{t-1}^T \cdots \gamma_0^T B \tag{165}$$
$$= (-1)^{t+1}\mu C\gamma_{t-1}C^{-1} \cdots (-1)^{t+1}\mu C\gamma_0 C^{-1}B \tag{166}$$
$$= (-1)^{t(t-1)}\mu^t C\gamma_{t-1} \cdots \gamma_0 C^{-1}B \tag{167}$$
$$= (-1)^{t(t-1)}\mu^t(-1)^{t-1+t-2+\cdots+1}C\gamma_0 \cdots \gamma_{t-1}C^{-1}B \tag{168}$$
$$= (-1)^{t(3t+1)/2}\mu^t CAC^{-1}B \tag{169}$$
$$= (-1)^{t(3t+1)/2}\mu^t CAA^{-1}B^{-T}B \tag{170}$$
$$= (-1)^{t(3t+1)/2}\mu^t C\nu B^{-1}B \tag{171}$$
$$= (-1)^{t(t-1)/2}\mu^t\nu C \tag{172}$$

$\therefore$ B and C may be symmetric or antisymmetric (independently). To see how this is relevant, consider the group, G, introduced earlier. In particular, consider the subset, $\{\Gamma_A\}_{A=0}^{2^D-1}$. Let $\sum_{A=0}^{2^D-1} C_A\Gamma_A = 0$ for some constants, $C_A \in \mathbb{C}$.

$$\therefore 0 = \sum_{A=0}^{2^D-1} C_A\Gamma_A\Gamma_B \tag{173}$$
$$\therefore 0 = \sum_{A=0}^{2^D-1} C_A\mathrm{tr}(\Gamma_A\Gamma_B) \tag{174}$$

However, I showed earlier that $\Gamma_A\Gamma_B = \pm\Gamma_C$ for some C and $\mathrm{tr}(\Gamma_C) = 0$ unless $\Gamma_C = I$ (in even dimensions).
$\therefore \mathrm{tr}(\Gamma_C) \neq 0 \implies \Gamma_B = (\Gamma_A)^{-1} = \pm\Gamma_A \implies A = B$.
$\therefore$ The sum in 174 collapses to $C_B = 0$.
As B is arbitrary, $\{\Gamma_A\}_{A=0}^{2^D-1}$ is a linearly independent set. The size of the set is $2^D = 2^{D/2}\times 2^{D/2}$, which is the dimension of the vector space of $2^{D/2}\times 2^{D/2}$ matrices.
$\therefore \{\Gamma_A\}_{A=0}^{2^D-1}$ is a basis for the set of $2^{D/2}\times 2^{D/2}$ matrices. This basis can be "antisymmetrised" to $\{\tilde{\Gamma}^{(n)}\}$, where $\tilde{\Gamma}^{(n)} = \gamma_{[a_1}\cdots\gamma_{a_n]}$, i.e. rather than $\gamma_{a_1}\cdots\gamma_{a_n}$ with $a_1 < a_2 < \cdots < a_n$, the indices are antisymmetrised. There are ${}^DC_n$ matrices of type, $\tilde{\Gamma}^{(n)}$. Furthermore, as C is invertible, $\{\Gamma_A\}_{A=0}^{2^D-1}$ is a basis $\implies \{C\Gamma_A\}_{A=0}^{2^D-1}$ is a basis $\implies \{C\tilde{\Gamma}^{(n)}\}$ is a basis.

$$
\begin{aligned}
(C\tilde{\Gamma}^{(n)})^T &= (\tilde{\Gamma}^{(n)})^T C^T && (175)\\
&= (\gamma_{[a_1}\cdots\gamma_{a_n]})^T(-1)^{t(t-1)/2}\mu^t\nu C && (176)\\
&= \gamma^T_{[a_n}\cdots\gamma^T_{a_1]}(-1)^{t(t-1)/2}\mu^t\nu C && (177)\\
&= (-1)^{t+1}\mu C\gamma_{[a_n}C^{-1}\cdots(-1)^{t+1}\mu C\gamma_{a_1]}C^{-1}(-1)^{t(t-1)/2}\mu^t\nu C && (178)\\
&= (-1)^{n(t+1)}\mu^{n+t}C(-1)^{t(t-1)/2}\nu(-1)^{n-1+n-2+\cdots+1}\gamma_{[a_1}\cdots\gamma_{a_n]} && (179)\\
&= (-1)^{(n^2+n+2nt-t+t^2)/2}\mu^{n+t}\nu C\tilde{\Gamma}^{(n)} && (180)
\end{aligned}
$$

$\therefore$ Each of the $C\tilde{\Gamma}^{(n)}$ is either symmetric or antisymmetric.
$\therefore$ Since every matrix can be decomposed into symmetric and antisymmetric parts, the antisymmetric $C\tilde{\Gamma}^{(n)}$ must form a basis for the antisymmetric $2^{D/2}\times 2^{D/2}$ matrices.
However, the set of antisymmetric matrices is known to have dimension,
${}^{2^{D/2}}C_2 = \frac{1}{2}2^{D/2}(2^{D/2}-1)$.
$\therefore$ There are $\frac{1}{2}2^{D/2}(2^{D/2}-1)$ antisymmetric $C\tilde{\Gamma}^{(n)}$. To count the number of antisymmetric $C\tilde{\Gamma}^{(n)}$, note that there are ${}^DC_n$ matrices of type, $C\tilde{\Gamma}^{(n)}$, and $\frac{1}{2}(1-(-1)^{(n^2+n+2nt-t+t^2)/2}\mu^{n+t}\nu) = 0$ for a symmetric $C\tilde{\Gamma}^{(n)}$ and 1 for an antisymmetric $C\tilde{\Gamma}^{(n)}$.

$$
\begin{aligned}
\therefore \frac{1}{2}2^{D/2}(2^{D/2}-1) &= \sum_{n=0}^{D}\frac{1}{2}(1-(-1)^{(n^2+n+2nt-t+t^2)/2}\mu^{n+t}\nu)\,{}^DC_n && (181)\\
\therefore 2^D - 2^D/2 &= \sum_{n=0}^{D}(1-(-1)^{(n^2+n+2nt-t+t^2)/2}\mu^{n+t}\nu)\,{}^DC_n && (182)\\
&= \sum_{n=0}^{D}{}^DC_n - \nu\mu^t(-1)^{t(t-1)/2}\sum_{n=0}^{D}\mu^n(-1)^{n(n+2t+1)/2}\ {}^DC_n && (183)\\
\therefore 2^{D/2}\mu^t(-1)^{t(t-1)/2} &= \nu\sum_{n=0}^{D}{}^DC_n\mu^n(-1)^{n(n+2t+1)/2} && (184)
\end{aligned}
$$

By sheer dumb luck, or otherwise, guess that

$$
(-1)^{n(n+2t+1)/2} = \frac{(-1)^{nt}}{2}((1+\mathrm{i})\mathrm{i}^n + (1-\mathrm{i})(-\mathrm{i})^n) \qquad (185)
$$

Because of the periodicity in powers of 1 and i, this expression only needs to hold for n, t mod 4, to hold in general. I have checked the equation really does hold for those 16 combinations on *Mathematica*.

$$\therefore 2^{D/2}\mu^t(-1)^{t(t-1)/2} = \frac{\nu}{2}\sum_{n=0}^{D}(\mu(-1)^t)^n((1+\mathrm{i})\mathrm{i}^n + (1-\mathrm{i})(-\mathrm{i})^n)^D C_n \tag{186}$$

$$= \frac{\nu(1+\mathrm{i})}{2}\sum_{n=0}^{D}(\mu(-1)^t)^n(\mathrm{i}^n - \mathrm{i}(-\mathrm{i})^n)^D C_n \tag{187}$$

$$= \frac{\nu(1+\mathrm{i})}{2}\left(\sum_{n=0}^{D}(\mathrm{i}\mu(-1)^t)^n\, {}^D C_n - \mathrm{i}\sum_{n=0}^{D}(-\mathrm{i}\mu(-1)^t)^n\, {}^D C_n\right) \tag{188}$$

$$= \frac{1}{2}\nu(1+\mathrm{i})((1+\mathrm{i}\mu(-1)^t)^D - \mathrm{i}(1-\mathrm{i}\mu(-1)^t)^D) \tag{189}$$

Since $1+\mathrm{i} = \sqrt{2}\mathrm{e}^{\mathrm{i}\pi/4}$ and $1-\mathrm{i} = \sqrt{2}\mathrm{e}^{-\mathrm{i}\pi/4}$, the last line can be re-written as

$$2^{D/2}\mu^t(-1)^{t(t-1)/2} = \frac{1}{2}\nu\sqrt{2}\mathrm{e}^{\mathrm{i}\pi/4}2^{D/2}(\mathrm{e}^{\mathrm{i}\mu(-1)^tD\pi/4} - \mathrm{e}^{\mathrm{i}\pi/2}\mathrm{e}^{-\mathrm{i}\mu(-1)^tD\pi/4}) \tag{190}$$

$$\therefore \nu = \frac{\sqrt{2}\mu^t(-1)^{t(t-1)/2}}{\mathrm{e}^{\mathrm{i}\pi/4}(\mathrm{e}^{\mathrm{i}\mu(-1)^tD\pi/4} - \mathrm{e}^{\mathrm{i}\pi/2}\mathrm{e}^{-\mathrm{i}\mu(-1)^tD\pi/4})} \tag{191}$$

Because of the periodicity of $\mathrm{e}^{\mathrm{i}x\pi/4}$ and $(-1)^x$, it only matters whether $\mu = 1$ or -1 and what s and t are modulo 8.
$\therefore$ There are only $2\times 8\times 8 = 128$ different cases. Again, by some miracle, one may guess that

$$\frac{\mathrm{e}^{\mathrm{i}\pi/4}(\mathrm{e}^{\mathrm{i}\mu(-1)^tD\pi/4} - \mathrm{e}^{\mathrm{i}\pi/2}\mathrm{e}^{-\mathrm{i}\mu(-1)^tD\pi/4})}{\sqrt{2}\mu^t(-1)^{t(t-1)/2}} = \cos\left(\frac{\pi}{4}(s-t)\right) - \mu\sin\left(\frac{\pi}{4}(s-t)\right) \tag{192}$$

To check that this equation really holds, one only needs to check the 128 different cases - a task I have completed with the aid of *Mathematica*. Finally, $\nu = \pm 1 \implies \nu = \frac{1}{\nu}$ and thus $\nu = \cos(\frac{\pi}{4}(s-t)) - \mu\sin(\frac{\pi}{4}(s-t))$. $\square$

Since equation 115, the discussion has been limited to even dimensions. It's now time to extend the results to odd dimensions. Let D be even and let the odd dimension of interest be $D+1$. If $D = s+t$, assume without loss of generality that $D+1 = (s+1)+t$, i.e. a space dimension is added. Let $\gamma_{D+1} = \gamma_0\cdots\gamma_{D-1}$ as before.

$$\therefore \gamma_{D+1}\gamma_a = \gamma_0\cdots\gamma_{D-1}\gamma_a \tag{193}$$

$$= \gamma_0\cdots\gamma_a\cdots\gamma_{D-1}\gamma_a \quad \text{(no sum)} \tag{194}$$

$$= \gamma_0\cdots\gamma_a\gamma_a\cdots\gamma_{D-1}(-1)^{D-a-1} \tag{195}$$

$$= (-1)^a\gamma_a\gamma_0\cdots\gamma_a\cdots\gamma_{D-1}(-1)^{D-a-1} \tag{196}$$

$$= (-1)^{D-1}\gamma_a\gamma_{D+1} \tag{197}$$

$$= -\gamma_a\gamma_{D+1} \quad \text{as } D \text{ is even} \tag{198}$$

$$\therefore \gamma_{D+1}\gamma_a + \gamma_a\gamma_{D+1} = 0 = -2\eta_{a,D}I \tag{199}$$

$$\text{Meanwhile, } (\gamma_{D+1})^2 = \gamma_0\cdots\gamma_{D-1}\gamma_0\cdots\gamma_{D-1} \tag{200}$$

$$= (-1)^{D-1+D-2+\cdots+1}(\gamma_0)^2\cdots(\gamma_{D-1})^2 \tag{201}$$

$$= (-1)^{D(D-1)/2}(-1)^s I \tag{202}$$

$$= (-1)^{D^2/2+(s-t)/2} I \tag{203}$$

$$= (-1)^{(s-t)/2} I \tag{204}$$

as $D^2/2$ is even, $(\gamma_a)^2 = I$ for timelike indices and $(\gamma_a)^2 = -I$ for spacelike indices.
$\therefore \{\gamma_a, \gamma_{D+1}\}_{a=0}^{D-1}$ satisfies the Clifford algebra for $s-t \equiv 2 \pmod 4$ and $\{\gamma_a, \mathrm{i}\gamma_{D+1}\}_{a=0}^{D-1}$ satisfies the Clifford algebra for $s-t \equiv 0 \pmod 4$ ($s-t \equiv 1, 3 \pmod 4$ are not possible for even D).
By theorems 1.2 and 1.3, in odd dimensions, there are two inequivalent representations, $\{\gamma_a, \gamma_{D+1}\}_{a=0}^{D-1}$ & $\{-\gamma_a, -\gamma_{D+1}\}_{a=0}^{D-1}$ and $\{\gamma_a, \mathrm{i}\gamma_{D+1}\}_{a=0}^{D-1}$ & $\{-\gamma_a, -\mathrm{i}\gamma_{D+1}\}_{a=0}^{D-1}$ respectively.
$\therefore$ Unlike the even case, $\{\gamma_a^*, \gamma_{D+1}^*\}_{a=0}^{D-1}$ & $\{-\gamma_a^*, -\gamma_{D+1}^*\}_{a=0}^{D-1}$ and $\{\gamma_a^*, -\mathrm{i}\gamma_{D+1}^*\}_{a=0}^{D-1}$ & $\{-\gamma_a^*, \mathrm{i}\gamma_{D+1}^*\}_{a=0}^{D-1}$ respectively are no longer equivalent.
$\therefore$ In $\gamma_a^* = \mu B \gamma_a B^{-1}$, μ can on longer be freely chosen as 1 or -1. Instead, μ will be fixed by forcing $\gamma_{D+1}^* = \mu B \gamma_{D+1} B^{-1}$ or $-\mathrm{i}\gamma_{D+1}^* = \mu B \mathrm{i}\gamma_{D+1} B^{-1}$.
First, consider $\gamma_{D+1}^* = \mu B \gamma_{D+1} B^{-1}$.

$$\begin{aligned}
\therefore \mu B \gamma_{D+1} B^{-1} &= \gamma_{D+1}^* && (205)\\
&= \gamma_0^* \cdots \gamma_{D-1}^* && (206)\\
&- \mu B \gamma_0 B^{-1} \cdots \mu B \gamma_{D-1} B^{-1} && (207)\\
&= \mu^D B \gamma_{D+1} B^{-1} && (208)\\
&= \mu B \gamma_0 B^{-1} \quad \text{as } D \text{ is even} && (209)\\
\therefore \mu &= 1 && (210)
\end{aligned}$$

Hence, when $s-t \equiv 2 \pmod 4$, $\mu = 1$. Similarly, $-\mathrm{i}\gamma_{D+1}^* = \mu B \mathrm{i}\gamma_{D+1} B^{-1} \implies \mu = -1$ when $s-t \equiv 0 \pmod 4$. These two equations can be summarised in one equation by
$\mu = (-1)^{(s-t+2)/2}$.
To proceed, not that $D+1$ odd, the irreducible representations still have dimension, $2^{D/2}$.
$\therefore \{\gamma_a\}_{a=0}^{D-1}$ can still be used to generate $\{\Gamma_A\}_{A=0}^{2^D-1}$, which will still be a basis for $2^{D/2} \times 2^{D/2}$ matrices. Furthermore, A's properties only depend on t, not s. Likewise, in finding $\nu = \pm 1$ and and the other results, I only needed $2^{D/2}$ is even, not D is even. In fact, looking back over the proofs, all the properties continue to hold. The only difference is $\mu = (-1)^{(s-t+2)/2}$ is fixed rather than free.
Thus far, I have written odd dimensions as $D+1 = (s+1)+t$. To write odd D as $s+t$, I will have to let $s \to s-1$ in the theorems for odd dimensions. Overall, one gets the following.

Theorem 1.9 (Summary of results)**.** *For $D = s+t$ (D may be odd or even) and $D > 1$,*

- $\mu = (-1)^{(s-t+1)/2}$ *in odd dimensions.*
- μ *can be freely chosen as* 1 *or* -1 *in even dimensions.*
- $\gamma_a^\dagger = (-1)^{t+1} A \gamma_a A^{-1}$ *where* $A = \gamma_0 \cdots \gamma_{t-1}$.
- $\exists$ *a matrix,* B*, such that* $\gamma_a^* = \mu B \gamma_a B^{-1}$.
- $\gamma_a^T = (-1)^{t+1} \mu C \gamma_a C^{-1}$ *where* $C = B^T A$.
- A*,* B *and* C *are all unitary,* $B^* B = \nu I$ *for* $\nu = \pm 1$*,* $B^T = \nu B$ *and* $C^T = \nu \mu^t (-1)^{t(t-1)/2} C$.
- $\nu = \cos\big(\frac{\pi}{4}(s-t)\big) - \mu \sin\big(\frac{\pi}{4}(s-t)\big)$ *in even dimensions.*
- $\nu = \cos\big(\frac{\pi}{4}(s-t-1)\big) - \mu \sin\big(\frac{\pi}{4}(s-t-1)\big)$ *in odd dimensions.*

Proof. See above □

I am now in a position to evaluate all possible combinations of ν, μ and $s-t$ (ν and μ only depend on $s-t$).
For $s-t \equiv 1, 3, 5, 7 \pmod 8$, $s-t-1 \equiv 0, 2, 4, 6 \pmod 8$ and hence $\mu = -1, 1, -1, 1$ and

$\nu = \cos 0 + \sin 0 = 1$, $\cos \pi/2 + \sin \pi/2 = -1$, $\cos \pi + \sin \pi = -1$, $\cos 3\pi/2 + \sin 3\pi/2 = 1$.
In the even cases, $\mu = \pm 1$ and $s - t \equiv 0, 2, 4, 6 \pmod 8$ imply $\nu = \cos 0 \mp \sin 0 = 1$, $\cos \pi/2 \mp \sin \pi/2 = \mp 1$, $\cos \pi \mp \sin \pi = -1$, $\cos 3\pi/2 \mp \sin 3\pi/2 = \pm 1$. These results are summarised in table 1.

ν	μ	Possible $s - t$ mod 8	Antiparticle related spinor
1	1	0, 6, 7	pseudo-Majorana
1	-1	0, 1, 2	Majorana
-1	1	2, 3, 4	SU(2) pseudo-Majorana
-1	-1	4, 5, 6	SU(2) Majorana

Table 1: The antiparticle related spinors possible in different spacetimes

Besides the suite of Majorana like spinors, another special type of spinor relevant to physics is the so-called Weyl spinor. Weyl spinors are defined as eigenvectors of γ_{D+1}. However, I already showed in equation 43 that in odd dimensions $\gamma_{D+1}\gamma_a = \gamma_a\gamma_{D+1} \ \forall a$
$\implies \gamma_{D+1} g = g\gamma_{D+1} \ \forall g \in G \implies \gamma_{D+1} \propto I$ by Schur's lemma.
$\therefore$ In odd dimensions, every spinor is an eigenvector of γ_{D+1} and so the concept of a Weyl spinor would be fruitless.
$\therefore$ Define Weyl spinors to exist only for even dimensional spacetimes.
Rather than $\gamma_{D+1}\Psi = \lambda\Psi$ however, it is more customary[7] to consider $(-1)^{(s-t)/4}\gamma_{D+1}\Psi = \lambda\Psi$ with $(-1)^{1/2}$ defined to be $-\mathrm{i}$ without loss of generality[8].

$$\lambda^2\Psi = (-1)^{(s-t)/4}\gamma_{D+1}(-1)^{(s-t)/4}\gamma_{D+1}\Psi \tag{211}$$
$$= (-1)^{(s-t)/2}\gamma_0 \cdots \gamma_{D-1}\gamma_0 \cdots \gamma_{D-1}\Psi \tag{212}$$
$$= (-1)^{(s-t)/2}(-1)^{D-1+D-2+\cdots+1}(\gamma_0)^2 \cdots (\gamma_{D-1})^2\Psi \tag{213}$$
$$= (-1)^{(s-t)/2}(-1)^{D(D-1)/2}(-1)^s I\Psi \tag{214}$$
$$= (-1)^{(s+t)^2/2+s-t}\Psi \tag{215}$$
$$\therefore \lambda = \pm(-1)^{(s+t)^2/4+(s-t)/2} \tag{216}$$

In even dimensions, $s-t$ is also even and thus $(s+t)^2/4+(s-t)/2$ is an integer $\implies \lambda = \pm 1$. Eigenvectors with eigenvalues, $+1$ and -1, are called left handed Weyl spinors and right handed Weyl spinors respectively.

Theorem 1.10. *The eigenspaces of left handed and right handed Weyl spinors both have dimension,* $2^{D/2-1}$*, and hence their direct sum is the entire representation space.*

Proof. In proving theorem 1.4, I showed that $\gamma_a^\dagger = \gamma_a$ for $0 \le a \le t-1$ and $\gamma_a^\dagger = -\gamma_a$ for $t \le a \le s+t-1$.

$$\therefore \gamma_{D+1}^\dagger\gamma_{D+1} = \gamma_{D-1}^\dagger \cdots \gamma_0^\dagger\gamma_0 \cdots \gamma_{D-1} \tag{217}$$
$$= (-1)^s\gamma_{D-1} \cdots \gamma_0\gamma_0 \cdots \gamma_{D-1} \tag{218}$$
$$= (-1)^s(-1)^s I \tag{219}$$
$$= I \tag{220}$$

$\therefore \gamma_{D+1}^\dagger$ commutes with γ_{D+1}, i.e. γ_{D+1} is a "normal" operator and thus diagonalisable.
$\therefore$ The sum of the dimensions of eigenspaces of $\lambda = 1$ and $\lambda = -1$ equals the dimension of the full space, namely $2^{D/2}$.

[7] With the benefit of hindsight, the eigenvalues are nicer with this convention
[8] As opposed to $(-1)^{1/2} = \mathrm{i}$

Next, let $(-1)^{(s-t)/4}\gamma_{D+1}\Psi = \pm\Psi$. As D is even, by equation 198, $\{\gamma_a, \gamma_{D+1}\} = 0$.
$\therefore (-1)^{(s-t)/4}\gamma_{D+1}\gamma_a\Psi = -(-1)^{(s-t)/4}\gamma_a\gamma_{D+1}\Psi = \mp\gamma_a\Psi$.
$\therefore$ If Ψ is in the $\pm$ eigenspace, then $\gamma_a\Psi$ is in the $\mp$ eigenspace. However, all the γ_a are invertible.
$\therefore \gamma_a$ induces a bijection between the $\pm$ eigenspace to the $\mp$ eigenspace.
$\therefore$ The $\pm$ eigenspaces must have the same dimension, namely $\frac{1}{2}2^{D/2} = 2^{D/2-1}$. $\square$

The component of an arbitrary spinor, Ψ, in each of these eigenspaces can be found by the projection operators, $P_\pm = \frac{1}{2}(I \pm (-1)^{(s-t)/4}\gamma_{D+1})$, since $P_+ + P_- = I$ and (using equation 204 and $s - t$ being even)

$$\begin{aligned}
(-1)^{(s-t)/4}\gamma_{D+1}P_\pm\Psi &= \frac{1}{2}(-1)^{(s-t)/4}\gamma_{D+1}(I \pm (-1)^{(s-t)/4}\gamma_{D+1})\Psi && (221)\\
&= \frac{1}{2}(-1)^{(s-t)/4}\gamma_{D+1}\Psi \pm \frac{1}{2}(-1)^{(s-t)/2}(\gamma_{D+1})^2\Psi && (222)\\
&= \frac{1}{2}(-1)^{(s-t)/4}\gamma_{D+1}\Psi \pm \frac{1}{2}(-1)^{(s-t)/2}(-1)^{(s-t)/2}\Psi && (223)\\
&= \frac{1}{2}(-1)^{(s-t)/4}\gamma_{D+1}\Psi \pm \frac{1}{2}\Psi && (224)\\
&= \pm\frac{1}{2}(\Psi \pm (-1)^{(s-t)/4}\gamma_{D+1}\Psi) && (225)\\
&= \pm P_\pm\Psi && (226)
\end{aligned}$$

Since Weyl spinors can be constructed in any even dimension and (by table 1) Majorana spinors can be constructed when $s-t \equiv 0, 1, 2 \pmod 8$, the double of a Majorana-Weyl spinor is possible when $s - t \equiv 0, 2 \pmod 8$.

2 Three space and one time dimension

Up to now, I've considered spinors very generally. For a specific example, consider the case most relevant to physics, namely $s = 3$ and $t = 1$.
$\therefore D = 4$, $2^{D/2} = 4$ and there is a unique irreducible representation[9] of the Clifford algebra (up to equivalence).
$\therefore$ It suffices to guess this representation (and thereby prove its existence too). I will use the so-called "Weyl representation,"

$$\gamma_a \equiv \begin{bmatrix} 0 & \sigma_a \\ \tilde{\sigma}_a & 0 \end{bmatrix} \text{ where } \sigma_a \equiv (I, \sigma_1, \sigma_2, \sigma_3),\ \tilde{\sigma}_a \equiv (I, -\sigma_1, -\sigma_2, -\sigma_3) \tag{227}$$

and σ_1, σ_2 & σ_3 are the Pauli matrices.

$$\begin{aligned}
\therefore \gamma_a\gamma_b + \gamma_b\gamma_a &= \begin{bmatrix} 0 & \sigma_a \\ \tilde{\sigma}_a & 0 \end{bmatrix}\begin{bmatrix} 0 & \sigma_b \\ \tilde{\sigma}_b & 0 \end{bmatrix} + \begin{bmatrix} 0 & \sigma_b \\ \tilde{\sigma}_b & 0 \end{bmatrix}\begin{bmatrix} 0 & \sigma_a \\ \tilde{\sigma}_a & 0 \end{bmatrix} && (228)\\
&= \begin{bmatrix} \sigma_a\tilde{\sigma}_b + \sigma_b\tilde{\sigma}_a & 0 \\ 0 & \tilde{\sigma}_a\sigma_b + \tilde{\sigma}_b\sigma_a \end{bmatrix} && (229)\\
&= \begin{bmatrix} -2\eta_{ab}I & 0 \\ 0 & -2\eta_{ab}I \end{bmatrix} && (230)\\
&= -2\eta_{ab}I \implies \text{the Clifford algebra is satisfied} && (231)
\end{aligned}$$

Next, it must be shown that the chosen representation is irreducible. Let S be a non-empty suspace of $\mathbb{C}^4$ invariant under all γ_a.

[9] Thus far, I have only proven theorems about the uniqueness of representations, not existence.

$\therefore \forall v \in \mathbb{C}^4$ and $\forall a \in \{0,1,2,3\}$, $\gamma_a v \in S$.
$\therefore \gamma_a \gamma_b v \in S$ as $\gamma_b v = v'$ for some $v' \in S$ and thus $\gamma_a v' \in S$.
Likewise, $\forall \lambda_1, \lambda_2 \in \mathbb{C}$, $(\lambda_1 \gamma_a + \lambda_2 \gamma_b) v \in S$ as $\gamma_a v, \gamma_b v \in S$ and S is closed under linear combinations by virtue of being a subspace.
$\therefore S$ is invariant under all products and linear combinations of γ_a and thus invariant under all linear combinations of elements in $G = \{\pm\Gamma_A\}_{A=0}^{15}$. By direct evaluation (on *Mathematica*),

$$\{\Gamma_A\}_{A=0}^{15} = \left\{ \begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{bmatrix}, \begin{bmatrix} 0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0&1\\0&0&1&0\\0&-1&0&0\\-1&0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0&-\mathrm{i}\\0&0&\mathrm{i}&0\\0&\mathrm{i}&0&0\\-\mathrm{i}&0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&1&0\\0&0&0&-1\\-1&0&0&0\\0&1&0&0 \end{bmatrix}, \right.$$
$$\begin{bmatrix} 0&-1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&1&0 \end{bmatrix}, \begin{bmatrix} 0&\mathrm{i}&0&0\\-\mathrm{i}&0&0&0\\0&0&0&-\mathrm{i}\\0&0&\mathrm{i}&0 \end{bmatrix}, \begin{bmatrix} -1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1 \end{bmatrix}, \begin{bmatrix} -\mathrm{i}&0&0&0\\0&\mathrm{i}&0&0\\0&0&-\mathrm{i}&0\\0&0&0&\mathrm{i} \end{bmatrix},$$
$$\begin{bmatrix} 0&1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&-1&0 \end{bmatrix}, \begin{bmatrix} 0&-\mathrm{i}&0&0\\-\mathrm{i}&0&0&0\\0&0&0&-\mathrm{i}\\0&0&-\mathrm{i}&0 \end{bmatrix}, \begin{bmatrix} 0&0&-\mathrm{i}&0\\0&0&0&\mathrm{i}\\-\mathrm{i}&0&0&0\\0&\mathrm{i}&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0&1\\0&0&-1&0\\0&1&0&0\\-1&0&0&0 \end{bmatrix},$$
$$\left. \begin{bmatrix} 0&0&0&-\mathrm{i}\\0&0&-\mathrm{i}&0\\0&-\mathrm{i}&0&0\\-\mathrm{i}&0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&-\mathrm{i}&0\\0&0&0&-\mathrm{i}\\\mathrm{i}&0&0&0\\0&\mathrm{i}&0&0 \end{bmatrix}, \begin{bmatrix} \mathrm{i}&0&0&0\\0&\mathrm{i}&0&0\\0&0&-\mathrm{i}&0\\0&0&0&-\mathrm{i} \end{bmatrix} \right\} \tag{232}$$

However, by inspection, complex linear combinations of these matrices can produce any 4×4 complex matrix (e.g. look at the 4 matricex subsets $\{0, 7, 8, 15\}$, $\{1, 4, 11, 14\}$, $\{2, 3, 12, 13\}$ and $\{5, 6, 9, 10\}$ with matrices labelled as per the order in which they are listed above).
$\therefore S$ is invariant under all 4×4 matrices.
$\therefore S = \mathbb{C}^4$
$\therefore$ The Weyl representation of the Clifford algebra is indeed irreducible.
The Weyl representation is also unitary under the standard inner product of $\mathbb{C}^4$ since $\gamma_0^\dagger = \gamma_0$ and $\gamma_i^\dagger = -\gamma_i$. As for Weyl spinors,

$$(-1)^{(s-t)/4}\gamma_5 = (-1)^{1/2}\gamma_0 \cdots \gamma_3 \tag{233}$$
$$= -\mathrm{i}\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}\begin{bmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{bmatrix}\begin{bmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{bmatrix}\begin{bmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{bmatrix} \tag{234}$$
$$= -\mathrm{i}\begin{bmatrix} -\sigma_1 & 0 \\ 0 & \sigma_1 \end{bmatrix}\begin{bmatrix} -\mathrm{i}\sigma_1 & 0 \\ 0 & -\mathrm{i}\sigma_1 \end{bmatrix} \tag{235}$$
$$= \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \tag{236}$$
$$\therefore (-1)^{(s-t)/4}\gamma_5 \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} w \\ x \\ -y \\ -z \end{bmatrix} \tag{237}$$

$\therefore$ span$(\{(1,0,0,0),(0,1,0,0)\})$ and span$(\{(0,0,1,0),(0,0,0,1)\})$ are the eigenspaces of left handed and right handed Weyl spinors respectively. To reflect this, the 4-component spinor, Ψ, can be written as $\Psi = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}$, where ψ_α and $\bar{\chi}^{\dot{\alpha}}$ are 2-component Weyl spinors. Undotted and dotted indices are left handed and right handed respectively.

As shown by equation 90, $M_{ab} = -\frac{1}{4}[\gamma_a, \gamma_b]$ are Lorentz generators in spinor space. Let $\sigma_{ab} = -\frac{1}{4}(\sigma_a\tilde{\sigma}_b - \sigma_b\tilde{\sigma}_a)$ and $\tilde{\sigma}_{ab} = -\frac{1}{4}(\tilde{\sigma}_a\sigma_b - \tilde{\sigma}_b\sigma_a)$. σ_{ab} and $\tilde{\sigma}_{ab}$ are called left handed and right handed Lorentz generators respectively because

$$M_{ab} = -\frac{1}{4}\left(\begin{bmatrix} 0 & \sigma_a \\ \tilde{\sigma}_a & 0 \end{bmatrix}\begin{bmatrix} 0 & \sigma_b \\ \tilde{\sigma}_b & 0 \end{bmatrix} - \begin{bmatrix} 0 & \sigma_b \\ \tilde{\sigma}_b & 0 \end{bmatrix}\begin{bmatrix} 0 & \sigma_a \\ \tilde{\sigma}_a & 0 \end{bmatrix}\right) \tag{238}$$

$$= -\frac{1}{4}\begin{bmatrix} \sigma_a\tilde{\sigma}_b - \sigma_b\tilde{\sigma}_a & 0 \\ 0 & \tilde{\sigma}_a\sigma_b - \tilde{\sigma}_b\sigma_a \end{bmatrix} \tag{239}$$

$$= \begin{bmatrix} \sigma_{ab} & 0 \\ 0 & \tilde{\sigma}_{ab} \end{bmatrix} \tag{240}$$

$$\therefore M_{ab}\Psi = \begin{bmatrix} \sigma_{ab} & 0 \\ 0 & \tilde{\sigma}_{ab} \end{bmatrix}\begin{bmatrix} \psi_\alpha \\ \overline{\chi}^{\dot{\alpha}} \end{bmatrix} = \begin{bmatrix} \sigma_{ab}\psi_\alpha \\ \tilde{\sigma}_{ab}\overline{\chi}^{\dot{\alpha}} \end{bmatrix} \tag{241}$$

$M_{ab}\Psi$ must still be a spinor of the same type as Ψ.
$\therefore \sigma_{ab}\psi_\alpha$ must be a left handed Weyl spinor and $\tilde{\sigma}_{ab}\overline{\chi}^{\dot{\alpha}}$ must be a right handed Weyl spinor.
$\therefore$ Since M_{ab} only induces a linear transformation, the spinor indices of σ_{ab} and $\tilde{\sigma}_{ab}$ must be $(\sigma_{ab})_\alpha{}^\beta$ and $(\tilde{\sigma}_{ab})^{\dot{\alpha}}{}_{\dot{\beta}}$ respectively $\implies M_{ab}\Psi = \begin{pmatrix} (\sigma_{ab})_\alpha{}^\beta\psi_\beta \\ (\tilde{\sigma}_{ab})^{\dot{\alpha}}{}_{\dot{\beta}}\overline{\chi}^{\dot{\beta}} \end{pmatrix}$.
This gives the so-called $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representations of the Lie algebra, $\mathfrak{so}^\uparrow(3,1)$, namely $M_{ab}(\psi_\alpha) = (\sigma_{ab})_\alpha{}^\beta\psi_\beta$ and $M_{ab}(\overline{\chi}^{\dot{\alpha}}) = (\tilde{\sigma}_{ab})^{\dot{\alpha}}{}_{\dot{\beta}}\overline{\chi}^{\dot{\beta}}$ respectively. Furthermore, for σ_{ab} and $\tilde{\sigma}_{ab}$ to have the indices they do (in type and position), the spinor indices of the extended Pauli matrices must be $(\sigma_a)_{\alpha\dot{\alpha}}$ and $(\tilde{\sigma}^a)^{\dot{\alpha}\alpha}$. Finally, by direct evaluation, one finds

$$(\sigma_{ab})_\alpha{}^\beta \equiv \frac{1}{2}\begin{bmatrix} 0 & \sigma_1 & \sigma_2 & \sigma_3 \\ -\sigma_1 & 0 & \mathrm{i}\sigma_3 & -\mathrm{i}\sigma_2 \\ -\sigma_2 & -\mathrm{i}\sigma_3 & 0 & \mathrm{i}\sigma_1 \\ -\sigma_3 & \mathrm{i}\sigma_2 & -\mathrm{i}\sigma_1 & 0 \end{bmatrix} \text{ and } (\tilde{\sigma}_{ab})^{\dot{\alpha}}{}_{\dot{\beta}} \equiv \frac{1}{2}\begin{bmatrix} 0 & -\sigma_1 & -\sigma_2 & -\sigma_3 \\ \sigma_1 & 0 & \mathrm{i}\sigma_3 & -\mathrm{i}\sigma_2 \\ \sigma_2 & -\mathrm{i}\sigma_3 & 0 & \mathrm{i}\sigma_1 \\ \sigma_3 & \mathrm{i}\sigma_2 & -\mathrm{i}\sigma_1 & 0 \end{bmatrix} \tag{242}$$

This was all at the level of the Lie algebra. To get to the Lie group, one must use the exponential map. The universal covering group of $\mathrm{SO}^\uparrow(3,1)$ is $\mathrm{SL}(2,\mathbb{C})$ and thus the exponential map will generate representations of $\mathrm{SL}(2,\mathbb{C})$, not $\mathrm{SO}^\uparrow(3,1)$.
Let $I + M \in \mathrm{SL}(2,\mathbb{C})$ for infinitesimal M. Thus, $1 = \det(I+M) = 1 + \mathrm{tr}(M) \implies \mathrm{tr}(M) = 0$.
$\therefore$ Since the Pauli matrices are a basis for traceless 2×2 matrices, $\mathfrak{sl}(2,\mathbb{C}) = \{z_i\sigma_i | z_i \in \mathbb{C}^3\}$. However, that's the complex Lie algebra. To get the real Lie algebra, let

$$z_1 = \frac{1}{2}(K^{01} + \mathrm{i}K^{23}),\ z_2 = \frac{1}{2}(K^{02} + \mathrm{i}K^{31}) \text{ and } z_1 = \frac{1}{2}(K^{03} + \mathrm{i}K^{12}) \tag{243}$$

$$\implies z_i\sigma_i = \frac{1}{2}((K^{01} + \mathrm{i}K^{23})\sigma_1 + (K^{02} + \mathrm{i}K^{31})\sigma_2 + (K^{03} + \mathrm{i}K^{12})\sigma_3) \tag{244}$$

for $K^{ab} \in \mathbb{R}$. Not all the K^{ab} have been defined yet; that is most conveniently accomplished[10] by letting $K^{ab} = -K^{ba}$.

$$\therefore \frac{1}{2}K^{ab}\sigma_{ab} = K^{01}\sigma_{01} + K^{02}\sigma_{02} + K^{03}\sigma_{03} + K^{12}\sigma_{12} + K^{13}\sigma_{13} + K^{23}\sigma_{23} \tag{245}$$

$$= \frac{1}{2}(K^{01}\sigma_1 + K^{02}\sigma_2 + K^{03}\sigma_3 + K^{12}\mathrm{i}\sigma_3 - K^{13}\mathrm{i}\sigma_2 + K^{23}\mathrm{i}\sigma_1) \tag{246}$$

$$= \frac{1}{2}((K^{01} + \mathrm{i}K^{23})\sigma_1 + (K^{02} + \mathrm{i}K^{31})\sigma_2 + (K^{03} + \mathrm{i}K^{12})\sigma_3) \tag{247}$$

$$= z_i\sigma_i \tag{248}$$

[10] with the benefit of hindsight

$\therefore \mathfrak{sl}(2,\mathbb{C}) = \left\{\frac{1}{2}K^{ab}(\sigma_{ab})_\alpha{}^\beta \middle|\ K^{ab} = -K^{ba} \in \mathbb{R}\right\}$
$\therefore$ As $\mathrm{SL}(2,\mathbb{C})$ is simply connected, $\{N_\alpha{}^\beta = \mathrm{e}^{K^{ab}(\sigma_{ab})_\alpha{}^\beta/2} |\ K^{ab} = -K^{ba} \in \mathbb{R}\}$ is a dense subset of $\mathrm{SL}(2,\mathbb{C})$.

Via equation 96, I showed that under $\gamma'^a = \Lambda^a{}_b\gamma^b = T(\Lambda)^{-1}\gamma^a T(\Lambda)$, $\Psi'(x) = T(\Lambda)\Psi$. I commented that representation of the Lorentz group, $T(\Lambda)$, could be extended to a representation of the universal covering group. This is exactly what I'll do now using the exponential map. As the Lorentz generators when acting on 4-component spinors are M_{ab}, $T(N) = \mathrm{e}^{K^{ab}M_{ab}/2}$. The factor of a half is necessary in the exponential because $\mathfrak{so}^\uparrow(3,1)$ is only 6 dimensional, where as $K^{ab}M_{ab}$ double counts the 6 independent M_{ab} via $K^{ba}M_{ba} = (-K^{ab})(-M_{ab})$.

$$\therefore T(N) = \mathrm{e}^{K^{ab}M_{ab}/2} \tag{249}$$
$$= \mathrm{e}^{\frac{1}{2}K^{ab}\begin{bmatrix}(\sigma_{ab})_\alpha{}^\beta & 0\\ 0 & (\tilde{\sigma}_{ab})^{\dot\alpha}{}_{\dot\beta}\end{bmatrix}} \tag{250}$$
$$= \sum_{n=0}^\infty \frac{1}{n!}\left(\frac{K^{ab}}{2}\right)^n \begin{bmatrix}(\sigma_{ab})_\alpha{}^\beta & 0\\ 0 & (\tilde{\sigma}_{ab})^{\dot\alpha}{}_{\dot\beta}\end{bmatrix}^n \tag{251}$$
$$= \sum_{n=0}^\infty \frac{1}{n!}\left(\frac{K^{ab}}{2}\right)^n \begin{bmatrix}((\sigma_{ab})_\alpha{}^\beta)^n & 0\\ 0 & ((\tilde{\sigma}_{ab})^{\dot\alpha}{}_{\dot\beta})^n\end{bmatrix} \tag{252}$$
$$= \begin{bmatrix}\mathrm{e}^{K^{ab}(\sigma_{ab})_\alpha{}^\beta/2} & 0\\ 0 & \mathrm{e}^{K^{ab}(\tilde{\sigma}_{ab})^{\dot\alpha}{}_{\dot\beta}/2}\end{bmatrix} \tag{253}$$

I've already shown $\mathrm{e}^{K^{ab}(\sigma_{ab})_\alpha{}^\beta/2} = N_\alpha{}^\beta$. Let $M = \mathrm{e}^{K^{ab}(\tilde{\sigma}_{ab})^{\dot\alpha}{}_{\dot\beta}/2}$.

$$\frac{1}{2}K^{ab}\tilde{\sigma}_{ab} = K^{01}\tilde{\sigma}_{01} + K^{02}\tilde{\sigma}_{02} + K^{03}\tilde{\sigma}_{03} + K^{12}\tilde{\sigma}_{12} + K^{13}\tilde{\sigma}_{13} + K^{23}\tilde{\sigma}_{23} \tag{254}$$
$$= \frac{1}{2}(-K^{01}\sigma_1 - K^{02}\sigma_2 - K^{03}\sigma_3 + \mathrm{i}K^{12}\sigma_3 - \mathrm{i}K^{13}\sigma_2 + \mathrm{i}K^{23}\sigma_1) \tag{255}$$
$$= \frac{1}{2}((-K^{01} + \mathrm{i}K^{23})\sigma_1 + (-K^{02} + \mathrm{i}K^{31})\sigma_2 + (-K^{03} + \mathrm{i}K^{12})\sigma_3) \tag{256}$$
$$= -z_i^*\sigma_i \tag{257}$$
$$\therefore M = \mathrm{e}^{-z_i^*\sigma_i} \tag{258}$$
$$\therefore M^\dagger = \mathrm{e}^{-z_i\sigma_i^\dagger} = \mathrm{e}^{-z_i\sigma_i} = N^{-1} \iff M = N^{-\dagger} \tag{259}$$
$$\therefore T(N)\Psi = \mathrm{e}^{K^{ab}M_{ab}/2}\Psi \tag{260}$$
$$= \begin{bmatrix}N_\alpha{}^\beta & 0\\ 0 & (N^{-\dagger})^{\dot\alpha}{}_{\dot\beta}\end{bmatrix}\begin{bmatrix}\psi_\beta\\ \bar{\chi}^{\dot\beta}\end{bmatrix} \tag{261}$$
$$= \begin{bmatrix}N_\alpha{}^\beta\psi_\beta\\ (N^{-\dagger})^{\dot\alpha}{}_{\dot\beta}\bar{\chi}^{\dot\beta}\end{bmatrix} \tag{262}$$

$\therefore$ Under the $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ representations of $\mathrm{SL}(2,\mathbb{C})$, left and right handed Weyl spinors respectively transform as $\psi'_\alpha = N_\alpha{}^\beta\psi_\beta$ and $\bar{\chi}'^{\dot\alpha} = (N^{-\dagger})^{\dot\alpha}{}_{\dot\beta}\bar{\chi}^{\dot\beta} = \bar{\chi}^{\dot\beta}(N^{-*})_{\dot\beta}{}^{\dot\alpha}$. One subtlety of this result (in particular the block diagonal form of $\mathrm{e}^{K^{ab}M_{ab}/2}$) is that although the representation of the CLifford algebra is irreducible, the induced $\mathrm{SL}(2,\mathbb{C})$ representation is not. The latter's irreducible components are the spaces of left handed and right handed spinors.

Since $N \in \mathrm{SL}(2,\mathbb{C}) \implies \det(N) = 1$, $N_\alpha{}^\mu N_\beta{}^\nu\varepsilon_{\mu\nu} = \varepsilon_{\alpha\beta}$ and $\varepsilon^{\mu\nu}(N^{-1})_\mu{}^\alpha(N^{-1})_\nu{}^\beta = \varepsilon^{\alpha\beta}$

where $\varepsilon_{\alpha\beta}$ and $\varepsilon^{\alpha\beta}$ are antisymmetric tensors with $\varepsilon_{12} = -1$ and $\varepsilon^{12} = 1$.
$\therefore$ As $\varepsilon_{\alpha\beta}$ and $\varepsilon^{\alpha\beta}$ are invariant tensors of $\mathrm{SL}(2,\mathbb{C})$ and $\varepsilon^{\alpha\gamma}\varepsilon_{\gamma\beta} = \delta^{\alpha}{}_{\beta}$, they can be used to raise and lower indices.
$\therefore \overline{\chi}'_{\dot{\alpha}} = \varepsilon_{\dot{\alpha}\dot{\beta}}\overline{\chi}'^{\dot{\beta}} = \varepsilon_{\dot{\alpha}\dot{\beta}}\overline{\chi}^{\dot{\gamma}}(N^{-*})_{\dot{\gamma}}{}^{\dot{\beta}}$.
$N_{\alpha}{}^{\mu}N_{\beta}{}^{\nu}\varepsilon_{\mu\nu} = \varepsilon_{\alpha\beta} \iff \varepsilon = N\varepsilon N^T$ in matrix notation.
$\therefore N^{-1}\varepsilon = \varepsilon N^T \implies -N^{-1}\varepsilon = -\varepsilon N^T \implies N^{-1}\varepsilon^T = \varepsilon^T N^T \implies \varepsilon_{\dot{\alpha}\dot{\beta}}(N^{-*})_{\dot{\gamma}}{}^{\dot{\beta}} = \varepsilon_{\dot{\beta}\dot{\gamma}}(N^*)_{\dot{\alpha}}{}^{\dot{\beta}}$.
$\therefore \overline{\chi}'_{\dot{\alpha}} = \varepsilon_{\dot{\alpha}\dot{\beta}}\overline{\chi}^{\dot{\gamma}}(N^{-*})_{\dot{\gamma}}{}^{\dot{\beta}} = \varepsilon_{\dot{\beta}\dot{\gamma}}(N^*)_{\dot{\alpha}}{}^{\dot{\beta}}\overline{\chi}^{\dot{\gamma}} = (N^*)_{\dot{\alpha}}{}^{\dot{\beta}}\overline{\chi}_{\dot{\beta}}$
Similarly, raising the index on the left handed spinor, $\psi'^{\alpha} = \varepsilon^{\alpha\beta}\psi'_{\beta} = \varepsilon^{\alpha\beta}N_{\beta}{}^{\gamma}\psi_{\gamma}$.
$\varepsilon^{\mu\nu}(N^{-1})_{\mu}{}^{\alpha}(N^{-1})_{\nu}{}^{\beta} = \varepsilon^{\alpha\beta} \implies \varepsilon = N^{-T}\varepsilon N^{-1} \implies \varepsilon N = N^{-T}\varepsilon \implies \varepsilon^{\alpha\beta}N_{\beta}{}^{\gamma} = (N^{-1})_{\beta}{}^{\alpha}\varepsilon^{\beta\gamma}$.
$\therefore \psi'^{\alpha} = \varepsilon^{\alpha\beta}N_{\beta}{}^{\gamma}\psi_{\gamma} = (N^{-1})_{\beta}{}^{\alpha}\varepsilon^{\beta\gamma}\psi_{\gamma} = \psi^{\beta}(N^{-1})_{\beta}{}^{\alpha}$.
Having established these transformation properties, one can noew develop the 2-component spinor formalism via tensor products, index raising/lowering etc. like for other tensor types.

The 2-component spinor formalism was based on writing the full spinor space as a direct sum of left handed and right handed Weyl spinors. However, I also spent many pages earlier considering Majorana spinors and it would be incomplete of me not not consider them in the special case of $s-t = 3-1 = 2$ where (by table 1) they do exist.
By definition 1.6 and theorem 1.7, a 4-component spinor is Majorana if and only if $\Psi = \nu\mu^t(-1)^{t(t-1)/2}C^{-1}\overline{\Psi}^T = 1\times(-1)^1(-1)^{1\times 0/2}C^{-1}\overline{\Psi}^T = -C^{-1}\overline{\Psi}^T \implies \overline{\Psi}^T = -C\Psi$.
It suffices to guess C by forcing it to satisfy theorem 1.5 and equation 172. With $\nu = 1,\ \mu = -1$ and $t = 1$, they say $C^\dagger C = I$, $\gamma_a^T = -C\gamma_a C^{-1}$ and $C^T = -C$. Guided by the antisymmetry and the block diagonal nature of the Weyl representation, try

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{263}$$

$$\therefore C^\dagger C = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I \tag{264}$$

$\therefore C^{-1} = -C$ by the previous line and thus $-C\gamma_a C^{-1} = C\gamma_a C$. Also, C can also be written slightly more compactly as

$$C = \begin{bmatrix} \varepsilon & 0 \\ 0 & -\varepsilon \end{bmatrix} \text{ where } \varepsilon = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{265}$$

$$\therefore -C\gamma_a C^{-1} = \begin{bmatrix} \varepsilon & 0 \\ 0 & -\varepsilon \end{bmatrix}\begin{bmatrix} 0 & \sigma_a \\ \tilde{\sigma}_a & 0 \end{bmatrix}\begin{bmatrix} \varepsilon & 0 \\ 0 & -\varepsilon \end{bmatrix} = \begin{bmatrix} 0 & -\varepsilon\sigma_a\varepsilon \\ -\varepsilon\tilde{\sigma}_a\varepsilon & 0 \end{bmatrix} \tag{266}$$

$$\varepsilon\sigma_0\varepsilon = \varepsilon I\varepsilon = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -\sigma_0^T \tag{267}$$

$$\varepsilon\sigma_1\varepsilon = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \sigma_1^T \tag{268}$$

$$\varepsilon\sigma_2\varepsilon = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \mathrm{i} & 0 \\ 0 & \mathrm{i} \end{bmatrix} = \begin{bmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{bmatrix} = \sigma_2^T \tag{269}$$

$$\varepsilon\sigma_3\varepsilon = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \sigma_3^T \tag{270}$$

$$\therefore -C\gamma_a C^{-1} = \begin{bmatrix} 0 & \tilde{\sigma}_a^T \\ \sigma_a^T & 0 \end{bmatrix} = \gamma_a^T \text{ since } \tilde{\sigma}_a = (I, -\sigma_i) \tag{271}$$

$\therefore$ The chosen matrix for C can indeed be used as the charge conjugation matrix.

$$\therefore -C\Psi = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -x \\ w \\ z \\ -y \end{bmatrix} \tag{272}$$

$$\text{Meanwhile, } \overline{\Psi}^T = (\Psi^\dagger A)^T \tag{273}$$

$$= A^T \Psi^* \tag{274}$$

$$= \gamma_0^T \Psi^* \tag{275}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} w^* \\ x^* \\ y^* \\ z^* \end{bmatrix} \tag{276}$$

$$= \begin{bmatrix} y^* \\ z^* \\ w^* \\ x^* \end{bmatrix} \tag{277}$$

$$\therefore -C\Psi = \overline{\Psi}^T \implies \Psi = \begin{bmatrix} w \\ x \\ -x^* \\ w^* \end{bmatrix} \tag{278}$$

In the 2-component spinor notation, $\begin{pmatrix} w \\ x \end{pmatrix}$ would be denoted as ψ_α.

$$\therefore \psi^\alpha = \varepsilon^{\alpha\beta}\psi_\beta \equiv \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} w \\ x \end{bmatrix} = \begin{bmatrix} -x \\ w \end{bmatrix} \implies \begin{bmatrix} -x^* \\ w^* \end{bmatrix} = (\psi^\alpha)^* \tag{279}$$

Conjugation swaps dotted and undotted spinor indices since $\psi'_\alpha = N_\alpha{}^\beta \psi_\beta$
$\implies (\psi'_\alpha)^* = (N^*)_\alpha{}^\beta (\psi_\beta)^*$ (and likewise for conjugating an initially dotted spinor) which is the transformation of right handed Weyl spinor as shown earlier. For this reason, $(\psi_\alpha)^*$ can be denoted as $\bar{\psi}_{\dot{\alpha}}$.
$\therefore$ The most general Majorana spinor for $s = 3$ and $t = 1$ is $\Psi = \begin{pmatrix} \psi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix}$.

Finally, it's worth checking that despite appearances, spinor representations are not the same as vector representations. It is often remarked (e.g. by quoting Michael Atiyah) that spinors are like the square root of a vector. That is because of arguments like the one below.
Let $\delta^a{}_b + X^a{}_b \in \mathrm{SO}^\uparrow(3,1)$ for infinitesimal $X^a{}_b$.
$\therefore 1 = \det(\delta^a{}_b + X^a{}_b) = 1 + \mathrm{tr}(X) \implies X^a{}_a = 0$.
Also, $\eta_{ab} = \eta_{cd}(\delta^c{}_a + X^c{}_a)(\delta^d{}_b + X^d{}_b) = \eta_{ab} + X_{ba} + X_{ab} \implies X_{ba} = -X_{ab}$. Antisymmetry automatically implies tracelessness; thus $\mathfrak{so}^\uparrow(3,1)$ consists of all 4×4 antisymmetric matrices.
$\therefore \Lambda = \mathrm{e}^{K^{ab}S_{ab}/2} \in \mathrm{SO}^\uparrow(3,1)$ where S_{ab} is a basis (with 6 independent elements) for 4×4 antisymmetric matrices. The corresponding group action on 4-component spinors is

$T(\Lambda) \equiv T(N) = \mathrm{e}^{K^{ab}M_{ab}/2}$. The standard basis for 4×4 antisymmetric matrices is

$$S_{ab} \equiv \left\{ \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \right\} \tag{280}$$

It can be checked that S_{ab} satisfies the Lie algebra generator commutation relations for $\mathfrak{so}^{\uparrow}(3,1)$. By Rodrigues' formula and other related identities, if (n_x, n_y, n_z) is a unit vector of $\mathbb{R}^3$, then $\mathrm{e}^{\theta A}$, where

$$A = \begin{bmatrix} 0 & n_z & -n_y \\ -n_z & 0 & n_x \\ n_y & -n_x & 0 \end{bmatrix}, \tag{281}$$

is a rotation of θ about $\vec{n}$. A can be represented in term of 4×4 matrices via

$$A = n_z \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - n_y \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} + n_x \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \tag{282}$$

$$= n_z S_{12} - n_y S_{13} + n_x S_{23} \tag{283}$$

and thus $\mathrm{e}^{\theta A} = \mathrm{e}^{\theta(n_z S_{12} - n_y S_{13} + n_x S_{23})} \in \mathrm{SO}^{\uparrow}(3,1)$. The corresponding representation on spinor space is

$$T(N) = \mathrm{e}^{\theta(n_z M_{12} - n_y M_{13} + n_x M_{23})} \tag{284}$$

$$= \mathrm{e}^{\theta\left(n_z \begin{bmatrix} \sigma_{12} & 0 \\ 0 & \tilde{\sigma}_{12} \end{bmatrix} - n_y \begin{bmatrix} \sigma_{13} & 0 \\ 0 & \tilde{\sigma}_{13} \end{bmatrix} + n_x \begin{bmatrix} \sigma_{23} & 0 \\ 0 & \tilde{\sigma}_{23} \end{bmatrix}\right)} \tag{285}$$

$$= \mathrm{e}^{\frac{\mathrm{i}\theta}{2} \begin{bmatrix} n_x\sigma_1 + n_y\sigma_2 + n_z\sigma_3 & 0 \\ 0 & n_x\sigma_1 + n_y\sigma_2 + n_z\sigma_3 \end{bmatrix}} \tag{286}$$

$$= \begin{bmatrix} \mathrm{e}^{\mathrm{i}\theta\vec{n}\cdot\vec{\sigma}/2} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\theta\vec{n}\cdot\vec{\sigma}/2} \end{bmatrix} \tag{287}$$

$$(\vec{n}\cdot\vec{\sigma})^2 = \begin{bmatrix} n^z & n_x - \mathrm{i}n_y \\ n_x + \mathrm{i}n_y & -n_z \end{bmatrix} \begin{bmatrix} n^z & n_x - \mathrm{i}n_y \\ n_x + \mathrm{i}n_y & -n_z \end{bmatrix} \tag{288}$$

$$= \begin{bmatrix} n_z^2 + n_x^2 + n_y^2 & 0 \\ 0 & n_x^2 + n_y^2 + n_z^2 \end{bmatrix} \tag{289}$$

$$= I \quad \text{as } ||\vec{n}|| = 1 \tag{290}$$

$$\therefore \mathrm{e}^{\mathrm{i}\theta\vec{n}\cdot\vec{\sigma}/2} = \sum_{m=0}^{\infty} \frac{1}{m!}\left(\frac{\mathrm{i}\theta}{2}\right)^m (\vec{n}\cdot\vec{\sigma})^m \tag{291}$$

$$= I \sum_{m=0}^{\infty} \frac{1}{(2m)!}\left(\frac{\mathrm{i}\theta}{2}\right)^{2m} + (\vec{n}\cdot\vec{\sigma}) \sum_{m=0}^{\infty} \frac{1}{(2m+1)!}\left(\frac{\mathrm{i}\theta}{2}\right)^{2m+1} \tag{292}$$

$$= \cos(\theta/2) I + \mathrm{i}\sin(\theta/2)\vec{n}\cdot\vec{\sigma} \tag{293}$$

Notice that a rotation of θ has lead to a rotation of only $\theta/2$ in the cos and sin terms acting on spinor space.
e.g. Let $\theta = 2\pi \implies \Lambda = \mathrm{e}^{\theta A} = I$ as a 2π rotation does nothing. However,

$$T(N) = \begin{bmatrix} \cos(\pi)I + \mathrm{i}\sin(\pi)\vec{n}\cdot\vec{\sigma} & 0 \\ 0 & \cos(\pi)I + \mathrm{i}\sin(\pi)\vec{n}\cdot\vec{\sigma} \end{bmatrix} \tag{294}$$

$$= -I \tag{295}$$

$\therefore T(N)\Psi = -\Psi$ under a 2π rotation.
$\therefore$ One needs to do a full 2π rotation twice to return the spinor, Ψ, to its original state.
$\therefore$ The spinor representation really is different to the vector representation. This essentially reflects the fact that the spinor is transforming under $\mathrm{SL}(2,\mathbb{C})$, not $\mathrm{SO}^{\uparrow}(3,1)$. The $\mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2 \cong \mathrm{SO}^{\uparrow}(3,1)$ isomorphism means N and $-N$ both correspond to the same Lorentz transformation, Λ. That is why $\Lambda = I$ can still lead to $T(N) = -I$; the two are related by the $\mathbb{Z}_2$ quotienting.

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